Radiative corrections to the masses of compound particles in the Ising field theory

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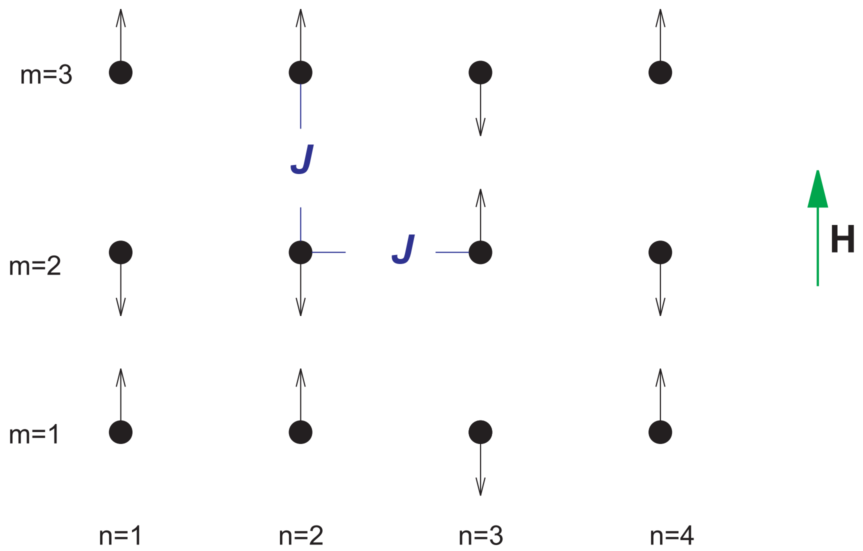
Ising model on the square lattice

$$\mathcal{E}[\sigma] = -\sum_{n=1}^{\mathcal{N}} \sum_{m=1}^{\mathcal{M}} \left(J\sigma_{m,n}\sigma_{m+1,n} + J\sigma_{m,n}\sigma_{m,n+1} + H\sigma_{m,n} \right)$$



$$\sigma_{m,n} = \pm 1$$

The lattice has $\mathcal{M}$ rows and $\mathcal{N}$ columns, $J > 0$ are the coupling constants, H is the external magnetic field.

Ising Field Theory

Definition. Ising field theory is the relativistic $(1+1)$-dimensional quantum field theory, which gives the scaling limit of the two-dimensional Ising model on the lattice in the critical region $T \to T_C$, $H \to 0$.

Euclidean action:

$$\mathcal{A}_{IFT} = \mathcal{A}_{CFT} + 2\pi m \int \varepsilon(x)\, d^2x - h \int \sigma(x)\, d^2x. \tag{1}$$

Action $\mathcal{A}_{CFT}$ corresponds to Conformal Field Theory (CFT) with central charge $c = 1/2$ and describes free massless Majorana fermions.

Primary operators: energy density $\varepsilon(x)$, spin density $\sigma(x)$.

Scaling dimensions: $X_\epsilon = 1$, $X_\sigma = 1/8$.

Parameters m and h are proportional to the deviations of the temperature and magnetic field from their critical point values in the original lattice Ising model: $m \sim\ T_c - T$, $h \sim H$.

Scaling parameter: $\eta = \frac{m}{h^{8/15}}$.

Quantum Hamiltonian of the Ising field theory

$$\mathcal{H} = \mathcal{H}_{FF} + V.$$

Free fermionic Hamiltonian:

$$\mathcal{H}_{FF} = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \omega(p)\, a_p^\dagger a_p,$$

with the dispersion law $\omega(p) = \sqrt{p^2 + m^2}$, and fermionic operators a_p, $a_p^\dagger$:

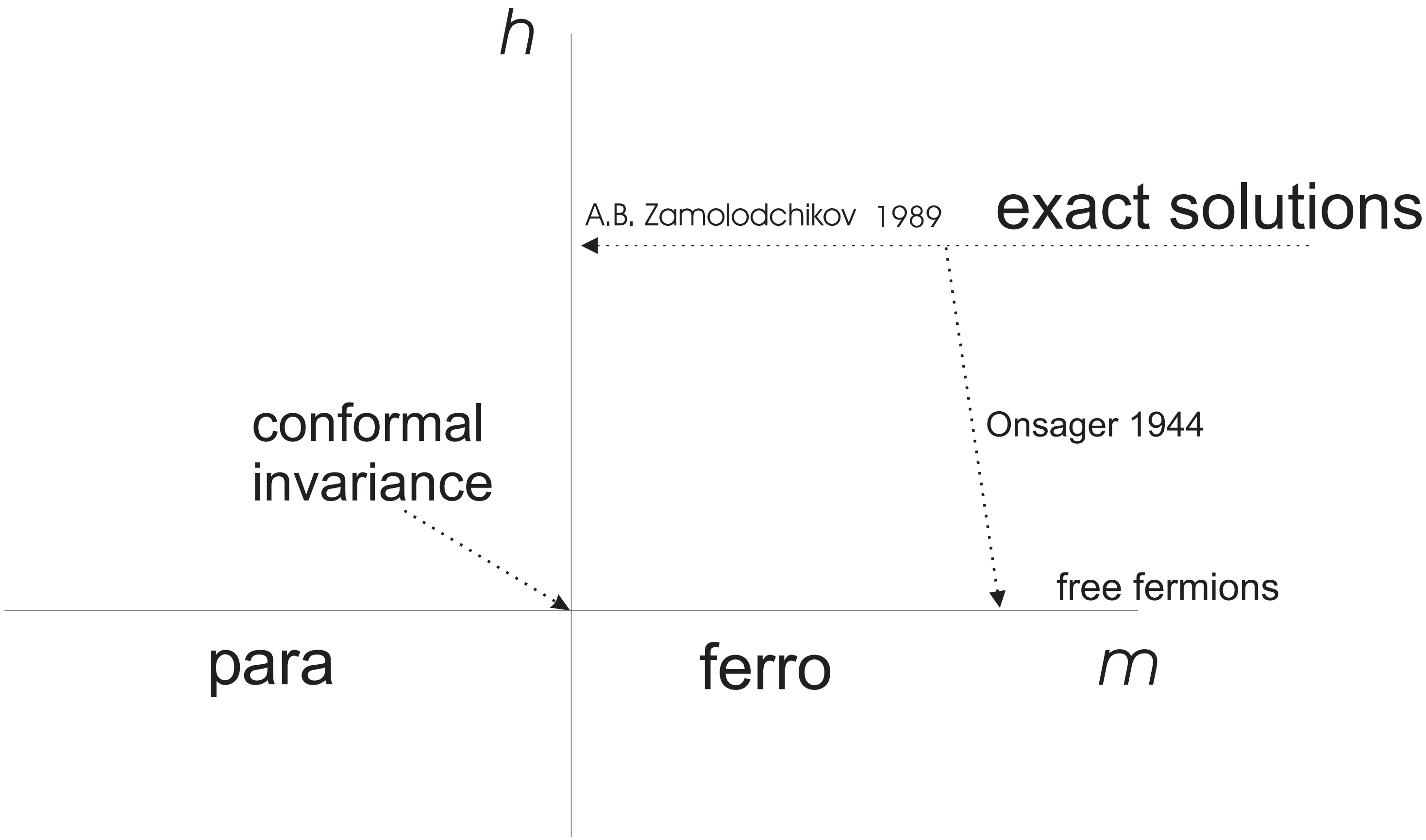
$$\{a_p, a_{p'}^\dagger\} = 2\pi\delta(p - p'), \quad \{a_p, a_{p'}\} = \{a_p^\dagger, a_{p'}^\dagger\} = 0.$$

