

Random percolation as a gauge theory

Stefano Lottini

Westfälische Wilhelms-Universität, Münster
Università di Torino & INFN, sez. di Torino

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The random percolation model is well known since long time in statistical mechanics; here, its interpretation as an outright gauge theory is described. After showing its connection to more traditional gauge theories, the model and its basic observables are defined on a purely topological basis, and a number of key features, emerging from Monte Carlo calculations, are presented. Even though the focus is on mechanisms shared with the generic gauge theory on a qualitative level, some quantitatively surprising features appear; finally, some words are spent on the currently ongoing investigations.

Plan of the talk

- Introduction to LGTs
- The road to percolation as a gauge theory
- Numerical results
 - Confinement, string tension
 - Glueball spectrum
 - Effective string
 - Magnetic monopoles
- Algorithms and performance
- What's going on

Introduction

One of the most prominent features of the strong interaction is *quark confinement*: if the temperature is below T_c , only colourless objects are seen as asymptotic states.

Since an analytical derivation of this property (among others) is currently well out of reach, various alternative techniques have been devised to gain some insight.

Three basic ingredients: *simplify* (sometimes dramatically) the theory, regularise it on a *lattice*, and (subsequently) perform *Monte Carlo* simulations.

Here a very simplified theory will be presented (*the most basic gauge theory ever*); nevertheless, the results provide evidence for the key properties we expect from a “realistic” gauge theory.

Pure gauge

Confinement is still observed if matter is removed from the system: only the gauge field remains, and quarks are inserted as external, **static probes**:
 \Rightarrow *pure gauge theory*.

On the lattice, gauge field \mapsto links, and Y-M action \sim Wilson action:

$$S_{\square} = 1 - \frac{1}{2\text{Tr } \mathbb{1}} (\text{Tr } U_{\square} + \text{Tr } U_{\square}^{-1}) \quad ; \quad Z = \int \mathcal{D}[U] \prod_{\square} e^{-\beta S_{\square}}$$

The plaquette operator U_{\square} is built with **link variables**, which are the fundamental degrees of freedom $\in \mathbb{G}$ (and $\beta \propto g^{-2}$).

QCD has $\mathbb{G} = SU(3)$, but any group (even discrete) can be chosen.

Observables at $T = 0$ and interquark potential

The $T = 0$ pure gauge observable is a generalisation of U_{\square} :

$$\langle W(R, T) \rangle = \langle \text{Tr} \prod_{\ell \in \gamma} U_{\ell} \rangle \stackrel{T \rightarrow \infty}{\simeq} A e^{-TV(R)} \quad , \quad \text{Wilson loop}$$

and is used to extract the $V(R)$ for a $q\text{-}\bar{q}$ pair at distance R .

Confinement means that a large $\langle W \rangle$ decays with **area law**:

$$V(R) \stackrel{R \rightarrow \infty}{\simeq} \sigma R \quad ; \quad \sigma = \text{string tension}$$

If $\sigma = 0$, the system is deconfined (**perimeter law** for large loops).

Finite temperature

$T > 0$: the time direction is **compactified**, with radius $\propto T^{-1}$.

A new class of observables, corresponding to **noncontractible** loops:

$$\langle P(\mathbf{x}) \rangle = \langle \text{Tr} \sum_t U^{\hat{t}}(\mathbf{x}, t) \rangle, \quad \text{Polyakov loop}$$

This observable is an order parameter for the global $\mathcal{C}(\mathbb{G})$ symmetry.

It measures the **free energy of a quark**: $\langle P \rangle \neq 0 \leftrightarrow$ deconfinement.

From $\langle P(0)P(\mathbf{x}) \rangle$, the **finite- T string tension** can be obtained.

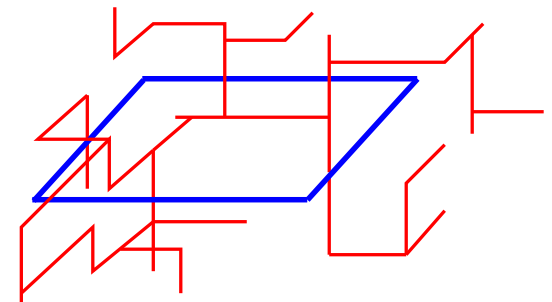
Interpretations of confinement - I

In a given gauge configuration, there is a network of **center vortices**, extended objects related to “singular gauge transformations”. [original idea: Nielsen, Olesen '79]

A vortex gives a multiplicative contribution to $W(R, T)$ if it pierces the loop: if the vortices are able to **disorder enough the gauge field**, there is confinement (i. e. area law for $\langle W \rangle$).

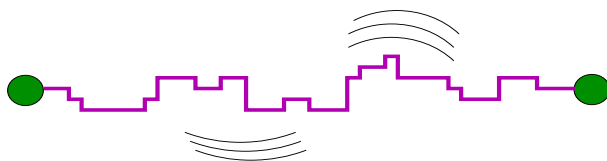
Supporting facts:

- Experimental observation of *center dominance*.
- Sensitivity of finite-temperature observables to the *center* of \mathbb{G} .



Interpretations of confinement - II

The vacuum acts on the (chromo-)electric charges as a **dual superconductor**, keeping all the flux between the sources squeezed in a **string-like tube** whose energy is proportional to its length \Rightarrow linear growth of $V(R)$ [Parisi '75, 't Hooft '75, Mandelstam '76].



A strong-coupling expansion of $\langle W \rangle$ can be made in terms of string worldsheet surfaces (bounded by the loop contour); this approach fails for too weak couplings.

The physics takes place in the **rough phase** (the string fluctuates quantistically on any length scale): assuming the $R \rightarrow \infty$ behaviour is massless and bosonic, the **Nambu-Goto type actions** yield quantitative predictions that can be tested on the lattice.

Simplifying the theory

The aim is to go down to the most basic formulation, while keeping – at least qualitatively – the same mechanisms as the original theory. This might allow to study them more clearly, besides providing huge computational advantages.

