Exact Renormalisation Group

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Kadanoff Scaling

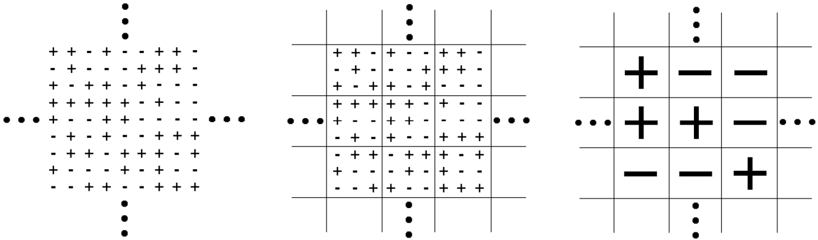
1966 **Kadanoff** introduced a complete new concept of scaling. He maps a near critical or critical system onto itself by a reduction of the degrees of freedom.
We consider a lattice of Ising Spins s with a lattice space *a* and the Hamiltonian:

$$H = -\frac{1}{2} \sum_{<i,j>} J s_i s_j + \sum_j h s_j . \tag{1}$$

The partition function is:

$$Z(T) = \int \frac{1}{h^{3N} N!} e^{-\frac{H(q,p)}{k_B T}} d^{3N} q d^{3N} p . \tag{2}$$

- Divide the lattice into blocks with the size $L \times L \times L$. If $L = n \cdot a$ each block contains n^d spins.
- Associate with each block a new effective block spin S. The new system of the block spins should have the same attributes as the old one.
- Renormalise the coupling coordinates in such a way, that the partition function stays the same.
- Rescale the spacial coordinates.



Kadanoff Scaling

The BAD news:

- It is impossible to realize a Kadanoff step on macroscopic systems.
- The Hamiltonoperator gets more and more difficult with every step of rescaling.

$$H = -\frac{1}{2}\sum_{<i,j>} J s_i s_j + \sum_{<<i,j>>} J_2 s_i s_j + \sum_{<<i,j,n,m>>} J_3 s_i s_j s_n s_m + \ldots + \sum_j h s_j \,.$$

The GOOD news:

- Kadanoff was able to obtain the hyperscaling laws.
- The Kadanoff Scaling Picture was the foundation of the exact renormalisation groups (RG).

The Renormalisation Group by Wilson

From now on we want to work in scalar field theory.
We change to momentum space and consider the action:

$$S[\phi] = \sum_{n=2}^{\infty} \int \prod_{j=1}^{n} \left(d^d p_j \lambda_n(\vec{p_1}...\vec{p_n}) \phi\left(\vec{p}_1\right) ... \phi\left(\vec{p}_n\right) \delta(\vec{p_1} + ... + \vec{p_n}) \right) .$$

We consider the generating functional:

$$Z[J] = \frac{1}{Z_0} \int \mathcal{D}\phi \exp\{-S[\phi] + J \cdot \phi\},$$

with the source term: $J \cdot \phi = \int d^d x J(\vec{x})\phi(\vec{x})$.

The Renormalisation Group by Wilson

Λ is defined as the biggest absolute value of the momentum of the system. In a RG-Step the momentum of the action is integrated out between Λ and a momentum Λ' with: $\Lambda' = \Lambda - d\Lambda$.
The RGT has the following attributes:

- It preserves the generating functional Z.
- It maps one action onto another one.
- It consists of integrating out degrees of freedom of short distances to obtain an effective action for the degrees of freedom of long distances.

Transformation + Rescaling and Renormalisation = Renormalisation Group Transformation (RGT)

The Wilson Renormalisation Group

We divide the field: $\phi\left(\vec{p}\right)$ in a ϕ_{gr} where the absolute momentum is bigger and another part ϕ_{sm} where it is smaller than Λ'. The generating functional becomes:

$$Z[J] = \frac{1}{Z_0}\int \mathcal{D}\phi \exp\{F[\phi,\lambda_i,\Lambda]\} = \frac{1}{Z_0}\int \mathcal{D}\phi_{gr}\mathcal{D}\phi_{sm} \exp\{F[\phi_{gr},\phi_{sm},\lambda_i,\Lambda]\}. \tag{3}$$

With: $F[\phi,\lambda_i,\Lambda] = -S[\phi] + J\cdot\phi$.
After the RG-step we obtain a generating functional that does not depend on momentums with $\left|\vec{p}\right| > \Lambda'$:

$$Z[J] = \frac{1}{Z_0'}\int \mathcal{D}\phi_{sm} \exp\{F[\phi_{sm},\lambda_i',\Lambda']\}. \tag{4}$$

Transformation of the Field Variable

We consider a general transformation of the field which leaves the partition function invariant: $\phi'(\vec{p}) = \phi(\vec{p}) + \sigma\Psi[\phi,\vec{p}]$.

We get: $S[\phi'] = S[\phi] + \sigma \int \Psi[\phi,\vec{p}]\frac{\delta S[\phi]}{\delta\phi(\vec{p})} d^d p$.