Interaction:

$$V = -h \int_{-\infty}^{\infty} d\mathrm{x}\, \sigma(\mathrm{x}).$$

Formfactors $\langle p_1 ... p_N | \sigma(x) | p_1' ... p_{N'}' \rangle \equiv \langle 0 | a_{p_1} ... a_{p_N} \sigma(x) a_{p_1'}^\dagger ... a_{p_{N'}'}^\dagger | 0 \rangle$ of the order spin operator $\sigma(x)$ are known due to Berg, Karowski, and Weisz (1979).

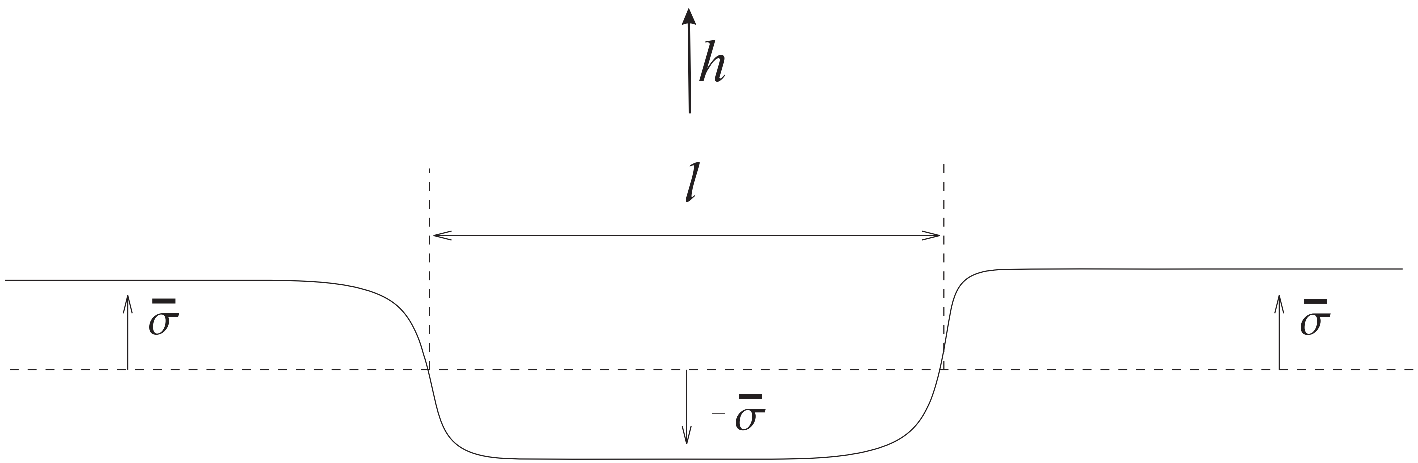
Ising field theory phase diagram



Small-h ferromagnetic regime. Confinement of fermions.

1. $h = 0$ Two ferromagnetic ground states $|\Phi_\uparrow(0)\rangle$ and $|\Phi_\downarrow(0)\rangle$ with spontaneous magnetizations $\langle\sigma\rangle = +\bar{\sigma}$ and $\langle\sigma\rangle = -\bar{\sigma}$ have the same energy $E_\uparrow(0) = E_\downarrow(0)$. Elementary excitations (free fermions) are the domain walls, interpolating between the two degenerate vacua.

2. $h > 0$. Degeneration is removed: $|\Phi_\uparrow(h)\rangle$ is the ground state, $|\Phi_\downarrow(h)\rangle$ is the metastable state; $E_\uparrow(h) - E_\downarrow(h) \approx -2\bar{\sigma}h\,L$. To generate a domain of a metastable phase in the stable surrounding, one needs to add the energy proportional to the length of the domain. Two domain walls **attract** one another with the energy $2\bar{\sigma}h\,|x_2 - x_1|$. This leads to confinement of fermions: an isolated domain wall gains infinite energy, elementary excitations now are the coupled pairs of fermions.



Motivations.

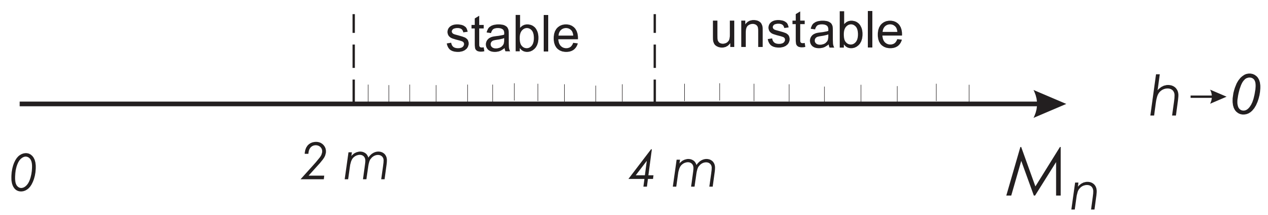
- 2D statistical mechanics: IFT describes the scaling behavior in the 2D Ising universality class;
- (1+1)-dimensional Quantum Field Theories: Described mechanism of confinement is quite common in (1+1)-dimensional QFT;
- 1D condense matter: IFT describes exotic 'bound-spinon' excitations in antiferromagnetic spin chains,
  Experiment: Kenzelmann *et al*, Phys. Rev. B **71**, 094411 (2005);
- high-energy physics: IFT gives a nice relativistic model of quark confinement. Fermions are the 'quarks'. Coupled pairs of fermions are the 'mesons'.

