

Heavy Quark Effective Theory and non-perturbative Matching

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Outline

- 1 Introduction
- 2 HQET in the continuum
- 3 HQET on the lattice
- 4 Renormalization and non-perturbative matching
- 5 Outlook

What is HQET and why do we need it?

hierachy of scales

$$\Lambda_{IR} = \frac{1}{L} \ll m_\pi, \dots, m_D, m_B \ll \frac{1}{a} = \Lambda_{UV}$$

Simulations of the c-quark on the lattice $\Rightarrow L/a \approx 60 \dots 120$
Problem with the b-quark $\Rightarrow$ Increase of lattice points by factor four

SOLUTION: HQET

asymptotic expansion of QCD quantities in powers of $1/m$, the heavy quark mass

effective continuum Lagrangian

- systems with one heavy quark
- b quark is static $m = m_b \rightarrow \infty$
- effective Lagrangian + correction terms $\sim 1/m$

$$\mathcal{L} = \mathcal{L}_h^{\text{stat}} + \frac{1}{2m} \mathcal{L}_h^{(1)} + \mathcal{L}_{\bar{h}}^{\text{stat}} + \frac{1}{2m} \mathcal{L}_{\bar{h}}^{(1)} + O\left(\frac{1}{m^2}\right)$$

$$\mathcal{L}_h^{\text{stat}} = \bar{\psi}_h(D_0 + m)\psi_h, \quad \mathcal{L}_h^{(1)} = -(\mathcal{O}_{\text{kin}} + \mathcal{O}_{\text{spin}})$$

$$\mathcal{O}_{\text{kin}}(x) = \bar{\psi}_h(x) \mathbf{D}^2 \psi_h(x) \quad \left\{ \begin{array}{l} \text{kinetic energy from} \\ \text{heavy quark's residual motion} \end{array} \right\}$$

$$\mathcal{O}_{\text{spin}}(x) = \bar{\psi}_h(x) \mathbf{B} \sigma \psi_h(x) \quad \left\{ \begin{array}{l} \text{chromomagnetic interaction} \\ \text{with the gluon field} \end{array} \right\}$$

Propagator

Continuum Propagator

$$G_h(x, y) = \Theta(x_0 - y_0) \delta(\mathbf{x} - \mathbf{y}) \exp(-m(x_0 - y_0)) \times \\ \mathcal{P} \exp \left\{ - \int_{x_0}^{y_0} dz_0 A_0(z_0, \mathbf{x}) \right\} P_+$$

m shifts all energies in the sector of the Hilbertspace containing only one heavy quark $\Rightarrow$ energy shift:

$$E_{h, \bar{h}}^{\text{QCD}} = E_{h, \bar{h}}^{\text{stat}} + m \\ \mathcal{L}_h^{\text{stat}} = \bar{\psi}_h (D_0 + \epsilon) \psi_h, \quad \mathcal{L}_{\bar{h}}^{\text{stat}} = \bar{\psi}_{\bar{h}} (-D_0 + \epsilon) \psi_{\bar{h}}$$

Symmetries

1. Flavor

Number of heavy quarks F . Add a flavorindex :

$$\psi_h \rightarrow \psi_h = (\psi_{h_1}, \dots, \psi_{h_F})^T, \quad \bar{\psi}_h \rightarrow \bar{\psi}_h = (\bar{\psi}_{h_1}, \dots, \bar{\psi}_{h_F})$$

$\Rightarrow$ symmetry:

$$\psi_h(x) \rightarrow V\psi_h(x), \quad \bar{\psi}_h(x) \rightarrow \bar{\psi}_h(x)V^\dagger, \quad V \in SU(F)$$

Symmetries

2. Spin

For each field exist two spin components, but $\mathcal{L}$ has no spin dependent interaction. SU(2) rotations are expressed through spin matrices σ_k :

$$\sigma_k = \frac{1}{2}\epsilon_{ijk}\sigma_{ij} \equiv \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

