Creutz fermions

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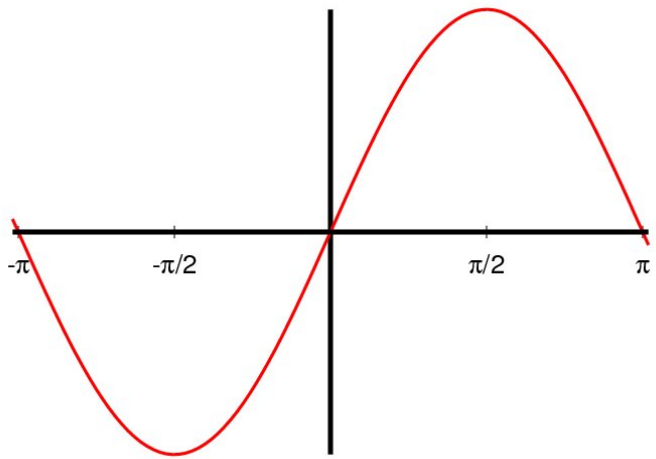
Fermions on the lattice

Naive discretization: replace derivatives by finite differences:

$$\partial_\mu \psi(x) \to \frac{1}{2a}\left(\psi(x + a\hat{\mu}) - \psi(x - a\hat{\mu})\right)$$

But:



- Additional zeros of lattice momenta $\sin(p_\mu)$ lead to 15 doublers.
- Nielsen-Ninomiya: Chirally symmetric doubler-free lattice fermion formulation is not possible. (1981)

Fermion formulations in use

- Wilson fermions:
  - Naive fermions + Wilson term.
  - Chiral symmetry explicitly broken.
  - Additive quark mass renormalization: $m_q = Z(m_0 - m_c(g))$.
- Staggered fermions:
  - Spreads spinor components over a 2^d hypercube.
  - Reduces number of doublers to 4 (“tastes“).
  - $U(1)_R \otimes U(1)_L$ chiral symmetry.
  - Further reduction of doublers by rooting.
- Ginsparg-Wilson fermions:
  - Chiral symmetry in terms of Ginsparg-Wilson relation: $\{D, \gamma_5\} = a\frac{1}{\rho} D\gamma_5 D$
  - Overlap and Domain Wall fermions.

An ancestor of Creutz fermions

Wilczek 1987: `Phys. Rev. Lett. 59 2397-2400`

$$\mathcal{L} = \sum_{\mu} \gamma_\mu f_\mu(p)$$

- Find functions $f_\mu(p)$ to minimize doubling (to two).
- Possible choice with zeros at $(0,0,0,0)$ and $(0,0,0,\pi)$:

$$f_j = \sin(ap_j)/a \qquad f_4 = \left(\sin(ap_4) + \lambda \sum_{j=1}^{3} \sin^2(ap_j/2) \right) /a$$

- Requires fine tuning of additional relevant operator.

