Thermodynamics of Lattice Gauge Theory

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The thermodynamics of color theory is:

- based on a theory with gauge symmetry over the non-abelian group $SU(N_c)$,
- "asymptotic freedom", the gluons behave as a gas of free particles at high energy,
- "confinement" at low energy, $\alpha_s(q^2)$ increases to larger and larger values and a single isolated color charge cannot exist.

**Failure of perturbative series**

Due to the confinement and $\alpha_s \sim 1$, any thermodynamic quantity must be studied with non-perturbative methods.
How does the quantum chromodynamic reach the Stefan-Boltzmann limit?

1. Recent results have shown that there isn’t a gas of free gluons also when $T \sim 3T_c$. [Datta, Gupta: ArXiv:1006.0938, 12/2010]
2. Perturbative analytical previsions are difficult.
3. Non perturbative string models assume that $N_c = 3$ is “similar” to $N_c = \infty$; this hypothesis can be verified by lattice gauge theories.
The lattice spacing \( a \neq 0 \) regularizes the gauge theory.

On lattice, bosonic fields are site variable; gauge fields are link variables:

\[
\begin{align*}
\phi(x) & \rightarrow \phi_i \\
A_\mu(x) & \rightarrow \exp(ig_s a A_\mu(x)) \equiv U_\mu(x)
\end{align*}
\]

The Wilson Loop:

\[
W = \text{Tr} \left( \prod_{x \in C} U_\mu(x) \right)
\]

is invariant under gauge transformations \( \Omega(x) \) of the field \( U_\mu(x) \):

\[
U_\mu(x) \rightarrow \Omega(x) U_\mu(x) \Omega^\dagger(x + \mu)
\]
Wilson Action
Yang-Mills action on lattice

The action can be written in the simplest way using only the "plaquette", the smallest Wilson loop:

$$S_W = -\frac{\beta}{N} \sum_{x, \mu \neq \nu} \text{Re} \text{Tr} \left( U_\mu(x) U_\nu(x + \mu) U_\mu^\dagger(x + \nu) U_{-\nu}(x + \nu) \right)$$


When $a \rightarrow 0$, in the "naive" continuum limit, $S_W$ tends to the usual Yang-Mills action:

$$S_W \rightarrow \left\{-\frac{1}{4} \int d^4x E F^{\mu\nu} F_{\mu\nu} \right\} \left(1 + O(a^2)\right)$$

if $\beta \equiv \frac{2N}{g_s^2}$. 
The partition function $Z$ of a generic quantum system with Hamiltonian $\hat{H}$ is:

$$Z = \text{Tr} \left\{ \exp \left( -\frac{\hat{H}}{T} \right) \right\} = \sum_i \exp \left( -\frac{E_i}{T} \right)$$

The partition function is equivalent to the Feynman functional, but with the temporal direction compactified:

$$Z = \int d\phi \exp \left( -\beta \int_0^{1/T} dt d^3x L(\phi) \right)$$

**Warning!**

In the exponential $\beta$ is related to $g_s$, the temperature is related to the length of the temporal direction $L_t = 1/T$!
The expectation value of any observable $\mathcal{O}$:

$$
\langle \mathcal{O} \rangle = \frac{\int \prod_{\mu,x} dU_{\mu}(x) \{ \mathcal{O} \exp (-S) \}}{\int \prod_{\mu,x} dU_{\mu}(x) \{ \exp (-S) \}}
$$

can be easily computed on a set of gauge configuration $\Phi_i$ generated with the Monte-Carlo sampling:

$$
\langle \mathcal{O} \rangle = \frac{1}{N_{\text{conf}}} \sum \mathcal{O}(\Phi_i)
$$

and a Markov chain can be easily defined due to the locality of the pure gauge action.

The Polyakov Loop
The order parameter of the transition of deconfinement

In a finite temperature lattice a Wilson line can be wrapped around the compactified temporal direction:

\[ L(\vec{x}) = \text{Tr} \left( \prod_{t=0}^{N_T-1} U_0(\vec{x}, t) \right) \]

This operator is called Polyakov Loop [Phys. Lett B 72: 477, 1978] and it is the order parameter of the deconfinement transition.

Confinement [L. G. Yaffe, B. Svetitsky, Physical Review D 26:963, 1982]
If \( \langle |L| \rangle \neq 0 \), then the lattice is in the deconfined phase, otherwise in those confined.
The deconfinement transition can be found on the peak of the susceptibility of the Polyakov Loop $\chi_L$:

$$\frac{\chi_L}{N_s^3} = (\langle |L|^2 \rangle - \langle |L| \rangle^2)$$
The deconfinement transition for a gauge group $SU(4)$ in $4D$ is of the first order, coexistence of many phases:

Latent heat differs from zero: $L_h/T_c^4 \sim 7.6!$
When $N_t$ increases, the peak of $\chi_L$ changes position:

Susceptivity $\chi = (\langle|L|^2\rangle - \langle|L|^2\rangle^2)$

lattice: $12^3 \times 4$

Susceptivity $\chi = (\langle|L|^2\rangle - \langle|L|^2\rangle^2)$

lattice: $15^3 \times 5$

Why? How is “$T_C$” defined?
The results of simulations with the Wilson action show that $\beta_c$ is independent from $N_s$, but not clearly from $N_t$:

