Westfälische Wilhelms-Universität Münster

Introduction to mass determination and variational smearing

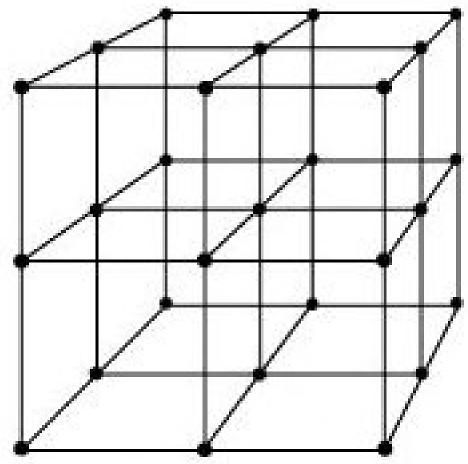
using the example of the 0^{++} glueball

wissen.leben WWU Münster

Dirk Sandbrink

Plan of the talk

- Introduction
- Measurements
  - Monte Carlo methods
  - Correlation function
- Improvements
  - Smearing methods
  - Variational Smearing
- Conclusions

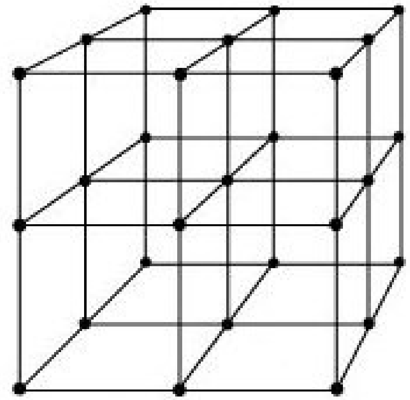


Ising spin system

- model of ferromagnetism
- 4D lattice, spin variables on its sites with values +1 or -1
- 2^{N^4} spin configurations *{s}*

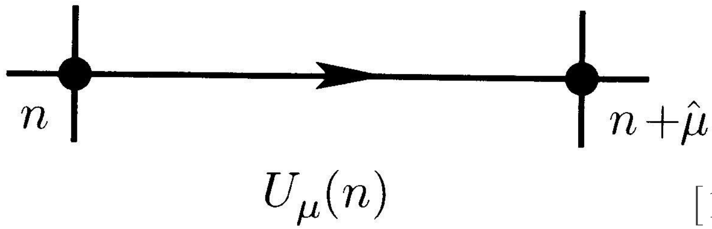
$$\langle O \rangle = \frac{1}{Z} \sum_{\{s\}} e^{-\beta H[s]} O[s] \quad with \quad Z = \sum_{\{s\}} e^{-\beta H[s]}$$

- prolem: e.g. $N = 16 \rightarrow 2^{65536} \approx 10^{19728}$
- solution: approximate by only a subset of configurations

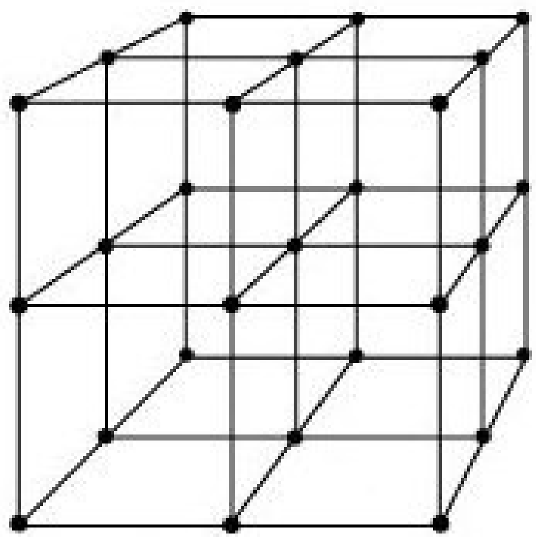


The gauge field on the lattice

- gauge fields $U_\mu(n)$ as link variables



[1]



Pure gauge field theory

- vacuum expectation value of an observable on the lattice

$$\langle O\rangle=\frac{1}{Z}\int D[U]e^{-S_G[U]}O[U] \quad \textit{with} \quad Z=\int D[U]e^{-S_G[U]}$$

- integration measure $\int D[U]=\prod_{n\in\Lambda}\prod_{\mu=1}^{4}\int dU_{\mu}(n)$

- Monte Carlo algorithm: approximate integral by an average of the observable evaluated on *N* sample gauge field configurations

$$\langle O\rangle\approx\frac{1}{N}\sum_{U_n}O[U_n] \quad \textit{with} \quad U_n\propto e^{-S[U_n]}$$

Monte Carlo method and importance sampling

- create a subset of *N* configurations
- uncertainty behaves like $O(1/\sqrt{N})$

- in the path integral the Boltzmann factor $\exp(-S)$ gives different importance to different field configurations
- choose gauge field configurations U_n according to the Gibbs measure

