Pressure of hot QCD up to $g^6 \ln\left(\frac{1}{g}\right)$

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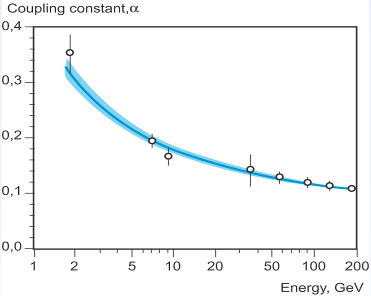
Asymptotic freedom

Nobel Prize 2004 for Wilczek, Politzer and Gross



$$\alpha_s = \frac{g^2}{4\pi} = \frac{12\pi}{(11N - 2N_f)\ln\left(\frac{M^2}{\Lambda^2}\right)}$$

- quarks at short distance $\hat{=}$ weakly interacting point particles
- exploring quark-gluon-plasma
  (neutron stars, heavy-ion collisions, $10^{-5}s$ after the big bang):
  1. lattice gauge theory
  2. **perturbation theory (but: only to a finite order !!!)**

Review of quantum statistical mechanics

- partition function: $Z = Tr\left(e^{-\beta H}\right)$
- expectation value: $\langle O\rangle = \frac{1}{Z} Tr\left(e^{-\beta H} O\right)$

pressure

$$P = \frac{T}{V}\ln Z$$

- noninteracting gas: $\ln Z = V\int \frac{d^3p}{(2\pi)^3}\ln\left(1 \pm e^{-\beta(\omega-\mu)}\right)^{\pm 1}$

$\Rightarrow$ massless bosons with $\mu = 0$: $P = \frac{\pi^2}{90}T^4$

path integral formulation

$$Z = \int \mathcal{D}A^a_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp\left(-\int_0^\beta d\tau\int d^dx\,\mathcal{L}_{QCD}\right)$$

$$\mathcal{L}_{QCD} = \frac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu} + \bar{\psi}\gamma_\mu D_\mu\psi + m\bar{\psi}\psi$$

Feynman rules

basic quantity of interest

$$p_{QCD}(T) = \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}A^a_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left(-\int_0^{\beta=1/T} d\tau \int d^d x \mathcal{L}_{QCD}\right)$$

Feynman rules $\hat{=}$ zero T rules with following substitutions:

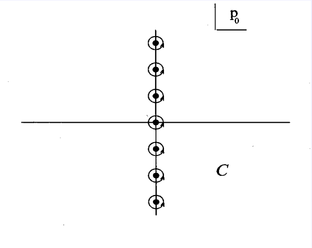
$$p_0 \longrightarrow i\omega_n$$
$$\int d^4p \longrightarrow T \sum_{\omega_n} \int d^3p$$

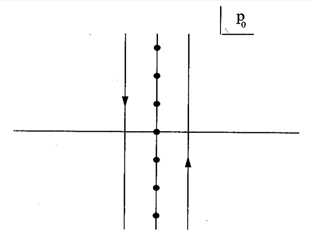
- boundary conditions over compact time-like direction:
  - $\Rightarrow$ periodic for bosons: $\omega_n = 2\pi T n$
  - $\Rightarrow$ anti-periodic for fermions: $\omega_n = (2n+1)\pi T$
- propagator:

$$\Delta(\omega_n, \mathbf{p}) = \frac{\mathbf{1}}{\omega_\mathbf{n}^2 + \mathbf{p}^2 + \mathbf{m}^2}$$

Contour integrals

- frequency sum: $T\sum_{n=-\infty}^{\infty} f\left(p_0 = i\omega_n = 2\pi nTi\right)$
- contour integral: $\frac{T}{2\pi i}\int_C dp_0 f(p_0)\frac{1}{2}\beta\coth\left(\frac{1}{2}\beta p_0\right)$

