Symmetries On The Lattice

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- Background, character theory of finite groups
- The cubic group on the lattice O_h
- Representation of O_h on Wilson loops
- Double group ${}^2O$ and spinor
- Construction of operator on the lattice

MOTIVATION

- Spectrum of non-Abelian lattice gauge theories ?
  - Create gauge invariant spin j states on the lattice
  - Irreducible operators
  - Monte Carlo calculations
  - Extract masses from time slice correlations

Character theory of point groups

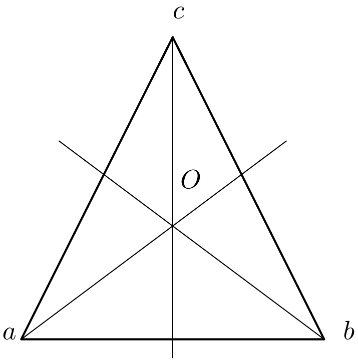
- **Groups, Axioms** A set $G = \{a, b, c, \dots\}$

  A$_1$: Multiplication $\circ : G \times G \to G$.
  A$_2$: Associativity $a, b, c \in G$, $(a \circ b) \circ c = a \circ (b \circ c)$.
  A$_3$: Identity $e \in G$, $a \circ e = e \circ a = a$ for all $a \in G$.
  A$_4$: Inverse, $a \in G$ there exists $a^{-1} \in G$, $a \circ a^{-1} = a^{-1} \circ a = e$.

- Groups with finite number of elements $\to$ the *order* of the group G : n_G.

The point group C_{3v}

- The point group C_{3v}
  (Symmetry group of molecule NH_3)



$G = \{R_a(\pi), R_b(\pi), R_c(\pi), E(2\pi), R_{\vec{n}}(2\pi/3), R_{\vec{n}}(-2\pi/3)\}$
noted
$G = \{A, B, C, E, D, F\}$ respectively.

Structure of Groups

- **Subgroups:**

Definition

A subset H of a group G that is itself a group with the same multiplication operation as G is called a *subgroup* of G.

Example: a subgroup of C_{3v} is the subset E, D, F

- **Classes:**

Definition

An element $g\prime$ of a group G is said to be "**conjugate**" to another element g of G if there exists an element h of G such that

$$g\prime = hgh^{-1}$$

Example: on can check that
$B = DCD^{-1}$

- Conjugacy Class

Definition

A class of a group G is a set of mutually conjugate elements of G.

The group C_{3v} has three conjugate classes:
$\phi_1 = E$
$\phi_2 = A, B, C$
$\phi_3 = D, F$

- Invariant subgroup

Definition

A subgroup H of a group G is said to be "invariant" subgroup if

$$ghg^{-1} \in H$$

for every $h \in H$ and every $g \in G$.

All subgroups are invariant if the multiplication operation is *commutative*.

- **Group Isomorphism:**

Definition

If Φ is *one-to-one* mapping of a group G onto a group $G\prime$ of the same order such that

$$\Phi(g_1)\Phi(g_2) = \Phi(g_1 g_2)$$

for all $g_1, g_2 \in G$, then Φ is said to be an "isomorphic" mapping.

- Direct product groups

Theorem

The set of pairs (g_1, g_2) *(for* $g_1 \in G_1, g_2 \in G_2$*) form a group with the group multiplication*

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$$

for all $g_1, g_1' \in G_1$ *and* $g_2, g_2' \in G_2$*.*
This group is denoted by $G_1 \otimes G_2$*, and is called the "direct product" of* G_1 *with* G_2*.*

Representation theory of finite groups

- Representation

Definition

The Homomorphism $\mathcal{R} : G \rightarrow \mathrm{Aut}(V)\,,\; g \mapsto \mathcal{R}(g)$ of G onto an operator group $\mathcal{R}$ of linear vector space V is called *Representation* of the group G where V as the *Representation space*, if the representation operators fullfill the multiplication operation as for the group G, i.e

$$\mathcal{R}\left(g_1\right)\mathcal{R}\left(g_2\right) = \mathcal{R}\left(g_1 g_2\right).$$

$$d_{\mathcal{R}} = d_V$$

One says the set $\mathcal{R}\left(g\right), g \in G$ forms a linear d_V-dimensional representation of the group G. if the correspondence is one-to-one "isomorphism", then the representation is called *faithful*.