$\Rightarrow$ symmetry:

$$\psi_h(x) \rightarrow e^{i\alpha_k\sigma_k}\psi_h(x), \quad \bar{\psi}_h(x) \rightarrow \bar{\psi}_h(x)e^{-i\alpha_k\sigma_k}$$

acts on each flavor component of the fields

Symmetries

3. Local Flavor Number

The static Lagrangian does not contain spacial derivatives

$\Rightarrow$ symmetry:

$$\psi_h(x) \rightarrow e^{i\eta(\mathbf{x})} \psi_h(x), \quad \bar{\psi}_h(x) \rightarrow \bar{\psi}_h(x) e^{-i\eta(\mathbf{x})}$$

$\eta(\mathbf{x})$ local phase

Renormalization

UV divergences can be cancelled by adding counter terms:

- composite fields with mass dimension $\leq d$
- have to share the symmetry of the bare $\mathcal{L}$

$$\mathcal{L}_h(x) = c_1 \mathcal{O}_1(x) + c_2 \mathcal{O}_2(x)$$
$$\mathcal{O}_1(x) = \bar{\psi}_h(x) \psi_h(x), \quad \mathcal{O}_2(x) = \bar{\psi}_h D_0 \psi_h$$

$c_2 = 1$ fixes only the unphysical field normalization

$c_1 = \delta m$ corresponds to an additive mass renormalization

Renormalization

further QCD renormalization of coupling and light quark mass

→ all divergences can be absorbed in δm

flavor symmetry → with several heavy flavors, δm is proportional to the unit matrix in flavor space

energies of all states are:

$$E_{h,\bar{h}}^{\text{QCD}} = E_{h,\bar{h}}^{\text{stat}}|_{\delta m=0} + m_{\text{bare}}$$

$$m_{\text{bare}} = \delta m + m$$

m is finite, m_{bare} and δm compensate the linear divergences of the static theory

HQET on the lattice - structure

- 1 discussion of the lattice HQET in the **static approximation**
- 2 discussion of the renormalization
- 3 Symanzik $O(a)$ improvement
- 4 correlation functions on the Schrödinger functional
- 5 next chapter: adding **$1/m$ terms**

lattice Lagrangian

Wilson discretization $\rightarrow$ be careful with the doublers
 continuum observables are transcribed to the lattice:

$$D_0 \gamma_0 \rightarrow \frac{1}{2} ((\nabla_0 + \nabla_0^*) \gamma_0 - a \nabla_0^* \nabla_0)$$

lattice Lagrangian:

$$\mathcal{L}_h = \frac{1}{1 + a\delta m} \bar{\psi}_h(x) [\nabla_0^* + \delta m] \psi_h(x)$$

$$\mathcal{L}_{\bar{h}} = \frac{1}{1 + a\delta m} \bar{\psi}_{\bar{h}}(x) [-\nabla_0 + \delta m] \psi_{\bar{h}}(x)$$

gauge invariant derivatives of observables acting on the quark fields:

$$\nabla_\mu \psi(x) = \frac{1}{a} [\lambda_\mu U(x, \mu) \psi(x + a\hat{\mu}) - \psi(x)], \quad \nabla_\mu^* \psi(x) = \frac{1}{a} [\psi(x) - \lambda_\mu^{-1} U(x - a\hat{\mu}, \mu)^{-1} \psi(x - a\hat{\mu})]$$

$$\lambda_\mu = e^{ia\theta_\mu/L}, \quad \theta_0 = 0, \quad -\pi < \theta_k < \pi$$

Action, Propagator

lattice action

$$S_h^W = a^4 \frac{1}{1 + a\delta m} \sum_x \bar{\psi}_h(x) (\nabla_0^* + \delta m) \psi_h(x)$$

quark propagator

$$G_h(x, y) = \theta(x_0 - y_0) \delta(\mathbf{x} - \mathbf{y}) \exp(-\delta\hat{m}(x_0 - y_0)) \mathcal{P}(y, x; 0)^\dagger P_+$$

$\Rightarrow$ energy shift:

$$E_{h,\bar{h}}^{\text{QCD}} = E_{h,\bar{h}}^{\text{stat}}|_{\delta m=0} + m_{\text{bare}}$$

$$m_{\text{bare}} = \delta\hat{m} + m$$

$$\delta\hat{m} = \frac{1}{a} \ln(1 + a\delta m)$$

Symmetries

the symmetries of the continuum theory are preserved on the lattice

- Spin
- Flavor
- local flavor number

HQET symmetries are defined in terms of transformations of the heavy quark fields
light quark fields do not change