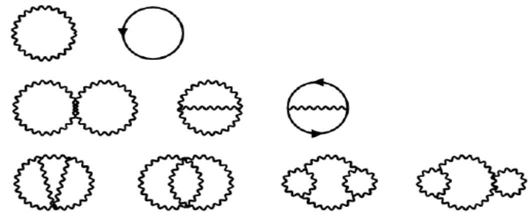




'switching from imaginary to real time'

$$T\sum_{n=-\infty}^{\infty} f\left(p_0 = i\omega_n\right) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dp_0 \frac{1}{2}\left[f(p_0) + f(-p_0)\right] + \frac{1}{2\pi i}\int_{-i\infty+\epsilon}^{i\infty+\epsilon} dp_0 \left[f(p_0) + f(-p_0)\right]\frac{1}{e^{\beta p_0} - 1}$$

Perturbative evaluation



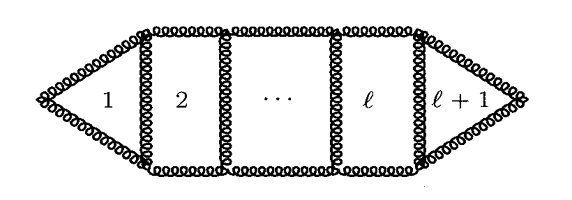
- 1-loop: QCD plasma $\hat{=}$ ideal gas of gluons and quarks

$$P_0 = \frac{\pi^2}{45} N_g T^4 + N \left(\frac{7\pi^2 T^4}{180} + \frac{\mu_f^2 T^2}{6} + \frac{\mu_f^4}{12\pi^2} \right)$$

- 2-loop: result = contribution of order g^2
- next contributions: $\Rightarrow g^3$, $g^4 \ln\left(\frac{1}{g}\right)$, g^4, g^5 and $g^6 \ln\left(\frac{1}{g}\right)$
  reason: infrared divergences

Infrared problems (Linde, [1980])

- $(l+1)$-loop diagram:



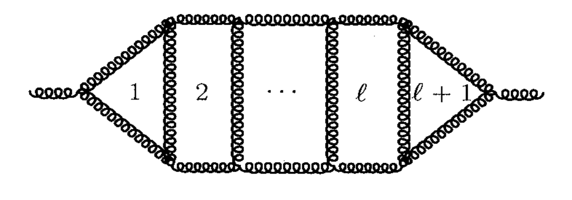
$$g^{2l}\left(T\int d^3p\right)^{l+1} p^{2l}(p^2+m^2)^{-3l}$$

$$\begin{aligned} g^{2l} &\quad \text{for} \quad l=1,2 \\ g^6T^4\ln(T/m) &\quad \text{for} \quad l=3 \\ g^6T^4(g^2T/m)^{l-3} &\quad \text{for} \quad l>3 \end{aligned}$$

- magnetic mass $m_{mag}=g^2T \Rightarrow$ **all loops $(l>3)$** contribute to g^6

Infrared problems (Linde, [1980])

- gluon self-energy:



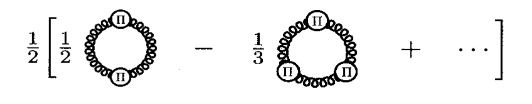
$$g^{2l+2}\left(T\int d^3p\right)^{l+1} p^{2l+2}(p^2+m^2)^{-3l-2}$$

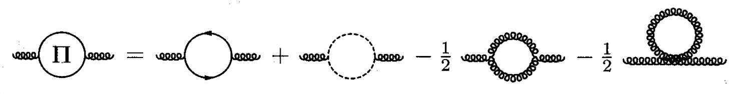
$$g^4T^2\ln(T/m) \quad \textit{for} \quad l=1$$
$$g^4T^2(g^2T/m)^{l-1} \quad \textit{for} \quad l>1$$

- $\Rightarrow$ infrared problem arises at g^4

Ring diagrams

- set of ring diagrams:





perturbative evaluation

$$\Rightarrow P = T^4 \left[c_0 + c_2 g^2 + c_3 g^3 + \left(c_4' \ln\left(\frac{1}{g}\right) + c_4 \right) g^4 + c_5 g^5 + c_6' \ln\left(\frac{1}{g}\right) + O(g^6) \right]$$