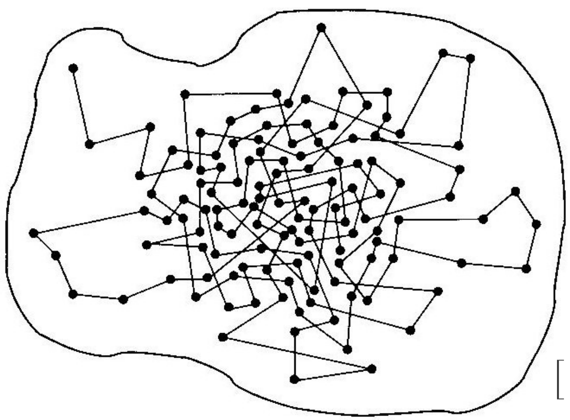
$$dP(U)=\frac{e^{-S[U]}D[U]}{\int D[U]e^{-S[U]}}$$

Markov chain

- start from some arbitrary configuration U_0
- construct a stochastic sequence of configurations

$$U_0 \rightarrow U_1 \rightarrow U_2 \rightarrow \ldots$$

- they have to follow the equilibrium distribution *P(U)*



[1]

Blocking and jackknifing

- configurations are not totally independent
- **blocking:** divide the data into sub-blocks of size *K*, compute mean value for each block, and take them as new variables

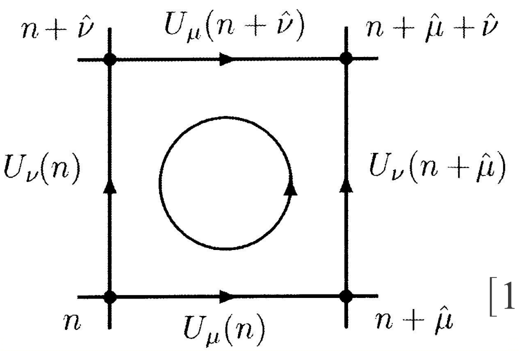
$$\underbrace{U_0 \quad U_1 \quad U_2}_{B_0} \quad \underbrace{U_3 \quad U_4 \quad U_5}_{B_1} \quad U_6 \quad \cdots$$

- **jackknifing:** *N* configurations, observable θ with mean value $\langle\theta\rangle$
  - construct *N* subsets by removing the *n*th element $\rightarrow \theta_n$

$$\sigma_\theta^2 = \frac{N-1}{N} \sum_{n=1}^{N} \left(\theta_n - \langle\theta\rangle\right)^2$$

0^{++} glueball interpolation operator

- bound states of gluons
- not yet observed in experiments (expect mass 1.7 GeV - 2.6 GeV)
- glueballs $J^{PC}=0^{++}$ and 0^{-+} are expected to occur in the lower supermultiplet of the low-energy spectrum of the SU(2) SYM
- interpolation operator $$O_{0^{++}}(U;x)=Tr[U_{12}+U_{23}+U_{31}]$$



[1]