Improvement

to obtain a meaningful result after extrapolating to the continuum limit

→ have to reduce the $O(a)$ effects

$O(a)$ Symanzik Improvement

can reduce the discretization errors by adding higher dimensional operators to the lattice action and composite fields

Symanzik Theory

effective continuum theory

→ describes the Wilson lattice action with finite a and the approach to the continuum limit

→ expansion in lattice constant a

effective action for a static quark:

$$S_{\text{eff}} = S_0 + aS_1 + a^2S_2 + \dots, \quad S_k = \int d^4x \mathcal{L}_k(x)$$

with the static continuum Lagrangian: $\mathcal{L}_0(x) = \mathcal{L}_h^{\text{stat}}(x)$

local operators of dimension $k + 4$: $\mathcal{L}_k$, which are built from products of gluon and quark fields

$\mathcal{O}(a)$ terms of the effective theory

consider all operators with dimension 5 and the same symmetry as the lattice theory:

$$\mathcal{L}_1(x) = \sum_{i=3}^5 c_i \mathcal{O}_i(x)$$

$\mathcal{L}_1$ will contain additional derivatives or powers of quark masses m to have $d = 5$:

$$\mathcal{O}_3(x) = \bar{\psi}_h D_0 D_0 \psi_h$$

$$\mathcal{O}_4(x) = m_l \bar{\psi}_h D_0 \psi_h$$

$$\mathcal{O}_5(x) = m_l^2 \bar{\psi}_h \psi_h$$

$O(a)$ terms of the effective theory

reduce some operators using field equations: $D_0\psi_h = 0$

→ resulting operator in $O(a)$: $\mathcal{O}_5(x) = m_l^2 \bar{\psi}_h \psi_h$

→ induces a redefinition of the mass counter-term δm (thus depends on m_l)

$$S_{\text{eff}} = \int d^4x \{ \mathcal{L}_h^{\text{stat}}(x) + ac_5 m_l^2 \bar{\psi}_h \psi_h + \dots \}$$

Renormalization of the axial current

time component of the axial current in Symanzik's effective theory:

$$(A_0^{\text{stat}})_{\text{eff}} = A_0^{\text{stat}}(x) + a \sum_{k=1}^4 \omega_k (\delta A_0^{\text{stat}})_k$$

$$A_0^{\text{stat}}(x) = \bar{\psi}(x) \gamma_0 \gamma_5 \psi_h$$

a basis for the dimension four fields:

$$(\delta A_0^{\text{stat}})_1 = \bar{\psi} \overleftarrow{D}_j \gamma_j \gamma_5 \psi_h$$

$$(\delta A_0^{\text{stat}})_2 = \bar{\psi} \gamma_5 D_0 \psi_h$$

$$(\delta A_0^{\text{stat}})_3 = \bar{\psi} \overleftarrow{D}_0 \gamma_5 \psi_h$$

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Renormalization of the axial current

for on-shell quantities the effective theory representation is:

$$(A_0^{\text{stat}})_{\text{eff}} = A_0^{\text{stat}}(x) + a\tilde{\omega}_1(\delta A_0^{\text{stat}})_1$$

To reduce the lattice $O(a)$ effects we add corresponding combinations of correction terms to the axial current:

$$(A_R^{\text{stat}})_0 = Z_A^{\text{stat}}(g_0, a\mu)(A_I^{\text{stat}})_0$$

$$(A_I^{\text{stat}})_0 = A_0^{\text{stat}} + ac_A^{\text{stat}}(g_0)\bar{\psi}\gamma_j\gamma_5\frac{1}{2}\left(\overleftarrow{\nabla}_j - \overleftarrow{\nabla}_j^*\right)\psi_h$$

$$\bar{\psi}(x)\overleftarrow{\nabla}_\mu = \frac{1}{a}[\bar{\psi}(x+a\hat{\mu})U^{-1}(x,\mu)\lambda_\mu^{-1} - \bar{\psi}(x)]$$

$$\bar{\psi}(x)\overleftarrow{\nabla}_\mu^* = \frac{1}{a}[\bar{\psi}(x) - \bar{\psi}(x-a\hat{\mu})U(x-a\hat{\mu},\mu)\lambda_\mu]$$