- Lindé [1980]: **barrier** at g^6 (pressure); g^4 (gluon self-energy)

Dimensional reduction (Ginsparg [1980], Appelquist, Pisarski [1981])

basic quantity of interest

$$p_{QCD}(T) \quad = \quad \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}A_\mu^a \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp(-\int_0^{\beta=1/T} d\tau \int d^d x \mathcal{L}_{QCD})$$

- finite T Euclidean field theories: volume with one compact dim.
- boundary condition: periodic for bosons, anti-periodic for fermions
- high T: Matsubara modes frozen
  $\Rightarrow$ finite T equilibrium field theory $\hat{=}$ zero T Euclidean field theory with a compact 4th dim.

dimensional reduction

$$\int_0^\beta d\tau \int d^d x \mathcal{L}_{QCD} \Rightarrow \int d^d x \mathcal{L}_E$$

The basic setting (Braaten, Nieto [1995])

- isolating infrared problems to 3D effective field theory (no fermions) $\Rightarrow$ lattice simulations necessary
- converting results from 3D lattice to 3D continuum regularization to 4D physical theory $\Rightarrow$ perturbative matching computations

pressure

$$p_{QCD}(T) = \lim_{V\to\infty} \frac{T}{V} \ln \int \mathcal{D}A^a_\mu \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp(-\int_0^\beta d\tau \int d^d x \mathcal{L}_{QCD})$$

- high T and small g $\Rightarrow$ hierarchy of 3 momentum scales for a QGP:
  1. $2\pi T$ of a typical momentum of a particle in the plasma
  2. gT associated with color-electric screening
  3. $g^2 T$ associated with color-magnetic screening

Contributions from the momentum scales (Kajantie et al. [2003])

- dimensional reduction: isolating infrared problems (Lindé, [1980])
  $\Rightarrow$ important: g small, otherwise: $g^2 T, gT << 2\pi T$ not possible

dimensional reduction

$$p_{QCD}(T) = p_E(T) + \frac{T}{V} \ln \int \mathcal{D}A_k^a \mathcal{D}A_0^a \exp\left(-\int d^d x \mathcal{L}_E\right)$$

$$\mathcal{L}_E = \frac{1}{2} Tr F_{kl}^2 + Tr\,[D_k, A_0]^2 + m_e^2 Tr A_0^2 + \lambda_E^{(1)} (Tr A_0^2)^2 + \lambda_E^{(2)} Tr A_0^4 + \ldots$$

- 5 matching coefficients: p_E, m_E^2, g_E^2, $\lambda_E^{(1)}$ and $\lambda_E^{(2)}$
- dimensional reduction:
  - A_0^a becomes a Higgs field
  - $\mathcal{L}_E$: super-renormalizable Lagrangian of 3D effective theory
  - Now: find relation between physical parameters of QCD ($\Lambda_{\bar{MS}}$, $m_i(\mu)$) and matching coefficients
  - still two dynamical scales: $g^2 T, gT$

Contributions from the momentum scales (Kajantie et al. [2003])

scale gT (integrating out A_0)

$$
\begin{aligned}
\frac{T}{V}\ln\int \mathcal{D}A_k^a \mathcal{D}A_0^a \exp\left(-\int d^d x \mathcal{L}_E\right) &= p_M(T) \\
&+ \frac{T}{V}\ln\int \mathcal{D}A_k^a \exp\left(-\int d^d x \mathcal{L}_M\right) \\
\mathcal{L}_M &= \frac{1}{2} Tr F_{kl}^2 + \ldots
\end{aligned}
$$