- Study the **(2 + 1) case** instead of the (3 + 1): “minor” changes happen (e. g. sheet-like vortices → string-like ones); this opens the way to exploiting **dual spin models**.
- Theories with a **2nd order transition** (enabling use of the **Svetitsky-Yaffe** idea to pin down an effective dimensionally-reduced theory).
- **Abelian** and/or **discrete** gauge groups.

These changes affect the results in a somewhat mild way, e. g. the paradigmatic quantity $\frac{T_c}{\sqrt{\sigma}}$ is **weakly dependent** on the gauge group!

Towards percolation

Center vortices have a \mathbb{G} -dependent behaviour (flux conservation), but are weakly correlated; one can think of a model whose starting degrees of freedom are \sim fully uncorrelated vortices.

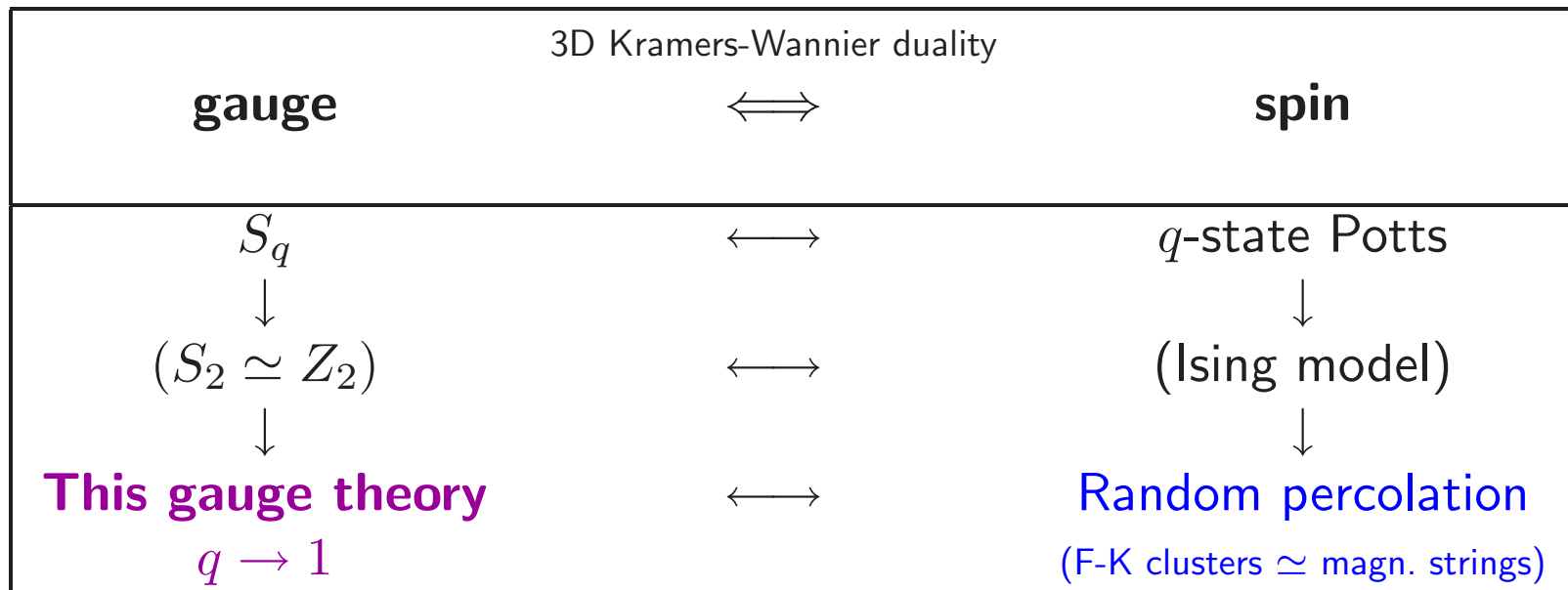
Learn from the **topological features** of the symmetric-group S_q theories (especially \mathbb{Z}_2), and look for a further simplification, with the insight provided by:

- **3D Kramers-Wannier duality** \Rightarrow a **spin** model (link action $\sim \sigma_x \sigma_y$), with a one-to-one mapping for coupling ($\beta_G \rightarrow \beta_I$) and observables.
- **Fortuin-Kasteleyn representation**: describe Z in terms of all possible partitions of the lattice into connected components, with weight dictated by a probability p encoding the original coupling.

Pedigree of the theory

The q -state F-K partition function works with any real non-negative q :

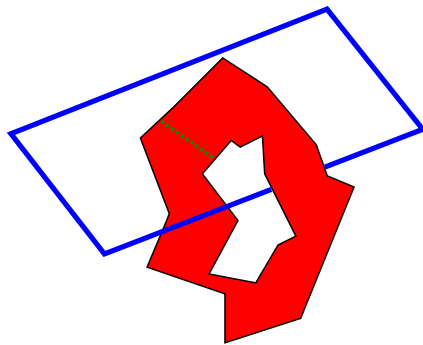
$$Z_q = \sum_{\mathcal{G}} p^{b_{\mathcal{G}}} (1-p)^{N_L - b_{\mathcal{G}}} \cdot q^{N_c(\mathcal{G})} \quad , \quad p = 1 - e^{-\beta I}$$



A closer look at the \mathbb{Z}_2 case

Wilson loops W in gauge theory become 't Hooft loops \widetilde{W} in Ising model: they measure the **linking of magnetised clusters** to the loop contour.

No F-K cluster can form a **closed line linked** to \widetilde{W} : it is incompatible with the electric flux running on the probe loop $\Rightarrow W = 0$.



$$\langle W \rangle \simeq \frac{\text{no. of configurations without linking to } W}{\text{total no. of configurations}}$$

If the Ising system is magnetised (= high β_I , low β_G), a large loop is suppressed with **area law**: confinement, with $\sigma_{\text{gauge}} \leftrightarrow \sigma_{\text{surface}}$.

Pure percolation formulation [Gliozzi, S.L., Panero, Rago, '05]

Work on the $q \rightarrow 1$ case of the spin-models family, and think of it as the dual of the gauge theory under study.