Since the partition function must stay the same we obtain:

$$Z_0 = \int \mathcal{D}\phi' \exp\{-S[\phi']\} = \int \mathcal{D}\phi \exp\{-S[\phi] - \sigma T_T[\Psi]S[\phi]\}.$$

As we consider infinitesimal transformations we obtain:

$$\frac{dS}{d\sigma} = T_T[\Psi]S.$$

Rescaling

We consider an infinitesimal change of the (momentum) scale:

$$p' = (1+\sigma)p. \tag{5}$$

The partition function has to stay the same:

$$Z_0 = \int \mathcal{D}\phi' \exp\{-S[\phi']\} = \int \mathcal{D}\phi \exp\{-S[\phi] - \sigma T_R S[\phi]\}. \tag{6}$$

As we consider infinitesimal transformations we obtain:

$$\frac{dS}{d\sigma} = T_R S. \tag{7}$$

We have to consider changes in:

- The differential volume.
- The couplings.
- The field itself.

The Wilson Renormalisation Group

A complete RG-Step is given by:

$$\dot{S} = T_T S + T_R S \tag{8}$$

We have to introduce the RG-time:

$$\frac{\partial}{\partial t} = -\Lambda \frac{\partial}{\partial \Lambda} \qquad \rightarrow \qquad \dot{S} = \frac{\partial S}{\partial t} = -\Lambda \frac{\partial S}{\partial \Lambda}. \tag{9}$$

Wilson was the first one who published a complete RG-Equation:

$$\dot{S} = \int (c+2\vec{p}^2)\{\frac{\delta^2 S}{\delta\phi(-\vec{p})\delta\phi(\vec{p})} - \frac{\delta S}{\delta\phi(\vec{p})}\frac{\delta S}{\delta\phi(-\vec{p})} + \phi(\vec{p})\frac{\delta S}{\delta\phi(\vec{p})}\} d^d p + T_R S.$$

The equation was introduced with a smooth Cutoff.

Approximations and Truncations

A first approximation could be the truncation of the action at a fixed order of the fields. Another form is the so called derivative expansion:

$$S[\phi] = \int d^d x \{V(\phi) + \frac{1}{2} Z(\phi)(\partial\phi)^2 + O(\partial^4)\}. \tag{10}$$

With a arbitrary Potential: $V(\phi)$.
If we consider the terms of the action to be independent of the fields ($Z(\phi) = 1.0$) we obtain the Local Potential Approximation:

$$S[\phi] = \int d^d x \frac{1}{2}(\partial\phi)^2 + V(\phi). \tag{11}$$

The Potential is given by:

$$V(\phi) = \sum_{n=2}^{\infty} \int \prod_{j=1}^{n} \left(d^d x_j\right) u_n \phi_1 ... \phi_n \delta(\vec{x_1} + ... + \vec{x_n}). \tag{12}$$

- The RG-Flow can be observed in the coupling space.
- We find a fixed point if all couplings stay constant within a RG-Transformation.
- A critical point in the statistical physics leads to a fixed point in the coupling space.
- In the scalar Φ^4-Theory and $3 < d < 4$ there should exists the Gaussian and the Wilson-Fisher fixed point.
- The existence of nontrivial fixed points depends on the dimension of the system.

We define the β-Function:

$$\beta_i(\vec{\lambda}) = \frac{\partial \lambda_i}{\partial t}. \tag{13}$$

At a fixed point we have:

$$\beta_i(\vec{\lambda}^*) = 0 \tag{14}$$

for all $1 \leq i \leq N$. If we consider the point $\vec{\tilde{\lambda}}$ in the vicinity of the fixed point $\vec{\lambda}^*$ with $a_i = \tilde{\lambda}_i - \lambda_i^*$ for every component. The β-Function becomes:

$$\beta_i(\vec{\tilde{\lambda}}) = \sum_{j=1}^{N} \frac{d\beta_i}{d\lambda_j}|_{\vec{\lambda}^*} \cdot a_j + O(a_j^2). \tag{15}$$

We define the matrix: $M_{ij} = \frac{d\beta_i}{d\lambda_j}|_{\vec{\lambda}^*}$ and assume that it is diagonalizable with the eigenvalues l_i and the eigenvectors: $\vec{e}_i$. With the eigenvectors you can write:

$$\vec{a} = \sum_{i=1}^{N} s_i \vec{e}_i. \tag{16}$$

We obtain in first order:

$$\vec{\beta}(\tilde{\lambda}_i) = \sum_{i=1}^{N} l_i s_i \vec{e}_i. \tag{17}$$

With a iteration length b, we obtain:

$$b\frac{d\vec{a}}{db} = \sum_{i=1}^{N} s_i(b) M \vec{e}_i = \sum_{i=1}^{N} l_i s_i(b) \vec{e}_i \tag{18}$$

and the differential equation:

$$b\frac{ds_i\,(b)}{dt} = l_i s_i\,(b)\,, \tag{19}$$

which is solved by:

$$s_i(b) = s_i(1) b^{l_i}. \tag{20}$$

The Equation of Wegner and Houghton

Wegner and Houghton introduced a sharp cutoff and created a new RG-Equation:

$$\dot{S} = \frac{1}{2}\int' \left\{ -\frac{\delta S}{\delta\phi(\vec{p})}\left(\frac{\delta^2 S}{\delta\phi(\vec{p})\delta\phi(-\vec{p})}\right)^{-1}\frac{\delta S}{\delta\phi(-\vec{p})} + \mathrm{Tr}\left(\ln\left(\frac{\delta^2 S}{\delta\phi(\vec{p})\delta\phi(-\vec{p})}\right)\right)\right\} d^d p + T_R S.$$