Schrödinger Functional

space-time is described through a cylinder of size $T \times L^3$

Dirichlet boundary conditions for time component

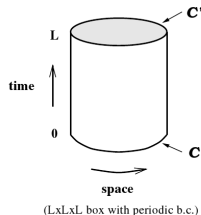
periodic boundary conditions for space component

$$\psi(x + \hat{k}L) = e^{i\theta} \psi(x), \quad \bar{\psi}(x + \hat{k}L) = e^{-i\theta} \bar{\psi}(x)$$

boundary fields:

$$\zeta(\mathbf{x}) = \frac{\delta}{\delta \bar{\rho}(\mathbf{x})}, \quad \bar{\zeta}(\mathbf{x}) = -\frac{\delta}{\delta \rho(\mathbf{x})},$$

$$\zeta'(\mathbf{x}) = \frac{\delta}{\delta \bar{\rho}'(\mathbf{x})}, \quad \bar{\zeta}'(\mathbf{x}) = -\frac{\delta}{\delta \rho'(\mathbf{x})}$$

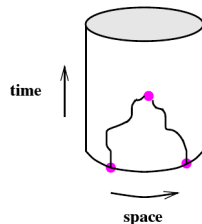


correlation functions on the SF

In QCD: relevant correlation functions in pseudoscalar and vector channel:

$$f_A(x_0, \theta) = -\frac{a^6}{2} \sum_{\mathbf{y}, \mathbf{z}} \langle (A_I)_0(x) \bar{\zeta}_b(\mathbf{y}) \gamma_5 \zeta_I(\mathbf{z}) \rangle$$

$$k_V(x_0, \theta) = -\frac{a^6}{6} \sum_{\mathbf{y}, \mathbf{z}, k} \langle (V_I)_k(x) \bar{\zeta}_b(\mathbf{y}) \gamma_k \zeta_I(\mathbf{z}) \rangle$$



With $O(a)$ improved axial and vector currents:

$$(A_I)_\mu = \bar{\psi}_I \gamma_\mu \gamma_5 \psi_b + ac_A \tilde{\partial}_\mu \bar{\psi}_I \gamma_5 \psi_b$$

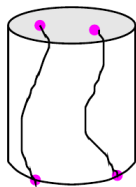
$$(V_I)_\mu = \bar{\psi}_I \gamma_\mu \psi_b + ac_V \tilde{\partial}_\mu \bar{\psi}_I i \sigma_{\mu\nu} \psi_b$$

correlation functions on the SF

In QCD: boundary to boundary correlation functions:

$$f_1(\theta) = -\frac{a^{12}}{2L^6} \sum_{\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}} \langle \bar{\zeta}'_l(\mathbf{u}) \gamma_5 \zeta'_b(\mathbf{v}) \bar{\zeta}_b(\mathbf{y}) \gamma_5 \zeta_l(\mathbf{z}) \rangle$$

$$k_1(\theta) = -\frac{a^{12}}{6L^6} \sum_{\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}, k} \langle \bar{\zeta}'_l(\mathbf{u}) \gamma_k \zeta'_b(\mathbf{v}) \bar{\zeta}_b(\mathbf{y}) \gamma_k \zeta_l(\mathbf{z}) \rangle$$



the SF correlation functions: gauge invariant, multiplicative
 renormalizable, shallow behavior in perturbative area $\rightarrow$ small
 cutoff effects

Renormalization and non-perturbative matching - structure

→ discuss correlation function $C_{AA}(x_0)$ in QCD and HQET

- 1 in leading order $1/m$ (massindepend)
- 2 add $1/m$ correction terms
- 3 discuss matching in next to leading order

→ Principle: fix the finite part of the renormalization coefficients in the way the effective theory describes the underlying QCD

- 1 Parameters of HQET ω_i
- 2 Matching observables Φ_i
- 3 Step Scaling Function

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correlation function C_{AA}^{QCD}

lattice regularized

$$C_{AA,R}^{\text{QCD}}(x_0) = Z_A^2 a^3 \sum_{\mathbf{x}} \langle A_0(\mathbf{x}) A_0^\dagger(0) \rangle_{\text{QCD}}$$

$A_\mu = \bar{\psi} \gamma_\mu \gamma_5 \psi_b$ axial current in QCD

ψ_b relativistic b-quark field

$Z_A(g_0)$ renormalization factor, fixed by chiral ward identities
(independent of renormalization scale)