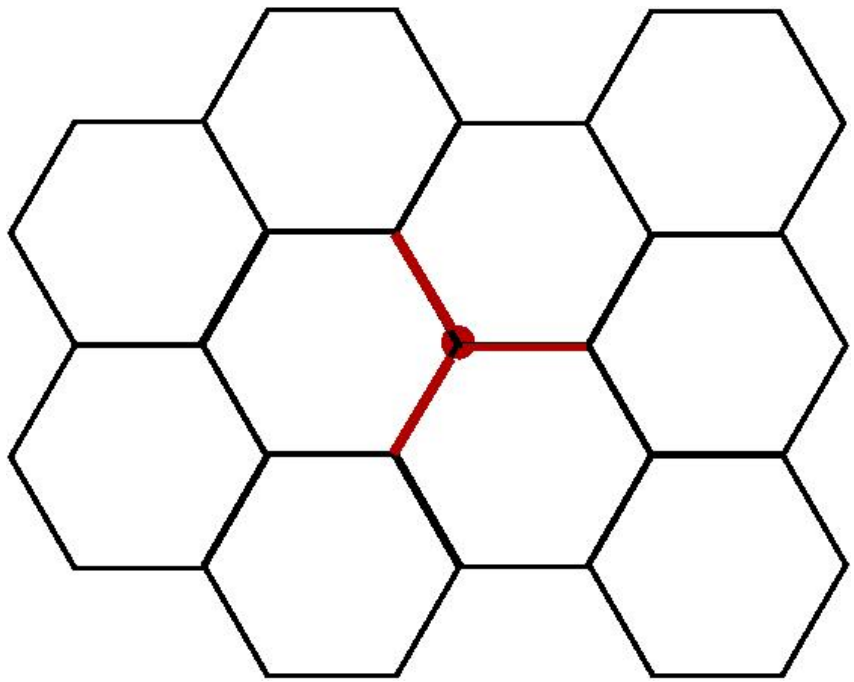
Graphene

Castro-Nieto *et. al.*, `arXiv:0709.1163`

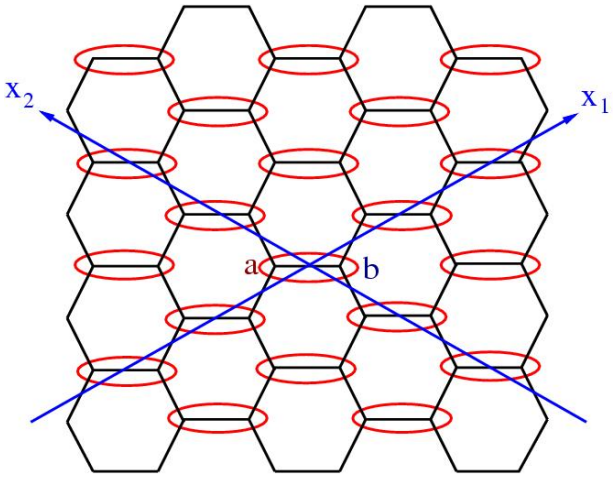
- 2d lattice of Carbon atoms.
- Interesting because graphene is building block of graphite, fullerens, and nanotubes.
- sp^2 hybridization: σ-bonds between lattice neighbours.
- The ”additional” electrons form π orbitals.

Low energy excitations:

- Nearest neighbour hopping of electrons in π orbitals.
- Described by a massless Dirac equation.



2d formulation



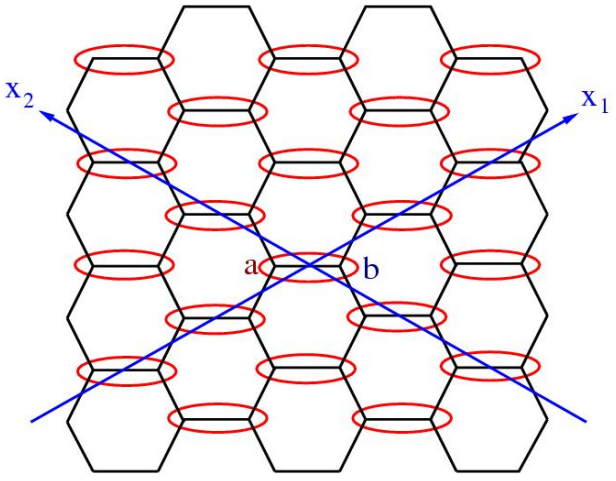
arXiv:0712.1201

- Hopping between a and b sites.
- Choose coordinate system as indicated with sites (x_1, x_2).
- Introduce creation operators $a^\dagger_{x_1,x_2}$, $b^\dagger_{x_1,x_2}$ that obey anticommutation relations.