- 2 matching coefficients: p_M and g_M^2

scale $g^2 T$

$$
p_G(T) = \frac{T}{V}\ln\int \mathcal{D}A_k^a \exp\left(-S_M\right)
$$

$$\Rightarrow p_{QCD}(T) = p_E(T) + p_M(T) + p_G(T)$$

The complete result up to order g^6 (Kajantie et al. [2003])

$$
\begin{aligned}
\frac{p_{QCD}(T)}{T^4\mu^{-2\epsilon}} &= \frac{p_E(T)+p_M(T)+p_G(T)}{T^4\mu^{-2\epsilon}} = g^0\left[\alpha_{E1}\right] + g^2\left[\alpha_{E2}\right] + \frac{g^3}{(4\pi)}\left[\frac{d_A}{3}\alpha_{E4}{}^{3/2}\right] \\
&+ \frac{g^4}{(4\pi)^2}\left[\alpha_{E3} - d_A C_A\left(\alpha_{E4}\left(\frac{1}{4\epsilon}+\frac{3}{4}+\ln\frac{\bar\mu}{2gT\alpha_{E4}{}^{1/2}}\right)+\frac{1}{4}\alpha_{E5}\right)\right] \\
&+ \frac{g^5}{(4\pi)^3}\left[d_A\alpha_{E4}{}^{1/2}\left(\frac{1}{2}\alpha_{E6} - C_A^2\left(\frac{89}{24}+\frac{\pi^2}{6}-\frac{11}{6}\ln 2\right)\right)\right] \\
&+ \frac{g^6}{(4\pi)^4}\left[\beta_{E1} - \frac{1}{4}d_A\alpha_{E4}\left((d_A-2)\beta_{E4}+\frac{2d_A-1}{N_c}\beta_{E5}\right)\right. \\
&\qquad - d_A C_A\left(\frac{1}{4}(\alpha_{E6}+\alpha_{E5}\alpha_{E7}+3\alpha_{E4}\alpha_{E7}+\beta_{E2}+\alpha_{E4}\beta_{E3})\right. \\
&\qquad\qquad \left.+ (\alpha_{E6}+\alpha_{E4}\alpha_{E7})\left(\frac{1}{4\epsilon}+\ln\frac{\bar\mu}{2gT\alpha_{E4}{}^{1/2}}\right)\right) \\
&\qquad + d_A C_A^3\left(\beta_M+\beta_G+\alpha_M\left(\frac{1}{\epsilon}+8\ln\frac{\bar\mu}{2gT\alpha_{E4}{}^{1/2}}\right)\right. \\
&\qquad\qquad \left.\left.+\alpha_G\left(\frac{1}{\epsilon}+8\ln\frac{\bar\mu}{2g^2TC_A}\right)\right)\right]
\end{aligned}
$$

Contribution to the pressure of order $O(g^6 \ln(1/g))$ (Kajantie et al. [2003])

- number of unknown matching coefficients remain (the β's)
- result must be scale independent

- How can we fix the contribution of order $O(g^6 ln(1/g))$?
  1. $\Rightarrow \beta_{E1} = d_A C_A(\alpha_{E6} + \alpha_{E4}\alpha_{E7})\frac{1}{4\epsilon} - d_A C_A^3(\alpha_M + \alpha_G)\frac{1}{\epsilon} + \beta_{E6}$
  2. adding and subtracting $\ln\left[\frac{\bar{\mu}}{2\pi T}\right]$'s