B-meson mass and decay constant

at large time the B-meson state dominates its spectral representation via

$$C_{AA,R}^{\text{QCD}} = Z_A^2 \frac{1}{2} \langle 0 | A_0^\dagger(0) | B \rangle \langle B | A_0(0) | 0 \rangle e^{-x_0 m_B} [1 + O(e^{-x_0 \Delta})]$$

obtain B-meson mass and decay constant:

$$\begin{aligned} \Gamma_{AA}^{\text{QCD}} &= -\tilde{\partial}_0 \ln(C_{AA}^{\text{QCD}}(x_0)) \\ &= m_B + O(e^{-x_0 \Delta}) \end{aligned}$$

$$\begin{aligned} [\phi^{\text{QCD}}]^2 &\equiv f_B^2 m_B \\ &= | \langle B | Z_A A_0 | 0 \rangle |^2 \\ &= 2 \lim_{x_0 \rightarrow \infty} \exp(x_0 \Gamma_{AA}^{\text{QCD}}(x_0)) C_{AA}^{\text{QCD}}(x_0) \end{aligned}$$

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correlation function C_{AA}^{stat}

we get the correlation function in the static approxi by replacing:
 $Z_A A_0 \rightarrow Z_A^{\text{stat}}(g_0, a\mu) A_0^{\text{stat}}$ with the scale dependent
 renormalization factor:

$$Z_A^{\text{stat}}(g_0, a\mu) = 1 + g_0^2(B_0 - \gamma_0 \ln(a\mu)) + O(g_0^2)$$

$$\begin{aligned} C_{AA,R}^{\text{stat}}(x_0) &= (Z_A^{\text{stat}})^2 C_{AA}^{\text{stat}}(x_0) \\ &= (Z_A^{\text{stat}})^2 a^3 \sum_x \left\langle A_0^{\text{stat}}(x) (A_0^{\text{stat}})^{\dagger}(0) \right\rangle_{\text{stat}} \end{aligned}$$

obtain B-meson mass and decay constant in the same way: $\Gamma_{AA}^{\text{stat}}$,
 $[\Phi^{\text{stat}}(\mu)]^2$

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obtain B-meson mass and decay constant in the same way: $\Gamma_{AA}^{\text{stat}}$,
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Principle of Matching

$C_{AA,R}^{\text{QCD}}, \Phi^{\text{QCD}} \rightarrow$ independent of $\mu \rightarrow$ depend on quark mass m
 $C_{AA,R}^{\text{stat}}, \Phi^{\text{stat}} \rightarrow$ depend on $\mu \rightarrow$ independent of quark mass m

Matching: put the HQET and QCD observables in relation in a finite renormalization scale up to order $1/m$

$$\Phi^{\text{QCD}}(m) = \tilde{C}_{\text{match}}(m, \mu) \Phi(\mu) + O\left(\frac{1}{m}\right)$$

- $\Rightarrow$ obtain mass dependence of HQET observables
- $\Rightarrow$ change of renormalization scale in effective theory with $m = m_b$
- $\Rightarrow$ perturbative expansion of matching factor:

$$\tilde{C}_{\text{match}}(m, \mu) = 1 + c_1\left(\frac{m}{\mu}\right) \bar{g}^2(\mu) + \dots$$

Principle of Matching

$C_{AA,R}^{\text{QCD}}, \Phi^{\text{QCD}} \rightarrow$ independent of $\mu \rightarrow$ depend on quark mass m
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- $\Rightarrow$ change of renormalization scale in effective theory with $m = m_b$
- $\Rightarrow$ perturbative expansion of matching factor:

$$\tilde{C}_{\text{match}}(m, \mu) = 1 + c_1\left(\frac{m}{\mu}\right) \bar{g}^2(\mu) + \dots$$

include $1/m$ corrections

action in HQET beyond static approxi:

$$S_{\text{HQET}} = a^4 \sum_x \{ \mathcal{L}_h^{\text{stat}}(x) + \sum_{k=1}^n \mathcal{L}^{(k)}(x) \}$$

$$\mathcal{L}^{(k)}(x) = \sum_i \omega_i^{(k)} \mathcal{L}_i^{(k)}(x)$$

$$\mathcal{L}_h^{\text{stat}} = \bar{\psi}_h (\nabla_0^* + \delta m) \psi_h$$

$$\mathcal{L}_h^{(1)} = -(\omega_{\text{kin}} \mathcal{O}_{\text{kin}} + \omega_{\text{spin}} \mathcal{O}_{\text{spin}})$$