Hamiltonian in position space given by sum over all possible (nearest neighbour) hoppings:

$$H = K \sum_{(x_1,x_2)} \Big(a^\dagger_{x_1,x_2} b_{x_1,x_2} + b^\dagger_{x_1,x_2} a_{x_1,x_2} + a^\dagger_{x_1+1,x_2} b_{x_1,x_2} + b^\dagger_{x_1-1,x_2} a_{x_1,x_2} + a^\dagger_{x_1,x_2-1} b_{x_1,x_2} + b^\dagger_{x_1,x_2+1} a_{x_1,x_2} \Big)$$

2d formulation



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- Hopping between a and b sites.
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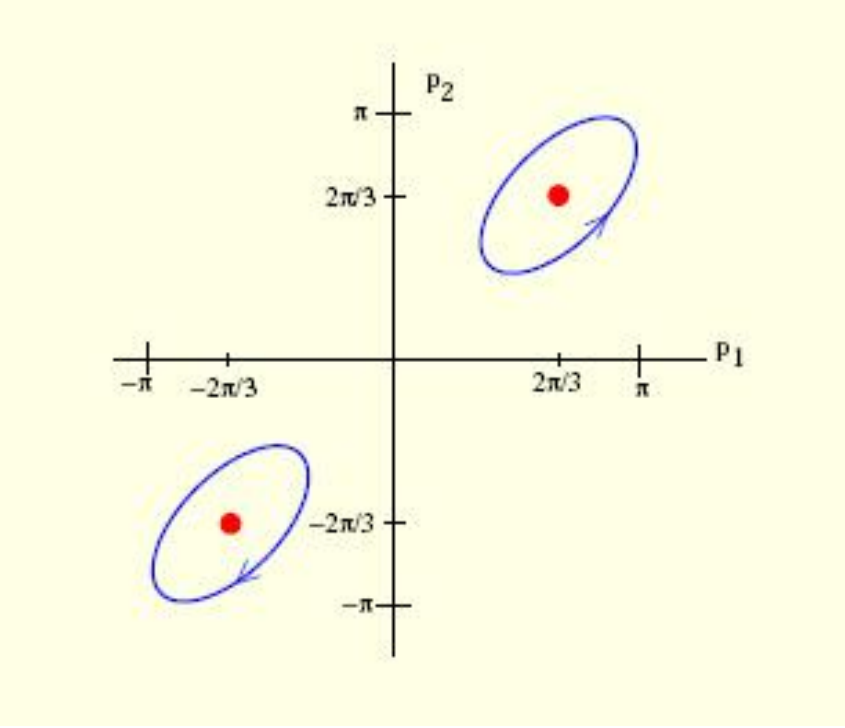
In momentum space: $a_{(x_1,x_2)} = \int_{-\pi}^{\pi} \frac{\mathrm{d}^2 p}{(2\pi)^2}\, e^{i(p_1 x_1 + p_2 x_2)} a_{(p_1,p_2)}$

$$H = K \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \left(a^\dagger_{(p_1,p_2)}, b^\dagger_{(p_1,p_2)} \right) \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \begin{pmatrix} a_{(p_1,p_2)} \\ b_{(p_1,p_2)} \end{pmatrix}$$

$$z = 1 + e^{-ip_1} + e^{+ip_2} \qquad E(p_1, p_2) = \pm K|z|$$

Thus $E = 0 \Leftrightarrow |z| = 0 \Leftrightarrow p_1 = p_2 = \pm 2\pi/3$.

Dirac points



Creutz, `http://latticeguy.net/slides/grapheneslides.pdf` (May 15, 2008)

- $E = 0 \Leftrightarrow |z| = 0 \Leftrightarrow p_1 = p_2 = \pm 2\pi/3$.
- Chiral symmetry: $\{H, \sigma_3\} = 0$
- Two fermion flavours of opposite chirality.

2d $\longrightarrow$ 4d

- Construction of 4d Hamiltonian $H = \gamma_5 D$.
- Starting point mimics 2d case:

$$\begin{aligned} z = & B(4C - \cos(p_1) - \cos(p_2) - \cos(p_3) - \cos(p_4)) \\ & + i\sigma_x(\sin(p_1) + \sin(p_2) - \sin(p_3) - \sin(p_4)) \\ & + i\sigma_y(\sin(p_1) - \sin(p_2) - \sin(p_3) + \sin(p_4)) \\ & + i\sigma_z(\sin(p_1) - \sin(p_2) + \sin(p_3) - \sin(p_4)) \end{aligned}$$

- Local action, not unique.
- $|z| = 0 \quad \Rightarrow \quad C < 1$
- For $C > \frac{1}{2}$: Two solutions $\pm p_j = \tilde{p} = \operatorname{acos}(C)$.