The prime on the integral symbol indicates that the momenta are integrated out between Λ' and Λ.
We want to work with the action:

$$S = \frac{1}{2}\int d^d p \vec{p}^2 \phi(\vec{p})\phi(-\vec{p}) + \sum_{i=2}^{\infty}\int \lambda_i \left(\prod_{j=1}^{i} d^d p_j \phi(\vec{p}_j)\right)\delta(\sum_{j=1}^{i}\vec{p}_j).$$

The equation of Wegner and Houghton is divided in a Linkterm, a Loopterm and a term that comes from the rescaling.

$$\dot{\lambda}_i = \dot{\lambda}_{Li,i} + \dot{\lambda}_{Lo,i} + \dot{\lambda}_{Re,i}.$$

The Linkterm

The derivations of the action are calculated as:

$$\frac{\delta S}{\delta\phi(\vec{p})} = \left(\int d^d p \Lambda^2 \phi(-\vec{p}) + i \int \lambda_i \left(\prod_{n=1}^{i-1} d^d p_n \phi(\vec{p}_n) \right) \delta((\sum_{n=1}^{i-1} \vec{p}_n) + \vec{p}) \right) .$$

$$\frac{\delta S}{\delta\phi(\vec{p})\delta\phi(-\vec{p})} = \Lambda^2 + \sum_{i=2}^{\infty} i(i-1) \int \lambda_i \left(\prod_{j=1}^{i-2} d^d p_j \phi(p_j) \right) \delta((\sum_{j=1}^{i-2} \vec{p}_j) + \vec{p} - \vec{p}).$$

The Deltafunktions can be rewritten:

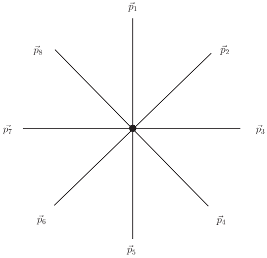
$$\delta\left((\sum_{n=1}^{i-1} \vec{p}_n) + \vec{p} \right) \cdot \delta\left((\sum_{m=1}^{j-1} \vec{p}_m) - \vec{p} \right) = \delta\left((\sum_{n=1}^{i-1} \vec{p}_n) + (\sum_{m=1}^{j-1} \vec{p}_m) \right) \cdot \delta\left((\sum_{m=1}^{j-1} \vec{p}_m) - \vec{p} \right) .$$

With the geometric series: $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ we can write:

$$\left(\frac{\delta S}{\delta\phi(\vec{p})\delta\phi(-\vec{p})}\right)^{-1} = \frac{1}{\Lambda^2 + 2\lambda_2 + 4\cdot 3\lambda_4\phi^2 + 6\cdot 5\lambda_6\phi^4 + \dots}$$
$$= \frac{1}{\Lambda^2 + 2\lambda_2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{4\cdot 3\lambda_4\phi^2 + 6\cdot 5\lambda_6\phi^4 + \dots}{\Lambda^2 + 2\lambda_2}\right)^k .$$

The terms of the action can be considered as vertices:

$$\int \lambda_8 \left(\prod_{i=1}^{8} d^d p_i \phi(\vec{p}_i)\right)\delta\left(\sum_{j=1}^{8}\vec{p}_j\right) \rightarrow$$



The renormalized propagator is introduced by:

$$\text{Prop} = \frac{1}{\Lambda^2 + 2\lambda_2}. \tag{21}$$

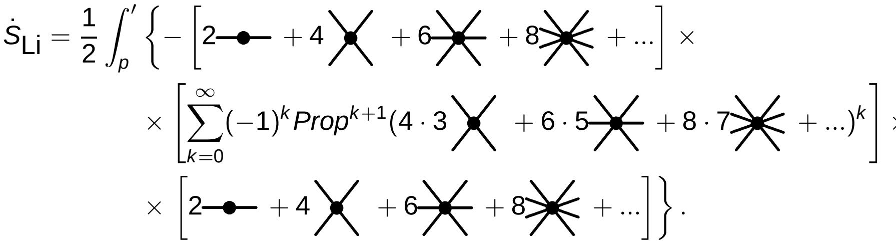
We can use: $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ and obtain:

$$\text{Prop} = \frac{1}{\Lambda^2} \frac{1}{1 + \frac{2\lambda_2}{\Lambda^2}} = \frac{1}{\Lambda^2} \sum_{k=0}^{\infty} \left(\frac{2\lambda_2}{\Lambda^2} \right)^k . \tag{22}$$

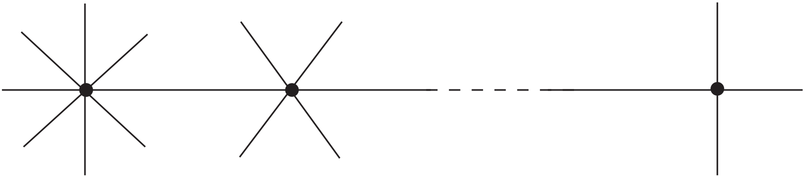
The change of the action is given by:

$$\dot{S}_{\text{Li}} = \frac{1}{2}\int_p^{\prime} \Big(-\left[\Lambda^2\phi + 2\lambda_2\phi + 4\lambda_4\phi^3 + 6\lambda_6\phi^5 + \ldots\right] \times \\ \times \frac{1}{\Lambda^2 + 2\lambda_2}\sum_{k=0}^{\infty}(-1)^k\left(\frac{4\cdot 3\lambda_4\phi^2 + 6\cdot 5\lambda_6\phi^4 + \ldots}{\Lambda^2 + 2\lambda_2}\right)^k \times \\ \times \left[\Lambda^2\phi + 2\lambda_2\phi + 4\lambda_4\phi^3 + 6\lambda_6\phi^5 + \ldots\right] \Big)\,. \tag{23}$$