Representation theory of finite groups

- 2d representation of C_{3v}

Example

$$\mathcal{R}(E) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathcal{R}(A) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathcal{R}(B) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \ \mathcal{R}(C) = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

$$\mathcal{R}(D) = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}, \ \mathcal{R}(F) = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}.$$

- equivalence:

Definition

Two d-dimensional representations $\mathcal{R}$ and $\mathcal{R}'$ of G are said to be *equivalent* if a "similarity transformation" exists,i.e

$$\mathcal{R}'(g) = S^{-1}\mathcal{R}(g)S \tag{1}$$

for every $g \in G$ and where S is non-singular $d \times d$ matrix.

Irreducible Representation

Definition

Let V be a representation space associated to the representation $\mathcal{R}$ of the group G, and let V_1 be a subspace of V of less dimension. V_1 is said to be *invariant subspace* of V, if for every $\mathbf{x} \in V_1$ and $g \in G \quad \mathcal{R}(g)(\mathbf{x}) \in V_1$. If such (non-trivial) invariant subspace exists then $\mathcal{R}$ is a "*Reducible*" representation of G.

Definition

A representation of a group G is said to be "irreducible" (*irrep.*) if it is not reducible.

- in several case, the reducible representation is decomposable into direct sum of irreps, where the invariant subspace are orthogonal. In this case,

Theorem

If G is a finite group then every reducible representation is G is completely reducible.

Character of a representation

Definition

The *Character* $\chi^{\mathcal{R}}(g)$ of $g \in G$ in a given $\mathcal{R}$ representation is defined by

$$\chi^{\mathcal{R}}(g) = \mathrm{Tr}\,\mathcal{R}(g)\,.$$

Theorem

In a given representation of a group G all the elements in the same class have the same character.

- due to the cyclic invariance of the trace

$$\mathrm{Tr}\left(D_{\mathcal{R}}(p)\,D_{\mathcal{R}}(g)\,D_{\mathcal{R}}\left(p^{-1}\right)\right) = \mathrm{Tr}D_{\mathcal{R}}(g)$$

  for every $p \in G$.

- **Example:**
  in the 2-dimensional representation of the point group C_{3v} the character system is given by:

$$\chi^{\mathcal{R}}(\phi_1) = 2$$
$$\chi^{\mathcal{R}}(\phi_2) = 0$$
$$\chi^{\mathcal{R}}(\phi_3) = -1$$

Great Orthogonality Theorem

- $$\frac{d_\mu}{n_G}\sum_g D^{\mu\dagger}(g)_{ki} D^\nu(g)_{jl} = \delta_{\mu\nu}\delta_{ij}\delta_{kl}$$

  where D^μ is the matrix representation of dimension d_μ of the group G.

Theorem

The dimension parameters d_μ of non-equivalent irreps satisfy

$$\sum_\mu {d_\mu}^2 = n_G \ ,$$

consequently the irreps characters satisfy

Theorem

$$\begin{aligned}\sum_i \frac{n_i}{n_G}\chi^{\mu\dagger}(C_i)\,\chi^\nu(C_i) &= \delta_{\mu\nu} \\ \sum_\mu \frac{n_i}{n_G}\chi^\mu(C_i)\,\chi^{\mu\dagger}(C_j) &= \delta_{ij}\ .\end{aligned}$$

Great Orthogonality Theorem

- Decomposition of reducible representation onto irreps

Theorem

given one reducible representation $\mathcal{R}$ of a finite group G, it decomposes onto direct sum with multiplicity a_μ of irreps $\mathcal{R}^\mu$ part of the group G,

$$a_\nu = \frac{1}{n_G} \sum_i n_i \, \chi^{\nu\dagger}(C_i) \, \chi^{\mathcal{R}}(C_i) \ .$$

The six important rules

1:

$$\sum_{\mu} {d_{\mu}}^{2} = n_G \ .$$

2:

$$\chi^{\mu}(\mathsf{E}) = d_{\mu}$$

3: The characters of irreps act as a set of orthonormal vectors,

$$\sum_{i} \frac{n_i}{n_G} \chi^{\mu\dagger}(C_i) \, \chi^{\nu}(C_i) = \delta_{\mu\nu} \ .$$

4 :The characters of all elements of the same class are equal.

5 :

$$n_{\text{irreps}} = n_{C_i} \ .$$

6 :The number of elements in a class is a divisor of n_G.

