Non-perturbative Determination of HQET Parameters

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HQET

Heavy Quark Effective Theory

- HQ → (here) system of a single b-(anti-)quark and (anti-)light (e.g.: u, d) quark(s)
- Compare with sun/earth, proton/electron...
  heavy constituent nearly at rest in rest frame
  (for $m_b \to \infty$ "static approximation")
- ET → expansion in $1/m_b$

(Eichten, 1987; Eichten and Hill, 1990)

...on the lattice

$$\Lambda_{IR} = L^{-1} \ll m_\pi, ..., m_B \ll a^{-1} = \Lambda_{UV}$$

Here: requirement $L/a \approx O(10^2)$, solution later!

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How to derive the Lagrangian?

Strategy

- Euclidean action
- What are the dominant degrees of freedom for our kin. situation for b/u,d?
- Decouple large and small components order by order in D_k/m
  $\overline{\psi}_{\mathrm{h}}\, D_k/m\, \psi_{\mathrm{h}} \ll \overline{\psi}_{\mathrm{h}}\psi_{\mathrm{h}}$

Definitions

- Projector for quark velocity $\mathbf{u} = 0$ is given by $P_{\pm} = \frac{1 \pm \gamma_0}{2}$
- Gamma matrices in Dirac representation

$$\gamma_k = \begin{pmatrix} 0 & -i\sigma_k \\ i\sigma_k & 0 \end{pmatrix} \quad \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- $\mathcal{D} := \gamma_\mu D_\mu \qquad \sigma_{\mu\nu} := \frac{i}{2}\left[\gamma_\mu, \gamma_\nu\right]$

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Definitions for the lattice

Derivatives on the lattice

- Derivative acting on colour singlet function
  $\partial_\mu f(x) = \frac{1}{a}\left[f(x + a\hat{\mu}) - f(x)\right]$
  $\partial^*_\mu f(x) = \frac{1}{a}\left[f(x) - f(x - a\hat{\mu})\right]$ $\qquad \tilde{\partial}_\mu = \frac{1}{2}\left(\partial_\mu + \partial^*_\mu\right)$
- Covariant derivative acting on quark field
  $\nabla_\mu \psi(x) = \frac{1}{a}\left[U(x,\mu)\psi(x + a\hat{\mu}) - \psi(x)\right]$
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Wilson-Dirac operator

$$D_W = \frac{1}{2}\left[\gamma_\mu\left(\nabla^*_\mu + \nabla_\mu\right) - a\nabla^*_\mu\nabla_\mu\right]$$

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1st Fouldy Wouthuysen-Tani transformation

- Count all fields, derivatives of fiels (e.g. $F_{\mu\nu}$) as $O(1)$, except for $D_0\psi_h = O(m)\psi_h$
- Transformed Lagrangian $\mathcal{L} = \overline{\psi}' \left(\mathcal{D}' + m\right) \psi'$
  $S = \frac{1}{2m} D_k \gamma_k = -S^\dagger$

  $\psi \to \psi' = \mathrm{e}^S \psi \qquad \overline{\psi} \to \overline{\psi}' = \overline{\psi} \mathrm{e}^{-\overleftarrow{S}}$
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  Expand $e^{-S} \Rightarrow \mathcal{D}' + m$
  $= \mathcal{D} + m + \{-S, \mathcal{D} + m\} + \frac{1}{2}\{-S, \{-S, \mathcal{D} + m\}\} + \ldots$

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- After some algebra

  $\mathcal{D}' = D_0\gamma_0 - \frac{1}{2m}[\underbrace{\gamma_k\gamma_0 F_{k0}}_{\text{off-diagonal}} + \frac{1}{2i}\sigma_{kl}F_{kl} + D_k D_k] + O(1/m^2)$