Correlation functions

- correlation function of an interpolating operator

$$C(x,y)=\langle O^{+}(x)O(y)\rangle$$

- fix momentum via Fourier transformation of the spatial dimensions

$$C(x^0,p)=\frac{1}{L^3}\sum_x C(x,0)e^{ipx}$$

- go to rest frame by using the zero-momentum time-slice operator

$$S_t=\frac{1}{L^{3/2}}\sum_x O(x,t)$$

- correlation function at time-seperation $\Delta t=x^0-y^0$

$$C(\Delta t)=\langle S^{+}_{t+\Delta t}S_t\rangle$$

Spectral decomposition of the correlator

- inserting a complete set if energy-eigenstates with zero-momentum leads to

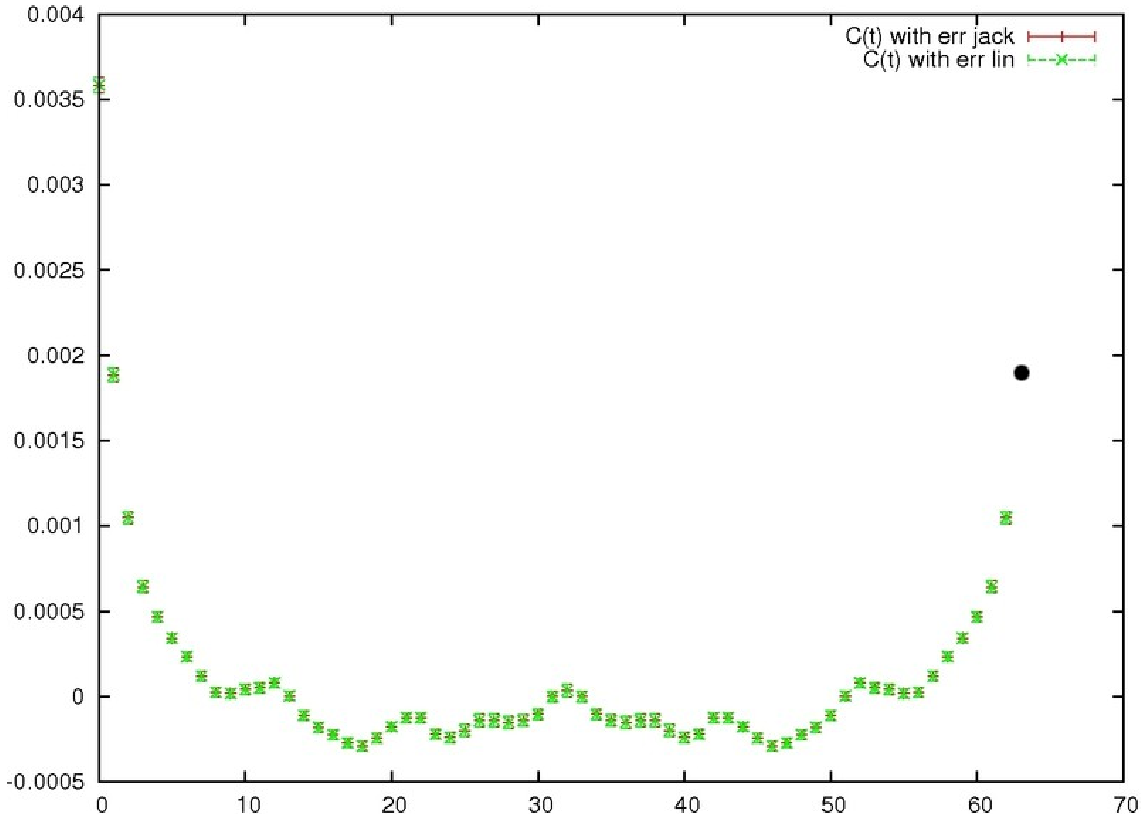
$$C(\Delta t)=\sum_{n=0}|\langle n|S_t|0\rangle|^2 e^{-E_n t}\pm|\langle 0|S_t^+|n\rangle|^2 e^{-E_n(T-\Delta t)}$$

$$C(\Delta t)=a_0^2+\sum_{n=1} a_n^2 e^{-E_n t}\pm a_n^2 e^{-E_n(T-\Delta t)}$$

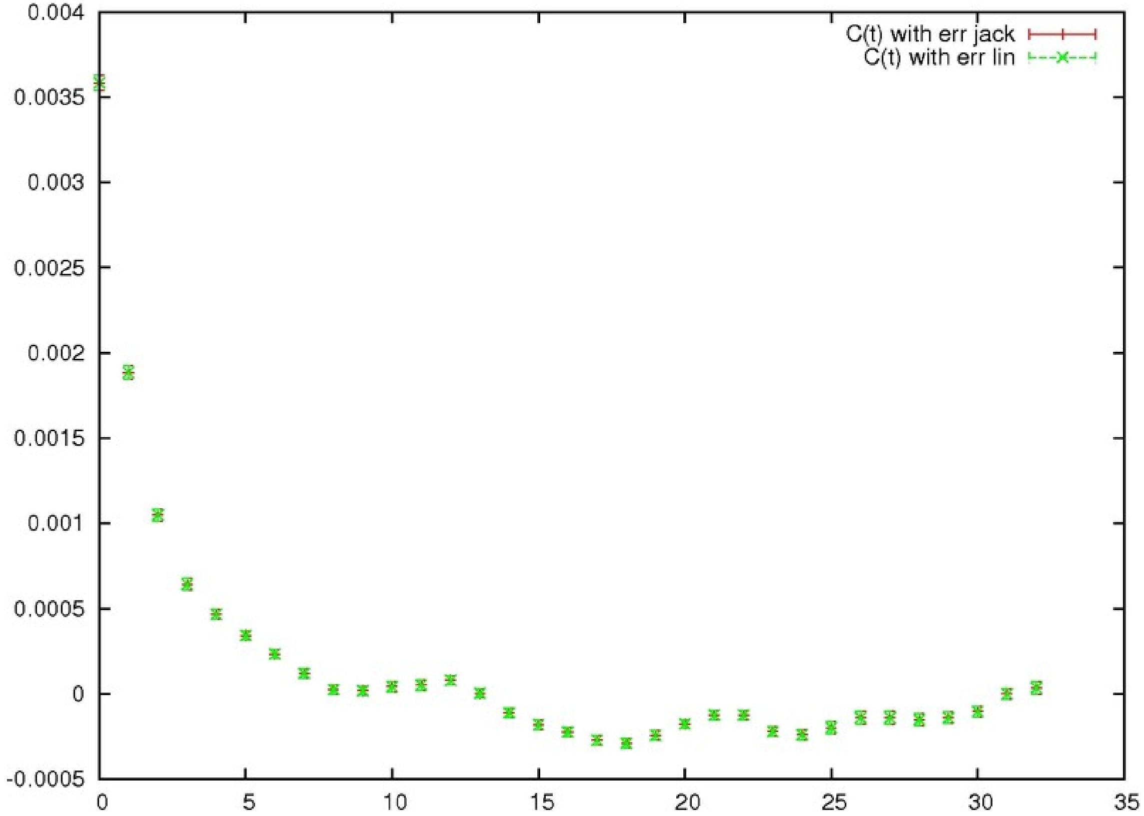
- the lightest bound state (n=1) dominates the correlator
- remove constant a_0 by substrating the vacuum expectation value