With Feynman-Diagrams we have to consider:



The graphs that contribute to the Linkterm have the form:



The influence of the Linkterm on the change of the couplings within a RGT is given by:

$$\dot{\lambda}_{Li} = \sum_{\mathrm{Li}} \frac{1}{2} (-Prop)^{N-1} l \lambda_l \ldots m(m-1)\lambda_m \ldots n\lambda_n .$$

The Loopterm

$$\mathrm{Tr}\left(\ln\left(\frac{\delta S}{\delta\phi(\vec{p})\delta\phi(-\vec{p})}\right)\right) = \mathrm{Tr}\left(\ln\left(\Lambda^2 + 2\lambda_2 + 4\cdot 3\lambda_4\phi^2 + 6\cdot 5\lambda_6\phi^4 + \dots\right)\right)$$
$$= \mathrm{Tr}\left(\ln\left(\Lambda^2 + 2\lambda_2\right) + \ln\left(1 + \frac{4\cdot 3\lambda_4\phi^2 + 6\cdot 5\lambda_6\phi^4 + \dots}{\Lambda^2 + 2\lambda_2}\right)\right).$$

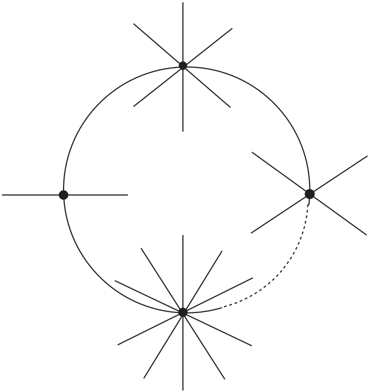
The first logarithm drops out when the action is renormalized.
For the second logarithm we use the series:

$$\ln(1+x) = \sum_{i=1}^{\infty}(-1)^{i+1}\frac{x^i}{i}.$$

We obtain:

$$\mathrm{Tr}\left(\ln\left(\frac{\delta S}{\delta\phi(\vec{p})\delta\phi(-\vec{p})}\right)\right) = \mathrm{Tr}\left(\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\left(\frac{4\cdot 3\lambda_4\phi^2 + 6\cdot 5\lambda_6\phi^4 + \dots}{\Lambda^2 + 2\lambda_2}\right)^i\right).$$

The graphs that contribute to the Loopterm have the form:



The integral over the the surface of a d-dimensional sphere is given by: $S_d = R^{d-1}\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.
Furthermore, we get a factor: $\frac{1}{(2\pi)^d}$ from the Fourier transformation.
The influence of the Loopterm on the change of the couplings within a RGT is given by:

$$\dot{\lambda}_{Lo} = \sum_{\text{Lo}} \frac{1}{2}\frac{1}{(2\pi)^d}\Lambda^{d-1}\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}\frac{(-1)^{N+1}}{N} Prop^N \prod_{i=1}^{N}\left(i(i-1)\lambda_i\right)^{k_i}.$$

Rescaling

We have to fix the dependence of the running momentum scale to the initial momentum scale:

$$\Lambda = \Lambda_0 \exp\{-t\}. \tag{24}$$

The ratio between two momentums has to be constant:

$$\frac{p}{\Lambda} = \text{konst.} \tag{25}$$

The momentums should scale in the form:

$$\dot{p} = -p. \tag{26}$$

The terms of the action have the form:

$$\int \lambda_i \left(\prod_{j=1}^{i} d^d p_j \phi(\vec{p}_j) \right) \delta(\sum_{j=1}^{i} \vec{p}_j). \tag{27}$$

The Rescaling of the differential volume:

$$+dS - d\left(\int \phi \frac{\delta S}{\delta \phi} d^d p\right). \tag{28}$$

The rescaling of the fields:

$$-(D_\phi - \frac{\eta_\phi}{2})\left(\int \phi \frac{\delta S}{\delta \phi} d^d p\right) \tag{29}$$

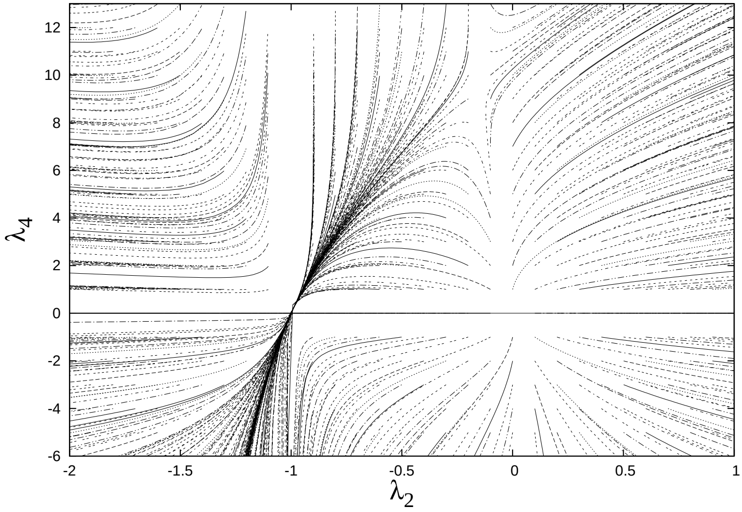
The rescaling of the momentum:

$$-\int \phi \vec{p} \frac{\partial'}{\partial \vec{p}} \frac{\delta S}{\delta \phi} d^d p \tag{30}$$