contribution of $O(g^6 \ln(1/g))$

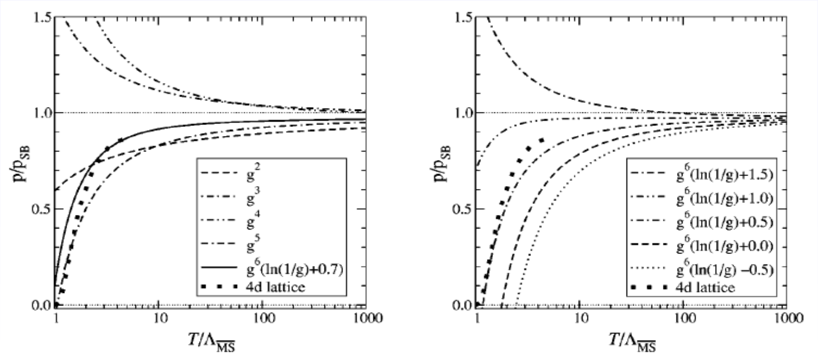
$$\left.\frac{p_{QCD}(T)}{T^4\mu^{-2\epsilon}}\right|_{g^6\ln(1/g)} = g^6\frac{d_A C_A}{(4\pi)^4}\Big[(\alpha_{E6} + \alpha_{E4}\alpha_{E7})\ln(g\alpha_{E4}^{1/2}) - 8C_A^2(\alpha_M \ln(g\alpha_{E4}^{1/2}) + 2\alpha_G \ln(gC_A^{1/2}))\Big]$$

$$N_c = 3 \Rightarrow p_{QCD}(T) = \frac{8\pi^2}{45}T^4\left[\sum_{i=0}^{6} p_i\left(\frac{\alpha_s(\bar{\mu})}{\pi}\right)^{i/2}\right]$$

The numerical convergence (Kajantie et al. [2003])

genuine $O(g^6 \ln(1/g) + g^6)$ contribution

$$\delta\left[\frac{p_{QCD}(T)}{T\mu^{-2\epsilon}}\right]_{g^6\ln(1/g)} = 8 d_A C_A^3 \frac{g_E^6}{(4\pi)^4}\left[(\alpha_M + 2\alpha_G)\ln\frac{m_E}{g_E^2} + \delta\right]$$



Outlook

- different scales in the system ⇒ various types of perturbatively computable logarithms in the 4-loop expansion:
  1. Logarithms of the type $g^6 \ln[(2\pi T)/(g^2 T)]$
  2. Logarithms of the type $g^6 \ln[(2\pi T)/(gT)]$
  3. Logarithms related to the running of the coupling constant in the 3-loop expression of order $O(g^4 \ln[(2\pi T)/(gT)]$.
- to complete the pressure from the current level $O\left(g^6 \ln\left(\frac{1}{g}\right)\right)$ to the full level $O(g^6)$:
  - 4-loop finite T sum-integrals
  - 4-loop vacuum integrals in $d = 3 - 2\epsilon$
  - 4-loop vacuum integrals in 3D lattice regularization
  - lattice simulations of the pure 3D gauge theory
- ⇒ analytic, non-perturbative resummation scheme

Resummation method

rearranging perturbation series

$$S_{eff} = \frac{1}{l}\left(S_G(\sqrt{l}A_k) + S_m(\sqrt{l}A_k)\right) - S_m(\sqrt{l}A_k)$$

- perturbative calculations as a power series in l
- goal: find particular size of tree-level mass term $m = Cg^2$ leading to a convergent perturbation series
- gap equation $\hat{=}$ self-consistent condition for the vector boson mass

$$\Delta_T(p^2) = \frac{1}{p^2 + m^2 - \Pi_T(p^2)} \sim \frac{Z}{p^2 + m^2} \quad for \quad p^2 \sim -m^2$$

$$\Rightarrow \Pi_T(p^2 = -m^2)\left(1 + \frac{\partial \Pi_T}{\partial p^2}(p^2 = -m^2)\right) \quad = \quad 0$$

- n-th order of resummed perturbation theory: computing gap equation for the self-energy up to l^n and solve for m
- 1-loop:

$$\Pi_T^{1-loop}(p^2 = -m^2) = 0$$

Resummation method

Lagrangian of massive YM (resummed non-linear σ-model ($\xi \to \infty$))

$$\mathcal{L} = \frac{1}{4} F^a_{kl} F^a_{kl} + \frac{1}{2} m^2 A^a_k A^a_k - \frac{l}{2} m^2 A^a_k A^a_k$$