Non-Perturbative QCD Correlation Functions in the Lattice Schrödinger Functional

Christian Wittemeier

Institut für Theoretische Physik – WWU Münster

30 May 2011

History and Applications of the Schrödinger Functional

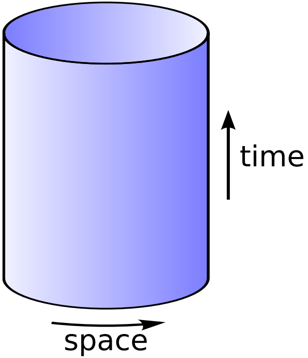
Symanzik, 1981 proved SF to be renormalizable, ϕ^4 theory

Lüscher et al., 1992 application to pure gauge theories

Sint, 1994 inclusion of fermions

ALPHA Collaboration various applications investigated by the ALPHA collaboration

- can be used to study the running coupling, masses etc.
- to extract hadron masses and matrix elements
- to determine parameters of heavy quark effective theory ("matching")



Correlation Functions

- expectation values of field products, e.g. $\langle \ldots \psi(x) \ldots \bar{\psi}(y) \ldots \rangle$
- some CF from the Schrödinger functional: $f_{\mathrm{P}}(x_0)$, $f_{\mathrm{A}}(x_0)$, f_1

Applications

Hadron Masses

- pion mass m_π can be extracted from $f_{\mathrm{P}}(x_0)$ or $f_{\mathrm{A}}(x_0)$

$$f_{\mathrm{P/A}} \propto \mathrm{e}^{-m_\pi \cdot x_0} \quad \text{for } x_0 \to \infty$$

Pion Decay Constant

- F_π determines lifetime of pion $\tau_\pi \propto F_\pi^{-2}$.
- can be calculated from $f_{\mathrm{A}}(x_0)$

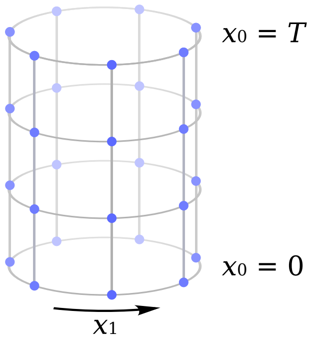
$$\frac{Z_{\mathrm{A}} f_{\mathrm{A}}}{\sqrt{f_1}} \approx \frac{1}{2} \cdot F_\pi \cdot \sqrt{m_\pi L^3} \cdot \mathrm{e}^{-(x_0 - T/2) m_\pi}$$

Contents

The Lattice Structure

- 4-dimensional Euclidean lattice
- set lattice spacing $a = 1$ ("lattice units")
- integer coordinates x_0, x_1, x_2, x_3

$$0 \leq x_0 \leq T$$
$$0 \leq x_k \leq L - 1 \qquad (k = 1, 2, 3)$$



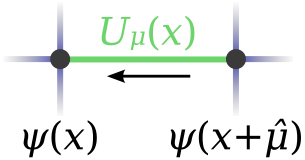
Boundary Conditions (BC) in the SF

- periodic BC in space: $f(x + L\hat{k}) = f(x)$
- Dirichlet BC in time, i.e. $f(0, \mathbf{x})$ and $f(T, \mathbf{x})$ have predefined values

Gauge Field

- represented by **link** matrices $\in \mathrm{SU}(3)$
- direction index $\mu \in \{0,1,2,3\}$

$$U_\mu(x)$$



- $U_\mu(x)\psi(x+\hat{\mu})$ is $\psi(x+\hat{\mu})$ parallel transported to x.

Boundary Conditions

$$U_k(0,\mathbf{x}) = \mathrm{e}^{\mathrm{i}C_k} \qquad U_k(T,\mathbf{x}) = \mathrm{e}^{\mathrm{i}C'_k} \qquad (k=1,2,3)$$

- C_k and C'_k are traceless Hermitian 3×3 matrices ($\in \mathfrak{su}(3)$).
- often set to zero

Fermionic Fields

$$\psi^{(i)}_{A\alpha}(x) \qquad \bar{\psi}^{(i)}_{A\alpha}(x)$$

- Dirac index $A \in \{1, 2, 3, 4\}$
- color index $\alpha \in \{1, 2, 3\}$
- flavor index i
  $\longrightarrow 4 \times 3 = 12$ components per flavor
- Graßmann-valued

Problem

- Dirac equation is of first order.
  $\longrightarrow$ Only half the components may be specified at the boundaries.