  $F_{kl} = [D_k, D_l]$

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After 1st transformation

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- $\psi'' = e^{S'}\psi'$ for cancelling off-diagonal term
- $\{-S', \mathcal{D}' + m\} \stackrel{!}{=} \frac{1}{2m}\gamma_k\gamma_0 F_{k0} + O(1/m^2)$
  works with $S' = \frac{1}{4m^2}\gamma_0\gamma_k F_{k0}$

(classical) Lagrangian

$$\mathcal{L} = \mathcal{L}_{\mathrm{h}}^{\mathrm{stat}} + \frac{1}{2m}\mathcal{L}_{\mathrm{h}}^{(1)} + \mathcal{L}_{\bar{\mathrm{h}}}^{\mathrm{stat}} + \frac{1}{2m}\mathcal{L}_{\bar{\mathrm{h}}}^{(1)} + O(\frac{1}{m^2})$$

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Lagrangian up to order $1/m$

$$\mathcal{L}_{\rm h}^{\rm stat} = \overline{\psi}_{\rm h}(m + D_0)\psi_{\rm h} \qquad P_+\psi_{\rm h} = \psi_{\rm h} \qquad \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h}$$

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$$\mathcal{L}_{\rm h}^{(1)} = -(\mathcal{O}_{\rm kin} + \mathcal{O}_{\rm spin}) \qquad \mathcal{L}_{\bar{\rm h}}^{(1)} = -(\overline{\mathcal{O}}_{\rm kin} + \overline{\mathcal{O}}_{\rm spin})$$

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Propagator (static approximation)

From $(\partial_{x_0} + A_0(x) + m)\, G_{\rm h}(x,y) = \delta(x-y)P_+$, get solution

$$G_{\rm h}(x,y) = \theta(x_0-y_0)\, {\rm e}^{-m(x_0-y_0)}\, \mathcal{P}\, \exp\left[-\int_{y_0}^{x_0} {\rm d}z_0 A_0(z_0,\mathbf{x})\right] \delta(\mathbf{x}-\mathbf{y})\, P_+$$

- Similar result for the anti-quark's propagator
- For any gauge field A_μ mass as factor ${\rm e}^{-m|x_0-y_0|}$

Energy shift

After path integration over gauge fields of any 2-point function with b-quark still exp. decay

$\rightarrow C_{\rm h}(x,y;m) = C_{\rm h}(x,y;0){\rm e}^{-m|x_0-y_0|}$

Remove m from $\mathcal{L}^{\rm stat}_{\bar{\rm h}/{\rm h}}$, treat

$$E^{\rm QCD}_{\bar{\rm h}/{\rm h}} = E^{\rm stat}_{\bar{\rm h}/{\rm h}} + m$$

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Toal quantum Lagrangian (static)

- Renormalizability of static theory: by adding counter terms $\rightarrow$ absorb UV divergences
- Symmetry of static Lagrangian has to be conserved $\rightarrow$ add $c_1 \overline{\psi}_{\mathrm{h}} \psi_{\mathrm{h}}$ with $c_1 = \delta m = (e_1 g_0^2 + e_2 g_0^4 + ...)\Lambda_{\mathrm{cut}}$
- Chiral symmetry would forbid this, but not present in static theory

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$$E_{\bar{\mathrm{h}}/\mathrm{h}}^{\mathrm{QCD}} = E_{\bar{\mathrm{h}}/\mathrm{h}}^{\mathrm{stat}}\Big|_{\delta m=0} + \delta m + m = E_{\bar{\mathrm{h}}/\mathrm{h}}^{\mathrm{stat}}\Big|_{\delta m=0} + m_{\mathrm{bare}}$$

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Static Lagrangian

With suitable normalization factor:

$$\mathcal{L}_{\mathrm{h}} = \frac{1}{1+a\delta m}\,\overline{\psi}_{\mathrm{h}}(x)\,[\nabla_0^* + \delta m]\,\psi_{\mathrm{h}}(x)$$

$$\mathcal{L}_{\bar{\mathrm{h}}} = \frac{1}{1+a\delta m}\,\overline{\psi}_{\bar{\mathrm{h}}}(x)\,[-\nabla_0 + \delta m]\,\psi_{\bar{\mathrm{h}}}(x)$$