$\delta m, \omega_i(g_0, m)$ have to be determined
 from the matching condition

Problem: renormalization of the pathintegral

$$P_{NRQCD} \propto \exp \left(-a^4 \sum_x [\mathcal{L}_{\text{light}}(x) + \mathcal{L}_h^{\text{stat}}(x) + \mathcal{L}_h^{(1)}] \right)$$

is not renormalizable

in perturbation theory $\rightarrow$ new divergences in one loop expansions
 $\rightarrow$ new counter terms $\rightarrow$ CL of lattice QCD does not exist!!

HQET $\Rightarrow$ expand P in powers of $1/m$ and count the terms
 $\propto \omega_{\text{kin}} = O(1/m) = \omega_{\text{spin}}$

$$P_{\text{HQET}} \equiv \exp(-a^4 \sum_x [\mathcal{L}_{\text{light}}(x) + \mathcal{L}_h^{\text{stat}}(x)]) \left\{ 1 - a^4 \sum_x \mathcal{L}_h^{(1/m)}(x) \right\}$$

is now renormalizable

HQET expectation values

up to order $1/m$:

$$\begin{aligned}\langle \mathcal{O} \rangle &= \langle \mathcal{O} \rangle_{\text{stat}} + \omega_{\text{kin}} a^4 \sum_x \langle \mathcal{O} \mathcal{O}_{\text{kin}}(x) \rangle_{\text{stat}} + \omega_{\text{spin}} a^4 \sum_x \langle \mathcal{O} \mathcal{O}_{\text{spin}}(x) \rangle_{\text{stat}} \\ &\equiv \langle \mathcal{O} \rangle_{\text{stat}} + \omega_{\text{kin}} \langle \mathcal{O} \rangle_{\text{kin}} + \omega_{\text{spin}} \langle \mathcal{O} \rangle_{\text{spin}}\end{aligned}$$

with the pathintegral average:

$$\langle \mathcal{O} \rangle_{\text{stat}} = \frac{1}{Z} \int_{\text{fields}} \mathcal{O} \exp(-a^4 \sum_x [\mathcal{L}_{\text{light}}(x) + \mathcal{L}_h^{\text{stat}}(x)])$$

HQET expansion of SF correlation functions

Expansion in first order $O(1/m)$ of the correlation functions:

$$\begin{aligned} [f_A]_R &= Z_A^{\text{HQET}} Z_{\zeta_h} Z_{\zeta} e^{-m_{\text{bare}} x_0} \left\{ f_A^{\text{stat}} + c_A^{(1)} f_{\delta A}^{\text{stat}} + \omega_{\text{kin}} f_A^{\text{kin}} + \omega_{\text{spin}} f_A^{\text{spin}} \right\} \\ [k_V]_R &= Z_V^{\text{HQET}} Z_{\zeta_h} Z_{\zeta} e^{-m_{\text{bare}} x_0} \left\{ k_V^{\text{stat}} + c_V^{(1)} k_{\delta V}^{\text{stat}} + \omega_{\text{kin}} k_V^{\text{kin}} + \omega_{\text{spin}} k_V^{\text{spin}} \right\} \\ [f_1]_R &= Z_{\zeta_h}^2 Z_{\zeta}^2 e^{-m_{\text{bare}} T} \left\{ f_1^{\text{stat}} + \omega_{\text{kin}} f_1^{\text{kin}} + \omega_{\text{spin}} f_1^{\text{spin}} \right\} \\ [k_1]_R &= Z_{\zeta_h}^2 Z_{\zeta}^2 e^{-m_{\text{bare}} T} \left\{ k_1^{\text{stat}} + \omega_{\text{kin}} k_1^{\text{kin}} + \omega_{\text{spin}} k_1^{\text{spin}} \right\} \end{aligned}$$

obtain this directly from the expansion of the expectation values, except for $f_{\delta A}^{\text{stat}}$:

$$f_{\delta A}^{\text{stat}}(x_0, \theta) = -\frac{a^6}{2} \sum_{\mathbf{y}, \mathbf{z}} \langle A_0^{(1)}(x) \bar{\zeta}_h(\mathbf{y}) \gamma_5 \zeta_l(\mathbf{z}) \rangle$$