'Meson' energy spectrum: $E_n(P) = (P^2 + M_n^2)^{1/2}$

What masses M_n have the 'mesons' in IFT?

Meson mass spectrum

At weak magnetic field, the messon masses $M_1, M_2, \ldots, M_n, \ldots$ should fill dense the region above the two kink masses $2m$.



Mesons with masses $M_n < 2M_1$ are stable.

Mesons with masses $M_n > 2M_1$ can decay in two (or more) light mesons.

Perturbation theory for the ferromagnetic Ising field theory at small $h > 0$.

$$\mathcal{H} = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, \omega(p)\, a_p^\dagger a_p - h \int dx\, \sigma(x),$$

with free fermionic spectrum $\omega(p) = (p^2 + m^2)^{1/2}$.

The meson energy spectrum $E_n(P) = (M_n^2 + P^2)^{1/2}$ is determined by the eigenvalue problem:

$$\mathcal{H} \;|\, \Phi_n(P)\rangle = [E_n(P) + E_{\text{vac}}]\,|\, \Phi_n(P)\rangle, \quad \hat{P}\,|\, \Phi_n(P)\rangle = P\,|\, \Phi_n(P)\rangle,$$

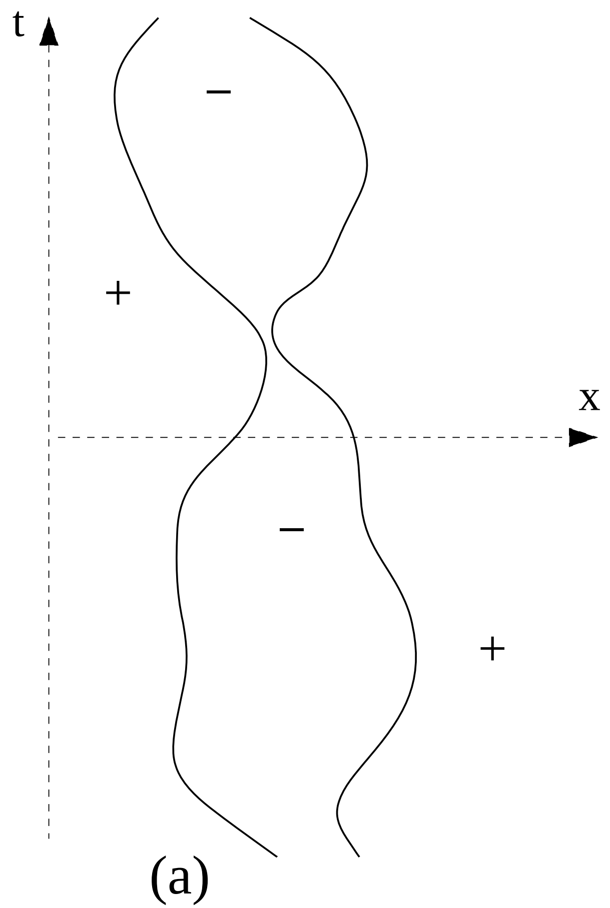
where $\hat{P}$ is the momentum operator, E_{vac} is the ground state energy, and M_n is the meson mass.

Two-quark approximation:

$$\mathcal{P}_2 \mathcal{H} \mathcal{P}_2 \,|\, \widetilde{\Phi}_n(P)\rangle = [\widetilde{E}_n(P) + \widetilde{E}_{\text{vac}}]\,|\, \widetilde{\Phi}_n(P)\rangle, \quad \hat{P}\,|\, \widetilde{\Phi}_n(P)\rangle = P\,|\, \widetilde{\Phi}_n(P)\rangle,$$

where $\mathcal{P}_2$ is the orthogonal projector onto the two-quark subspace $\mathcal{F}_2$ of the Fock space $\mathcal{F}$, $\quad |\, \widetilde{\Phi}_n(P)\rangle \in \mathcal{F}_2$.

In the two-quark approximation $\widetilde{E}_n(P) \neq (\widetilde{M}_n^2 + P^2)^{1/2}$.



Possible world lines of quarks in a meson. (a) Both quarks propagate forwards in time. (b) Creation an annihilation of virtual pairs lead to the presence of more then two quarks in the intermediate state. (From F Z 2006)

Bethe-Salpeter equation (Fonseca, Zamolodchikov 2001, 2006)

In the momentum representation:

$$\Big[\omega(P/2+p)+\omega(P/2-p)-\Delta\widetilde{E}(P)\Big]\,\Psi_P(p)=f_0\int_{-\infty}^{\infty}G_P(p|k)\,\Psi_P(k)\,\frac{dk}{2\pi},$$

where $f_0=2h\bar{\sigma}$ is the 'bare string tension',

$$\langle P'/2-p,P'/2+p|\widetilde{\Phi}(P)\rangle=2\pi\delta(P'-P)\,\Psi_P(p),$$

$$G_P(p|k)=\frac{1}{(p-k)^2}-\frac{1}{(p+k)^2}+G_P^{(reg)}(p|k),$$

and $G_P^{(reg)}(p|k)$ is regular at real p and k.

In the coordinate representation:

$$\Big[\omega(P/2+i\partial_x)+\omega(P/2-i\partial_x)+f_0\,|x|-\Delta\widetilde{E}(P)\Big]\,\Psi_P(x)$$

$$=f_0\int_{-\infty}^{\infty}dx'\,U_P^{(loc)}(x|x')\,\Psi_P(x').$$

Perturbative solutions of the Bethe-Salpeter equation

$$\left[\omega(P/2+i\partial_x)+\omega(P/2-i\partial_x)+f_0\,|x|-\Delta\widetilde{E}(P)\right]\Psi_P(x)$$

$$=f_0\int_{-\infty}^{\infty}dx'\,U_P^{(loc)}(x|x')\,\Psi_P(x').$$

at $f_0\to 0.$

Meson momentum P is a free parameter. For technical reasons, it is convenient to put $P\to\infty$. Then the meson mass in the two-quark approximation $\widetilde{M}$ is defined by

$$\Delta\widetilde{E}(P)=|P|+\frac{\widetilde{M}^2}{2|P|}+O(|P|^{-3}).$$

$$\widetilde{M}_1 < \widetilde{M}_2 < \ldots < \widetilde{M}_n < \widetilde{M}_{n+1} < \ldots$$

Low energy expansion in $\lambda = f_0/m^2 = 2h\bar{\sigma}/m^2 \to 0$, $n \ll \lambda^{-1}$.

