Exact Renormalisation Group

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Contents

Statistical Physics

- In statistical physics a system has $\approx 10^{23}$ degrees of freedom
- It seems to be impossible to calculate these systems exactly

$\longrightarrow$ The reduction of the degrees of freedom is inevitable!

Statistical Physics

- It's known that continuous phase transitions can be characterized by parameters known as critical exponents:
  - Heat capacity: $C_V \propto |T_c - T|^{-\alpha}$
  - Susceptibility: $\chi \propto |t|^{-\gamma}$
  - Correlation Length: $\xi \propto |t|^{-\nu}$
  - Pair Correlation Function: $G(\boldsymbol{r}) =< \Psi(\boldsymbol{0})\Psi(\boldsymbol{r}) >\propto \frac{1}{r^{d-2+\eta}}$
- These critical exponents obey scaling relations like:
  - $\beta = \frac{\gamma}{\delta - 1}$
  - $\nu = \frac{\gamma}{2-\eta}$

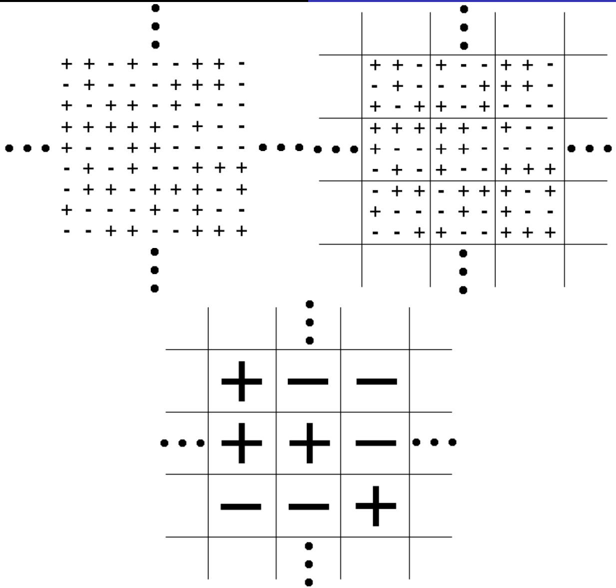
Statistical Physics

- The statistical systems are divided into distinct **universality** classes, each with a characteristic long range behaviour if the order parameter has the same symmetry and dimension.
- **Landau** introduced his **theory** and defined the order parameter. If we know something about the order parameter we get to know something about the macroskopic nature of the system.

Kadanoff Scaling

1966 **Kadanoff** introduced a complete new concept of scaling. He maps a near critical or critical system onto itself by a reduction of the degrees of freedom:

- Consider a lattice of spacing with $s = \frac{1}{2}$ Ising Spins with the values ± 1 ($H = -\frac{1}{2} J \sum_{\boldsymbol{\delta}} s_{\mathbf{x}} s_{\mathbf{x}+\boldsymbol{\delta}}$).
- Divide the lattice into blocks of $L \times L \times L$ with $L = b \cdot a$. So each block containes b^d spins.
- Associate with each block a new effettive block spin $s_{x'}$ and rescale all spatial coordinates $\mathbf{x} \rightarrow \mathbf{x'} = \frac{\mathbf{x}}{b}$.
- The condition is, that the statistical properties and the coupling to the external field has to stay the same.



Kadanoff Scaling

- The new spin on the renormalised lattice takes the value of the majority of the spins in the block
  $s_{\{s_i\}} = sign(\sum_i s_i^I) = \pm 1$ or we use: $\bar{s}_I = \frac{1}{b^d} \sum_{i \in I} s_i \cong \zeta(b) s_I'$
- Kadanoff introduced a spin and a thermal renormalisation factor:
  $$t' \approx \vartheta(b) t \qquad\qquad h' \approx \zeta(b) h$$
- The correlation function should renormalise as:
  $G(\boldsymbol{x}; t, h) \equiv < S_{\boldsymbol{0}} S_{\boldsymbol{x}} > \approx \zeta^2(b) G(\boldsymbol{x'}; t', h')$
- But how should one define ζ and ϑ?

Kadanoff Scaling

- We know if: $t = h = 0 \rightarrow t' = h' = 0$ and at the critical point we have: $G_C(\boldsymbol{r}) \propto \frac{1}{r^{d-2+\eta}}$
- One finds that ζ must be a power of b and Kadanoff proposed the form: $\zeta(b) = b^{-\omega}$ and $\vartheta(b) = b^{\lambda}$
- The previous scaling laws are still valid and the analysis leads to new exponent relations, the hyperscaling laws like: $d\nu = 2 - \alpha$
- All the exponents are now determined by ω and λ.