Application to C_{3v} point group

- group : $n_G = 6$
- irreps: $n_{C_i} = 3$ Classes $\rightarrow$ 3 irreps :Γ_1, Γ_2, Γ_3,
- dimensions of irreps: 1, 1, 2
  $1^2 + 1^2 + 2^2 = 6$
- the elements:

C_{3v}	E	A	B	C	D	F
Γ_1	1	1	1	1	1	1
Γ_2	1	-1	-1	-1	1	1
Γ_3	M_1	M_2	M_3	M_4	M_5	M_6

- where

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$M_3 = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \; M_4 = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$
$$M_5 = \begin{pmatrix} \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}, \; M_6 = \begin{pmatrix} \frac{-1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} \end{pmatrix}.$$

- Characters table :

C_{3v}	ϕ_1	$3\phi_2$	$2\phi_3$
Γ_1	1	1	1
Γ_2	1	-1	1
Γ_3	2	0	-1

Hints

- **Q**: how to compute the characters without the knowledge of the representation matrices ?
  A: resolve the equation system given by the orthogonality theorem of the character . . .

  (non linear system !)

  Q: why are the characters are necessary ?
  A: allowing decomposition of given Reducible rep.

Example: 3-dimensional rep. of C_{3v} group

- 3-d rep $\Longrightarrow$ *Reducible..*

in $\mathbb{R}^3$ we choose a basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

- hi
- $F = Rot_{\vec{e}_3}(\frac{2\pi}{3})$

$$F(\vec{e}_1) = \sqrt{\frac{3}{4}}\vec{e}_2 - \frac{1}{2}\vec{e}_1$$
$$F(\vec{e}_2) = -\sqrt{\frac{3}{4}}\vec{e}_1 - \frac{1}{2}\vec{e}_2$$
$$F(\vec{e}_3) = \vec{e}_3$$

then the matrix representation of the element F is

$$\mathcal{D}(F) = \begin{pmatrix} \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- The characters in the reducible representation $\mathcal{D}$ for the three classes are:

$$\chi^{\mathcal{D}}(\phi_1) = 3,$$
$$\chi^{\mathcal{D}}(\phi_2) = -1,$$
$$\chi^{\mathcal{D}}(\phi_3) = 0.$$

the multiplicity factors are then,

$$a_{\Gamma_1} = \frac{1}{6}(1.1.3 + 3.1.(-1) + 2.1.0) = 0,$$
$$a_{\Gamma_2} = \frac{1}{6}(1.1.3 + 3.(-1).(-1) + 2.1.0) = 1,$$
$$a_{\Gamma_3} = \frac{1}{6}(1.2.3 + 3.0.(-1) + 2.(-1).0) = 1.$$

then,

$$\mathcal{D} = \Gamma_2 \oplus \Gamma_3.$$

The lattice cubic group O

- 24 elements
- non abelian
- geometrical symmetric shape: The cube
- one **point of symmetry** (the center of the cube) fixed under action of all elements
- 13 axis of symmetry
- elements $\Longrightarrow$ rotations of: π, $\pm\frac{\pi}{2}$ and $\pm\frac{2\pi}{3}$

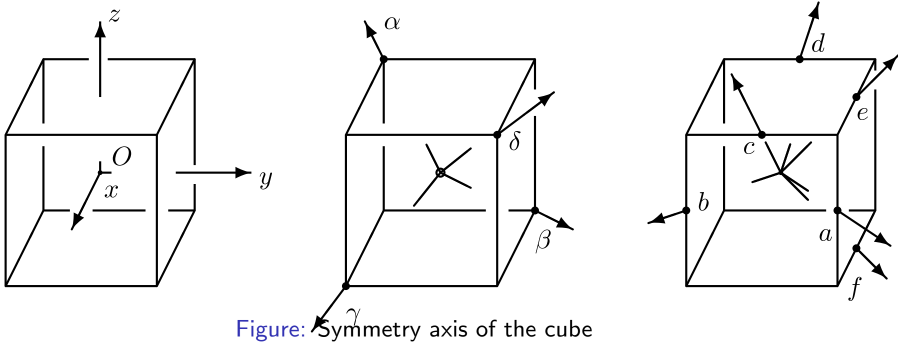


Figure: Symmetry axis of the cube

Order of the axis

- Ox,Oy and Oz : 4
- $O\alpha$, $O\beta$, $O\gamma$ and $O\delta$: 3
- Oa, Ob, Oc, Od, Oe and Of : 2