Define the model in the following way:

- The lattice links in a configuration are set to *on* or *off*, independently, with probability p and $1 - p$.
- p plays the rôle of the **coupling constant**: the chain of relations between S_q models implies $\beta_G = -\log(p)$.
- There is **no dynamics** at all; action and partition function are trivial:

$$S = 0 \quad , \quad Z = \sum_n \binom{N_L}{n} p^n (1 - p)^{N_L - n} \equiv 1$$

Percolation in statistical mechanics

Take a finite lattice of size N , and define $\Pi_N(p)$ as the probability that a configuration generated at p contains a connected *on* cluster joining two opposite boundaries.

This wrapping probability approaches a step function $\theta(p - p_c)$ in the thermodynamic limit $N \rightarrow \infty$: p_c is a genuine **second-order critical point**, related to the appearance of an “infinite cluster” in the system.

This is the analogue of the infinite F-K cluster signalling a nonzero magnetisation e. g. in the Ising model.

Depending only on the lattice dimensionality, percolation defines its own **universality class** (e. g. $\nu_{2D} = 4/3$, $\nu_{3D} \simeq 0.8765 \dots$).

[see e. g. Stauffer, Aharony '94]

Gauge observables in percolation

From what has been seen in the \mathbb{Z}_q family of theories, one is led to define any loop expectation value $\langle W \rangle$ as follows:

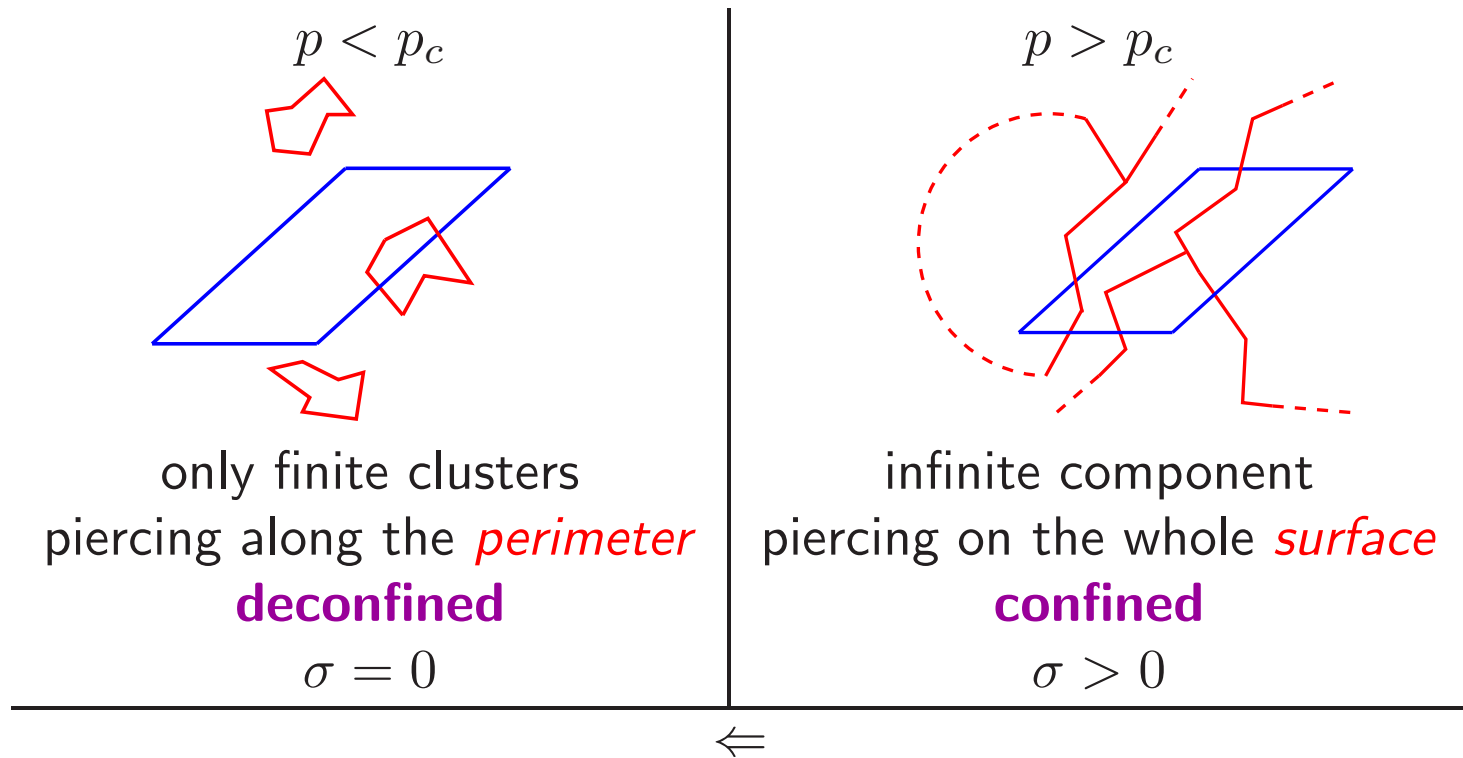
$$\langle W \rangle = \frac{\text{no. of configurations with no } on \text{ clusters linked to } W}{\text{total no. of configurations}}$$

This purely **topological** definition applies to **plaquettes**, **Wilson loops**, and even **Polyakov-Polyakov** correlators.

The only feature of interest in the configuration is the topology of **closed paths** in *on*-clusters: this is a sort of “**gauge invariance**” (the observables do not change if the simply-connected parts of the configuration are modified).

Confinement in percolation

What about the expectation value of a very large Wilson loop?



There is a **genuine deconfinement transition** occurring at p_c !