Fermionic Fields – Boundary Conditions

- Use Projectors P_+, P_-

$$P_+ = \frac{1+\gamma_0}{2} \qquad\qquad P_- = \frac{1-\gamma_0}{2}$$

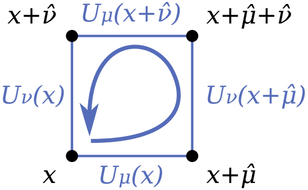
BC for Fermionic Fields

$$P_+\psi(0,\mathbf{x}) = \rho(\mathbf{x}) \qquad\qquad \bar{\psi}(0,\mathbf{x})P_- = \bar{\rho}(\mathbf{x})$$
$$P_-\psi(T,\mathbf{x}) = \rho'(\mathbf{x}) \qquad\qquad \bar{\psi}(T,\mathbf{x})P_+ = \bar{\rho}'(\mathbf{x})$$

- $\rho, \bar{\rho}, \rho', \bar{\rho}'$ are usually set to zero.

Lattice Action – Gauge Part

- basic building block: **plaquette**
- product of link matrices along 1×1 square



$$P_{x,\mu,\nu} = U_\mu(x) \cdot U_\nu(x+\hat{\mu}) \cdot U_\mu(x+\hat{\nu})^{-1} \cdot U_\nu(x)^{-1}$$

Gauge Action

$$S_{\mathrm{G}}[U] = \frac{1}{g_0^2} \sum_{x,\mu,\nu} \mathrm{tr}(1 - P_{x,\mu,\nu})$$

- approaches continuum action for $a \to 0$

Lattice Action – Fermionic Part

Wilson–Dirac Action

$$S_{\mathrm{F}}[U,\psi,\bar{\psi}] = \sum_x \bar{\psi}(x)\left[\gamma_\mu \cdot \frac{1}{2}(\nabla_\mu + \nabla^*_\mu) + m + \underbrace{\frac{1}{2}\nabla^*_\mu\nabla_\mu}_{\text{Wilson term}}\right]\psi(x)$$

- ∇_μ, ∇^*_μ: forward and backward differences

$$\nabla_\mu\psi(x) = \mathrm{e}^{i\theta_\mu/L} \cdot U_\mu(x) \cdot \psi(x+\hat{\mu}) - \psi(x)$$

- Wilson term removes "doublers"

- in addition, $\mathcal{O}(a)$ counterterms, which reduce lattice errors to $\mathcal{O}(a^2)$ ("clover" term, boundary terms)

Define Dirac operator D:

$$S_{\mathrm{F}}[U,\psi,\bar{\psi}] = \sum_{x,y} \bar{\psi}(x)D(x,y)\psi(y)$$

- matrix in Euclidean, spinor and color space

The Schrödinger Functional

Schrödinger Functional

- partition function dependent on the boundary fields

$$\mathcal{Z}[C, C', \rho, \bar{\rho}, \rho', \bar{\rho}'] = \int [\mathrm{d}U, \mathrm{d}\psi, \mathrm{d}\bar{\psi}] \; \mathrm{e}^{-S_{\mathrm{G}}[U]-S_{\mathrm{F}}[U,\psi,\bar{\psi}]}$$

- describes transition amplitude from primed to unprimed field configuration

Expectation Values

$$\langle O \rangle = \frac{\int [\mathrm{d}U, \mathrm{d}\psi, \mathrm{d}\bar{\psi}] \;\; O[U, \psi, \bar{\psi}] \, \mathrm{e}^{-S_{\mathrm{G}}-S_{\mathrm{F}}}}{\int [\mathrm{d}U, \mathrm{d}\psi, \mathrm{d}\bar{\psi}] \;\; \mathrm{e}^{-S_{\mathrm{G}}-S_{\mathrm{F}}}}$$

Evaluation of Expectation Values

- first only fermionic EV (can be evaluated via **Wick theorem**)

$$\langle O\rangle_{\mathrm{F}} = \frac{\int[\mathrm{d}\psi,\mathrm{d}\bar{\psi}]\;\; O\,\mathrm{e}^{-S_{\mathrm{F}}}}{\int[\mathrm{d}\psi,\mathrm{d}\bar{\psi}]\;\; \mathrm{e}^{-S_{\mathrm{F}}}} = \frac{\int[\mathrm{d}\psi,\mathrm{d}\bar{\psi}]\;\; O\,\mathrm{e}^{-S_{\mathrm{F}}}}{\det D}$$