Propagator:

$$G_{\mathrm{h}}(x,y) = \theta(x_0-y_0)\,\delta(\mathbf{x}-\mathbf{y})\,\exp\left(-\widehat{\delta m}(x_0-y_0)\right)\mathcal{P}(y,x;0)^{\dagger}\,P_{+}$$

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Symanzik $O(a)$ improvement

- Add terms to lattice action S to reduce cutoff effects
- Terms must vanish in continuum limit
- $S_{\mathrm{eff}} = S_0 + aS_1 + \ldots \qquad S_{\mathrm{i}} = \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{i}}(x)$
  $(S_0 = \int \mathrm{d}^4x\, \mathcal{L}_0^{\mathrm{stat}}(x))$

(Kurt and Sommer, 2001)

Mass dim. 5 terms (fields and masses) to add that conserve symmetry:

- $\mathcal{O}_3 = \overline{\psi}_{\mathrm{h}} D_0 D_0 \psi_{\mathrm{h}}$ (vanishes due to eq. of motion $D_0\psi_{\mathrm{h}} = 0$)
- $\mathcal{O}_4 = m_{\mathrm{l}} \overline{\psi}_{\mathrm{h}} D_0 \psi_{\mathrm{h}}$ (vanishes as well)
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Axial current

In Symanzik's effective theory

$$\left(A_0^{\rm stat}\right)_{\rm eff} = A_0^{\rm stat} + a \sum_{k=1}^{4} \omega_k \left(\delta A_0^{\rm stat}\right)_k$$

$$\left(A_0^{\rm stat}\right)_1 = \overline{\psi}\overleftarrow{D}_{\rm j}\gamma_{\rm j}\gamma_5\psi_{\rm h} \qquad \left(A_0^{\rm stat}\right)_2 = \overline{\psi}\gamma_5 D_0\psi_{\rm h}$$

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- Eq. of motion for $\psi_{\rm h}$ excludes 2nd term!
- Neglect 4th term because of assumption $am_{\rm l} \ll 1$

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Renormalized improved axial current on the lattice

$$\begin{aligned}
\left(A_{\rm R}^{\rm stat}\right)_0 &= Z_{\rm A}^{\rm stat}(g_0, a\mu)\left(A_{\rm I}^{\rm stat}\right)_0 \\
\left(A_{\rm I}^{\rm stat}\right)_0 &= A_0^{\rm stat} + ac_{\rm A}^{\rm stat}(g_0)\overline{\psi}\gamma_j\gamma_5\frac{1}{2}\left(\overleftarrow{\nabla}_j + \overleftarrow{\nabla}_j^*\right)\psi_{\rm h}
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$$\left(A_{\rm R}^{\rm stat}\right)_0 = Z_{\rm A}^{\rm stat}(g_0, a\mu)\left(A_{\rm I}^{\rm stat}\right)_0$$

$$\left(A_{\rm I}^{\rm stat}\right)_0 = A_0^{\rm stat} + ac_{\rm A}^{\rm stat}(g_0)\overline{\psi}\gamma_{\rm j}\gamma_5\frac{1}{2}\left(\overleftarrow{\nabla}_{\rm j} + \overleftarrow{\nabla}_{\rm j}^*\right)\psi_{\rm h}$$

Improvement coefficient $c_{\rm A}^{\rm stat}$ can be determined that cutoff effects are $O(a^2)$

Axial current

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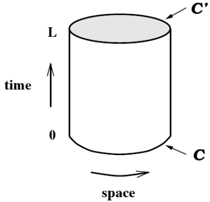
General features

- Cylindric finite space-time volume $T \times L^3$ (Lüscher et al., 1992)



(LxLxL box with periodic b.c.)