The Wilson Renormalisation Group

Again there is a lattice with Ising Spins: $H = -J \sum_{<i,j>} s_i s_j$

We now use the partition function:

$$Z[T,n] = \sum_{\{S_i\}} e^{-H[S_i,T,n,a]}$$

$$\rightarrow Z[T,n] = \sum_{\{s_I\}} \sum_{\{\sigma_I^\alpha\}} e^{-H[s_I,\sigma_I^\alpha,T,n,a]} = \sum_{\{s_I\}} e^{-H'[s_I,T,\frac{n}{b^d},ab]}$$

Since we have infinite many interactions we use a H with all possible couplings:

$$H[\boldsymbol{K},s_i,n] = -K_1 \sum_{<i,j>} s_i s_j + K_2 \sum_{<<i,j>>} s_i s_j + K_3 \sum_{<<i,j,k,l>>} s_i s_j s_k s_l + \dots$$

We now want to rescale the lattice spacing by $ab \to a$. And if we renormalise the couplings ($\boldsymbol{K} \to \boldsymbol{K}'$) we obtain a system that looks exactly like the old one.

Transformation + Rescaling and Renormalisation = Renormalisation Group Transformation (RGT)

The RGT has the following attributes:

- It preserves the partition function Z
- It maps a Hamiltonian onto another one
- It consists of integrating out short distance degrees of freedom to obtain an effective Hamiltonian for the long distance degree of freedom

We now consider a linear transformation between spins and block spins with a renormalisation by $\lambda(b)$:

$$S_I = \frac{\lambda(b)}{b^d} \sum_{i \in I} S_i^I$$

The relation among the stochastic variables S_i^I and S_I leads to relations among their thermodynamic averages:

$$\begin{aligned} < S_I S_J >= \quad & \frac{1}{Z} \sum_{\{S_I\}} S_I S_J e^{-H[\mathbf{K}', S_L]} = \frac{1}{Z} \sum_{\{S_I\}} S_I S_J \sum_{\{\sigma_L^\alpha\}} e^{-H[\mathbf{K}, S_L, \sigma_L^\alpha]} \\ = \quad & \frac{\lambda^2(b)}{b^{2d}} \frac{1}{Z} \sum_{\{S_I\}} \sum_{\{\sigma_I^\alpha\}} \sum_{i \in I, j \in J} S_i^I S_j^J e^{-H[\mathbf{K}, S_i]} = \frac{\lambda^2(b)}{b^{2d}} \sum_{i \in I, j \in J} G^{(2)}(\mathbf{x_i}, \mathbf{x_j}) \end{aligned}$$

With the definition of the two-point correlation function of the Spins S_i:
$G^{(2)}(\mathbf{x_i}, \mathbf{x_j}) = G^{(2)}(\mathbf{r_{ij}}, \mathbf{K}) =< S_i S_j >$

If we consider two spins S_I and S_J with large distance we see:

$$\sum_{i \in I, j \in J} G^{(2)}(\boldsymbol{x_i}, \boldsymbol{x_j}) \simeq b^{2d} G^{(2)}(\boldsymbol{x_i}, \boldsymbol{x_j})$$

And find that:

$$G^{(2)}(r_{IJ}, \boldsymbol{K'}) = \lambda^2(b) G^{(2)}(r_{ij}, \boldsymbol{K})$$

In three dimensions and at large distances a typical form of the two point correlation function is:

$$< S_i S_j >= G^{(2)}(\bar{r}, \bar{\xi}) \sim \frac{e^{-\bar{r}/\bar{\xi}}}{\bar{r}^{\theta}}$$

where: $\bar{r} = \frac{r}{a}$ $\bar{\xi} = \frac{\xi}{a}$.

We thus obtain: $G^{(2)}(\bar{r}', \bar{\xi}') \sim \frac{e^{-\frac{\bar{r}/b}{\bar{\xi}'}}}{\bar{r}^{\theta}} b^{\theta}$

where $\bar{r}' = \frac{r}{ba}$

Now we can see that: $\bar{\xi}' = \frac{\bar{\xi}}{b}$ and $\lambda(b) = b^{\theta/2}$ so we have a relation between the renormalisation and the critical exponents!!!