- integrals splitted into gauge and fermionic part:

$$\langle O\rangle = \frac{\int[\mathrm{d}U]\;\; \langle O\rangle_{\mathrm{F}}\cdot\det D\cdot\mathrm{e}^{-S_{\mathrm{G}}}}{\int[\mathrm{d}U]\;\; \det D\cdot\mathrm{e}^{-S_{\mathrm{G}}}}$$

Quenched Approximation

- Approximate determinant by one: $\det D = 1$.

$$\langle O\rangle_{\mathrm{quenched}} = \frac{\int[\mathrm{d}U]\;\; \langle O\rangle_{\mathrm{F}}\cdot\mathrm{e}^{-S_{\mathrm{G}}}}{\int[\mathrm{d}U]\;\; \mathrm{e}^{-S_{\mathrm{G}}}} = \langle\langle O\rangle_{\mathrm{F}}\rangle_{\mathrm{G}}$$

- means: no virtual quarks

Monte Carlo Integration

$$\langle\langle O\rangle_{\mathrm{F}}\rangle_{\mathrm{G}} = \frac{\int[\mathrm{d}U]\ \langle O\rangle_{\mathrm{F}}\cdot \mathrm{e}^{-S_{\mathrm{G}}}}{\int[\mathrm{d}U]\ \mathrm{e}^{-S_{\mathrm{G}}}}$$

Gauge integration is performed using Monte Carlo algorithm

- Choose random gauge fields U (probability proportional to $\mathrm{e}^{-S_{\mathrm{G}}[U]}$).
- Calculate $\langle O\rangle_{\mathrm{F}}$ for each gauge field U.
- Average over all results.

Hybrid Monte Carlo (HMC)

To generate a new U the HMC

- creates a random momentum field π
- evolves U and π according to classical Hamiltonian dynamics (with S_{G} as potential) $\Rightarrow$ candidate field U'
- finally decides whether to take U' or again U (to compensate for numerical errors).

Contents

Building Blocks for Observables

What can observables be built of?

The gauge field U and the fermionic fields ψ and $\bar{\psi}$.

What about the boundary fields $\rho, \bar{\rho}, \ldots$?

They are zero or at least constant anyway. Use derivatives instead!

Boundary Sources

Use derivatives of the action S to build observables with boundary sources:

$$\zeta(\mathbf{x}) = -\frac{\delta S}{\delta \bar{\rho}(\mathbf{x})} \qquad \bar{\zeta}(\mathbf{x}) = \frac{\delta S}{\delta \rho(\mathbf{x})}$$

$$\zeta'(\mathbf{x}) = -\frac{\delta S}{\delta \bar{\rho}'(\mathbf{x})} \qquad \bar{\zeta}'(\mathbf{x}) = \frac{\delta S}{\delta \rho'(\mathbf{x})}.$$

- combinations of U, ψ and $\bar{\psi}$ fields near the boundary

Symmetries

Symmetries must be taken into account!

Symmetries

- $\mathrm{SU}(3)$ color gauge symmetry
- $\mathrm{SU}(2)$ flavor symmetry (or $\mathrm{SU}(3)$)
- parity, charge conjugation
- spatial symmetries

- First, make up field combinations with defined transformation behaviour.
- Then, combine them into observables.

Symmetric Field Products – Example

Pseudo-Scalar Density

$$P_a(x) = \bar{\psi}(x) \cdot \gamma_5 \cdot \frac{\tau_a}{2} \cdot \psi(x)$$

in components:

$$P_a(x) = \bar{\psi}^{(i)}_{A\alpha}(x) \cdot (\gamma_5)_{AB} \cdot \frac{\tau_a^{(ij)}}{2} \cdot \psi^{(j)}_{B\alpha}(x)$$

(τ_a: Pauli matrices in flavor space)

γ_5 $\rightarrow$ spin 0 and negative parity
$\tau_a/2$ $\rightarrow$ iso-spin triplet

- represents a mixture of the pion and other, excited states with the same quantum numbers