C and C' : gauge potentials

- Periodic spatial boundary conditions

$$\psi(x + L\hat{k}) = e^{i\theta_k}\psi(x)$$
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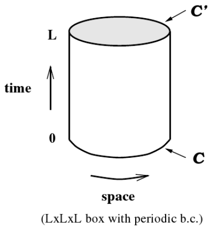

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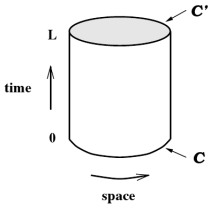

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Boundary fields

Definition

$$\zeta(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}(\mathbf{x})} \qquad \bar{\zeta}(\mathbf{x}) = -\frac{\delta}{\delta\rho(\mathbf{x})}$$

$$\zeta'(\mathbf{x}) = \frac{\delta}{\delta\bar{\rho}'(\mathbf{x})} \qquad \bar{\zeta}'(\mathbf{x}) = -\frac{\delta}{\delta\rho'(\mathbf{x})}$$

Can be used to build up correlation functions, 2 cases

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- Bulk to boundary correlation functions
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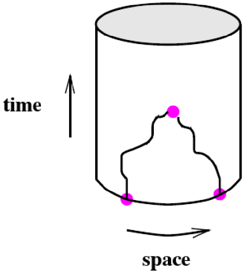

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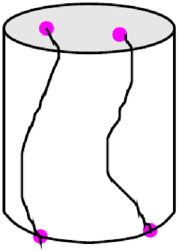

1st case in QCD

f_A

$$f_A(x_0,\theta) = -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}}\left\langle (A_\mathrm{I})_0\,(x)\,\bar{\zeta}_\mathrm{h}(\mathbf{y})\,\gamma_5\,\zeta_\mathrm{l}(\mathbf{z})\right\rangle$$

with $O(a)$ improved axial current $(A_\mathrm{I})_\mu$

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2nd case in QCD

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$$f_1(\theta) = -\frac{a^{12}}{2L^6} \sum_{\mathbf{u},\mathbf{v},\mathbf{y},\mathbf{z}} \left\langle \overline{\zeta}'_{\rm l}(\mathbf{u})\, \gamma_5\, \zeta'_{\rm h}(\mathbf{z})\, \overline{\zeta}_{\rm h}(\mathbf{y})\, \gamma_5\, \zeta_{\rm l}(\mathbf{z}) \right\rangle$$

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Renormalizion/matching problem at $O(1/m_{\mathrm{b}})$

- $\mathcal{O}_{\mathrm{kin}}(x)$ and $\mathcal{O}_{\mathrm{spin}}(x)$ of $O(5)$
- $W_{\mathrm{NRQCD}} \propto \exp\left(-a^4 \sum_x [\mathcal{L}_{\mathrm{light}}(x) + \mathcal{L}_{\mathrm{h}}^{\mathrm{stat}}(x) + \mathcal{L}_{\mathrm{h}}^{(1)}(x)]\right)$ is not renormalizable. (W_{NRQCD} : path integral weight)
- New divergences at each order in perturbation theory $\Rightarrow \nexists$ continuum limit of lattice theory

Solution

HQET is to reproduce $1/m$–expansion of observables
$\Rightarrow$ expand W in $1/m$ (counting $\omega_{\mathrm{kin}} = O(1/m) = \omega_{\mathrm{spin}}$)

$$W_{\mathrm{NRQCD}} \to W_{\mathrm{HQET}} := \exp\left(-a^4 \sum_x [\mathcal{L}_{\mathrm{light}}(x) + \mathcal{L}_{\mathrm{h}}^{\mathrm{stat}}(x)]\right) \times \left(1 - a^4 \sum_x \mathcal{L}_{\mathrm{h}}^{(1)}(x)\right)$$

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Correlation functions up to $O(1/m_b)$