$$S_t\rightarrow\tilde{S}_t=S_t-\langle S_t\rangle_U$$

Correlation function



Symmetrised correlation function



Effective mass

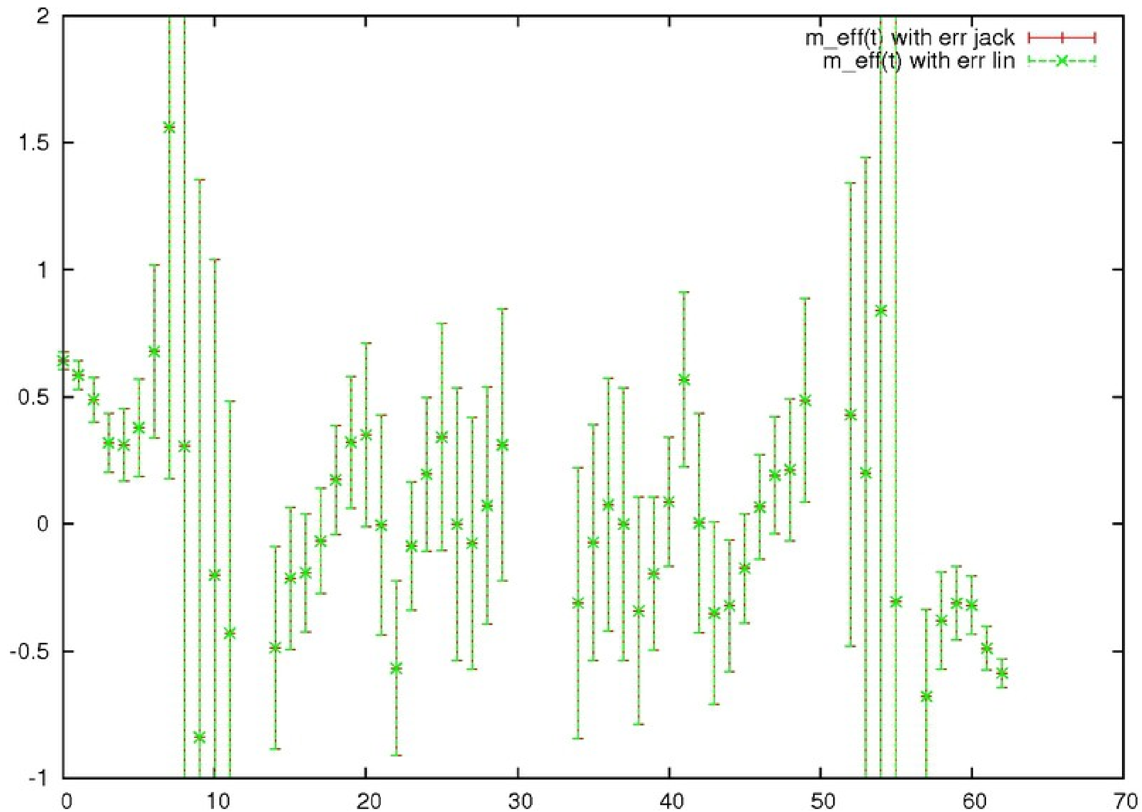
- depends only on pairs of time-slices $$m_{eff}(t) = \ln \frac{C(t)}{C(t+1)}$$

$$m_{eff}(t) = \ln \frac{C(t)}{C(t+1)} \approx \ln \frac{a * e^{-mt}}{a * e^{-m(t+1)}} = \ln e^{-mt+m(t+1)} = m$$

- shows a plateau where the lowest mass term dominates
- rough guide for choosing the fitting range