More Symmetric Field Products

- In the following we will use:

pseudo-scalar density: $$P_a(x) = \bar{\psi}(x) \cdot \gamma_5 \cdot \frac{\tau_a}{2} \cdot \psi(x)$$

axial current: $$A_{0a}(x) = \bar{\psi}(x) \cdot \gamma_0 \gamma_5 \cdot \frac{\tau_a}{2} \cdot \psi(x)$$

with boundary fields:

lower boundary: $$O_a = \sum_{\mathbf{yz}} \bar{\zeta}(\mathbf{y}) \cdot \gamma_5 \cdot \frac{\tau_a}{2} \cdot \zeta(\mathbf{z})$$

upper boundary: $$O'_a = \sum_{\mathbf{yz}} \bar{\zeta}'(\mathbf{y}) \cdot \gamma_5 \cdot \frac{\tau_a}{2} \cdot \zeta'(\mathbf{z}).$$

- All have spin 0 and negative parity.
- Sums over $\mathbf{y}$, $\mathbf{z}$ project to zero momentum states.
- Many other field combinations are possible, e.g. involving vector currents, baryonic sources etc.

Correlation Functions (CF)

- Observables must be invariant under symmetries.
- Combine appropriate field products.

Correlation Functions

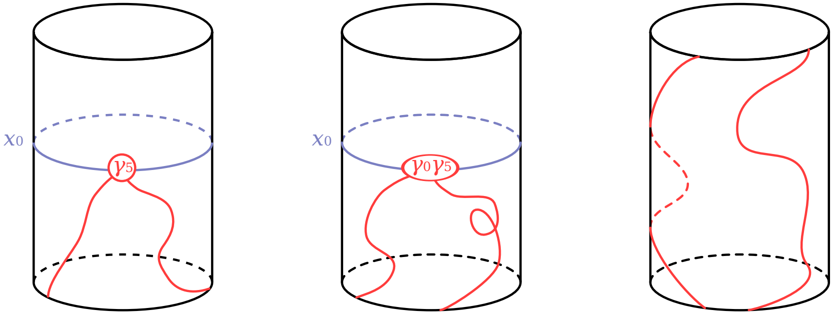
$$f_{\mathrm{P}}(x_0) = -\frac{1}{3L^3} \cdot \sum_{\mathbf{x}} \langle P_a(x_0, \mathbf{x}) \cdot O_a \rangle$$

$$f_{\mathrm{A}}(x_0) = -\frac{1}{3L^3} \cdot \sum_{\mathbf{x}} \langle A_{0a}(x_0, \mathbf{x}) \cdot O_a \rangle$$

$$f_1 = -\frac{1}{3L^6} \cdot \left\langle O'_a \cdot O_a \right\rangle$$

- additional summation over $\mathbf{x}$, since CF are independent of $\mathbf{x}$ anyway (translational symmetry)

Pictorial Representations of CF



$f_{\mathrm{P}} \propto \langle P_a O_a \rangle$ $\quad$ $f_{\mathrm{A}} \propto \langle A_{0a} O_a \rangle$ $\quad$ $f_1 \propto \langle O'_a O_a \rangle$

Fermionic Integration

Let us try to calculate $f_{\mathrm{P}} = -\frac{1}{3L^3}\sum_{\mathbf{x}}\langle P_a(x_0,\mathbf{x})\cdot O_a\rangle$.

$$f_{\mathrm{P}}(x_0) = \left\langle -\frac{1}{3L^3}\sum_{\mathbf{xyz}}\left\langle \bar{\psi}(x)\gamma_5\frac{\tau_a}{2}\psi(x)\cdot\bar{\zeta}(\mathbf{y})\gamma_5\frac{\tau_a}{2}\zeta(\mathbf{z})\right\rangle_{\mathrm{F}}\right\rangle_{\mathrm{G}}$$

- Fermionic integral can be evaluated by means of the *Wick theorem*.
- Only one way of contraction does not vanish:

$$\langle\ldots\rangle_F = -\frac{1}{3L^3}\sum_{\mathbf{xyz}}\bar{\psi}(x)\gamma_5\frac{\tau_a}{2}\psi(x)\cdot\bar{\zeta}(\mathbf{y})\gamma_5\frac{\tau_a}{2}\zeta(\mathbf{z})$$

Boundary-to-Bulk Propagator

Boundary-to-Bulk Propagator

$$\bar{S}(x) = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{y}} \left\langle \psi(x) \bar{\zeta}(\mathbf{y}) \right\rangle_{\mathrm{F}}$$