<table>
<thead>
<tr>
<th>$N_t$</th>
<th>$N_s$</th>
<th>Number of “sweeps”</th>
<th>Critical value $\beta_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>12</td>
<td>217500</td>
<td>10.486(5)</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>517500</td>
<td>10.4875(25)</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>379500</td>
<td>10.490(5)</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>144000</td>
<td>10.6352(3)</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
<td>264000</td>
<td>10.6352(3)</td>
</tr>
<tr>
<td>6</td>
<td>18</td>
<td>170000</td>
<td>10.7816(33)</td>
</tr>
<tr>
<td>7</td>
<td>21</td>
<td>64000</td>
<td>10.92(2)</td>
</tr>
</tbody>
</table>
The temperature of the lattice is:

\[ T = \frac{1}{aN_t} \]

The “scale fixing” is needed for knowing the lattice spacing \( a \). Starting from the knowledge of the couples \( (N_t, \beta_c) \):

\[ a(\beta_c) = \frac{1}{T_c N_t} \]

\( T_c \approx 260 \text{ MeV} \)

\( a(\beta) \) is obtained extrapolating the global behavior with a fit.


The scale change when the gauge coupling change too, so the quantum fluctuations break the conformal symmetry.
The data are well fitted by an exponential ($\chi_R = 0.23$):

\begin{align*}
\text{Non-perturbative scale evolution}
\end{align*}

\begin{align*}
\text{The fit gives the relation between } T \text{ and } \beta (\bar{\beta} = \beta - 10.71): \\
T = \frac{T_c}{N_t} \exp \left( c_0 + c_1 \bar{\beta} + c_2 \bar{\beta}^2 \right)
\end{align*}

where $c_0 = 1.707397(2), c_1 = 1.2491(4) \text{ e } c_2 = -0.81(1)$. 
The Wilson action can be “improved” by adding irrelevant operators (loop larger than plaquette) for increasing the convergence to the continuum limit and the symmetries of operators.


$$ S_{SY} = -\frac{\beta}{N_c} \text{ReTr} \sum_{\mu \neq \nu} \left( \frac{5}{3} U_{\mu\nu} - \frac{1}{12} R_{\mu\nu} \right) $$

$$ \rightarrow \left\{ -\frac{1}{4} \int d^4 x \, \varepsilon \, F_{\mu\nu} F_{\mu\nu} \right\} (1 + O(a^4)) $$

has relative discretization errors of order $O(a^4)$. 
The scaling for $\beta_c$ with the improved Symanzik action is better and it reaches faster the perturbative value:

$$ f(N_t) = \frac{\log(N_t+1) - \log(N_t)}{\beta_c(N_t+1) - \beta_c(N_t)} \xrightarrow{N_t \to \infty} \frac{48\pi^2}{44N_c^2} $$
The trace of the energy-momentum tensor $\Delta = (\epsilon - 3p)/T^4$ has a peak near the critical point ($\sim 1.04T_c$) and it slowly goes to zero:

![Graph of the trace of the energy-momentum tensor $\Delta/T^4$]
The trace of the energy-momentum tensor has small differences between $SU(4)$ and $SU(5)$:

Trace of the energy–momentum tensor $\Delta/(d_A T^4)$

- $SU(4)$ – lattice: $5 \times 15^3$
- $SU(5)$ – lattice: $4 \times 12^3$
The trace of the energy-momentum tensor has small differences between $SU(5)$ and $SU(6)$:

<table>
<thead>
<tr>
<th>$N_c$</th>
<th>$T_c$</th>
<th>Trace of the energy-momentum tensor $\Delta/(d_A T^4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$SU(5)$</td>
<td>1.2</td>
<td>0.88</td>
</tr>
<tr>
<td>$SU(6)$</td>
<td>1.61</td>
<td>0.86</td>
</tr>
</tbody>
</table>

The limit $N_c \to \infty$ exists!
The trace of the energy-momentum tensor

Open questions

Δ has many deviation from perturbative estimations. Why?

Trace of the energy–momentum tensor $\Delta/(d_A T^4)$

SU(4) – lattice: $5 \times 15^3$

$\sim \log(T/T_c)$ fails when $T > T_c$. [J. O. Andersen et al., hep-ph:1106-0514 (2011)]
\[ \Delta \text{ has many deviation from perturbative estimations. Why?} \]

Trace of the energy–momentum tensor \( \Delta/(d_A T^4) \)

SU(5) – lattice: \( 4 \times 16^3 \)

\[ T^2 \text{ contribution when } T > T_c. \quad [R. D. Pisarski, hep-ph:0612191 (2006)] \]
Conclusions
Thermodynamics of lattice gauge theory

1. Lattice simulations are an important non-perturbative tool
2. Lattice discretization errors can be “easily” reduced with improved actions
3. Answers:
   1. Phase transition of the same order when \( N_c \geq 3 \)
   2. No differences in thermodynamic variables for \( T > T_c \)
   3. Small linear differences for \( T \sim T_c \)
   4. The limit \( N_c \to 3^+ \) exists!
4. Questions:
   1. How does a color theory reach the Stephan-Boltzmann limit?
   2. Non perturbative effects are still present also after the deconfinement transition
   3. Where does the related mass scale come from?