McCoy and Wu (1978), FZ(2001, 2006)

$$\frac{\widetilde{M}_n^2}{4m^2} - 1 = z_n\, t^2 + \frac{z_n^2}{5}\, t^4 - \left(\frac{3\, z_n^3}{175} + \frac{57}{280}\right) t^6 + \left(\frac{23\, z_n^4}{7875} + \frac{1543\, z_n}{12600}\right) t^8 + \frac{13}{1120\,\pi}\, t^9 + \left(-\frac{1894\, z_n^5}{3031875} - \frac{23983\, z_n^2}{242550}\right) t^{10} + \frac{3313\, z_n}{10080\,\pi}\, t^{11} + \ldots,$$

where $t = \lambda^{1/3}$, and $(-z_n)$ is the zero of the Airy function, $\mathrm{Ai}(-z_n) = 0$.

Semiclassical (WKB) expansion in λ: $n \gg 1$

R (2005, 2009), FZ (2006).

$$\frac{\widetilde{M}_n^2}{4m^2} = \cosh^2 \theta_n,$$

where θ_n solves equation

$$\sinh 2\theta_n - 2\theta_n = 2\pi\lambda(n - 1/4) + 2\lambda^2 S_1(\theta_n) + 2\lambda^3 S_2(\theta_n) + O(\lambda^4),$$

Multi-quark corrections

Exact eigenvalue problem:

$$\mathcal{H} \;|\, \Phi_n(P)\rangle = [E_n(P) + E_{\rm vac}] \,|\, \Phi_n(P)\rangle, \quad \hat{P} \,|\, \Phi_n(P)\rangle = P \,|\, \Phi_n(P)\rangle,$$

The exact meson eigenvector $|\Phi_n(P)\rangle$ contains four-quark, six-quark, ... contributions, which were ignored in the two-quark approximation.

Renormalized Bethe-Salpeter equation:

$$[\varepsilon(P/2+p) + \varepsilon(P/2-p) - \Delta E(P)]\, \Psi_P(p) = f \int_{-\infty}^{\infty} \mathbb{G}_P(p|k)\, \Psi_P(k)\, \frac{dk}{2\pi},$$

$\Delta E(P) = (P^2 + M^2)^{1/2}$,

$\varepsilon(p)$ is the renormalized quark dispersion law, $\varepsilon(p) \neq (p^2 + m_q^2)^{1/2}$,

f is the the renormalized string tension,

$\mathbb{G}_P(p|k)$ is the renormalized kernel:

$$\mathbb{G}_P(p|k) = \frac{1}{(p-k)^2} - \frac{1}{(p+k)^2} + \mathbb{G}_P^{(reg)}(p|k),$$

where $\mathbb{G}_P^{(reg)}(p|k) = G_P^{(reg)}(p|k) + \Delta\mathbb{G}_P^{(reg)}(p|k)$.

Renormalization of the quark dispersion law:

$$\varepsilon(p) = (p^2 + m^2)^{1/2} + \delta\varepsilon(p) = (p^2 + m_q^2)^{1/2} + \Delta\varepsilon(p).$$

The second order term determines the leading correction to the quark energy $\delta_2\varepsilon(p)$

$$\delta_2 \langle \underline{p}|\mathcal{H}|\underline{k}\rangle = 2\pi\delta(p-k)\,[\delta_2\varepsilon(p) + \delta_2 E_{vac}].$$

Formfactor expansion for $\delta_2\varepsilon(p)$:

$$\delta_2\varepsilon(p) = \delta_{2,3}\varepsilon(p) + \delta_{2,5}\varepsilon(p) + \ldots,$$

$$\delta_{2,n}\varepsilon(p) = -\frac{h^2}{n!}\int_{-\infty}^{\infty} \frac{dq_1 \ldots dq_n}{(2\pi)^{n-1}} \frac{\delta(q_1 + \ldots + q_n - p)}{\omega(q_1) + \ldots + \omega(q_n) - \omega(p)}$$

$$\cdot \lim_{k \to p} \langle p|\sigma(0)|q_1, \ldots, q_n\rangle\langle q_n, \ldots, q_1|\sigma(0)|k\rangle,$$

Integral representation for $\delta_2\varepsilon(p)$ (FZ 2003):

$$\delta_2\varepsilon(p) = -h^2 \int_{-\infty}^{\infty} d\mathrm{x} \int_0^{\infty} d\mathrm{y}\, \lim_{k\to p} \langle p|\sigma(\mathrm{x},\mathrm{y})(1-\mathcal{P}_1)\sigma(0,0)|k\rangle.$$

where $\sigma(\mathrm{x},\mathrm{y}) = \exp(-i\mathrm{x}\hat{P} + \mathrm{y}\mathcal{H}_0)\sigma(0)\exp(i\mathrm{x}\hat{P} - \mathrm{y}\mathcal{H}_0)$.

Renormalized quark mass:

$$m_q^2 = m^2(1 + a_2\,\lambda^2 + a_3\,\lambda^3 + \ldots),$$
$$a_2 = a_{2,3} + a_{2,5} + \ldots,$$

Three-fermion contribution to the formfactor expansion of $\delta_2\varepsilon(p)$:

$$\delta_{2,3}\varepsilon(p) = -\frac{h^2}{3!}\int_{-\infty}^{\infty}\frac{dq_1dq_2dq_3}{(2\pi)^2}\,\frac{\delta(q_1+q_2+q_3-p)}{\omega(q_1)+\omega(q_2)+\omega(q_3)-\omega(p)}$$
$$\cdot \lim_{k\to p}\langle p|\sigma(0)|q_1,q_2,q_3\rangle\langle q_3,q_2,q_1|\sigma(0)|k\rangle.$$

Numerical values:

$$a_{2,3} \approx 0.07 \quad (\text{FZ}\ \ 2001),$$
$$a_{2,3} = \frac{1}{16} + \frac{1}{12\pi^2} = 0.07094\ldots \quad (\text{exact value}\ \ \text{R}\ 2009),$$
$$a_2 = 0.071010809 \quad (\text{FZ}\ \ 2003),$$
$$a_3 = 0 \quad (?)$$
$$\Delta\varepsilon(p) = -\frac{\lambda^2}{8}\frac{p^2m^4}{(p^2+m^2)^{5/2}} + O(\lambda^3), \quad (\text{FZ}\ \ 2006)$$

Renormalized string tension (FZ, 2006):

$$f = f_0 \left(1 + c_2\, \lambda^2 + c_4\, \lambda^4 + \cdots\right),$$

$$E_{vac} = L\, m^2 \left(-\frac{1}{2}\,\lambda + \tilde{g}_2\ \lambda^2 + \tilde{g}_3\ \lambda^3 + \tilde{g}_4\ \lambda^4 + \ldots \right),$$

$$c_{2k} = -2\,\tilde{g}_{2k+1}, \quad c_2 = -0.003889\ldots$$

Renormalization of short-range interaction between quarks $\Delta \mathbb{G}_P^{(reg)}(p|k)$.