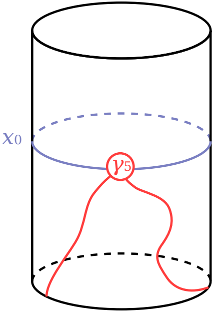
- matrix in Dirac and color space
- independent of flavor for mass-degenerate quarks
- dependent on gauge field U
- can be obtained as solution of Dirac equation with source

$$\sum_{x'} D(x, x') \cdot \bar{S}(x') = \frac{\tilde{c}_t}{\sqrt{L^3}} \cdot \delta_{x_0,1} \cdot U_0(0, \mathbf{x})^{-1} \cdot P_+$$

Fermionic Integration

$$\bar{S}(x) = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{y}} \left\langle \psi(x)\bar{\zeta}(\mathbf{y}) \right\rangle_{\mathrm{F}} \qquad \gamma_5 \bar{S}(x)^\dagger \gamma_5 = \frac{1}{\sqrt{L^3}} \sum_{\mathbf{z}} \left\langle \zeta(\mathbf{z})\bar{\psi}(x) \right\rangle_{\mathrm{F}}$$

$$\begin{aligned} f_{\mathrm{P}}(x_0) &= \left\langle -\frac{1}{3L^3} \sum_{\mathbf{xyz}} \bar{\psi}(x)\gamma_5 \frac{\tau_a}{2}\psi(x) \cdot \bar{\zeta}(\mathbf{y})\gamma_5 \frac{\tau_a}{2}\zeta(\mathbf{z}) \right\rangle_{\mathrm{G}} \\ &= \left\langle \frac{1}{3} \sum_{\mathbf{x}} \mathrm{tr}\left[\gamma_5 \bar{S}(x)^\dagger \gamma_5 \gamma_5 \frac{\tau_a}{2} \bar{S}(x) \gamma_5 \frac{\tau_a}{2} \right] \right\rangle_{\mathrm{G}} \\ &= \left\langle \frac{1}{2} \sum_{\mathbf{x}} \mathrm{tr}\left[\bar{S}(x)^\dagger \bar{S}(x) \right] \right\rangle_{\mathrm{G}} \end{aligned}$$



Explicit Form of Correlation Functions

$$f_{\mathrm{P}}(x_0) = \left\langle +\frac{1}{2} \sum_{\mathbf{x}} \mathrm{tr}\left[\bar{S}(x)^\dagger \bar{S}(x)\right] \right\rangle_{\mathrm{G}}$$

$$f_{\mathrm{A}}(x_0) = \left\langle -\frac{1}{2} \sum_{\mathbf{x}} \mathrm{tr}\left[\bar{S}(x)^\dagger \gamma_0 \bar{S}(x)\right] \right\rangle_{\mathrm{G}}$$

$$f_1 = \left\langle +\frac{1}{2} \sum_{\mathbf{x}} \mathrm{tr}\left[\bar{S}_{\mathrm{T}}^\dagger \bar{S}_{\mathrm{T}}\right] \right\rangle_{\mathrm{G}}$$

- $\bar{S}_{\mathrm{T}}$ ("boundary-to-boundary propagator") can be calculated from $\bar{S}(x)$.

This is what we have to compute!

Summary

1. Calculate boundary-to-bulk propagator $\bar{S}(x_0)$ by solving the Dirac equation

$$\sum_{x'} D(x,x') \cdot \bar{S}(x') = \frac{\tilde{c}_t}{\sqrt{L^3}} \cdot \delta_{x_0,1} \cdot U_0(0,\mathbf{x})^{-1} \cdot P_+.$$

2. Compute correlation functions via the formulas we have just obtained, e.g.

$$f_{\mathrm{P}}(x_0) = \frac{1}{2} \sum_{\mathbf{x}} \mathrm{tr}\left[\bar{S}(x)^\dagger \bar{S}(x)\right]$$

   (yet without gauge integration).

3. *Integrate over gauge fields by repeating the above steps for many gauge fields U and average over the results.* (not considered in the following)

Contents

Our Project

The APE Machines

- massively-parallel, located at DESY Zeuthen
- especially for QCD simulations
- special programming language: *TAO*
- started 1987, finally shut down this year

⇒ We have to write a new program.

- language: C → platform-independent
- based on elements of DD-HMC code by M. Lüscher for periodic BC

Contributors

Zeuthen	Hubert Simma, Andreas Athenodorou
Mainz	Michele Della Morte
Münster	Jochen Heitger and myself
	et al.