Counting the number of elements:

- 3 C_4-axis C_{4i} , $i \in \{x, y, z\}$ n-times rotations of $\frac{\pi}{2}$ $(n = 1, \ldots, 4)$
- 4 C_3-axis C_{3i} , $i \in \{\alpha, \ldots, \delta\}$ n-times rotations of $\frac{2\pi}{3}$ $(n = 1, 2, 3)$
- 6 C_2-axis C_{2i} , $i \in \{a, \ldots, f\}$ n-times rotations of π $(n = 1, 2)$

the number of elements $=$ $3.3 + 4.2 + 6.1 + 1 \ = \ 24$

Conjugate classes

- $E = \{id\}$: Identity
- $6C_2 = \{\mathsf{C}_{2i}(\varphi)\}$ with $i \in \{a, \ldots, f\}$ and $\varphi = \pi$
- $8C_3 = \{\mathsf{C}_{3i}(\varphi)\}$ with $i \in \{\alpha, \ldots, \delta\}$ and $\varphi = \pm\frac{2\pi}{3}$
- $6C_4 = \{\mathsf{C}_{4i}(\varphi)\}$ with $i \in \{x, y, z\}$ and $\varphi = \pm\frac{\pi}{2}$
- $3C_4^2 = \{\mathsf{C}_{4i}(\varphi)\}$ with $i \in \{x, y, z\}$ and $\varphi = \pi$

Irreps of O

- 5 conjugate classes $\Longrightarrow$ 5 Irreps $d_\mu = (d_1, d_2, d_3, d_4, d_5)$ where $\sum_{\mu=1}^{5} {d_\mu}^2 = 24$

  solution: dim $= (1, 1, 2, 3, 3) \Longrightarrow A_1, A_2, E, T_1, T_2$.

- **Table of characters**

Conj-Classes		E	C_2	C_3	C_4	C_4^2
		1	6	8	6	3
Representation	A_1	1	1	1	1	1
	A_2	1	-1	1	-1	1
	E	2	0	-1	0	2
	T_1	3	-1	0	1	-1
	T_2	3	1	0	-1	-1

The full cubic group O_h

- $$O_h = O \times \{e, I\}$$

  Inversions through the point symmetry of the cube.
- number of elements $= 48$
- 10 conjugate classes $\Longrightarrow$ 10 irreps

Table of characters

Conj. Cl.		E	C_2	C_3	C_4	C_4^2	I	IC_2	IC_3	IC_4	IC_4^2
		1	6	8	6	3	1	6	8	6	3
Rep.	A_1^+	1	1	1	1	1	1	1	1	1	1
	A_2^+	1	-1	1	-1	1	1	-1	1	-1	1
P $= 1$	E^+	2	0	-1	0	2	2	0	-1	0	2
	T_1^+	3	-1	0	1	-1	3	-1	0	1	-1
	T_2^+	3	1	0	-1	-1	3	1	0	-1	-1
	A_1^-	1	1	1	1	1	-1	-1	-1	-1	-1
	A_2^-	1	-1	1	-1	1	-1	1	-1	1	-1
P $= -1$	E^-	2	0	-1	0	2	-2	0	1	0	-2
	T_1^-	3	-1	0	1	-1	-3	1	0	-1	1
	T_2^-	3	1	0	-1	-1	-3	-1	0	1	1

The spin (bosonic) states

continuum SO(3) $\Longrightarrow$ subgroup O

$\mathcal{R}^j$; $j = 0, 1, 2, \ldots \Longrightarrow \mathcal{R}^j_{\mathrm{O}}$ subduced representation

Spin j	Decomposition into Irreps of O
0	A_1
1	T_1
2	$E \oplus T_2$
3	$A_2 \oplus T_1 \oplus T_2$
4	$A_1 \oplus E \oplus T_1 \oplus T_2$
5	$E \oplus 2T_1 \oplus T_2$
6	$A_1 \oplus A_2 \oplus E \oplus T_1 \oplus 2T_2$
7	$A_2 \oplus E \oplus 2T_1 \oplus 2T_2$
8	$A_1 \oplus 2E \oplus 2T_1 \oplus 2T_2$
9	$A_1 \oplus A_2 \oplus E \oplus 3T_1 \oplus 2T_2$
10	$A_1 \oplus A_2 \oplus 2E \oplus 2T_1 \oplus 3T_2$
11	$A_2 \oplus 2E \oplus 3T_1 \oplus 3T_2$
12	$2A_1 \oplus A_2 \oplus 2E \oplus 3T_1 \oplus 3T_2$

each Irrep. of O can describe spin components of spin-j irreps.