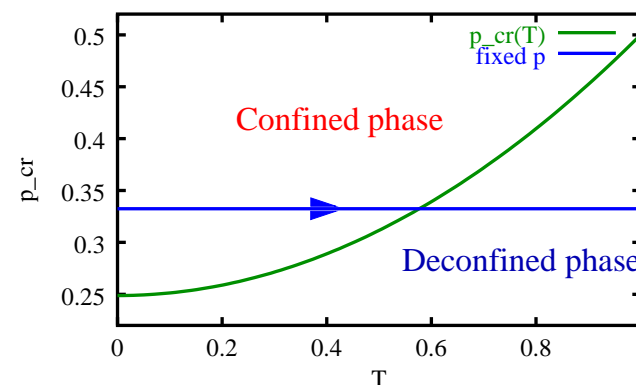
Finite temperature and deconfinement

At finite temperature T , the system is a 2D infinite slice with thickness $\propto 1/T$: the threshold probability $p_c(T)$ for the infinite cluster to appear is now a **function of T** and follows two-dimensional scaling laws.

Since $p_c(T)$ is an increasing function of T , by keeping a fixed probability $p_{cr}(0) < p < p_{cr}(\infty)$ and heating up the system, **the infinite cluster at some point T_c vanishes away**, leaving finite pieces that no longer give the area law to the loop behaviour: this is precisely a **finite temperature deconfinement transition**.

The amplitude ratio $T_c/\sqrt{\sigma}$ is well defined, with scaling laws

$$\begin{aligned}\sigma(p_c) &= S(p_c - p_c^{3D})^{2\nu} \\ T_c(p_c) &= \tau_0(p_c - p_c^{3D})^\nu\end{aligned}$$



String tension from Wilson loops

Preliminarily, measure the quantity $p_c(T)$ for a variety of lattices and temperatures, using the Newman-Ziff algorithm to evaluate the $\Pi(p)$.

Measure the expectation values of **rectangular Wilson loops** in 3D systems with the linking-detection prescription. A fit to an **area+perimeter** law can be tried, $\langle W(R, T) \rangle \propto \exp[-\sigma RT - p(R + T)]$.

The agreement is, however, much better if one includes the **Leading Order correction** coming from the **string rough fluctuations** (more on it later) :

$$\langle W(R, T) \rangle \propto \sqrt{\frac{\eta(i)\sqrt{R}}{\eta(iT/R)}} \cdot \exp[-\sigma RT - p(R + T)]$$

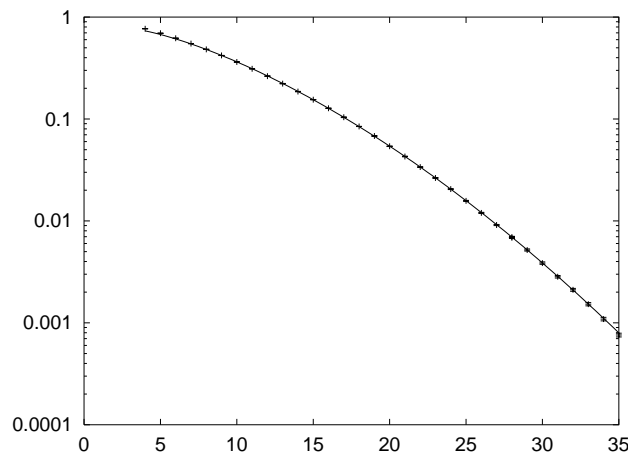
$\eta(\tau) = (e^{2i\pi\tau})^{1/24} \prod_{n=1}^{\infty} [1 - (e^{2i\pi\tau})^n]$ is the Dedekind eta function.

String tension from Wilson loops

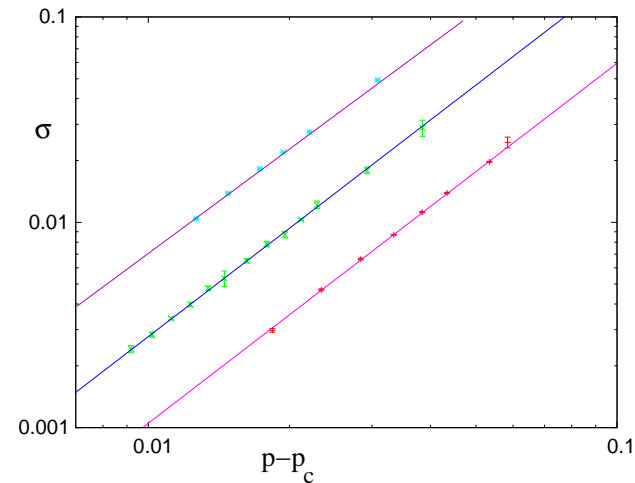
The values obtained for σ show, not too far from $p_{cr}(0)$, a **good scaling behaviour** and allow to extrapolate the amplitude S for each kind of lattice:

$$S_{SC,site} = 3.370(8) \quad S_{SC,bond} = 8.90(3) \quad S_{BCC,bond} = 22.07(2) \dots$$

The universal ratio $T_{cr}/\sqrt{\sigma} \simeq 1.5$ was calculated for seven different lattices and temperatures: its **universality** was proven within errors.



Typical $\langle W(R, R) \rangle$



Scaling of $\sigma(p)$; BCC, SC, SC_s

Polyakov-Polyakov correlators at critical temperature

Exactly at the **critical point**, the correlator between two Polyakov lines should exhibit a **power-law shape**, whose exponent is fixed by the dimensionality and universality class of the system.

Arguing that, at finite T , the system behaves according to 2D percolation universality class, one can use an **adapted version of the Svetitsky-Yaffe conjecture** to predict that:

$$\langle P(0)P(R) \rangle_{T_c} \propto R^{-\frac{5}{24}} .$$

Numerical measurement yield evidence that **this expectation is fulfilled**.

Pure gauge spectrum: glueballs

The plaquette-plaquette (zero-momentum projected) correlator shows a *multiple exponential* decay \Rightarrow it couples to a whole **tower of massive physical states** in the 0^+ spin/parity channel.

Dihedral D_4 time-slice symmetry \Rightarrow operators can be constructed for each channel with $J^P \in \{0^+, 0^-, 2^+, 2^-, 1/3\}$.