Definition of expectation values in HQET

$$\begin{aligned}\langle O\rangle = & \langle O\rangle_{\rm stat} + \omega_{\rm kin}\, a^4 \sum_x \langle O O_{\rm kin}(x)\rangle_{\rm stat} \\ & + \omega_{\rm spin}\, a^4 \sum_x \langle O O_{\rm spin}(x)\rangle_{\rm stat} \\ := & \langle O\rangle_{\rm stat} + \omega_{\rm kin}\langle O\rangle_{\rm kin} + \omega_{\rm spin}\langle O\rangle_{\rm spin}\end{aligned}$$

with path integral average to be taken with respect to the lowest order action

$$\langle O\rangle_{\rm stat} = \frac{1}{\mathcal{Z}} \int_{\rm fields} O \exp\left(-a^4 \sum_x [\mathcal{L}_{\rm light}(x) + \mathcal{L}_{\rm h}^{\rm stat}(x)]\right)$$

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$$

Renormalized expansion in $1/m_b$

$$[f_A]_R(x_0,\theta) \;=\; Z_A^{\rm HQET}\, Z_{\zeta_h}\, Z_\zeta\, e^{-m_b x_0} \;\times\; \left[f_A^{\rm stat} + c_A^{\rm HQET} f_{\delta A}^{\rm stat} + \omega_{\rm kin} f_A^{\rm kin} + \omega_{\rm spin} f_A^{\rm spin}\right]$$

$$f_{\delta A}^{\rm stat}(x_0,\theta) \;=\; -\frac{a^6}{2}\sum_{\mathbf{y},\mathbf{z}} \left\langle \delta A_0^{\rm stat}(x)\,\bar\zeta_h(\mathbf{y})\,\gamma_5\,\zeta_l(\mathbf{z})\right\rangle$$

Renormalized expansion in $1/m_{\rm b}$

$$[f_1]_{\rm R}(x_0,\theta) = Z_{\zeta_{\rm h}}^2\, Z_{\zeta}^2\, {\rm e}^{-m_{\rm b} T} \times \left[f_1^{\rm stat} + \omega_{\rm kin} f_1^{\rm kin} + \omega_{\rm spin} f_1^{\rm spin}\right]$$

$$[k_1]_{\rm R}(x_0,\theta) = Z_{\zeta_{\rm h}}^2\, Z_{\zeta}^2\, {\rm e}^{-m_{\rm b} T} \times \left[f_1^{\rm stat} + \omega_{\rm kin} f_1^{\rm kin} - \frac{1}{3}\omega_{\rm spin} f_1^{\rm spin}\right]$$

Non-perturbative matching

Renormalization at leading order in 1/m

- Consider matrix element Φ^{QCD}

  massless limit → chiral symmetry → no $\mu-$ but $m-$dependence
- In HQET

  arbitrary renorm. sceme → no $m-$ but $\mu-$ dependence
- Matching

  $\Phi^{\mathrm{QCD}}(m) = \tilde{C}_{\mathrm{match}}(m,\mu) \times \Phi(\mu) + O(1/m)$
- Perturbative expansion

  $\tilde{C}_{\mathrm{match}}(m,\mu) = 1 + c_1(m/\mu)\bar{g}^2(\mu) + ...$

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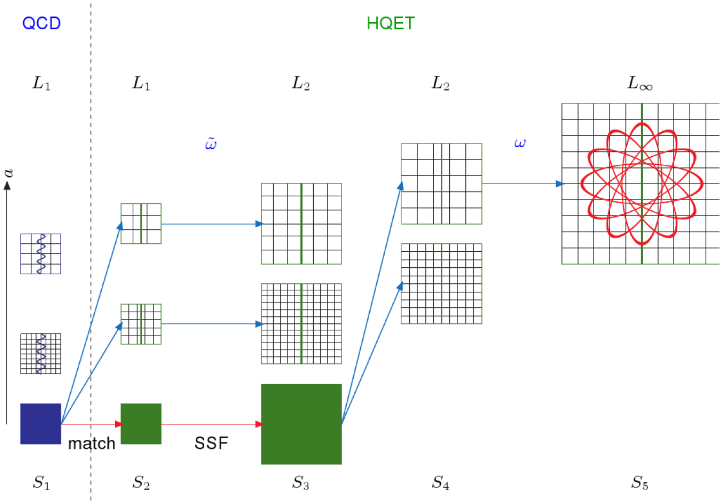
HQET parameters