Inverse problem !

Irreps $\mathcal{R}$ of cubic group	contribution to spin j in the continuum
A_1	0, 4, 6, 8, ...
A_2	3, 6, 7, 9, ...
E	2, 4, 5, 6, ...
T_1	1, 3, 4, 5, ...
T_2	2, 3, 4, 5, ...

$$\mathrm{O} \in \mathrm{SO}(3) \longrightarrow \mathcal{R}^{\mathrm{j}}_{\mathrm{O}} \text{ is "subduced representation" of } \mathcal{R}^j$$

- Spin 2

Example

spin 2 $\longrightarrow$ doublet E and triplet T_2 $\longrightarrow$ $m(E)$ and $m(T_2)$ in the continuum $\longrightarrow$ $m(E)/m(T_2) \approx 1$

Representation theory of O_h on the Wilson loops

- Gluball states: A physical state of a gauge theory on the lattice is created by the action of **lattice gauge invariant operators** on the vacuum, for example:

$$\psi(\vec{x},t) = \sum_i c_i \, \mathcal{O}_i(\vec{x},t) \mid \Omega\rangle \qquad \text{with} \qquad c_i \in \mathbb{C}$$

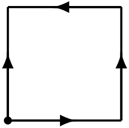
$$\mathcal{O}_i\left(\vec{x},t\right) = \text{Tr}\, U\left(\mathcal{C}_i\right) - \langle \text{Tr}\, U\left(\mathcal{C}_i\right)\rangle$$

  $\mathcal{C}_i$ is a closed links product: "**Wilson loops**".

- The zero-momentum time slice operators are defined as

$$\mathcal{O}_i\left(t\right) = \frac{1}{\sqrt{L^3}} \sum_{\vec{x}} \mathcal{O}_i\left(\vec{x},t\right) \; .$$

  The trivial Wilson loop: the plaquette variable



$U_{x;\mu}$,

- $$U_p = U_{x;\mu\nu} = U^{\dagger}_{x;\nu} U^{\dagger}_{x+a\hat{\nu};\mu} U_{x+a\hat{\mu};\nu} U_{x;\mu}$$
- one constructs six simple space plaquettes

$$U_{(\vec{x};12)}\ , U_{(\vec{x};31)}\ , U_{(\vec{x};23)}\ , U_{(\vec{x};21)}\ , U_{(\vec{x};13)}\ , U_{(\vec{x};32)}$$

- **Charge Conjugation:**
  it changes the orientation of the Wilson loops,

$$\mathrm{Tr}\,\{CU_p\} = (\mathrm{Tr}\,U_p)^* \ .$$

  for a given Wilson loop of length L it is represented by $L-$Tupel

$$(\mu_1, \dots, \mu_L) \quad \text{with} \quad \sum_{i=1}^{L} \hat{\mu}_i = 0$$

  Wilson loop is invariant under cyclic permutation of the tupel

$$[\mu_1, \dots, \mu_L]$$

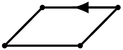
- C-**transformation**

$$C\left[\mu_1, \mu_2, \ldots, \mu_L\right] \equiv \left[-\mu_L, \ldots, -\mu_2, -\mu_1\right] \ .$$

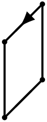
- P-**transformation**

$$P\left[\mu_1, \mu_2, \ldots, \mu_L\right] \equiv \left[-\mu_1, -\mu_2, \ldots, -\mu_L\right] \ .$$