Now we will have a look on the physics in the **momentum space**:

The Wilson RG procedure is carried out in two steps:

1. Integration of the high energy modes of the field: $p \in [\Lambda', \Lambda]$ with $\Lambda \propto a^{-1}$
2. And change of lenght scale by a factor $\boldsymbol{p} \to \boldsymbol{p}' = \lambda(b)\boldsymbol{p}$

Wilson devided $\phi(p)$ into rapid and slow modes ($\phi = \phi_< + \phi_>$):

$$Z = \int D\phi e^{-H[\phi, \boldsymbol{K}, \Lambda]} = \int D\phi_< D\phi_> e^{-H[\phi_<, \phi_>, \boldsymbol{K}, \Lambda]}$$

$\boldsymbol{K}'$ is the new coupling constant:

$$Z = \int D\phi_< e^{-H[\phi_<, \boldsymbol{K}', \Lambda/b]}$$

with: $$e^{-H[\phi_<, \boldsymbol{K}', \Lambda/b]} = \int D\phi_> e^{-H[\phi_<, \phi_>, \boldsymbol{K}, \Lambda]}$$

Transformation of the Field Variable

We consider a general transformation of the field which leaves the partition function invariant. $\phi'_p = \phi_p + \sigma \Psi_p[\phi]$

Then one has: $H[\phi'] = H[\phi] + \sigma \int\limits_p \Psi_p[\phi] \frac{\delta H[\phi]}{\delta \phi_p}$

Moreover we have: $D\phi' = \int D\phi \frac{\partial\{\phi'\}}{\partial\{\phi\}} = \int D\phi(1 + \sigma \int\limits_p \frac{\delta \Psi_p[\phi]}{\delta \phi_p})$

Since the partition function must stay the same we obtain:

$$Z = \int D\phi' exp\{-H[\phi']\} = \int D\phi exp\{-H[\phi] - \sigma G_{tra}[\Psi]H[\phi]\}$$

with: $G_{tra}[\Psi]H[\phi] = \int\limits_p (\psi_p \frac{\delta H}{\delta \phi_p} - \frac{\delta \Psi_p}{\delta \phi_p})$

If we consider the case of N Components the expression generalizes to: $G_{tra}[\Psi]H[\phi] = \sum_{\alpha=1}^{N} \int\limits_p (\psi_p^\alpha \frac{\delta H}{\delta \phi_p^\alpha} - \frac{\delta \Psi_p^\alpha}{\delta \phi_p^\alpha})$

Rescaling

We consider an infinitesimal change of (momentum) scale:
$p \rightarrow p' = bp = (1+\sigma)p$
The consequence on $H[\phi]$ is written as: $H \rightarrow H' = H + \sigma G_{dil} H$
Now there are changes introduced on various factors:

- The differential volume
- The couplings
- The field itself
- ..

We obtain something like:

$$G_{dil} H = -\left(\int_p \phi_p \mathbf{p} \partial_p \frac{\delta}{\delta \phi_p} + d_\phi \int_p \phi_p \frac{\delta}{\delta \phi_p}\right) H$$

If we choose the new Cutoff in infinitesimal distance to the old Cutoff ($\Lambda' = \Lambda - d\Lambda$) we obtain a differential equation as renormalization group equation. We define a so called renormalization group time by:

$$\dot{H} = -\Lambda \frac{\partial H}{\partial \Lambda} =: \frac{\partial H}{\partial t} \qquad \text{and} \qquad t = -ln(\frac{\Lambda}{\Lambda_0})$$

The general expression of the exact RG equation may formally be written as:

$$\dot{H} = G_{dil} H + G_{tra}[\Psi] H$$

We get the new Hamiltonian: $H \rightarrow H' = H + \dot{H}\Delta t$

Wilson used $\Psi_p = (c + 2p^2)(\phi_p - \frac{\delta H}{\delta \phi_{-p}})$ and obtained the RG equation:

$$\dot{H} = G_{dil} H + \int_p (c + 2p^2)[\frac{\delta}{\delta \phi_p} \frac{\delta H}{\delta \phi_{-p}} - \frac{\delta H_p}{\delta \phi_{-p}} \frac{\delta H}{\delta \phi_p} + \frac{\delta H}{\delta \phi_p} \phi_p]$$

Polchinskis Smooth Cutoff

Polchinski introduced a general ultraviolet Cutoff function: $K(p^2/\Lambda^2)$. He used the Hamiltonian:

$$H[\phi] \equiv \frac{1}{2}\int \phi_p \phi_{-p} p^2 K^{-1}(\frac{p^2}{\Lambda^2}) + H_{int}[\phi]$$