The corresponding third-order correction to the meson mass $M_n = 2m\cosh\theta$ is determined by its diagonal part $\Delta \mathbb{G}_P^{(reg)}(p|p)$.

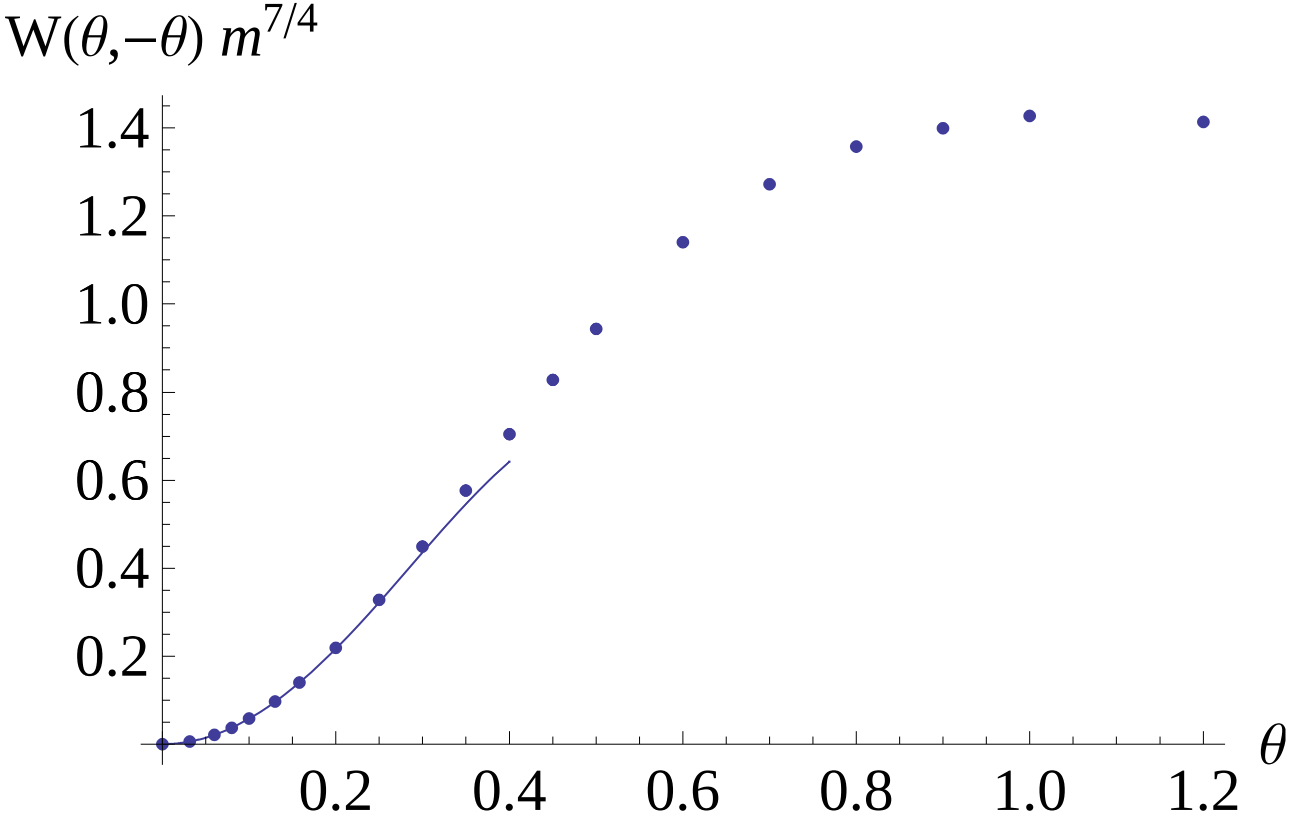
$$\frac{\delta_3 M_n^2}{m^2} = -\frac{\lambda^3}{4\sinh^2\theta}\,\frac{m^2\, W_{irr}(\beta+\theta, \beta-\theta)}{\bar{\sigma}^2},$$

where

$$W_{irr}(\beta_1, \beta_2) = \int_{\infty}^{\infty} d\mathrm{x} \int_0^{\infty} d\mathrm{y} \lim_{\substack{\beta_1' \to \beta_1 \\ \beta_2' \to \beta_2}} \langle \beta_2, \beta_1 | \sigma(\mathrm{x},\mathrm{y})\sigma(0,0) | \beta_1', \beta_2' \rangle_{irr},$$

$|\beta\rangle = (m\cosh\beta)^{1/2} |p(\beta)\rangle, \quad p(\beta) = m\sinh\beta,$

$W_{irr}(\beta_1 + \beta, \beta_2 + \beta) = W_{irr}(\beta_1, \beta_2).$



Small-θ fit:

$$W_{irr}(\theta, -\theta)\, m^{7/4} = 4\, \bar{s}^2 (B_2\, \theta^2 + B_4\, \theta^4) + O(\theta^6),$$

where $B_2 = 0.8, \quad B_4 = -1.6, \quad \bar{s} = \frac{\bar{\sigma}}{m^{1/8}} = 1.35783834\ldots$

Multi-quark correction to the meson mass in the low-energy region to λ^3-order compared with numerical TFFSA results (FZ, 2006)

$$\frac{M_n(\eta) - \widetilde{M}_n(m, f_0)}{h^{8/15}} = \eta \left[a_2\, t^6 + \frac{z_n}{6}(4c_2 - a_2)\, t^8 + \left(-\frac{B_2}{4} + a_3 \right) t^9 + O(t^{10}) \right],$$

$t = \lambda^{1/3}$, $a_2 = 0.0710809...$, $c_2 = -0.003889$, $B_2 = 0.8$. Conjectured: $a_3 = 0$.