$N_{\mathrm{HQET}} = 5$

ω_i	definition	classical value	static value
ω_1	m_{bare}	m_{b}	$m_{\mathrm{bare}}^{\mathrm{stat}}$
ω_2	$\ln(Z_A^{\mathrm{stat}})$	0	$\ln(Z_{A,RGI}^{\mathrm{stat}} C_{\mathrm{PS}})$
ω_3	$c_A^{(1)}$	$-1/(2m_{\mathrm{b}})$	ac_A^{stat}
ω_4	ω_{kin}	$1/(2m_{\mathrm{b}})$	0
ω_5	ω_{spin}	$1/(2m_{\mathrm{b}})$	0

Illustration of the strategy



(Heitger and Sommer, 2003)

S_1

- N_{HQET} observables $\Phi_i(L, M, a)$ needed
  (with M RGI mass of the heavy quark)
- Chosen from combinations of SF correlation function that they are renormalizable in a suitable way
- $\exists\, \Phi_i(L, M, 0)$
- Pick observables that have a linear expansion in ω_i for HQET

$$\Phi(L, M, a) = \eta(L, a) + \phi(L, a)\omega(M, a)$$

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S_1

Elements of $\Phi^{\mathrm{QCD}}(L_1, m, a)$

1. $L\Gamma^{\mathrm{P}} = -L\tilde{\partial}_0 \{\ln[-f_A(x_0 = T/2, \theta_0)]\}$
2. $\ln\left[\frac{-f_A(x_0=T/2,\theta_0)}{\sqrt{f_1(\theta_0)}}\right]$
3. $R_A = \ln[f_A(x_0 = T/2, \theta_1)/f_A(x_0 = T/2, \theta_2)]$
4. $R_1 = \frac{1}{4}\left\{\ln\left[f_1(\theta_1)k_1(\theta_1)^3\right] - \ln\left[f_1(\theta_2)k_1(\theta_2)^3\right]\right\}$
5. $\frac{3}{4}\ln\left[\frac{f_1(\theta_0)}{k_1(\theta_0)}\right]$

- θ_i from SF with $\theta_1 < \theta_2$

S_2 and S_3 (SSF)

Impose matching

$$\Phi(L_1 \approx 0.4\,\text{fm},\ M,\ a) = \Phi^{\text{QCD}}(L_1 \approx 0.4\,\text{fm},\ M,\ 0)$$

(Choose L_1 that $1/L_1 \ll m_b$ for precise HQET expansion
and while doing continuum extrapolation keep $am_b < 1/2$)

- Invert expression for $\Phi(L_1, M, a)$ which is linear in ω
  $\tilde{\omega} := \phi^{-1}(L_1, a)[\Phi^{\text{QCD}}(L_1, M, 0) - \eta(L_1, a)]$

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S_4 and S_5

Get ω

$$\omega(M,a) := \phi^{-1}(L_2,a)[\Phi(L_2,M,0) - \eta(L_2,a)]$$

(ϕ^{-1} and η from simulation S_4)

- Interpolate to desired value of β

$$\Phi(L_2,M,0) = \eta^{\mathrm{b}}(L_2,0) + \lim_{a/L_1\to 0}\left(\Sigma(L_1,a)[\Phi(L_1,M,0) - \eta^{\mathrm{b}}(L_1,a)] + \hat{\Sigma}(L_1,a)\right)$$

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Example: Simulations in quenched approx.