Fields on the Computer

Gauge Links

- represented as 3×3 complex matrices
- needs more memory than necessary for $\mathrm{SU}(3)$, but multiplication etc. is easier

```
typedef struct
{
   complex c11, c12, c13,
           c21, c22, c23,
           c31, c32, c33;
} su3;
```

Fermions

- 4 Dirac components
- 3 color components

$\Rightarrow$ 12 complex comp. at all

- Graßmann nature not explicitly regarded

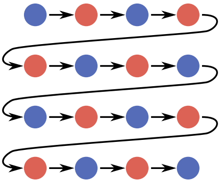
```
typedef struct
{
   su3_vector c1, c2,
              c3, c4;
} spinor;

typedef struct
{
   complex c1, c2, c3;
} su3_vector;
```

Storage Order of Fields

In which order should the fields be stored?

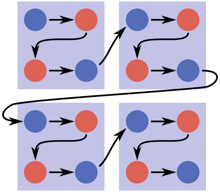
lexicographic order

$$\begin{array}{ccc} (0,0,0,0) & \dots & (0,0,0,L-1) \\ (0,0,1,0) & \dots & (0,0,1,L-1) \\ \vdots & \vdots & \vdots \\ (T,L-1,L-1,0) & \dots & (T,L-1,L-1,L-1) \end{array}$$



cacheblocks Divide lattice into smaller blocks and run through the blocks one by one.

$\Rightarrow$ Points close to each other on the lattice are likely to appear close to each other in memory. Can be *cached*.



- It can be useful to store even and odd points separately.

Other Methods for Gaining Speed

Parallelization

- distribute *spatial* lattice among N processes
- e.g. Dirac operator can be computed for N points at once
- “neighbour” processes need to communicate

```
if (rank==ip) {
  mpc_send_d(sw[2*ix].u,72,0,0);
}
if (rank==0) {
  mpc_recv_d(aux[0].u,72,ip,0);
  // ...
}
```

Assembler Instructions

- SSE instructions
- also used for graphics processing
- manipulate several floating point numbers at once

```
__asm__ ("mulpd %0, %%xmm0 \n\t"
         "mulpd %1, %%xmm1 \n\t"
         ...
         "m" ((*rpl).c1.c1),
         "m" ((*rpl).c1.c2),
         ...
```

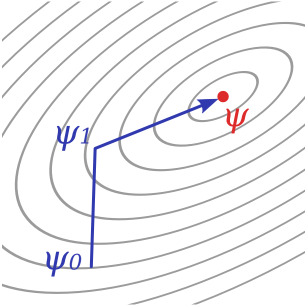
Solving the Dirac Equation

We must solve:

$$D\psi = \eta$$

- D is ca. $10^5 \times 10^5$ matrix.

$\Rightarrow$ Use approximate solver.



currently: conjugate gradient (CG) method

Stiefel, Hestenes, 1952

- needs Hermitian matrix: D not Hermitian, but $\gamma_5 \cdot D$!

$\Rightarrow$ Solve $(\gamma_5 D)\psi = \gamma_5 \eta$ instead.

- CG searches for the minimum of $F(\psi) \equiv \frac{1}{2}\langle\psi|\gamma_5 D|\psi\rangle - \mathrm{Re}\langle\psi|\gamma_5\eta\rangle$
- minimizes $F(\psi)$ along a certain direction in each step
- Directions are *conjugate* to each other, i.e. orthogonal w. r. t. the scalar product defined by $\gamma_5 D$.
- stopped when residuum is small enough

Computation of the Boundary-to-Bulk Propagator

1. Compute propagator $\bar{S}$.
2. Compute correlation functions.

$$\sum_{x'} (\gamma_5 D(x,x'))_{A\alpha,B\beta} \cdot \bar{S}_{B\beta,C\gamma}(x') = \left[\gamma_5 \frac{\tilde{c}_t}{\sqrt{L^3}} \delta_{x_0,1} U_0(0,\mathbf{x})^{-1} P_+\right]_{A\alpha,C\gamma}$$

for $C = 1$ to 2, $\gamma = 1$ to 3 **do**

prepare right-hand side $\gamma_5\eta$ for indices C,γ

$$(\gamma_5\eta(x))_{A\alpha} = \frac{\tilde{c}_t}{\sqrt{L^3}} \delta_{x_0,1} \left[U_0(0,\mathbf{x})^{-1}\right]_{\alpha,\gamma} \left[\gamma_5 P_+\right]_{A,C}$$

call CG to solve $\gamma_5 D \cdot \psi = \gamma_5\eta$

save propagator comp. for C,γ: $\bar{S}_{B\beta,C\gamma}(x) = \psi_{B\beta}(x)$

end for

- Propagator components $C = 3,4$ differ only by a minus sign.