Effective Mass

$$m_{eff}(t)=\ln\frac{C(t)}{C(t+1)}$$



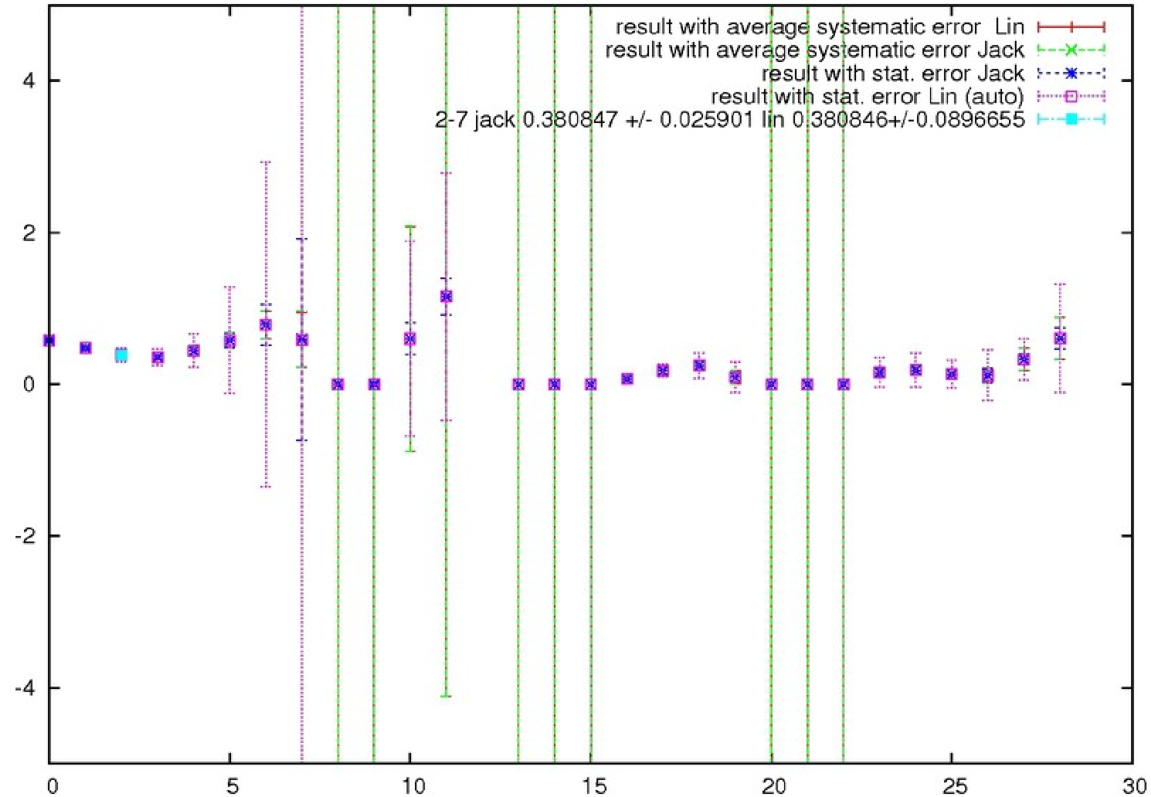
Mass fit

- fit the two-point correlation function to

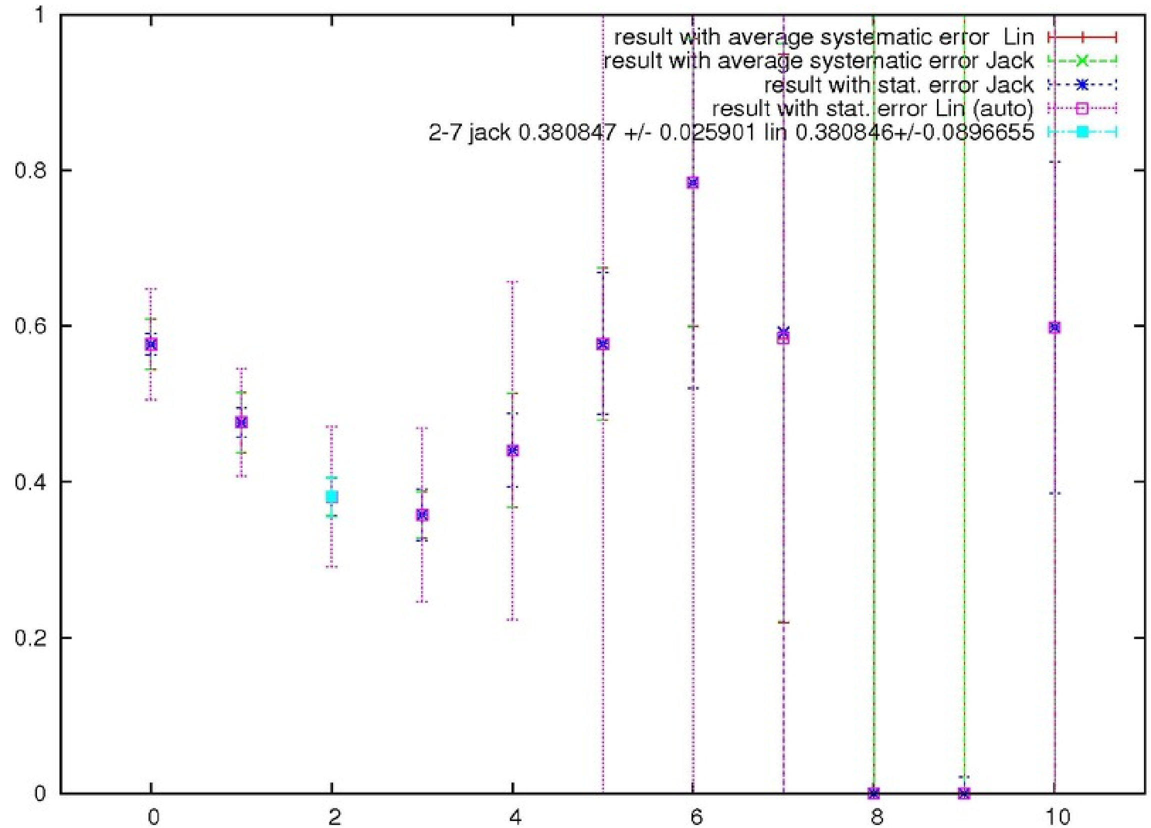
$$C(\Delta t)=a_1^2\left(e^{-m_1 \Delta t}+e^{-m_1(T-\Delta t)}\right)$$