The complete RG-Equation is given by:

$$\dot{S} = \frac{1}{2}\int' \left\{ -\frac{\delta S}{\delta \phi}\left(\frac{\delta^2 S}{\delta \phi \delta \phi}\right)^{-1} \frac{\delta S}{\delta \phi} + \mathrm{Tr}\left(\ln\left(\frac{\delta^2 S}{\delta \phi \delta \phi}\right)\right)\right\} d^d p - d\left(\int \phi \frac{\delta S}{\delta \phi} d^d p\right)$$
$$+dS - (D_\phi - \frac{\eta_\phi}{2})\left(\int \phi \frac{\delta S}{\delta \phi} d^d p\right) - \int \phi \vec{p} \frac{\partial'}{\partial \vec{p}} \frac{\delta S}{\delta \phi} d^d p. \tag{31}$$

The flow in the coupling space in scalar Φ^4-Theory:



From the RG-Equation we obtain:

$$\beta_4(\vec{\lambda}) = -\Lambda \frac{\partial \lambda_4}{\partial \Lambda} = \frac{\partial \lambda_4}{\partial t} = -\frac{3}{\Gamma(\frac{d}{2})} \cdot \frac{1}{2^d (\pi)^{\frac{d}{2}}} \left(\frac{1}{1+\lambda_2} \right)^2 (\lambda_4)^2 + (4-d)\lambda_4.$$

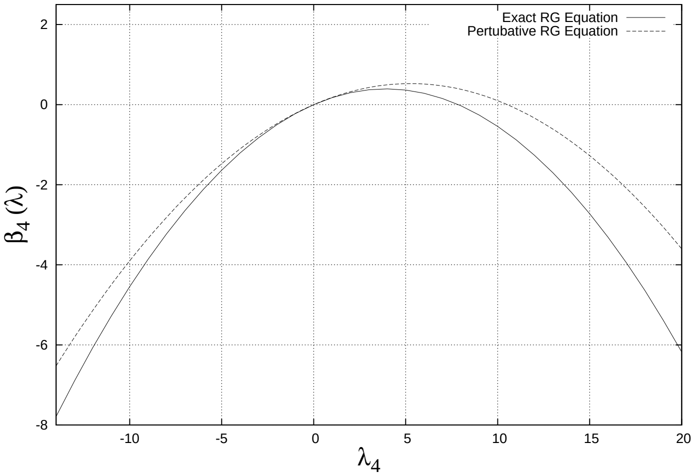
In a massless theory and $d \approx 4$ we can develop the equation:

$$\beta_4(\vec{\lambda}) = -\frac{3\lambda_4^2}{16\pi^2} + (4-d)\lambda_4.$$

In the literature we find:

$$M \frac{d\lambda}{dM} = \beta_4(\vec{\lambda}) \approx +\frac{3\lambda_4^2}{16\pi^2} - (4-d)\,\lambda_4,$$

The β function at d=3.8:



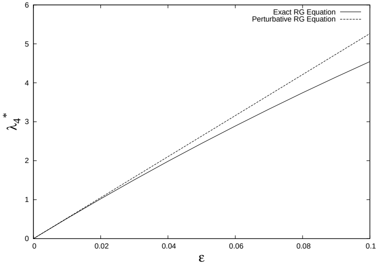
The Position of the Wilson-Fisher Fixed Point

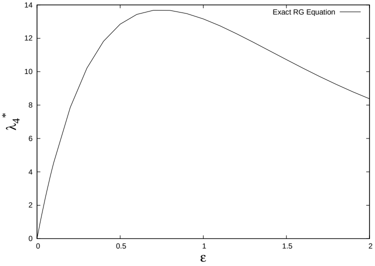
We define:

$$\epsilon = 4 - d. \tag{32}$$

In the massless theory the position of the Wilson-Fisher fixed point is given by:

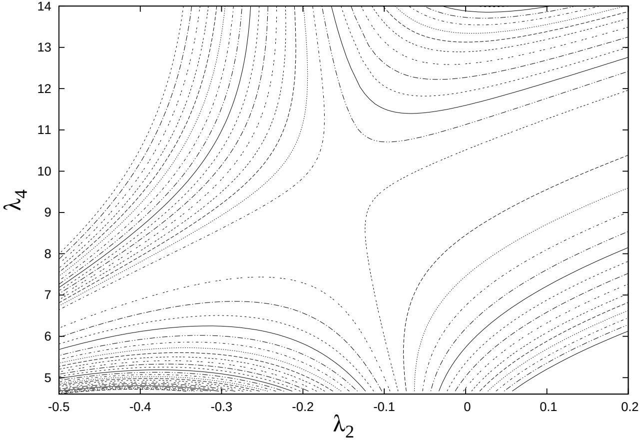
$$\lambda_4^* = \frac{(2^2\pi)^{\frac{4-\epsilon}{2}}}{3\Gamma(\frac{4-\epsilon}{2})}\epsilon. \tag{33}$$