Fermion action by Creutz

$$H = \gamma_5 D = K \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}$$

$$\gamma_k = \sigma_x \otimes \sigma_k \qquad \gamma_4 = -\sigma_y \otimes 1 \qquad \gamma_5 = \sigma_z \otimes 1$$

$$D_C = K \left(-4iBC\gamma_4 + iB\gamma_4 \sum_{\mu=1}^{4} \cos(p_\mu) + i \sum_{j=1}^{3} \gamma_j s_j(p) \right)$$

where

$$s_1(p) = \sin(p_1) + \sin(p_2) - \sin(p_3) - \sin(p_4)$$
$$s_2(p) = \sin(p_1) - \sin(p_2) - \sin(p_3) + \sin(p_4)$$
$$s_3(p) = \sin(p_1) - \sin(p_2) + \sin(p_3) - \sin(p_4)$$

Continuum behaviour

Expand $D_C(p)$ about one of its zeros: $p = \tilde{p} + q$:
$S = \sqrt{1 - C^2}$

$$\begin{aligned} D_C \approx \; & i\gamma_1 KC\,(q_1 + q_2 - q_3 - q_4) \\ & + i\gamma_2 KC\,(q_1 - q_2 - q_3 + q_4) \\ & + i\gamma_3 KC\,(q_1 - q_2 + q_3 - q_4) \\ & + iKBS\gamma_4\,(q_1 + q_2 + q_3 + q_3) \end{aligned}$$

Therefore choose new momenta k_μ to reproduce Dirac matrix:

$$\begin{aligned} k_1 &= KC(q_1 + q_2 - q_3 - q_4) \\ k_2 &= KC(q_1 - q_2 - q_3 + q_4) \\ k_3 &= KC(q_1 - q_2 + q_3 - q_4) \\ k_4 &= KBS(q_1 + q_2 + q_3 + q_4) \end{aligned}$$

Near $p = \tilde{p} - q$ one finds $k_4 \to -k_4$, i. e. opposite chirality.

Geometry

Lattice generated by k_μ should be orthogonal. This fixes the angles between the axes generated by the q_μ's.

$$\cos\theta = \frac{B^2S^2 - C^2}{B^2S^2 + 3C^2}$$

$$\begin{aligned} k_1 &= KC(q_1 + q_2 - q_3 - q_4) \\ k_2 &= KC(q_1 - q_2 - q_3 + q_4) \\ k_3 &= KC(q_1 - q_2 + q_3 - q_4) \\ k_4 &= KBS(q_1 + q_2 + q_3 + q_4) \end{aligned}$$

N.B.: q-lattice is orthogonal $\quad\Leftrightarrow\quad B = C/S$

Boriçi version

Use translated and rescaled Creutz action:

- $C = 1/\sqrt{2}$ and $B = C/S = 1$; $S = \sqrt{1 - C^2} = C$
- $\cos(\tilde{p} + q_\mu) = C\cos(q_\mu) - S\sin(q_\mu)$
- $\sin(\tilde{p} + q_\mu) = S\cos(q_\mu) + C\sin(q_\mu)$
- $s_4(q) = -\sum_{\mu=1}^{4} \sin(q_\mu)$

$$D_B(q) = i\sum_{\mu=1}^{4} \gamma_\mu \left(s_\mu(q) + c_\mu(q)\right) = i\left(\gamma, s(q) + c(q)\right)$$

$$c_1(q) = \cos(q_1) + \cos(q_2) - \cos(q_3) - \cos(q_4)$$

$$c_2(q) = \cos(q_1) - \cos(q_2) - \cos(q_3) + \cos(q_4)$$

$$c_3(q) = \cos(q_1) - \cos(q_2) + \cos(q_3) - \cos(q_4)$$

$$c_4(q) = \sum_{\mu=1}^{4} (\cos(q_\mu) - 1)$$

Boriçi version

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- $\cos(\tilde{p} + q_\mu) = C\cos(q_\mu) - S\sin(q_\mu)$
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$$D_B(q) = i\sum_{\mu=1}^{4} \gamma_\mu \left(s_\mu(q) + c_\mu(q)\right) = i\left(\gamma, s(q) + c(q)\right)$$