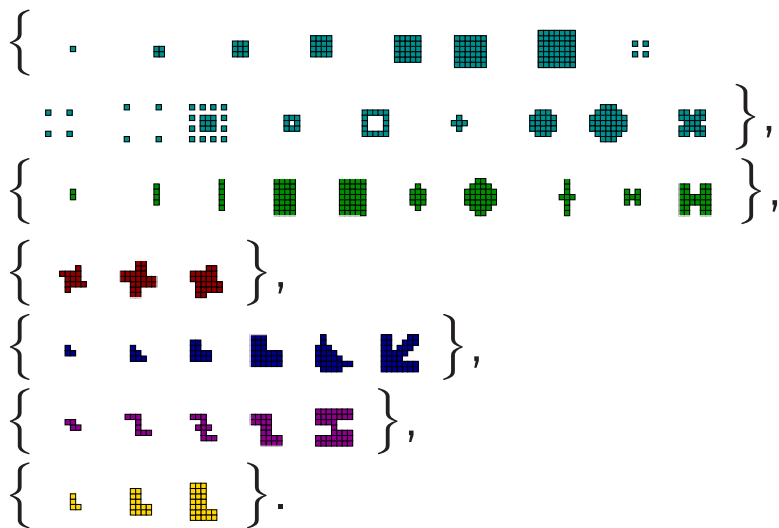
A **cross-correlation matrix** is constructed with:

$$\mathcal{C}_{ij}^{(JP)}(t) = \sum_{x,y}^{(y-x)_3=t} [\langle \mathcal{O}_i^{(JP)}(x) \mathcal{O}_j^{(JP)}(y) \rangle - \langle \mathcal{O}_i^{(JP)} \rangle \langle \mathcal{O}_j^{(JP)} \rangle]$$

and then diagonalised as $\mathcal{C}(t > t_0)\bar{\mathbf{x}} = \lambda^{t_0}(t)\mathcal{C}(t_0)\bar{\mathbf{x}}$, to extract **glueball masses** from each channel.

Glueballs operators' construction

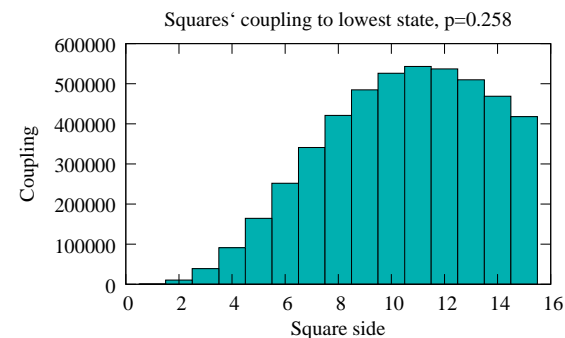
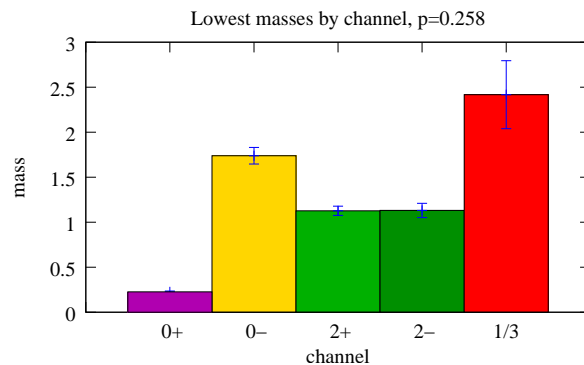
Choosing operators in various symmetry classes, construct spin/parity operators according to the dihedral group representations:



0^-	$\left(\begin{array}{c} \text{red cross} - \text{red cross} \\ \text{purple cross} - \text{purple cross} \end{array} \right) + \left(\begin{array}{c} \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \\ \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \end{array} \right)$
2^+	$\left(\begin{array}{c} \text{green vertical} - \text{green horizontal} \\ \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \end{array} \right) + \left(\begin{array}{c} \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \\ \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \end{array} \right)$
2^-	$\left(\begin{array}{c} \text{blue L} - \text{blue L} + \text{blue T} - \text{blue T} \\ \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \end{array} \right) - \left(\begin{array}{c} \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \\ \text{yellow L} - \text{yellow L} + \text{yellow T} - \text{yellow T} \end{array} \right)$
$1/3$	$\left\{ \begin{array}{l} \text{yellow L} - \text{yellow T} \\ \text{yellow T} - \text{yellow L} \\ \text{yellow L} - \text{yellow T} \\ \text{yellow T} - \text{yellow L} \end{array} \right.$

Glueballs, results [Glozzi, S.L.: '05]

- The 0^+ lightest glueball shows **good scaling**, and a second mass is visible.
- For each channel, the lowest state is easily recognizable: they follow the expected hierarchy, and $m_0^{0^+} / \sqrt{\sigma} \simeq 4.46$ (cf. the $SU(2)$ value 4.7).
- The square loop with highest coupling to the lightest glueball gives an estimate for its diameter: ~ 0.24 fm.



Effective string theory for the flux tube

$R \rightarrow \infty$: the fluctuating flux tube can be described by a **string model**.

Massless bosonic string \Rightarrow Nambu-Goto (+ P-S refinement) action.

One obtains a prediction for loop observables, with “more-than-universal” **LO** and **NLO** terms (besides **area/perimeter** “classical” part) :

$$\langle P(0)P^\dagger(R) \rangle = \frac{e^{-cL - \sigma RL - \frac{(D-2)\pi^2 L [2E_4(\tau) - E_2^2(\tau)]}{1152\sigma R^3}} + \mathcal{O}(1/R^5)}{\eta(\tau)^{D-2}}$$

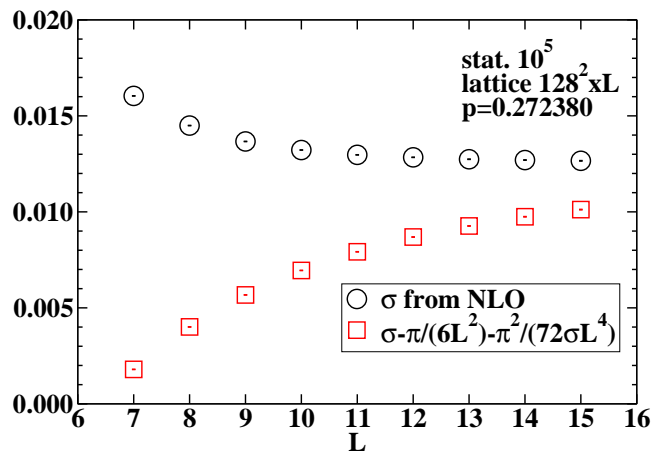
E_2, E_4 are the second and fourth Eisenstein functions, $L = 1/T$ and $\tau = iL/2R$.