The Exponents of the Wilson-Fisher Fixed Point

Wilson-Fisher fixed point at d=3.0:



We want to look at the scaling of the couplings along the direction of the Eigenvectors:

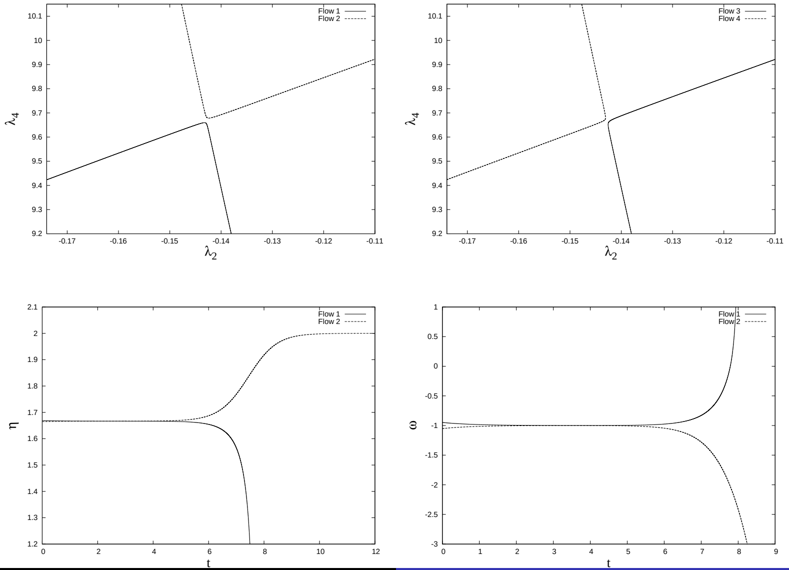
$$\left.\frac{\frac{\partial(\lambda_2-\lambda_2^*)}{\partial t}}{(\lambda_2-\lambda_2^*)}\right|_{\lambda_2\approx\lambda_2^*;\lambda_4=\lambda_4^*}=\eta \quad \text{and} \quad \left.\frac{\frac{\partial(\lambda_4-\lambda_4^*)}{\partial t}}{(\lambda_4-\lambda_4^*)}\right|_{\lambda_2=\lambda_2^*;\lambda_4\approx\lambda_4^*}=\omega$$

It is also possible to describe the exponents by:

$$\left.\frac{\partial\beta_2(\vec{\lambda})}{\partial\lambda_2}\right|_{\lambda_2\approx\lambda_2^*;\lambda_4=\lambda_4^*}=-\frac{\lambda_4^*}{2^d(\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\left(\frac{1}{1+\lambda_2}\right)^2+2=\eta,$$

$$\left.\frac{\partial\beta_4(\vec{\lambda})}{\partial\lambda_4}\right|_{\lambda_2=\lambda_2^*;\lambda_4\approx\lambda_4^*}=-\frac{6\lambda_4}{2^d(\pi)^{\frac{d}{2}}\Gamma(\frac{d}{2})}\left(\frac{1}{1+\lambda_2^*}\right)^2+(4-d)=\omega.$$

We obtain: $\eta=1.6\bar{6}$ and $\omega=-1.0$ for $d=3.0$.


Flow 1
Flow 2
λ_4
λ_2
Flow 3
Flow 4
λ_4
λ_2
Flow 1
Flow 2
η
t
Flow 1
Flow 2
ω
t

ϵ	η	ω
0.2	$1.9\bar{3}$	-0.2
0.4	$1.8\bar{6}$	-0.4
0.7	$1.7\bar{6}$	-0.7
1.0	$1.6\bar{6}$	-1.0

From the table we can educe the rules:

$$\eta = 2 - \frac{\epsilon}{3} \qquad \text{and} \qquad \omega = -\epsilon. \tag{34}$$

These relations are confirmed by the literature.

We consider three dimensional fields:

$$\vec{\phi}(\vec{p}) = \begin{pmatrix} \phi_1(\vec{p}) \\ \phi_2(\vec{p}) \\ \phi_3(\vec{p}) \end{pmatrix},$$

and the action:

$$S = \frac{1}{2}\int d\vec{p}\vec{p}^2\vec{\phi}(\vec{p})\vec{\phi}(-\vec{p})$$
$$+\sum_{i=2}^{\infty}\int \lambda_i \left(\prod_{j=1}^{i} d\vec{p}_j\right)\left(\vec{\phi}(\vec{p}_1)\vec{\phi}(\vec{p}_2)\right)\ldots\left(\vec{\phi}(\vec{p}_{i-1})\vec{\phi}(\vec{p}_i)\right)\delta\left(\sum_{j=1}^{i}\vec{p}_j\right).$$