Leaving $Z[\boldsymbol{K}]$ invariant he obtains the Polchinski equation:

$$\Lambda\frac{dH_{int}}{d\Lambda} = \frac{1}{2}\int_p p^{-2}\Lambda\frac{dK}{d\Lambda}(\frac{\delta H_{int}}{\delta\phi_p}\frac{\delta H_{int}}{\delta\phi_{-p}} - \frac{\delta^2 H_{int}}{\delta\phi_p\delta\phi_{-p}})$$

The Equation of Wegner and Houghton

If we introduce Polchinski's cutoff function $K(p^2/\Lambda^2)$ as a sharp cutoff we can get to the ER Group Equation derived by Wegner and Houghton:

$$\dot{H} = \lim_{t\to 0} \frac{1}{2t}\left[\int_p^{\prime} ln\left(\frac{\delta^2 H}{\delta\phi_p \delta\phi_{-p}}\right) - \int_p^{\prime} \frac{\delta H}{\delta\phi_p}\frac{\delta H_{int}}{\delta\phi_{-p}}\left(\frac{\delta^2 H}{\delta\phi_p \delta\phi_{-p}}\right)^{-1}\right] + G_{dil} H + const$$

The prime on the integral symbol indicates that the momenta are restricted to $p \in [\Lambda - d\Lambda, \Lambda]$

The Effective Average Action Method

We introduce an infrared cutoff for the rapid modes by giving the slow modes of the partition function a large mass.

$$Z_k[B] = \int D\phi(x) exp(-H[\phi] - \Delta H_k[\phi] + \int B\phi)$$

with: $\Delta H_k[\phi] = \frac{1}{2} \int_p R_k(p)\phi_p \phi_{-p}$

We now define: $W_k[Z] = log(Z_k[B])$

The magnetisation is by definition the average of $\phi(x)$ and is therefore:

$$M(x) = \frac{\delta W_k}{\delta B(x)}$$

We perform the Legendre transformation:

$$\Gamma_k[M] + W_k[B] = \int BM - \frac{1}{2}\int_p R_k(p) M_p M_{-p}$$

- If $k = 0$ all fluctuations are integrated out, we see that: $W_k \to W$; $Z_k \to Z$ and so we find: $\Gamma_{k=0}[M] = \Gamma[M]$
- If $k = \Lambda$ all fluctuations are frozen, we see that: $\Gamma_{k=\Lambda}[M] = H[\phi = M]$
- When $0 < k < \Lambda$ the rapid modes must be almost uneffected by the Cutoff function: $R_k(|p| > k) \simeq 0$
  The slow modes have a mass that allmost decouples from the long distance physics.

The Wetterich Equation

With the effective action one can build up a RG Equation called the Wetterich equation:

$$\partial_k \Gamma_k = \frac{1}{2} \int_p \partial_k R_k(p) (\Gamma_k^{(2)}[M] + R_k)^{-1}_{p,-p}$$

With: $\Gamma_k^{(2)}[M] = \frac{\delta^2 \Gamma_k}{\delta M_p \delta M_{-p}}$

Approximations and Truncations

Expansion of the field:

$$H = H_{kin} + \lambda_4 \int_p (\phi)^4 + \lambda_6 \int_p (\phi)^6 + \lambda_8 \int_p (\phi)^8$$

Expansion of the derivatives of the field:

$$H[\phi] = \int d^d x \{V(\phi) + \frac{1}{2} Z(\phi)(\partial_\mu \phi)^2 + O(\partial^4)\}$$

One special case of the derivative expansion is the Local Potential Approximation (LPA):

$$H[\phi] = \int d^d x (\frac{Z}{2}(\partial_\mu \phi)^2 + V(\phi))$$

Fixed Points and Renormalisation Group Flows

The RG flow takes place in the space of coupling constants $\boldsymbol{K}$.
We consider a second order phase transtion: $\bar{\xi} = \infty \rightarrow \bar{\xi}' = \infty$.
$\boldsymbol{T}(.;p)$ should be the function that maps $\boldsymbol{K} = \boldsymbol{K}^{(0)}$ onto $\boldsymbol{K}^{(1)}$ after a RG transformation: $\boldsymbol{K}^{(n+1)} = \boldsymbol{T}(\boldsymbol{K}^{(n)})$

We define the critical surface: Set of points $\boldsymbol{K}$ in the couplingconstant space for which $\bar{\xi} = \infty$

We define a fixed point in the space of coupling constants as a place $\boldsymbol{K}^*$ where we find: $\boldsymbol{T}(\boldsymbol{K}^*,p) = \boldsymbol{K}^*$

If a system $\boldsymbol{K}^{(0)}$ has $\bar{\xi} < \infty$ it moves away from the critial surface under RG transformations ($\bar{\xi}' = \bar{\xi}/b$).