Computation of Correlation Functions

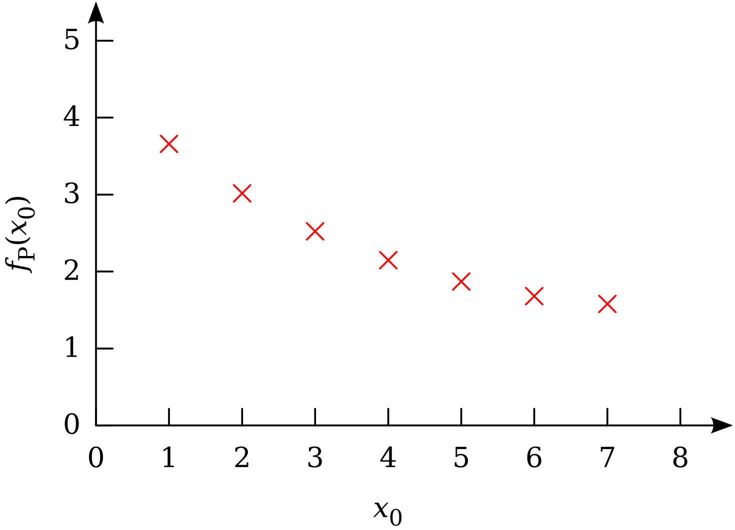
1. Compute propagator $\bar{S}$.
2. Compute correlation functions.

$$f_{\mathrm{P}}(x_0) = \left\langle \frac{1}{2} \sum_{\mathbf{x}} \mathrm{tr}\left[\bar{S}(x)^\dagger \bar{S}(x) \right] \right\rangle_{\mathrm{G}} = \left\langle \frac{1}{2} \sum_{\mathbf{x}} \bar{S}_{A\alpha,B\beta}(x)^* \bar{S}_{A\alpha,B\beta}(x) \right\rangle_{\mathrm{G}}$$

```
void f_p_rel(int field_1, int field_2, double *out) {
  ...
  for(x0=1; x0<TSIZE; x0++) {
    ...
    for(ind=(x0-1)*TSLICE/2; ind<x0*TSLICE/2; ind++) {
      for(dir_col=0; dir_col<6; dir_col++) {
        ...
        psi1 = psle[field_1+dir_col]+ind;  // psle: spinors at "even" points
        psi2 = psle[field_2+dir_col]+ind;
        ...
        realspinordagxspinor(&prod1, psi2, psi1);
        ...
        sum[0] += prod1+prod2;  // prod2: product of spinors at "odd" points
        ...

void realspinordagxspinor(double *r, spinor *psi1, spinor *psi2) {
  ...
  *r = psi1->c1.c1.re*psi2->c1.c1.re+psi1->c1.c1.im*psi2->c1.c1.im
      +psi1->c1.c2.re*psi2->c1.c2.re+psi1->c1.c2.im*psi2->c1.c2.im
      ...
```

The Result

$f_{\mathrm{P}}(x_0)$ computed for a unity gauge field

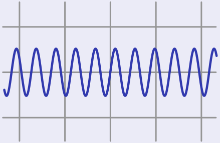


Contents

Planned Applications – Matching of HQET

Problem with Simulating Heavy Quarks

e.g. beauty quark, relevant length scale $\frac{1}{m_b} \ll$ lattice spacing



Solution: Heavy Quark Effective Theory (HQET) *Eichten & Hill, 1990*

- expansion in terms of $\frac{1}{m_b}$
- unknown parameters must be determined by a matching procedure, ideally non-perturbative in order to avoid power divergences
- Observables from HQET and full QCD are equated:

$$\Phi_k^{\mathrm{HQET}}(L, m_b) = \Phi_k^{\mathrm{QCD}}(L, m_b) \qquad k = 1, 2, \ldots, N.$$

N: number of unknown parameters

Heitger, Sommer et al. ALPHA Collaboration

Planned Applications – Matching of HQET

Matching of Axial Current

- number N of unknown parameters:

	3	parameters in the action
	2	parameters in the axial current
$\Rightarrow$	5	QCD observables needed for matching

- can be constructed from f_1, f_{A} and k_1 (a CF in the vector channel)
- other observables needed for vector current (e.g. involving k_{V})
- has been investigated perturbatively at tree level by *Samantha Dooling*

- Φ_k^{QCD} need to be computed only in a small volume ($\approx 0.5\,\mathrm{fm}$)
- larger volumes for extracting physical quantities reached by *step scaling method*

to be continued next week by Michael Topp...