And we want to look at the symmetric case:

$$\frac{\delta\phi_i(\vec{p})}{\delta\phi_j(\vec{p})} = \delta_{ij}.$$

We define a matrix A:

$$A_{ij} = \frac{\delta^2 S}{\delta\phi_i(\vec{p})\delta\phi_j(-\vec{p})}.$$

The equation of Wegner and Houghton becomes:

$$\dot{S} = \frac{1}{2}\int' \left\{ - \begin{pmatrix} \frac{\delta S}{\delta\phi_1(\vec{p})} & \frac{\delta S}{\delta\phi_2(\vec{p})} & \frac{\delta S}{\delta\phi_3(\vec{p})} \end{pmatrix} (A)^{-1} \begin{pmatrix} \frac{\delta S}{\delta\phi_1(-\vec{p})} \\ \frac{\delta S}{\delta\phi_2(-\vec{p})} \\ \frac{\delta S}{\delta\phi_3(-\vec{p})} \end{pmatrix} + \mathrm{Tr}\left(\ln(A)\right) \right\} d^d p$$
$$+ T_R S.$$

The Loopterm

The deviation of the action becomes:

$$\frac{\delta S}{\delta\phi_k(\vec{p})\delta\phi_l(-\vec{p})} = \vec{p}^2\delta_{kl} + \sum_{i=2}^{\infty}\Bigg[\delta_{kl} i \int \lambda_i \left(\prod_{j=1}^{i} d\vec{p}_j\right)\left(\vec{\phi}(\vec{p}_1)\vec{\phi}(\vec{p}_2)\right)\dots\left(\vec{\phi}(\vec{p}_{i-3})\vec{\phi}(\vec{p}_{i-2})\right)\delta\left(\sum_{j=1}^{i-2}\vec{p}_j\right)$$
$$+ i(i-2)\int \lambda_i \left(\prod_{j=1}^{i} d\vec{p}_j\right)\left(\vec{\phi}(\vec{p}_1)\vec{\phi}(\vec{p}_2)\right)\dots\left(\vec{\phi}(\vec{p}_{i-5})\vec{\phi}(\vec{p}_{i-4})\right)\times$$
$$\times\phi_l(\vec{p}_{i-3})\phi_k(\vec{p}_{i-1})\delta\left(\sum_{j=1}^{i-2}\vec{p}_j\right)\Bigg].$$

We define the placeholder:

$$C = \int \lambda_i \left(\prod_{j=1}^{i} d\vec{p}_j\right)\left(\vec{\phi}(\vec{p}_1)\vec{\phi}(\vec{p}_2)\right)\dots\left(\vec{\phi}(\vec{p}_{i-5})\vec{\phi}(\vec{p}_{i-4})\right)\delta\left(\sum_{j=1}^{i-2}\vec{p}_j\right).$$

We can write:

$$\begin{aligned} A_{kl} &= \delta_{kl}\vec{p}^2 + \delta_{kl}\sum_{i=2}^{\infty} iC\left(\vec{\phi}(\vec{p}_{i-3})\vec{\phi}(\vec{p}_{i-2})\right) + \sum_{i=2}^{\infty} i(i-2)C\phi_l(\vec{p}_{i-3})\phi_k(\vec{p}_{i-1}) \\ &= \left(\vec{p}^2 + 2\lambda_2\right)\left[\delta_{kl} + \frac{\delta_{kl}\sum_{i=4}^{\infty} iC\left(\vec{\phi}(\vec{p}_{i-3})\vec{\phi}(\vec{p}_{i-2})\right) + \sum_{i=4}^{\infty} i(i-2)C\phi_l(\vec{p}_{i-3})\phi_k(\vec{p}_{i-1})}{\vec{p}^2 + 2\lambda_2}\right] \\ &= \left(\vec{p}^2 + 2\lambda_2\right)\left[\delta_{kl} + \frac{B_{ij}}{\vec{p}^2 + 2\lambda_2}\right]. \end{aligned}$$

We can expand the logarithm again:

$$\begin{aligned} \mathrm{Tr}\ln(A) &= \mathrm{Tr}\left(\ln\left(\vec{p}^2 + 2\lambda_2\right) + \ln(\mathbb{1} + B)\right) \\ &= \mathrm{Tr}\left(\ln\left(\vec{p}^2 + 2\lambda_2\right) + \sum_{i=1}^{\infty}(-1)^{i+1}\left(\frac{1}{\vec{p}^2 + 2\lambda_2}\right)^i \frac{B^i}{i}\right). \end{aligned}$$

Than the Loopterm becomes:

$$\mathrm{Tr}\ln(A) = \sum_{i=1}^{\infty}\left(\frac{(-1)^{i+1}}{i}\left(\frac{1}{\vec{p}^2 + 2\lambda_2}\right)^i \mathrm{Tr}\left(B^i\right)\right).$$