- choose appropriate fitting interval $[t_{min}, t_{max}]$
- t_{min} must be high enough to ensure dominance of the lowest mass m_1

Mass fit


result with average systematic error Lin
result with average systematic error Jack
result with stat. error Jack
result with stat. error Lin (auto)
2-7 jack 0.380847 +/- 0.025901 lin 0.380846+/-0.0896655
4
2
0
-2
-4
0
5
10
15
20
25
30

Mass fit


result with average systematic error Lin
result with average systematic error Jack
result with stat. error Jack
result with stat. error Lin (auto)
2-7 jack 0.380847 +/- 0.025901 lin 0.380846+/-0.0896655
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10

Smearing methods

- typically one replaces the link variables by local averages over short paths connecting the link's endpoints
- goals:
    - improve overlap of an lattice operator with the physical state
    - better signal-to-noise ratio

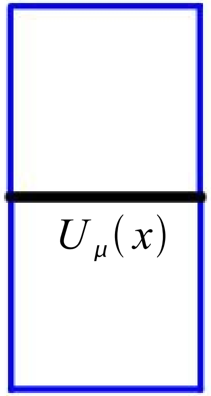
APE Smearing

- substitute a link by itself and the space-like staples surrounding it

$$U_\mu{}'(x) = U_\mu(x) + \epsilon_{ape} \sum_{\nu=\pm1,\,\nu\neq\mu}^{\pm3} U_\nu^+(x+\hat{\mu})\, U_\mu(x+\hat{\nu})\, U_\nu^+(x)$$

- repeat N_{ape} times

- choose N_{ape} and ε_{ape} carefully



Motivation for variational smearing

- increase overlap of the operator with the state
- better result for the mass
- separate excited states

Basic concept of variational smearing

- use several different interpolation operators $O_i, i=1,\ldots N$
- compute all possible cross correlation functions

$$C_{ij}(t)=\langle O_i(t)\,O_J(0)\rangle$$

- in Hilbert space these correlators have the spectral decomposition

$$C_{ij}(t)=\sum_n \langle 0|O_i|n\rangle\langle n|O_j^{+}|0\rangle\, e^{-tm_n}$$

- select optimal combination of operators $O=\sum_j c_j O_j$

  with correlator $C(t)=\sum_{ij} c_i c_j C_{ij}(t)$

Generalized eigenvalue problem

- solve generalized eigenvalue problem

$$C_{ij}(t)\,\boldsymbol{v}_j^{(\boldsymbol{k})} = \lambda^{(k)} C_{ij}(0)\,\boldsymbol{v}_j^{(\boldsymbol{k})}$$

- sort eigenvalues such that

$$\lambda^{(1)} \geqslant \lambda^{(2)} \geqslant \ldots \geqslant \lambda^{(N)}$$

- eigenvalues behave as

$$\lambda^{(k)}(t) \propto e^{-t m_k}\left[1 + O\left(e^{-t \Delta m_k}\right)\right]$$

- eigenvectors yield the coefficients for the Operator O which best overlaps the state