Via an orthogonal transformation:

$$D_B(q) = i\sum_{\mu=1}^{4} \gamma_\mu \sin(q_\mu) + i\sum_{\mu=1}^{4} \Gamma_\mu \left(\cos(q_\mu) - 1\right)$$

γ_μ, Γ_μ: two different sets of Dirac matrices.

Position space

$$S_C = \frac{1}{2}\sum_x \sum_{\mu=1}^{4} \Big(\overline{\psi}(x)(\tilde{\Gamma}_\mu + iB\gamma_4)\psi(x+\hat{\mu}) - \overline{\psi}(x)(\tilde{\Gamma}_\mu - iB\gamma_4)\psi(x-\hat{\mu}) \Big) - 4iBC \sum_x \overline{\psi}(x)\gamma_4\psi(x)$$

with

$$\tilde{\Gamma}_\mu = \sum_k \gamma_k \Xi_{k\mu}, \quad s_k(p) = \sum_\mu \Xi_{k\mu} \sin(p_\mu)$$

- $\tilde{\Gamma}_\mu$ is neither a Dirac matrix nor a vector!
- Chiral symmetry: $\{\gamma_5, D_C\} = 0$

CPT

”Ordinary“ C,P,T,CP symmetries are broken. arXiv:0801.3361
For example:

$$P: \quad \overline{\psi}\gamma_j \sin(p_4)\psi \quad \rightarrow \quad \overline{\psi}\gamma_4\,\gamma_j \sin(p_4)\gamma_4\psi = -\overline{\psi}\gamma_j \sin(p_4)\psi$$
$$T: \quad \overline{\psi}\gamma_4\psi \quad \rightarrow \quad \overline{\psi}\gamma_4\gamma_5\,\gamma_4\,\gamma_5\gamma_4\psi = -\overline{\psi}\gamma_4\psi$$

Therefore:

- New relevant operators demand fine-tuning of extra parameters.
- Are there possible new symmetries substituting the standard ones? arXiv:0804.1145

Permutation symmetries for zeros

Creutz' suggestion for a lattice with high symmetry:

$$C = \cos(\pi/5),\ B = \sqrt{5}$$

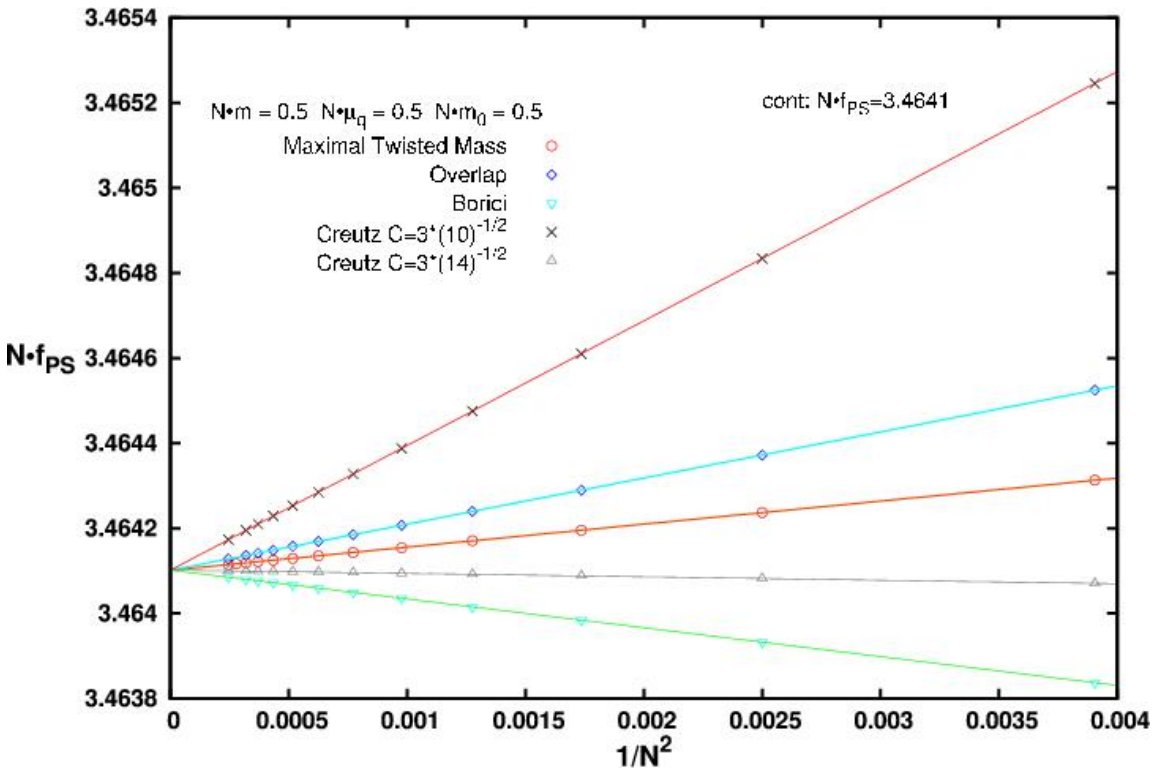
- Zeros in momentum space are distributed equally.
- Above lattice has 120 element symmetry.

in general:

- Action distinguishes $k_4 \sim (q_1 + q_2 + q_3 + q_4)$ (appears in the site diagonal term): There's only time-reversal in this direction.
- In space $4! = 24$ element symmetry group left.
  (3d diamond lattice)
- Thus 48 element symmetry group; possible difficulties because of special treatment of time?

Discretization effects

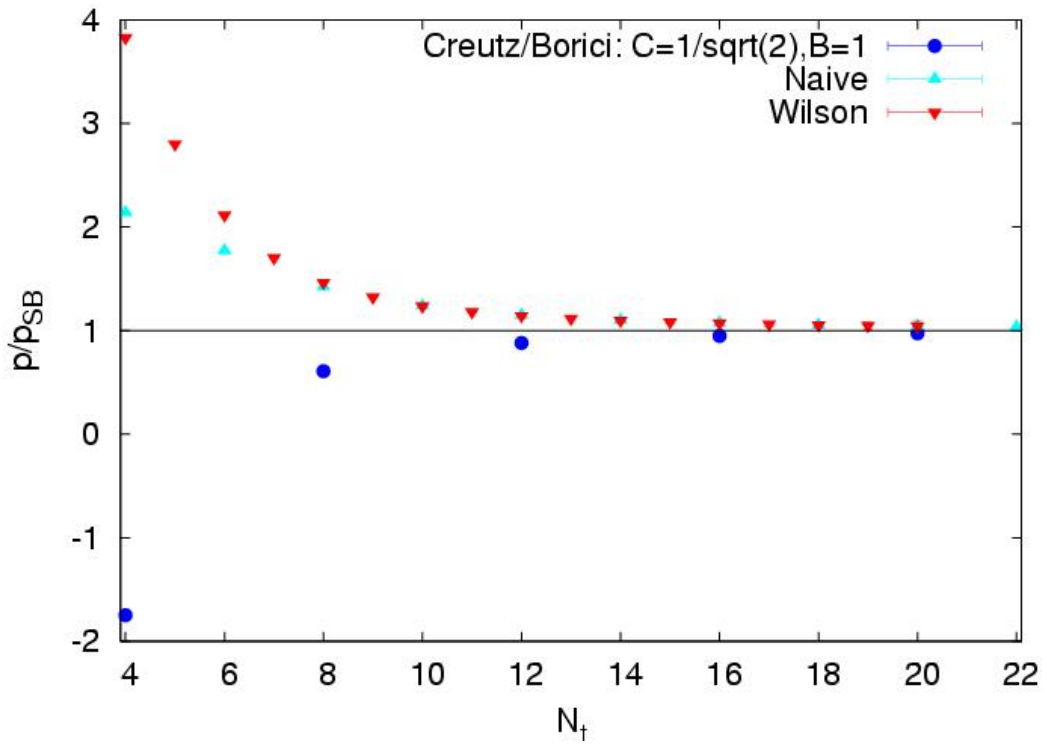
- Perturbation theory: m_p, m_{PS}, f_{PS}.
- $C = 3/\sqrt{10}$ and $B = 3$ for correct continuum limit of propagator. (also $C = 3/\sqrt{14}$)
- $\mathcal{O}(a^2)$ scaling behaviour.