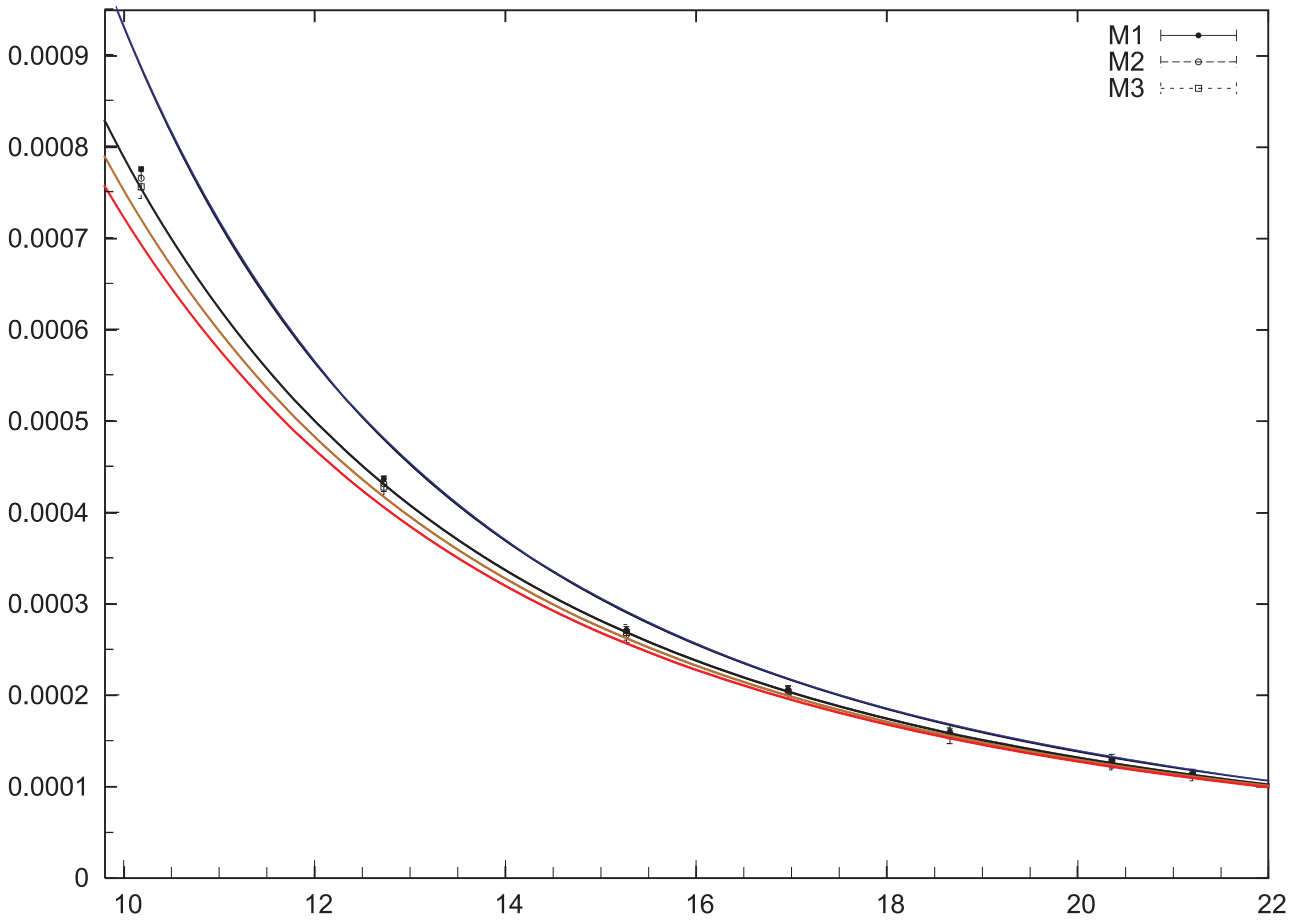
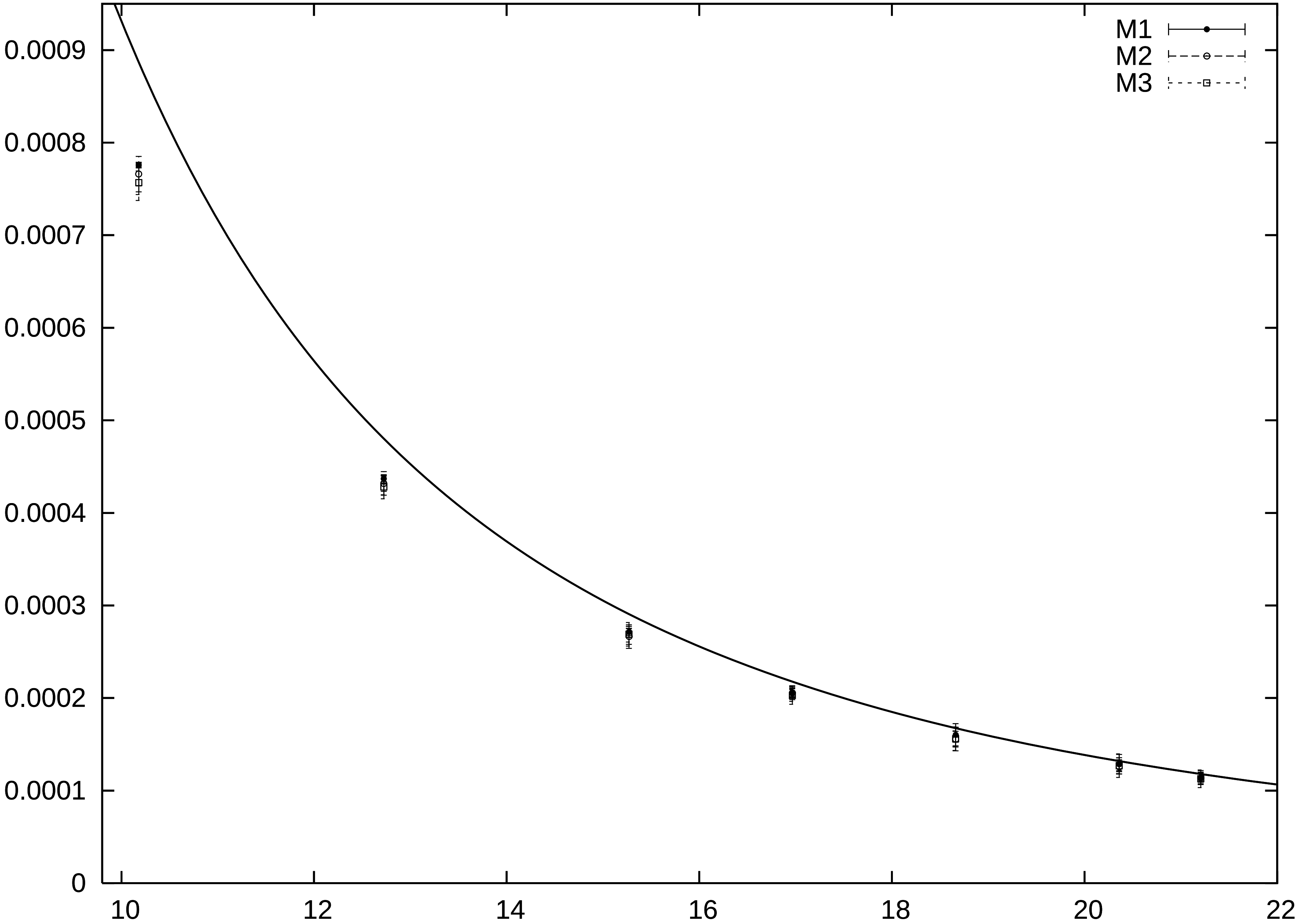