In terms of finite-temperature string tension this becomes:

$$\sigma(T) = \sigma_0 - \frac{\pi}{6}T^2 - \frac{\pi^2}{72\sigma_0}T^4 + (\text{non-universal terms. . .})$$

What is the underlying string theory?

From the Polyakov-Polyakov correlators the finite-temperature $\sigma(T)$ is extracted. [Giudice, Gliozzi, S.L., '07, '09]



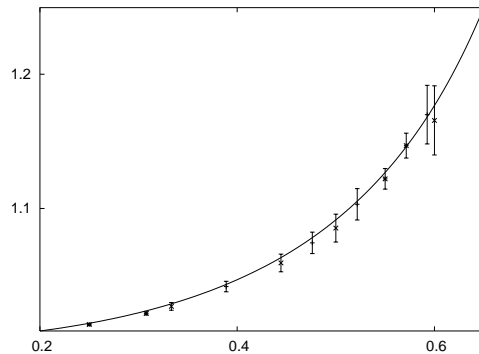
- Not only the $\langle PP \rangle$ is seen to follow the NLO prediction:

$$\sigma(L) = \underbrace{\sigma_0 - \frac{\pi}{6}T^2 - \frac{\pi^2}{72\sigma_0}T^4}_{\text{NLO}} + \underbrace{\frac{\pi^3}{C\sigma_0^2}T^6}_{\text{non-universal}} + \mathcal{O}(T^8);$$

- but in this system also the first model-dependent correction was clearly identified ($C \simeq 300$).
- The continuum limit of $T_c/\sqrt{\sigma_0} = 1.497(2)$ was also refined.

\Rightarrow The effective string of percolation realises a particular model in the Nambu-Goto class of effective theories.

Universal functions as further signals of the rough phase



String evidences in the universal large-distance behaviour of:

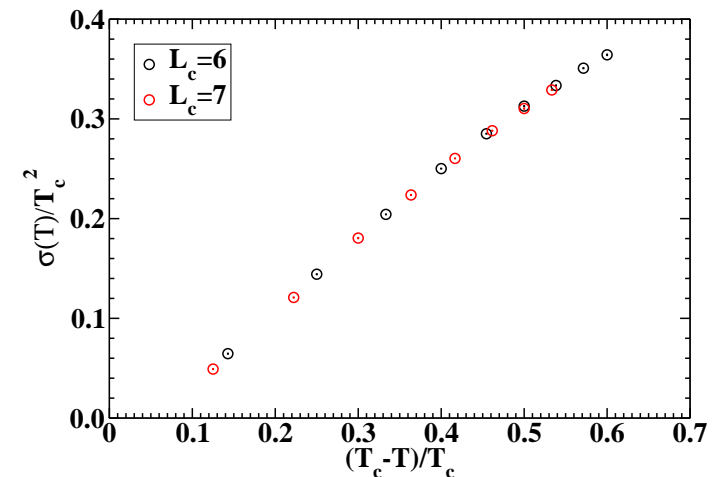
$$e^{n^2\sigma} \frac{W(R+n, R-n)}{W(R, R)} \rightarrow f(t) = \sqrt{\frac{\eta(i)\sqrt{1-t}}{\eta(i\frac{1+t}{1-t})}},$$

with $t = \frac{n}{R}$ and **no adjustable parameters**.

At finite temperature, expected **universality** of the ratio

$$g(t) = \frac{\sigma(T)}{T_c^2},$$

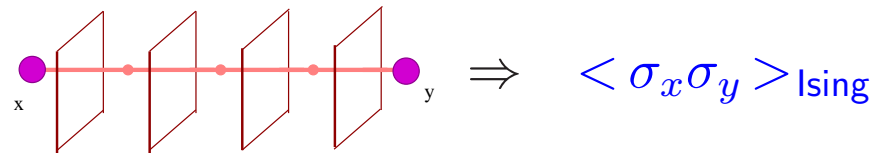
as a function of the reduced temperature $t = \frac{T_c - T}{T_c}$.



The quest for monopoles in percolation [Giudice, Gliozzi, S.L., '08]

The confined phase is signalled also by the disorder parameter $\langle \Phi_m \rangle \neq 0$. When the condensate melts down (deconfinement), monopoles survive in the plasma as well.

3D \mathbb{Z}_2 case: two external magnetic sources (and their 't Hooft line) \Rightarrow a train of flipped plaquettes ($\beta \rightarrow -\beta$): by duality, this means