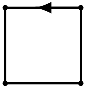
Example: Wilson loop of length 4



$= O_1$

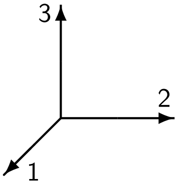


$= O_2$



$= O_3$

Adopting the convention

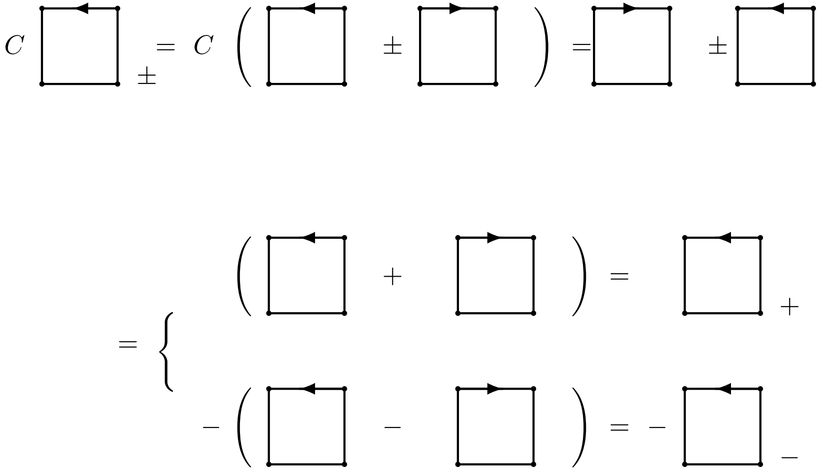


- the three plaquettes are labeled

$$O_1 = [1, 2, -1, -2]\,, \quad O_2 = [3, 1, -3, -1] \quad \text{and} \quad O_3 = [2, 3, -2, -3]\ .$$

- plaquettes are P-invariants, look for linear combinations to obtain C-eigenstates,these are defined by

$$[\mu_1, \ldots, \mu_L]_\pm \equiv [\mu_1, \ldots, \mu_L] \pm [-\mu_L, \ldots, -\mu_1]\,,$$

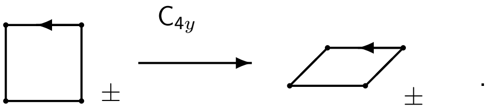


this translates

$$CO_{3\pm} = \pm O_{3\pm}.$$

Transformation under the cubic group

- behaviour O_3 under C_{4y}



- Representation $\tilde{\mathcal{R}}(g)$

$$\tilde{\mathcal{R}}(g)\left([\mu_1, \ldots, \mu_L]_\pm\right) \equiv [T(g)\,\hat{\mu}_1, \ldots, T(g)\,\hat{\mu}_L]_\pm$$

$T(g)$ are the canonical 3d vector representation matrices.
continue to the previous example: $O_{3\,\pm} = [2, 3, -2, -3]_\pm$ and the element $\mathsf{C}_{4y} \in \mathsf{O}$ with the representation matrix of T_1

$$T\left(\mathsf{C}_{4y}\right) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- $$\begin{aligned} \tilde{\mathcal{R}}(\mathsf{C}_{4y})\left([2,3,-2,-3]_{\pm}\right) &= \left[T(\mathsf{C}_{4y})\,\hat{2},\ldots,T(\mathsf{C}_{4y})\,\left(-\hat{3}\right)\right]_{\pm} \\ &= [2,-1,-2,1]_{\pm} \\ &= [1,2,-1,-2]_{\pm} \\ &= O_{1\,\pm}\,. \end{aligned}$$

- continuing to O_1 and O_2,$\mathsf{C}=+1$

$$D_{\tilde{\mathcal{R}}^{++}}(\mathsf{C}_{4y}) = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

Characters

- class E

  Id:

  $$\begin{aligned} O_{1\,\pm} &\longmapsto O_{1\,\pm} \\ O_{2\,\pm} &\longmapsto O_{2\,\pm} \\ O_{3\,\pm} &\longmapsto O_{3\,\pm} \end{aligned}$$

- class C_2

  C_{2a}:

  $$\begin{aligned} O_{1\,\pm} &\longmapsto CO_{1\,\pm} \\ O_{2\,\pm} &\longmapsto O_{3\,\pm} \\ O_{3\,\pm} &\longmapsto O_{2\,\pm} \end{aligned}$$

- class C_3

  $\mathrm{C}_{3\alpha}$:

  $$\begin{aligned} O_{1\,\pm} &\longmapsto CO_{2\,\pm} \\ O_{2\,\pm} &\longmapsto O_{3\,\pm} \\ O_{3\,\pm} &\longmapsto CO_{1\,\pm} \end{aligned}$$

- class C_4

  C_{4x}:

  $$O_{1\,\pm} \longmapsto O_{2\,\pm}$$
  $$O_{2\,\pm} \longmapsto CO_{1\,\pm}$$
  $$O_{3\,\pm} \longmapsto O_{3\,\pm}$$