Simulation	L	Theory	L/a
S_1	L_1	QCD	$40, 32, 24, 20$
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(Della Morte et al., 2008; Blossier et al., 2009)

- QCD
  f_A, f_1, k_1 and further correlation functions for calculating $f_A^{\text{impr.}}$
  (depend on θ, κ and on x_0 (f_A only))
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  (depend on θ, action (HYP1/HYP2) and on x_0 ($f_A s$ only))
- $\theta = 0, 0.5$ or 1
  Same value for all spatial dimensions!
  (remember $\psi(x + L\hat{k}) = e^{i\theta}\psi(x)$ from the SF)

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- HQET
  $f_A, f_{\delta A}, f_A^{\mathrm{kin}}, f_A^{\mathrm{spin}}, f_1, f_1^{\mathrm{kin}}, f_1^{\mathrm{spin}}$
  (depend on θ, action (HYP1/HYP2) and on x_0 ($f_A s$ only))
- $\theta = 0, 0.5$ or 1
  Same value for all spatial dimensions!
  (remember $\psi(x + L\hat{k}) = e^{i\theta}\psi(x)$ from the SF)

"Software code"

- Functions and define constants (n^{QCD}_{lattice}, used values of θ, ...)
- Read from input files: paths to folders with input data,
  tli coefficients for Σ, η, Φ
- Loop over n^{QCD}_{lattice}
  - Read from input files values for κ, ...
  - Loop over all values stored in input file (lines)
    - "If sequence" to check, if line in input file
      fits to relevant data for computation
      yes $\rightarrow$ store in array
  - Call function to perform binning and then jackknifing
  - Call function to compute $\Phi^{\text{QCD}}_{\text{i}}(L, M, a)$
- Take continuum limit
  (call function that uses Gauss-Jordan algorithm)

"Software code"

- Loop over $n_{\text{lattice}}^{HQET}$
  - Read from input files values for θ, ...
  - Loop over all values stored in input file (lines)
    - "If sequence" to check if line in input file fits to relevant data for computation
      yes $\rightarrow$ store in array
  - Call function to perform binning and then jackknifing
  - Call functions to compute functions build of correlation functions (e.g: combination with different values for θ)
  - Depending on lattice call function to calculate $\Sigma_{\text{ij}}, \eta_{\text{i}}, \ldots$
- Perform necessary continuum limits ($\rightarrow \Phi(L_2, M, 0)$)
- Loop over lattices from S_4
  - Compute ω for each combination of κs and HYP1/HYP2
- Interpolate
- Output of all computed data

Outlook

Up to now only reproducing results but

now use software for...

1. - Some data with $N_f = 2$

   (ALPHA Collaboration, 2010/2011)

2. - Investigation of the quark mass dependence of the mass splitting (between B and B^*) in the B-meson system from the charm region to the static limit
   - $\Delta m = m_{B^*} - m_B = \frac{2\lambda_2^{RGI}}{M_b} C_{spin}\left(M_b/\Lambda_{\overline{MS}}\right) + O(1/M_b^2)$
   - Analysis of new data from (quenched) simulations beyond the matching point at m_b to clarify the issue

3. - ...

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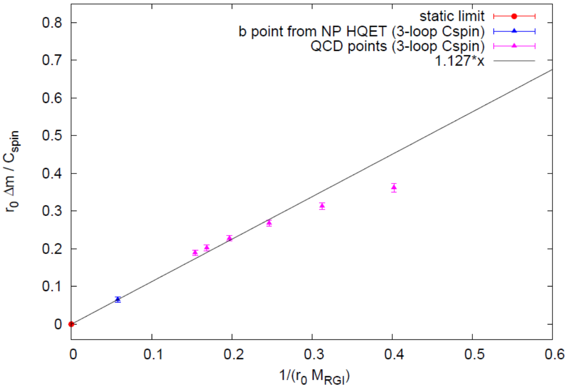
Spin Splitting - Expectation