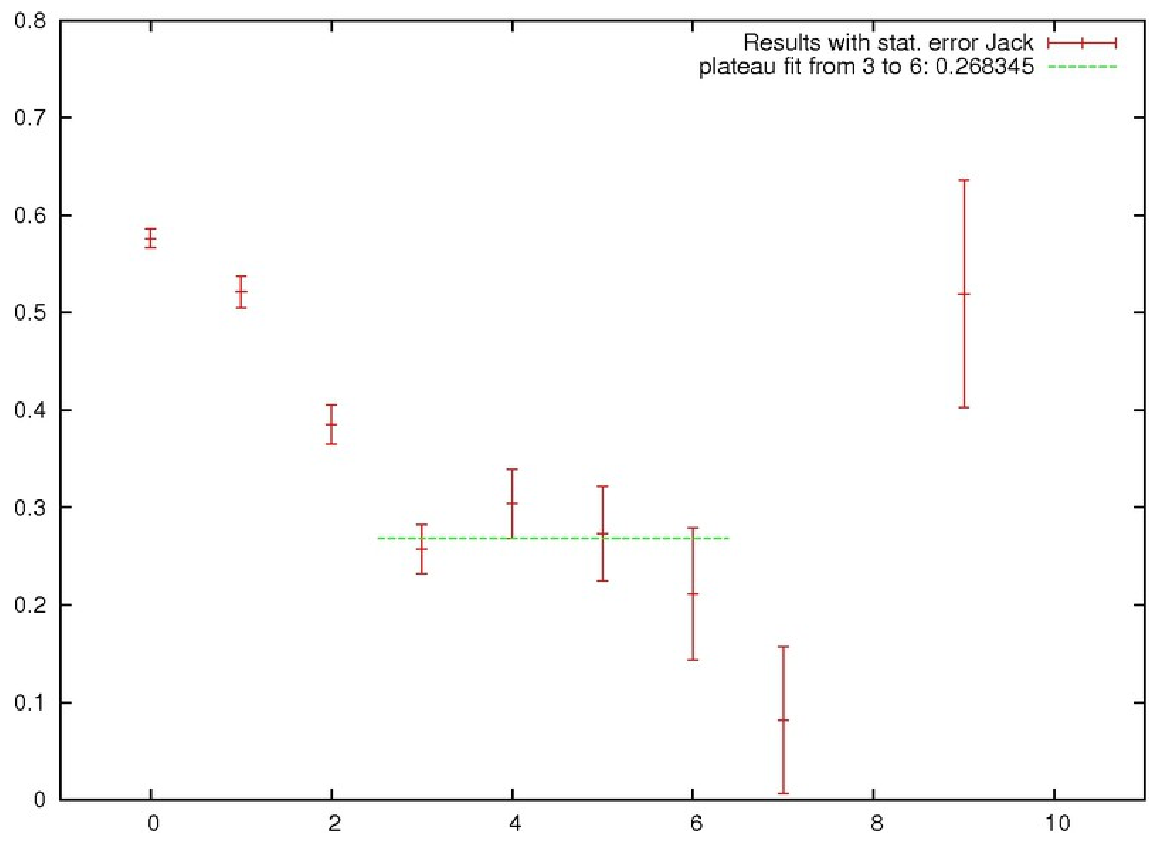
t-eigenvector method

- perform diagonalization at each *t* to get eigenvalues

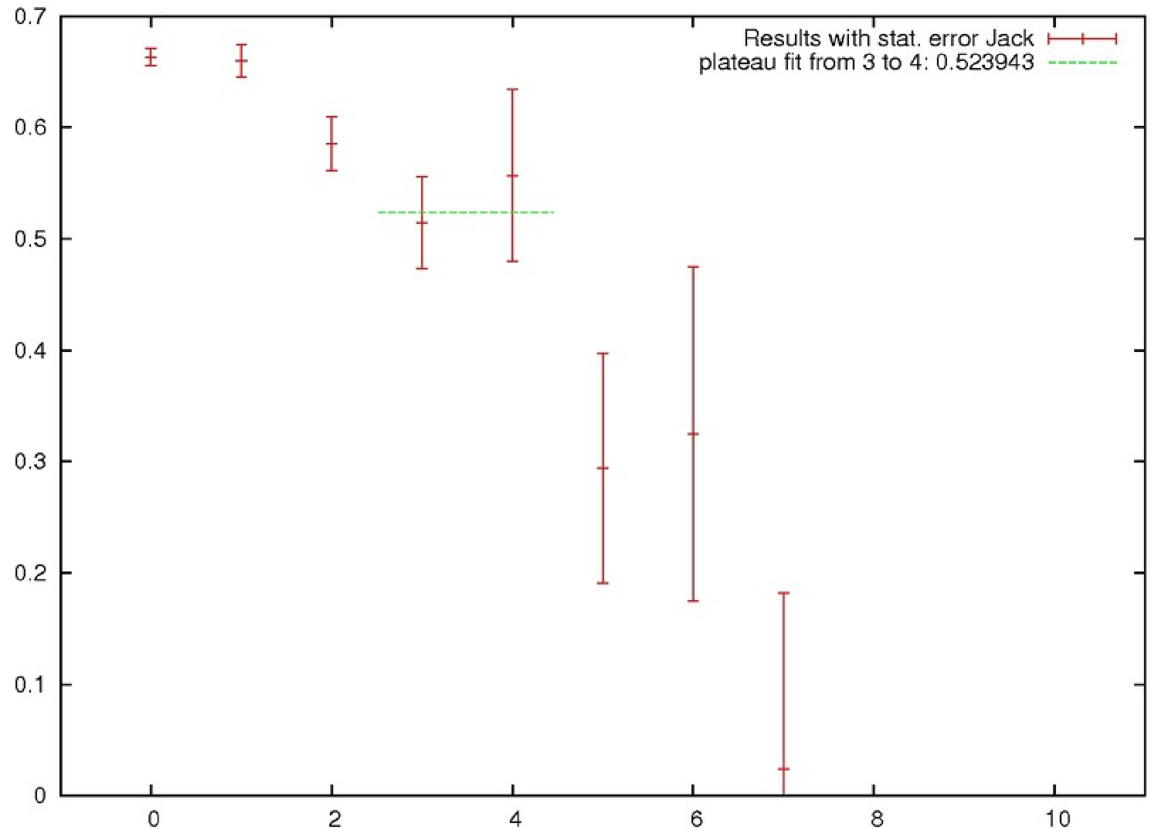
$$am_k(t) = \ln\left[\frac{\lambda^{(k)}(t)}{\lambda^{(k)}(t+1)}\right]$$