Figure 7 from FZ, hep-th/0612304, 2006



Conclusions

The meson masses $M_n(h)$ in the ferromagnetic Ising field theory in the weak confinement regime $h \to +0$ are studied to the third order in the magnetic field h.

- The third order correction to $\widetilde{M}_n(h)$ in the semiclassical case $n \gg 1$ coming from the perturbative solution of the 'bare' Bethe-Salpeter is calculated.
- It is conjectured, that the renormalized quark mass $m_r(\lambda) = m(1 + a_2\lambda^2 + a_4\lambda^4 + \ldots)$ expanded in λ has only even powers in $\lambda \sim h$.
- Contribution $a_{2,3}$ of the three-quark diagrams into the second-order radiative correction to the quark mass is calculated exactly.
- An integral representation for the third-order 'local' multi-quark correction to $M_n(h)$ coming from the renormalization of the regular part of the integral kernel in the Bethe-Salpeter equation is obtained.
- Obtained radiative corrections to masses of three lightest mesons are in good agreement with TFFSA data of Fonseca and Zamolodchikov.

Further developments

1. V.A. Fateev, S.L. Lukyanov, and A.B. Zamolodchikov

On the mass spectrum in the 't Hooft's 2D model of mesons

J. Phys. A **42** (2009) 304012

Bethe-Salpeter equation in IFT infinite momentum frame $P \to \infty$. FZ 2006

$$\left[\frac{4m^2}{1-u^2} - \tilde{M}^2\right]\Phi(u) = 2h\bar{\sigma}\int_{-1}^{1}\frac{dv}{\pi}\frac{\Phi(v)}{\sqrt{(1-u^2)(1-v^2)}}\left[\frac{1-uv}{(u-v)^2} + \frac{uv}{2}\right]$$

Bethe-Salpeter equation $U(N)$ QCD at $N \to \infty$ in two-dimensions in infinite momentum frame $P \to \infty$. 't Hooft 1974

$$\left[\frac{4m^2}{1-u^2} - \tilde{M}^2\right]\Phi(u) = 2g^2\int_{-1}^{1}\frac{dv}{\pi}\frac{\Phi(v)}{(u-v)^2}$$

Confinement in other two-dimensional models

- sine-Gordon model: Delfino and Mussardo (1998);
- trikritical Ising model: Lepori, Mussardo and Tótz (2008);
- a wide class of non-integrable 2D conformal field theories: Mussardo and Takács (2009);
- q-state Potts field theory: Delfino and Grinza (2008); R (2009); Lepori, Tótz, and Delfino (2009);

The mechanism of confinement (through breaking of the degeneracy of discrete vacua) in these model is the same as in the Ising field theory. However, these models describe interacting particles even at zero 'magnetic field' breaking the discrete symmetry.

How to calculate masses of compound particles at $h \to 0$ in such models?

S.B. Rutkevich, arXiv:0907.3697 (2009)

The q-state Potts model on the square lattice

Hamiltonian:

$$\mathcal{H} = -\frac{1}{T} \sum_{<x,y>} \delta_{s(x),s(y)} - H \sum_{x} \delta_{s(x),q}.$$

Here x and y enumerates the sites of the square lattice, $s(x) = 1, \ldots, q$ is the discrete spin variable, the first summation is over the nearest neighbour pairs, T is the temperature, H is the external magnetic field applied along the q-th direction, $\delta_{\alpha,\alpha'}$ is the Kroneker symbol.

Order parameter:

$$\sigma_\alpha(x) = \delta_{s(x),\alpha} - \frac{1}{q}, \quad \alpha = 1, \ldots, q.$$

Ferromagnetic phase transition at $T = T_c$:

$$T_c = \frac{1}{\log(1+\sqrt{q})}.$$

Transition is continuous at $2 \leq q \leq 4$.

The scaling limit at $H \to 0$, $T \to T_c$ is described by the action

$$\mathcal{A} = \mathcal{A}^{(q)}_{CFT} - \tau \int d^2x\, \varepsilon(x) - h \int d^2x\, \sigma_q(x) \ ,$$

$\mathcal{A}^{(q)}_{CFT}$ corresponds to the Conformal Field Theory, associated with the the critical point.

Central charge:

$$c(q) = 1 - \frac{6}{t(t+1)}, \quad \text{where} \quad \sqrt{q} = 2\sin\frac{\pi(t-1)}{2(t+1)}.$$

Coupling constants: $\tau \sim (T - T_c)$ and $h \sim H$.

At $h = 0$ the action is invariant under the permutation group S_q.

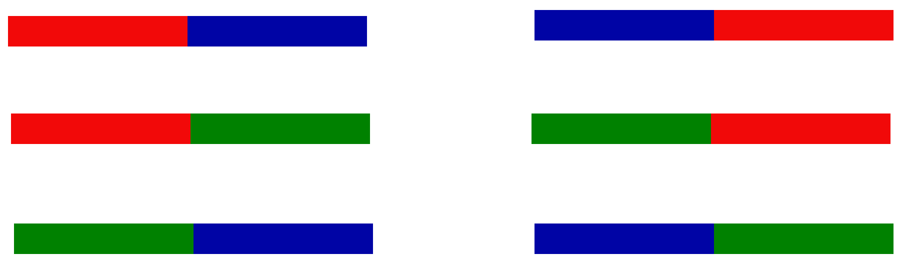
The vacuum at $\tau < 0$, $h = 0$ is q-fold degenerated.