Appendix

The following slides were not part of the actual presentation.

QCD Observables for Matching of HQET

- 5 QCD observables needed for matching
- a possible choice:

$$\left\{ L\Gamma^{\mathrm{P}}, \quad \ln(-f_{\mathrm{A}}/\sqrt{f_1}), \quad R_{\mathrm{A}}, \quad \frac{3}{4}\ln(f_1/k_1), \quad R_1 \right\}$$

with

$$\Gamma^{\mathrm{P}} = -\tilde{\partial}_0 \left(\ln(-f_{\mathrm{A}}(x_0,\theta_0))\right)\Big|_{(x_0=T/2, T=L)}$$

$$R_{\mathrm{A}} = \ln\left(\frac{f_{\mathrm{A}}(T/2,\theta_1)}{f_{\mathrm{A}}(T/2,\theta_2)}\right)$$

$$R_1 = \frac{1}{4}\left[\ln(f_1(\theta_1)k_1(\theta_1)^3) - \ln(f_1(\theta_2)k_1(\theta_2)^3]\right)$$

- k_1 is a CF similar to f_1, but in the vector channel. (Replace γ_5 by γ_k and sum over $k = 1, 2, 3$!)

References – Explanations

- [1, 2, 3] are the original articles by Symanzik, Lüscher et al. about the Schrödinger functional.
- About how to extract hadron masses and matrix elements from correlation functions, see [4].
- About the pion decay constant, see [5].
- [6] includes a description of the Hybrid Monte Carlo algorithm.
- [7] and [8] have introductions to lattice QCD, the Schrödinger functional, HQET and other topics. In the appendix of [8] it is also shown in detail how to express the correlation functions in terms of propagators.
- [9] gives an introduction to HQET.
- I have got the explicit form of the matching observables from [10].

References I

[1] K. Symanzik.
Schrödinger Representation and Casimir Effect in Renormalizable Quantum Field Theory.
Nucl. Phys., volume B190:page 1, 1981.
doi:10.1016/0550-3213(81)90482-X.

[2] Martin Lüscher, Rajamani Narayanan, Peter Weisz & Ulli Wolff.
The Schrödinger functional – a renormalizable probe for non-abelian gauge theories.
Nucl. Phys., volume B384:pages 168–228, 1992.
doi:10.1016/0550-3213(92)90466-O.
arXiv:hep-lat/9207009.

References II

[3] Stefan Sint.
On the Schrödinger functional in QCD.
Nucl. Phys., volume B421:pages 135–158, 1994.
doi:10.1016/0550-3213(94)90228-3.
arXiv:hep-lat/9312079.

[4] Marco Guagnelli, Jochen Heitger, Rainer Sommer & Hartmut Wittig (ALPHA).
Hadron masses and matrix elements from the QCD Schroedinger functional.
Nucl. Phys., volume B560:pages 465–481, 1999.
doi:10.1016/S0550-3213(99)00466-6.
arXiv:hep-lat/9903040.

References III

[5] J. Gasser & G. R. S. Zarnauskas.
On the pion decay constant.
Phys.Lett.B, volume 693:pages 122–128, August 2010.
arXiv:1008.3479.

[6] Martin Lüscher.
Computational Strategies in Lattice QCD.
2010.
arXiv:1002.4232.

[7] Patrick Fritzsch.
B-meson properties from non-perturbative matching of HQET to finite-volume two-flavour QCD.
Ph.D. thesis, Westfälische Wilhelms-Universität Münster, 2009.

References IV

[8] Roland Hoffmann.
Chiral properties of dynamical Wilson fermions.
Ph.D. thesis, Humboldt-Universität zu Berlin, 2005.

[9] Rainer Sommer.
Non-perturbative QCD: renormalization, O(a)-improvement and matching to Heavy Quark Effective Theory.
2006.
arXiv:hep-lat/0611020.

References V

[10] Benoit Blossier, Michele della Morte, Nicolas Garron & Rainer Sommer.
HQET at order 1/m: I. Non-perturbative parameters in the quenched approximation.
JHEP, volume 06:page 002, 2010.
doi:10.1007/JHEP06(2010)002.
arXiv:1001.4783.

[11] ALPHA collaboration.
Internal notes.