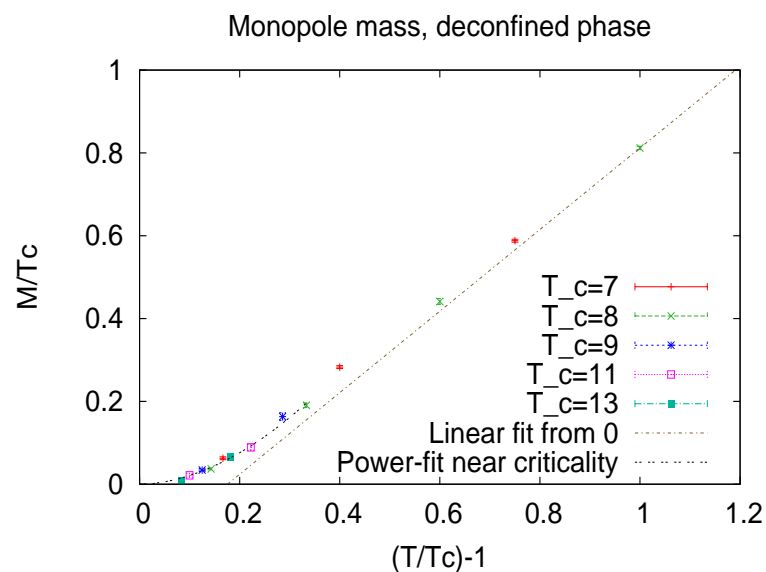
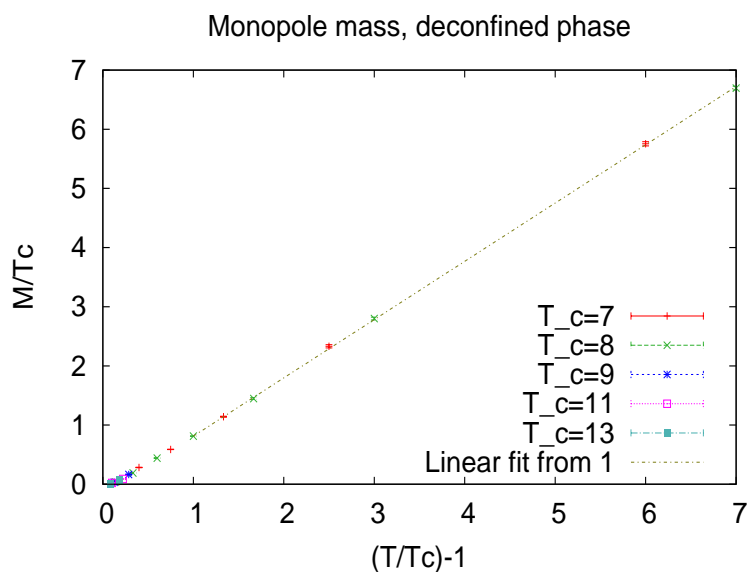
The spin-spin correlator, under the F-K representation, becomes a connectedness measurement, naturally extended to percolation: it is possible to look for monopoles.

Recipe: measure the zero-momentum projected connectedness correlator, then read mass(es) from its exponential decay with t .

Monopoles in percolation: deconfined phase

The deconfined phase displays a **single monopole**.

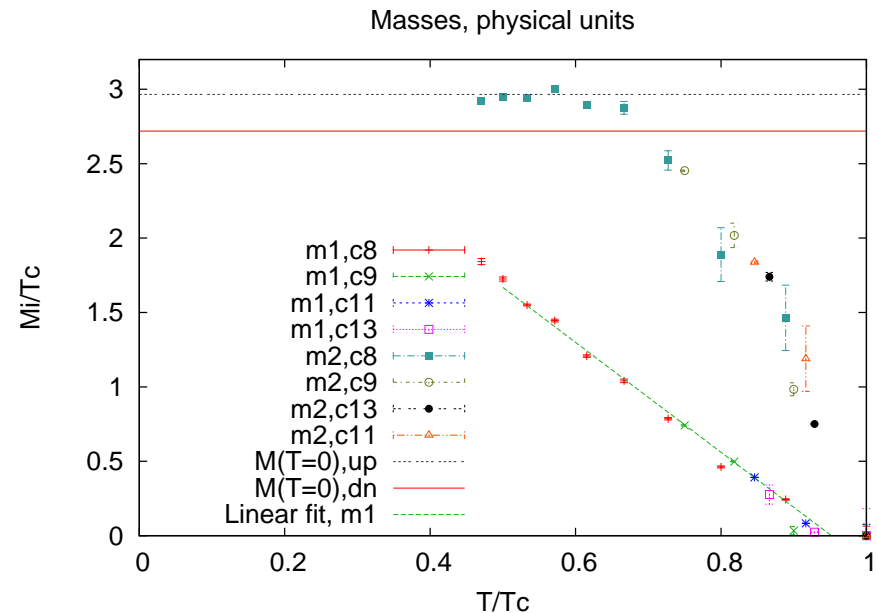
In the range $T \sim 2T_c$ to $7T_c$ the mass is **linear in $(T - T_c)$** , while near criticality a power-law increase is observed.



Monopoles in percolation: confined phase

Surprise: **two monopole masses**, with non-trivial behaviour in T .

- $T = 0$: it is $m_1 = m_2 = m_{\text{glueball}}$.
- $T = T_c$: all masses vanish.
- Inbetween: different behaviours!



Apparently, the masses reach their common $T = 0$ value at $T \sim \frac{1}{3}T_c$ and $T \sim \frac{2}{3}T_c$.

(and the condensate scales near zero-temperature as T^6 , after the physical rescaling to $\frac{\langle \Phi m \rangle}{T_c^{\nu/2}}$).

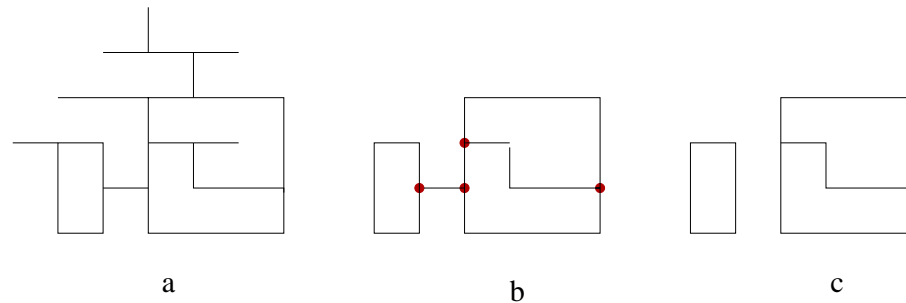
Computational considerations

The simple formulation of the model makes simulations very efficient:

- **No thermalisation issues**: each configuration is generated from scratch.
- **No update** process needed.
- **High statistics** attainable: the only expensive operation is cluster identification/winding detection.
- Large ($L \sim 8$ fm) and fine ($a \sim 0.05$ fm) lattices can be simulated.
- “**Microcanonical**” algorithms give whole continuous functions of p .