At $h \neq 0$ the action symmetry group reduces to S_{q-1}.

The 3-state Potts field theory at $\tau < 0$.

Three degenerate $|0_\alpha\rangle$ vacua at $h = 0$. $\alpha = 1, 2, 3$.

At $h = 0$ the particle sector consists of six kinks $K_{\alpha\beta}(\theta)$ at $\alpha, \beta = 1, 2, 3$.



$$E = m \cosh\theta, \quad p = m \sinh\theta.$$

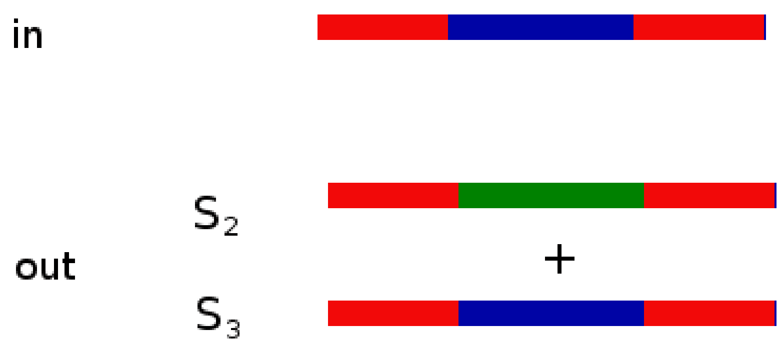
The q-state Potts field theory is integrable at $h = 0$
(Chim and A.B. Zamolodchikov, 1992)

Two-kink scattering at $h = 0$, $\tau < 0$:

$$K_{\alpha\gamma}(\theta_1)K_{\gamma\beta}(\theta_2) = S_0(\theta_{12}) \sum_{\delta \neq \gamma} K_{\alpha\delta}(\theta_2)K_{\delta\beta}(\theta_1) + S_1(\theta_{12})K_{\alpha\gamma}(\theta_2)K_{\gamma\beta}(\theta_1)$$

at $\quad \alpha \neq \beta$,

$$K_{\alpha\gamma}(\theta_1)K_{\gamma\alpha}(\theta_2) = S_2(\theta_{12}) \sum_{\delta \neq \gamma} K_{\alpha\delta}(\theta_2)K_{\delta\alpha}(\theta_1) + S_3(\theta_{12})K_{\alpha\gamma}(\theta_2)K_{\gamma\alpha}(\theta_1).$$

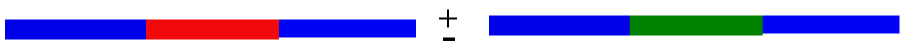


Application of the magnetic field $h > 0$ breaks the degeneracy of discrete vacua at $\tau < 0$.

Kinks are now bound into mesonic and baryonic states.

Two series of mesonic states

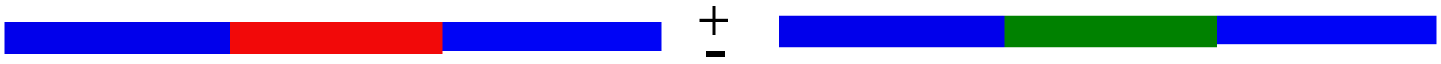
$$\pi_{0,1} = K_{31}K_{13} \pm K_{32}K_{23}.$$



Two series of baryonic states

$$p_{0,1} = K_{31}K_{12}K_{23} \pm K_{32}K_{21}K_{13}.$$

Meson masses in the limit $h \to +0$



At small h, the average distance between the quarks in the meson is large compared with the correlation length m^{-1}. Therefore, at $x \gg m^{-1}$:

$$[2\,\omega(\hat{p}) - M_n + \Delta\mathcal{E}\,x]\;\Phi^{(n)}(x) = 0.$$

Here $\Phi^{(n)}(x)$ is the wave function describing the relative motion of two quarks in a meson in its rest frame. After the Fourier transform:

$$[2\,\omega(p) - M_n + i\,\Delta\mathcal{E}\,\partial_p]\;\Phi^{(n)}(p) = 0,$$

providing

$$\Phi^{(n)}(x) = \int_{\infty}^{\infty} \frac{dp}{2\pi}\,\exp\left\{\frac{i[f(p) - p\,M_n]}{\Delta\mathcal{E}} + ipx\right\}$$

where

$$f(p) = 2\int_0^p dq\,\omega(q) = m^2\left[\theta + \frac{\sinh 2\theta}{2}\right].$$

At small distances $x \lesssim m^{-1}$ interaction between quarks is strong. But in the leading order in h it can be characterized by the $h = 0$ scattering matrix known due to Chim and Zamolodchikov. This determines the boundary condition at $x = 0$ for the meson wave function, $\Phi^{(n)}(x)$ and masses M_n of mesons.

For small enough n, equation for $\Phi^{(n)}(x)$ reduces to the Airy equation:

$$\left[2m - \frac{\partial_x^2}{m} - M_n + \Delta\mathcal{E}\, x\right] \Phi^{(n)}(x) = 0.$$

The boundary condition is 'bosonic' for π_0; $\Phi_0'(0) = 0$,
and 'fermionic' for π_1, $\Phi_1(0) = 0$, providing

$$M_n = 2m + \zeta^{2/3} z_n'\, m \quad \text{for} \quad \pi_0,$$
$$M_n = 2m + \zeta^{2/3} z_n\, m \quad \text{for} \quad \pi_1.$$

Here $\zeta = \Delta\mathcal{E}/m^2 \sim h$, $(-z_n)$ are the zeros of the Airy function, $\mathrm{Ai}(-z_n) = 0$, and $(-z_n')$ are the zeros of its derivative, $\mathrm{Ai}'(-z_n') = 0$.

These results were recently partly confirmed by Lepori, Tóth and Delfino (2009) by numerical calculation of particle spectrum of 3-state Potts field theory by means of Truncated Conformal Space Approach (TCSCA), invented by Yurov and Al.B. Zamolodchikov (1990).

For the masses of five lightest mesons they observe the power law

$$M_n - 2m \approx c^{(n)} h^{\gamma}$$

with $\gamma \approx 0.7$.
For the ratio of prefactor corresponding to the lightest π_0 and π_1 mesons they obtain the approximate value

$$c_1^{(1)}/c_0^{(1)} \approx 2,$$

to be compared with the analytical prediction $z_1/z_1' \approx 2.3$.