- only eigenvalues needed
- no basis t_b needed

result of t-eigenvector method $am_k(t)$



result of t-eigenvector method $am_k(t)$

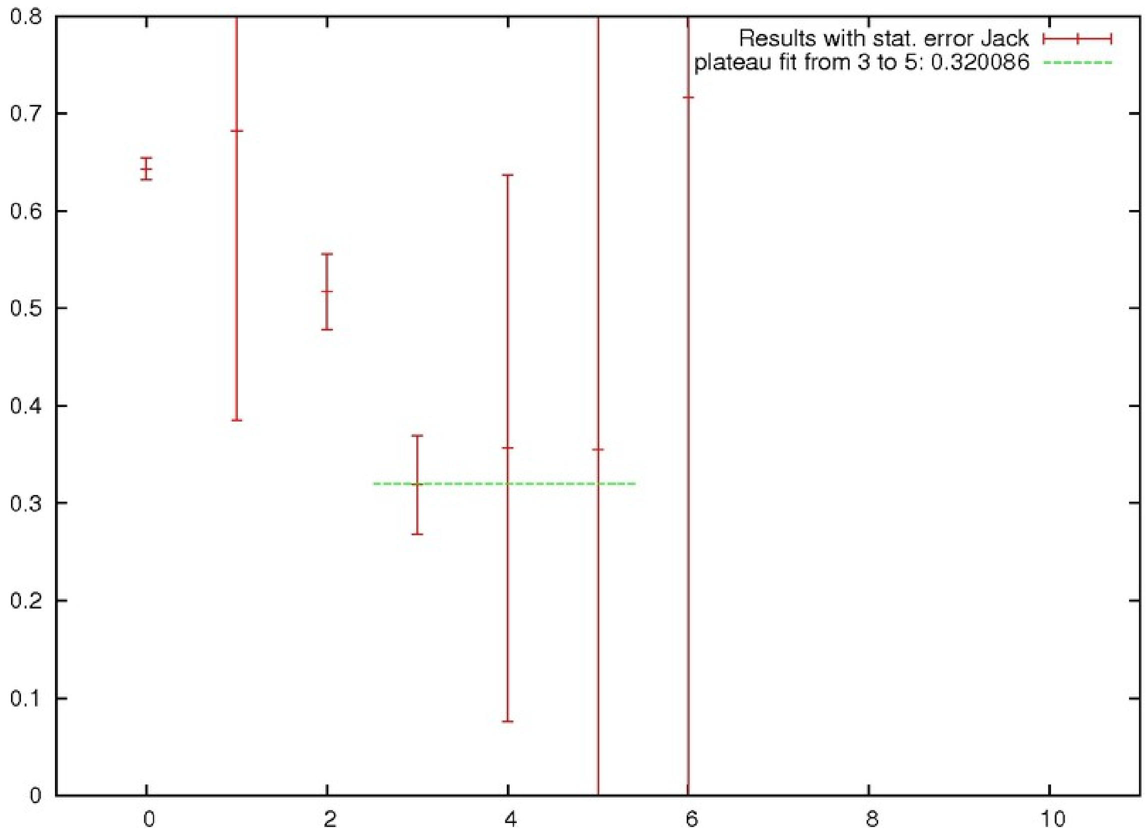


fixed-vector method

- first solve eigenvalue problem on a single time-slice $t = t_b$
- use resulting eigenvectors for rotating the cross correlators for each time-slice

$$am_k(t) = \ln\left[\frac{\sum_{ij} \boldsymbol{v}_{\boldsymbol{i}}^{(k)} \boldsymbol{v}_{\boldsymbol{j}}^{(k)} C_{ij}(t)}{\sum_{ij} \boldsymbol{v}_{\boldsymbol{i}}^{(k)} \boldsymbol{v}_{\boldsymbol{j}}^{(k)} C_{ij}(t+1)}\right]$$

result of fixed-vector method $am_k(t)$



Conclusion

- variational smearing leads to better results
  - fine tune the smearing levels

- glueballs have a very noisy signal because of their purely gluonic nature
  - more configurations needed

Bibliography

(1) Gattringer, Lang, *“Quantum Chromodynamics on the lattice”*, Springer 2010

(2) Kamel Demmouche, *N=1 SU(2) Supersymmetric Yang-Mills theory on the lattice with light dynamical Wilson gluinos”*, PhD thesis

(3) M. Lüscher and U. Wolff, *“How to calculate the elastic scattering matrix in two-dimensional quantum field theories by numerical simulations”*, Nucl. Phys. **B339** (1990) 222-252

(4) C. J. Morningstar and M. J. Peardon, *“The glueball spectrum from an anisotropic lattice study”*, Phys. Rev. **D47** (1993) 034509, arXiv:hep-lat/9901004