arXiv:0802.3637

Finite T: Stefan-Boltzmann-Limit

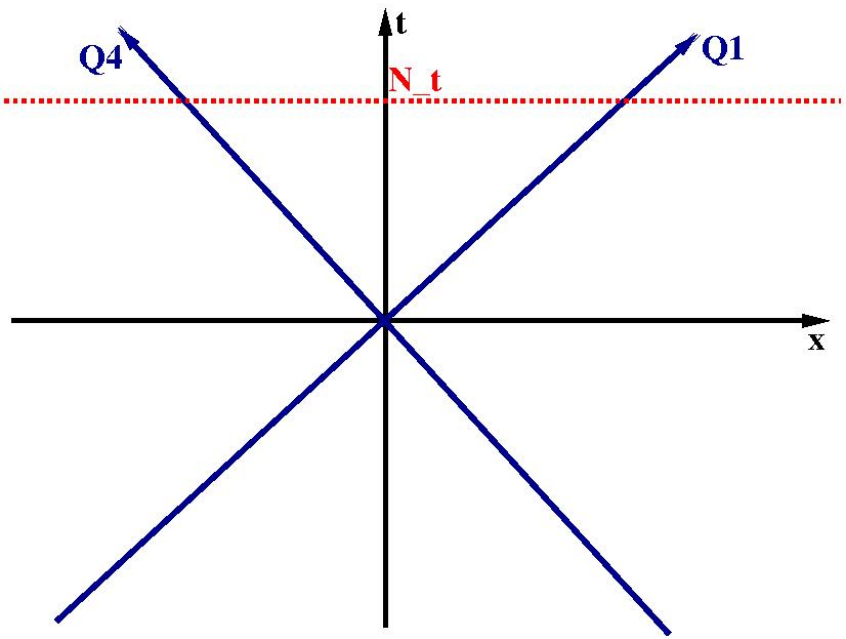
- Pressure for gas of ($n_f = 2$) free fermions: $p/p_{SB} = 90/(21\pi^2)\,\ln \mathrm{Det} M_F$
- $C = 1/\sqrt{2}$, $B = 1$ (only choice that seems to work).
- Use q_4-axis for Matsubara sum.



What is so special about Boriçi fermions?

Explanation attempts...

- Restriction to $N_t = 4k$, $k \in \mathbb{Z}$ in analogy to staggered fermions?!
- $B = C/S$: orthogonal lattice.
- $C = 1/\sqrt{2}$: q_4 axis as time axis (in general $k_4(q)$).



- k_μ generate x_μ axes.
- q_4 generates $Q_4 = \frac{1}{2}(-C, C, -C, C)$.
- Stepwidth in Q_4 direction: aC
- Demand same temperature $T = \frac{1}{aN_t} = \frac{1}{(aC)N_t\sqrt{2}}$ $\Rightarrow C = 1/\sqrt{2}$.

Summary and questions

- Creutz fermions are motivated by electronic structure of graphene.
- Local: only nearest neighbour interactions.
- Chiral symmetry, minimal doubling.
- $N_f = 2$ on a (non-hypercubic) lattice.
- Possible problems: C,P,T,CP are broken; new symmetries?
- How to perform a Matsubara sum for general Creutz lattices?
- Gauging for arbitrary C, B?