As a consequence of spin symmetry of the lattice action we obtain the identities: $f_A^{\text{kin}} = -k_V^{\text{kin}}$, $f_A^{\text{spin}} = 3k_V^{\text{spin}}$

free parameters of HQET

free parameters of the effective theory:

$$\omega_i, \quad i = 1, \dots, N_{\text{HQET}}$$

$$\omega^{\text{HQET}} = \left(\underbrace{m_{\text{bare}}, \ln(Z_A^{\text{HQET}})}_{\substack{\text{parameters of static} \\ \text{approx: } m_{\text{bare}}^{\text{stat}}, \ln(Z_A^{\text{stat}})}}, \underbrace{c_A^{(1)}, \omega_{\text{kin}}, \omega_{\text{spin}}}_{\substack{\text{parameters of effective} \\ \text{theory}}} \right)^t$$

in static approxi: $\omega_{\text{kin}} = \omega_{\text{stat}} = 0, \quad c_A^{\text{HQET}} \rightarrow ac_A^{\text{stat}}$

Matching observables

procedure to calculate HQET parameters up to order $1/m$
 → introduce observable which fulfill the matching condition:

$$\underbrace{\Phi_i^{\text{QCD}}(m)}_{\text{assumption:}} = \underbrace{\Phi_i^{\text{HQET}}(m, a)}_{\text{massdependence only in}} , \quad i = 1, \dots, N_{\text{HQET}}$$

continuum limit
 has already
 been taken

HQET observables,
 depends on finite a ,
 which defines the bare
 parameters

observables on the Schrödinger functional

SF with $L = T = L_1 \approx 0.5 fm$:

$$\Phi_i(L_1, M, a) = \Phi_i^{\text{QCD}}(L_1, M, 0), \quad i = 1, \dots, N_{\text{HQET}}$$

$\Rightarrow \Phi_i$ have a HQET expansion linear in ω_i :

$$\Phi(L, M, a) = \eta(L, a) + \phi(L, a)\omega(M, a)$$

η : N_{HQET} dimensional vector

$\phi(L, a)$: $N_{\text{HQET}} \times N_{\text{HQET}}$ block matrix

Problem of matching

HQET applicable for $Lm_b \gg 1$

only then we obtain a good expansion in $1/m$

LQCD requires $am_b \ll 1$

to have small $O(a)$ lattice artifacts

Strategy to obtain a larger volume

- start with QCD in a small volume ($L \simeq 0.5 fm$) $\rightarrow$ take the CL
- matching to HQET $\Rightarrow$ fixing parameter of effective theory
- cross gap to larger volumes with practical lattice spacings, on which physical observables (e.g. B-Meson mass, decay constant) can be determined

$\Rightarrow$ Step Scaling Function (SSF)

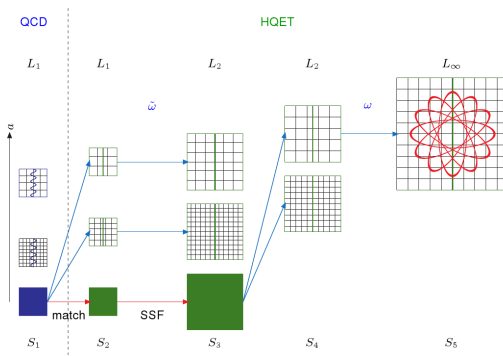
SSF

on the SF we use $L_2 = 2L_1$ ($2L = 2L_1 \simeq 1.0fm$):

$$\Phi_k^{\text{HQET}}(2L) = \sigma_k \left(\{ \Phi_j^{\text{HQET}}(L), j = 1, \dots, N \} \right)$$

after some applications of the SSF we achieve a lattice with $L \simeq 2fm$ with the same resolution as in the small volume

Strategy of the Step Scaling method



Strategy of the non perturbative matching of HQET and QCD parameters and the Step Scaling method