The Linkterm

We introduce the placeholder:

$$
\begin{aligned}
C_1 &= \vec{p}^2 + \sum_{i=2}^{\infty} i \int \lambda_i \left(\prod_{j=1}^{i} d\vec{p}_j \right) \left(\vec{\phi}(\vec{p}_1)\vec{\phi}(\vec{p}_2) \right) \dots \left(\vec{\phi}(\vec{p}_{i-3})\vec{\phi}(\vec{p}_{i-2}) \right) \delta \left(\sum_{j=1}^{i-2} \vec{p}_j \right), \\
C_2 &= i\,(i-2) \sum_{i=4}^{\infty} \int \lambda_i \left(\prod_{j=1}^{i} d\vec{p}_j \right) \left(\vec{\phi}(\vec{p}_1)\vec{\phi}(\vec{p}_2) \right) \dots \left(\vec{\phi}(\vec{p}_{i-5})\vec{\phi}(\vec{p}_{i-4}) \right) \delta \left(\sum_{j=1}^{i-2} \vec{p}_j \right), \\
C_3 &= \sum_{i=2}^{\infty} i \int \lambda_i \left(\prod_{j=1}^{i} d\vec{p}_j \right) \left(\vec{\phi}(\vec{p}_1)\vec{\phi}(\vec{p}_2) \right) \dots \left(\vec{\phi}(\vec{p}_{i-3})\vec{\phi}(\vec{p}_{i-2}) \right) \delta \left(\left(\sum_{j=1}^{i-1} \vec{p}_j \right) + \vec{p} \right).
\end{aligned}
\tag{35}
$$

At least we obtain:

$$
\begin{pmatrix} \frac{\delta S}{\delta \phi_1(\vec{p})} & \frac{\delta S}{\delta \phi_2(\vec{p})} & \frac{\delta S}{\delta \phi_3(\vec{p})} \end{pmatrix} (A)^{-1} \begin{pmatrix} \frac{\delta S}{\delta \phi_1(-\vec{p})} \\ \frac{\delta S}{\delta \phi_2(-\vec{p})} \\ \frac{\delta S}{\delta \phi_3(-\vec{p})} \end{pmatrix} = \frac{C_3 \phi(\vec{p}_{i-1}) C_3 \phi(\vec{p}_{i-1})}{C_1 + C_2 \vec{\phi}_{i-3} \vec{\phi}_{i-2}}.
$$

The Linkterm is given by:

$$\dot{S}_{Li} = \left[\int d\vec{p}\vec{p}^2 \phi_k(-\vec{p}) + 2\lambda_2 + 4\lambda_4\phi^3 + 6\lambda_6\phi^5 + ... \right] \times$$
$$\times \left[\frac{1}{P^{-}1 + 2\lambda_2 + 4 \cdot 3\lambda_4\phi^2 + 6 \cdot 5\lambda_6\phi^4 + ...} \right] \times$$
$$\times \left[\int d\vec{p}\vec{p}^2 \phi_k(-\vec{p}) + 2\lambda_2 + 4\lambda_4\phi^3 + 6\lambda_6\phi^5 + ... \right]$$

We can calculate the β function out of the RG-Equation:

$$\beta_4(\vec{\lambda}) = -\frac{11}{3}\frac{1}{\Gamma(\frac{d}{2})}\cdot\frac{1}{2^d(\pi)^{\frac{d}{2}}}\left(\frac{1}{1+\lambda_2}\right)^2(\lambda_4)^2+(4-d)\lambda_4.$$

For the massless theory we get for the Taylor expansion at $d=4$ in first order:

$$\beta_4(\vec{\lambda}) = -\frac{11}{3}\frac{\lambda_4^2}{16\pi^2}+(4-d)\lambda_4.$$

In the literature we find:

$$M\frac{d\lambda_4}{dM} = \beta \approx +\frac{11}{3}\frac{\lambda_4^2}{16\pi^2}-(4-d)\,\lambda_4,$$

Thank you for your attention!!!

Turbulent Systems

The hydrodynamic flow of a fluid is described by the Navier-Stokes equation:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 v_i}{\partial x_i \partial x_j}. \tag{36}$$

We normally assume, that the fluid is incompressible:

$$\frac{\partial v_i}{\partial x_i} = 0. \tag{37}$$

Until today a nontrivial solution of the Navier-Stokes Equation is not known. The reasons are:

- The NSE is a nonlinear partial differential equation.
- The NSE is nonlocal.
- The hydrodynamic flow has a high number of degrees of freedom.

K41-Theory

We should try to find a solution in a statistical sense.
In 1941 Kolmogorov created a Theory to describe the Turbulence. He postulated that you can find a non vanishing energy dissipation (ϵ) at microscopic length scales. And he defined three relevant length scales:

- The macroscopic length scale.
- The intermediate length scale x.
- The microscopic length scale $\eta = (\frac{\nu^3}{\epsilon})^{\frac{1}{4}}$

K41-Theory

He took the structure function $S_p(x)$ as a macroscopic measurable quantity:

$$S_p(x) \equiv < \left[\left(\vec{v}\left(\vec{r} + x\vec{e} \right) - \vec{v}\left(\vec{r} \right) \right) \cdot \vec{e} \right]^p > . \tag{38}$$

The structure function of the intermediate scale should be independent of macroscopic and of the microscopic length scale and it should be self similar. So Kolmogorov suggested:

$$S_p(x) = C_p \epsilon^{\frac{p}{3}} x^{\frac{p}{3}} . \tag{39}$$

But: The rule is valid only for the structure function of the second order.

Open Questions:

- How does the correct scaling behaves?
- Are there scaling exponents? Bifractal, multifractal,...?
- Can you find fixed points in the coupling space?

Dirk Homeier built up an action that describes the Navier-Stokes equation. He used the Martin Siggia Rose formalism and the Faddeev-Poppov Method. The action should be studied by Renormalisation Group Equations.