We now consider an infinitesimal RG transformation:
$\boldsymbol{K}_{b(1+\epsilon)} - \boldsymbol{K}_b = \boldsymbol{T}(\boldsymbol{K}_b, 1+\epsilon) - \boldsymbol{T}(\boldsymbol{K}_b, 1)$
We find the evolution of the couplings of the model with the scales: $b\frac{\partial \boldsymbol{K}_b}{\partial b} = \frac{\partial \boldsymbol{T}}{\partial b}|_{\boldsymbol{K}_b,1} = \beta(\boldsymbol{K}_b)$

We thus obtain in vincinity to the FP $\boldsymbol{K}*$:
$\beta(\boldsymbol{K}_b) - \beta(\boldsymbol{K}^*) = \frac{d\boldsymbol{\beta}}{d\boldsymbol{K}_b}|_{\boldsymbol{K}^*}\delta\boldsymbol{K}_b + O(\delta\boldsymbol{K}_b^2)$
with: $\delta\boldsymbol{K} = \boldsymbol{K}_b - \boldsymbol{K}^*$ and $M_{ij} = \frac{d\beta_i}{dK_{b,j}}|_{\boldsymbol{K}}$
We suppose that the set of Eigenvectors is a complete basis:
$M\boldsymbol{e_i} = \lambda_i\boldsymbol{e}_i$ and $\delta\boldsymbol{K}_b = \sum_i v_i(b)\boldsymbol{e}_i$

One now obtains: $b\frac{dv_i(b)}{db} = \lambda_i v_i(b) \rightarrow v_i(b) = v_i(1)b^{\lambda_i}$
We define:

- $\lambda_i > 0$: v_i is a **relevant** coupling and $\boldsymbol{e}_i$ is a **relevant** direction and goes away from $\boldsymbol{K}^*$
- $\lambda_i < 0$: v_i is a **irrelevant** coupling and $\boldsymbol{e}_i$ is a **irrelevant** direction and approaches $\boldsymbol{K}*$
- $\lambda_i = 0$: v_i is a **marginal** coupling and $\boldsymbol{e}_i$ is a **marginal** direction

With the Local Potential Approximation the following fixed points are found:

- $d \geq 4$: There is only the trivial Gausian fixed point in the origin.
- $3 \leq d < 4$: There is the trivial Gausian fixed point and the so called Wilson Fisher fixed point.
- A new nontrivial fixed point emanates from the origin below each dimensional threshold: $d_k = 2k/(k-1)$ and $k = 2{,}3{,}4{,}...\infty$

The exact Renormalisation Group and Turbulence

The hydrodynamic flow is described by the Navier Stokes equation:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{u}$$

Normaly one assumes that the fluid is incompressible: $\boldsymbol{\nabla}\boldsymbol{u} = 0$
You can describe the degree of turbulence by the so called Reynolds number: $R = \frac{LV}{\nu}$
If you want to describe the turbulence you have three problems:

- Nonlinear partial differential equation
- Nonlocal
- High number of degrees of freedom

Kolmogorov creates the K41 Theory to describe the Turbulence. He took the structure $S_p(x)$ function as macroskopic measurable quantity:

$$S_p(x) := \langle[(\boldsymbol{u}(\boldsymbol{r} + x_0\boldsymbol{l}_0) - \boldsymbol{u}(\boldsymbol{r})) * \boldsymbol{l}_0]^p\rangle$$

Kolmogorov postulated that you can find a nonvanishing energy dissipation (ϵ) at microscopic lenght scales. Now you have three relevant lenght scales in turulence:

- The macroscopic lenght scale L
- The intermediate lenght scale x
- The microskopic lenght scale $\eta = (\frac{\nu^3}{a})^{\frac{1}{4}}$

The structure function of the intermediate scale should be independent of makroscopic and of the microskopic lenght scale. It should be given by: $S_p(x) = C_p\epsilon^{\frac{p}{3}}x^{\frac{p}{3}}$

Summary

- Kadanoff Scaling
- Achitecture of the Renormalisation Group Equations
- Important examples of Renormalisation Group Equations:
  - The Wilson R.G.E.
  - The Polchinski R.G.E.
  - The R.G.E. of Wegner and Houghton
  - The Wetterich R.G.E.
- Approximations and Truncations
- Fixed Points and Renormalisation Group Flows
- The Exact Renormalisation Group and Turbulence