HQET expansion of the heavy-light vector current

semileptonic decays are important to determine the parameters of the standard model, especially the CKM matrix elements

$$V_0^{\text{HQET}}(x) = Z_V^{\text{HQET}} [V_0^{\text{stat}}(x) + \sum_{i=1}^2 c_V^{(i)} V_0^{(i)}(x)]$$

find all $O(1/m)$ correction terms:

- $d = 4$ operators
- right flavor structure, transformation under spatial lattice rotations and parity
- use equation of motion to eliminate operators
- cannot use symmetry conditions because they are broken at $O(1/m)$

$$V_0^{(1)}(x) = \bar{\psi}(x) \frac{1}{2} \gamma_i (\nabla_i^S - \overleftarrow{\nabla}_i^S) \psi_h(x), \quad V_0^{(2)}(x) = -\tilde{\partial}_i V_i^{\text{stat}}$$

HQET expansion of the heavy-light vector current

the spatial component of the vector current

$$V_k^{\text{HQET}}(x) = Z_V^{\text{HQET}} [V_k^{\text{stat}}(x) + \sum_{i=3}^6 c_V^{(i)} V_k^{(i)}(x)]$$

the $1/m$ correction terms:

$$V_k^{(3)}(x) = \bar{\psi}(x) \frac{1}{2} \gamma_k \gamma_i (\nabla_i^S - \overleftarrow{\nabla}_i^S) \psi_h(x)$$

$$V_k^{(4)}(x) = \bar{\psi}(x) \frac{1}{2} (\nabla_k^S - \overleftarrow{\nabla}_k^S) \psi_h(x)$$

$$V_k^{(5)}(x) = \tilde{\partial}_i (\bar{\psi}(x) \gamma_k \gamma_i \psi_h(x))$$

$$V_k^{(6)}(x) = \tilde{\partial}_k V_0^{\text{stat}}$$

Observables for the matching step

in previous computations the SF specific periodicity angles are the same for the heavy and light quark:

$$\theta_h = \theta_l$$

in order to get sensitivity to the coefficients $c_V^{(i)}, c_A^{(i)}$ use:

$$\theta_h \neq \theta_l$$

Observables for the matching step

HQET parameters can be determined by using two-point-functions and three-point-functions, which are inspired by semileptonic decays

$$\begin{aligned}
 & f_A^{(\theta_h=\theta_l)} \text{ for } c_A^{(1)}, & f_A^{(\theta_h=-\theta_l)} \text{ for } c_A^{(2)}, \\
 & k_V^{(\theta_h=\theta_l)} \text{ for } c_V^{(3)}, & k_V^{(\theta_h=-\theta_l)} \text{ for } c_V^{(5)}, \\
 & f_V^{(\theta_h=-\theta_l)} \text{ for } c_V^{(4)}, c_V^{(6)}, & h_V^{(\theta_h=-\theta_l)} \text{ for } c_V^{(4)}, c_V^{(6)}
 \end{aligned}$$

with three-point-functions:

$$\begin{aligned}
 f_V(x_0) &\propto \sum_{\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}, k} \left\langle \bar{\zeta}'_b(\mathbf{u}) \gamma_5 \zeta'_d(\mathbf{v}) \left(V_{bu}^I \right)_k(x) \bar{\zeta}_d(\mathbf{y}) \gamma_5 \zeta_u(\mathbf{z}) \right\rangle \\
 h_V(x_0) &\propto \sum_{\mathbf{u}, \mathbf{v}, \mathbf{y}, \mathbf{z}, k, i} \left\langle \bar{\zeta}'_b(\mathbf{u}) \gamma_5 \zeta'_d(\mathbf{v}) \left(V_{bu}^I \right)_k(x) \bar{\zeta}_d(\mathbf{y}) \gamma_i \zeta_u(\mathbf{z}) \right\rangle
 \end{aligned}$$

Summary

- HQET :
asymptotic expansion of QCD quantities in powers of $1/m_b$
- Schrödinger functional:
QCD in a finite euclidean space-time of size $T \times L^3$ with
specific boundary conditions
- Matching:
procedure to calculate free parameters of HQET
 $\omega_i^{\text{HQET}}, i = 1, \dots, 5$ up to order $1/m$
→ define matching observables Φ_i

$$\Phi_i^{\text{QCD}}(m) = \Phi_i^{\text{HQET}}(m, a)$$

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