Compilation of quenched data in the charm region and from HQET

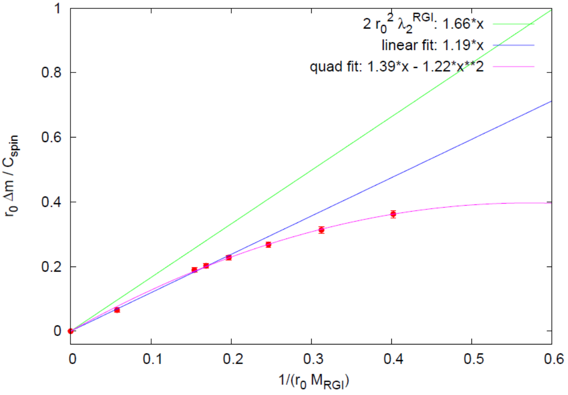
(ALPHA Collaboration, 2008-2010)



- Linear behavior expected when plotting $\Delta m / C_{\text{spin}}\left(M_{\text{b}}/\Lambda_{\overline{\text{MS}}}\right)$ against $1/M_{\text{b}}$

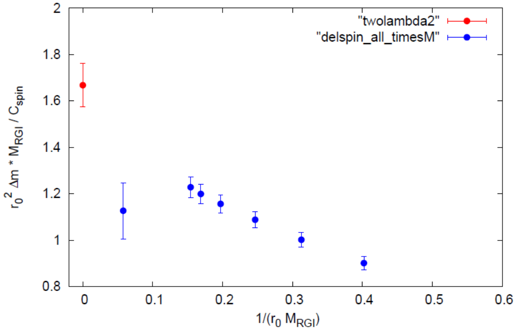
Spin Splitting - Discrepancy in the slope

Slope $\propto \lambda_2^{\mathrm{RGI}}$ also directly (and independently) computable in HQET



- Comparison of fits with calculated value for $r_0^2 \lambda_2^{\mathrm{RGI}}$

Spin Splitting - Investigation



- $\Delta m M_\mathrm{b} = 2\lambda_2^\mathrm{RGI} C_\mathrm{spin}\left(M_\mathrm{b}/\Lambda_{\overline{\mathrm{MS}}}\right) + O(1/M_\mathrm{b})$
- Roughly constant behaviour in b-quark region expected, if $O(1/M_b^2)$ corrections are small
- Large $O(1/M_b^2)$ corrections for the spin-splitting in the b-region?
- Investigate problematic region with data from mentioned further simulations

Summary

1. Why do we need HQET?
2. How can we derive $\mathcal{L}$ for HQET?
3. What does the SF offer us for HQET?
4. What are the parameters of HQET?
5. How can we compute them?
6. What is to do next?

Thank you for your attention!

Literature

- Overview
  - Sommer, R.:
    Introduction to Non-perturbative HQET.
    arXiv: hep-lat/1008.0710
  - Sommer, R.:
    Non-perturbative QCD: renormalization, O(a)-improvement and matching to Heavy Quark Effective Theory.
    arXiv: hep-lat/0611020
- $N_f = 0$
  - Heitger, J.; Sommer, R.:
    Non-perturbative Heavy Quark Effective Theory.
    arXiv: hep-lat/0310035
  - Blossier, B.; Della Morte, M.; Garron, N.; Sommer, R.:
    HQET at order 1/m: I. Non-perturbative parameters in the quenched approximation.
    arXiv: hep.lat/1001.4783.
  - Della Morte, M.; Garron, N.; Papinutto, M.; Sommer, R.:
    Heavy Quark Effective Theory computation of the mass of the bottom quark.
    arXiv: hep-ph/0609294
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  - ALPHA Collaboration:
    B meson spectrum and decay constant from Nf=2 simulations
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