Some words on percolation algorithms

Most of the measurements on the randomly-generated configurations involve looking for topological linking to some closed line. Before taking measures, however, the configuration is mapped to its loop skeleton (“loop gauge”) via *removal of dangling ends and bridges*.



The cluster structure is constructed with the *Hoshen-Kopelman algorithm*: each node has a *parent node*, up to the cluster’s root which points to itself. To evaluate winding numbers, a ± 1 offset is associated to links dual to the loop surface in reconstructing clusters.

In the special case of the 1×1 plaquette, this can be avoided if one first goes to the “loop gauge”.

A particularly optimised approach is implemented when (possibly a lot of) loops are to be measured in every spatial position.

Current activities going on. . .

- **Pressure $P(T)$** : adaptation of the integral method based on integration of $\langle \text{plaquette} \rangle$ over the coupling. Computationally a formidable task, but one gets a continuous function of p all at once. As usual, $T = 0$ is the most expensive part. [S.L.]
 - ⇒ currently $T = 0$ data are being produced (but time scales as $\sim L^9$!)
 - ⇒ statistical characterisation of loop structures in standard percolation
- **Finite-temperature glueball spectrum**, in order to compare with the monopole masses. [P. Giudice]
 - ⇒ Monopole masses split, glueball masses do not!

Conclusions

- Percolation has the status of an outright pure gauge theory.
- “No dynamics”, but with properly defined observables the features are far from trivial.
- In particular, extremely fine quantum string effects arise.
- Some properties are still to be completely understood.
- This perspective unveils new aspects of percolation as a statistical model.

Computing of the pressure $P(T)$ in percolation

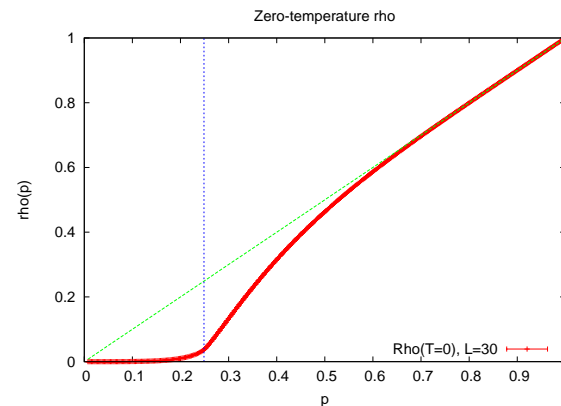
Standard approach: with a proper choice of β_0 ,

$$P_\beta(T) \sim \int_{\beta_0}^{\beta} (\square_T - \square_0) d\beta' .$$

In percolation $\rho(p) = 1 - \square(p)$, density of links *forming closed loops*.

Thus, with $\beta = -\log(p)$,

$$P_p(T) = \int_p^1 \frac{\rho_0(p) - \rho_T(p)}{p} dp .$$



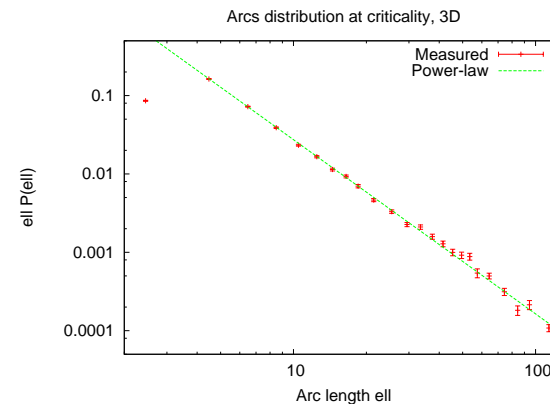
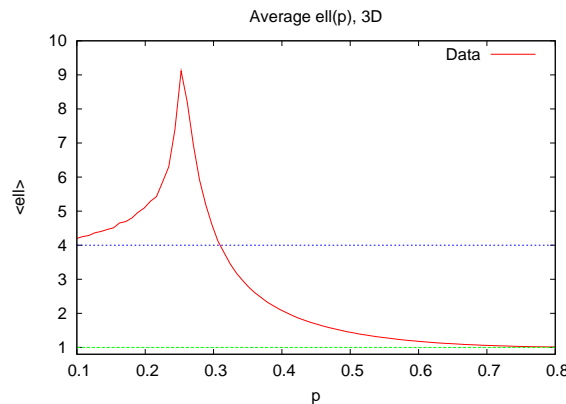
Distribution of loops in percolation

Key relation:

$$\frac{d \log(\rho/p)}{dp} = \ell(p) - 1 ,$$

with $\ell(p) = \int \mathcal{P}_p(\ell) \ell d\ell \simeq$ average length of uninterrupted loop segments.

The maximum of $\ell(p)$ is at the transition point: by scaling, all information on the loop structure is encoded in $\mathcal{P}_{p_c}(\ell) \sim \ell^{-c}$; $c \simeq 3.223(9)$.



Difficulties and payoff

- Take measures in “microcanonical” ensemble (fix the number n of on links rather than p), then convert to the “canonical” with:

$$O(p) = \sum_{n=0}^N \binom{N}{n} p^n (1-p)^{N-n} O_n ,$$

A **single experiment** $n : 0 \rightarrow N$ yields the observable as a function of p .

- This means keeping track of the links forming loops as the system is filled: a **highly nonlocal** task, careful optimisation is needed!
- On the other hand, no need for interpolation of a finite set of sampled points along the coupling.

The algorithm

The configuration is stored in the RAM on three levels simultaneously as the system is filled:

- A physical layer, with single links and their on/off status.
- First abstract layer (“black” clusters): keeps track of the connected components, incrementally.
- Second (“red”) abstract layer: clusters of the loop-only projected configuration.

Resort to expensive non-local operations only when needed: e. g. when the new link joins two points belonging *to the same black component but to different red components*.