- class C_4^2

  C_{2x}:

  $$O_{1\,\pm} \longmapsto CO_{1\,\pm}$$
  $$O_{2\,\pm} \longmapsto CO_{2\,\pm}$$
  $$O_{3\,\pm} \longmapsto O_{3\,\pm}$$

Decomposition

- $$a_\mu = \frac{1}{n_G} \sum_i n_i \, \chi^\mu (C_i) \, \chi^{\tilde{\mathcal{R}}} (C_i) \ ,$$

The case: $C = +1$

- $$\begin{aligned}
a_{A_1} &= \frac{1}{24}(1\cdot 1\cdot 3 + 6\cdot 1\cdot 1 + 8\cdot 1\cdot 0 + 6\cdot 1\cdot 1 + 3\cdot 1\cdot 3) = 1\\
a_{A_2} &= \frac{1}{24}(1\cdot 1\cdot 3 + 6\cdot (-1)\cdot 1 + 8\cdot 1\cdot 0 + 6\cdot (-1)\cdot 1 + 3\cdot 1\cdot 3) = 0\\
a_{E} &= \frac{1}{24}(1\cdot 2\cdot 3 + 6\cdot 0\cdot 1 + 8\cdot (-1)\cdot 0 + 6\cdot 0\cdot 1 + 3\cdot 2\cdot 3) = 1\\
a_{T_1} &= \frac{1}{24}(1\cdot 3\cdot 3 + 6\cdot (-1)\cdot 1 + 8\cdot 0\cdot 0 + 6\cdot 1\cdot 1 + 3\cdot (-1)\cdot 3) = 0\\
a_{T_2} &= \frac{1}{24}(1\cdot 3\cdot 3 + 6\cdot 1\cdot 1 + 8\cdot 0\cdot 0 + 6\cdot (-1)\cdot 1 + 3\cdot (-1)\cdot 3) = 0\,.
\end{aligned}$$

  The representation $\tilde{\mathcal{R}}^{++}$ decomposes into

  $$\tilde{\mathcal{R}}^{++} = A_1^{++} \oplus E^{++}$$

the case: C = −1

- $$\begin{aligned} a_{A_1} &= 0 \\ a_{A_2} &= 0 \\ a_E &= 0 \\ a_{T_1} &= 1 \\ a_{T_2} &= 0\ , \end{aligned}$$

  one has

  $$\mathcal{R}^{\tilde{+}-} = T_1^{+-}\ ,$$

Summarising the results

irreps of O	A_1	A_2	E	T_1	T_2	A_1	A_2	E	T_1	T_2
dim of irreps	1	1	2	3	3	1	1	2	3	3
dim of $\tilde{\mathcal{R}}$: 3	$C = +1$					$C = -1$				
	1	0	1	0	0	0	0	0	1	0

Wave functions, The orthonormal basis

- One has at the end to determine the orthonormal basis corresponding to each invariant subspace of the 3-d representation space, these basis are formed by linear combinations of the Wilson loop operators treated above.
  for this purpose we look for a basis where each $D_{\tilde{\mathcal{R}}}(g)$ for all g have a block diagonal form, this happens when one finds a diagonalized matrix C which commutes with all $D_{\tilde{\mathcal{R}}}(g)$ and when A is the matrix which diagonalizes C, one can read off directly the orthonormal basis of the invariant subspaces from the columns of the matrix A^{-1}.
  one can sum all the matrices of each class to obtain such a C matrix.

the case: C = +1

- $$C(E) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad C(C_2) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$C(C_3) = \begin{pmatrix} 0 & 4 & 4 \\ 4 & 0 & 4 \\ 4 & 4 & 0 \end{pmatrix} \qquad C(C_4) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$C(C_4^2) = \begin{pmatrix} 3 & & \\ & 3 & \\ & & 3 \end{pmatrix} .$$

ONB

- eigenvectors from A^{-1}

$$A^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix} ,$$

- Finally the orthonormal basis are summarized in the following table:

Representation $\tilde{\mathcal{R}}$ of O	Plaquettes linear combinations
A_1^{++}	$O_{1\,+} + O_{2\,+} + O_{3\,+}$
E^{++}	$-2O_{1\,+} + O_{2\,+} + O_{3\,+}$,
	$O_{2\,+} - O_{3\,+}$

the case: $C = -1$

In this case the 3-d representation $\tilde{\mathcal{R}}$ is irrep of O and it is exactly the vector representation T_1 of O, and it has an eigenspace spanned by three Wilson loop operators, consequently the C matrices are diagonals

$$C(E) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad C(C_2) = \begin{pmatrix} -2 & & \\ & -2 & \\ & & -2 \end{pmatrix}$$

$$C(C_3) = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix} \qquad C(C_4) = \begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$$

$$C\left(C_4^2\right) = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} .$$

ONB

- Table

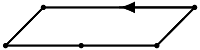
Rep. $\tilde{\mathcal{R}}$ of O	Plaquettes linear combinations
T_1^{+-}	$O_{1\,-}$,
	$O_{2\,-}$,
	$O_{3\,-}$

- Plauettes In $SU(2)$ gauge theory the matrices (links) have real trace, the trace of the plaquette is also real, a representation T_1^{+-} does not exist, in the case of $SU(3)$ the ONB are given by the imaginary part of the Plaquette trace.

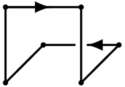
Wilson loop of length greater than 4

The same strategy is to be followed, for example: loops of length 6, there are three different prototypes which can be distinguished:

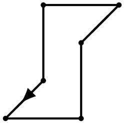
- double plaquettes (6):



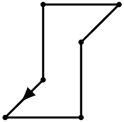
- bent plaquettes (12):



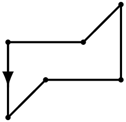
- twisted plaquettes (4):



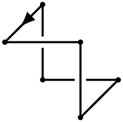
Twisted plaquettes



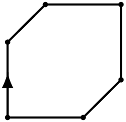
$= O_1$



$= O_2$



$= O_3$



$= \mathcal{O}_4$

Summary: The Irreps content of the cubic group representation on the Wilson loop up to length 6

- simple plaquette
- double plaquette
- twisted plaquette

A_1^{++}	A_2^{++}	E^{++}	T_1^{++}	T_2^{++}	A_1^{+-}	A_2^{+-}	E^{+-}	T_1^{+-}	T_2^{+-}
1	1	2	3	3	1	1	2	3	3
$PC = +1, +1$					$PC = +1, -1$				
1	0	1	0	0	0	0	0	1	0
1	1	2	0	0	0	0	0	1	1
1	0	0	0	1	0	1	0	1	0

The orthonormal basis of twisted loops

- $\tilde{\mathcal{R}}^{PC}$

	Loop-Op.	$O^{\pm}_{1\,\pm}$	$O^{\pm}_{2\,\pm}$	$O^{\pm}_{3\,\pm}$	$O^{\pm}_{4\,\pm}$
$\tilde{\mathcal{R}}^{PC}$	$A_1^{++},\ A_2^{+-}$	1	1	1	1
	$T_2^{++},\ T_1^{+-}$	1	-1	1	-1
		-1	1	1	-1
		1	1	-1	-1

The double group ^{2}O of the cubic group O

- **Fermionic states**
  Double space groups come into play when fermion spin functions are introduced. Consider the $l-$th irrep of $SO(3)$, the character satisfies

$$\chi^l(\alpha + 2\pi) = (-)^{2l}\chi^l(\alpha) \ ,$$

- for integer l one finds the expected results. For half-integer (such as occurs for $SU(2)$ the *universal covering group* of $SO(3)$) this gives $\chi^l(\alpha + 2\pi) = -\chi^l(\alpha)$. However,

$$\chi^l(\alpha \pm 4\pi) = \chi^l(\alpha) \ ,$$

  then the rotation in this case through 4π is the Identity E, and through 2π is the symmetry operation J such that $J^2 = E$.
  - O is finite subgoup of $SO(3) \Longrightarrow$ ^{2}O is finite subgoup of $SU(2)$. bosonic (tensor) irreps $\Longrightarrow$ bosonic and fermionic (spinor) irreps.

double group elements

- The rotation of the cube through 2π produces a negative identity.

$$\Downarrow$$

- the doubling of the order of the symmetry axis.

$$\Downarrow$$

- The number of the double group elements is twice the number elements of the cubic group 2×24.

$$\Downarrow$$

- but the number of the **irreps** is not necessary twice **!**

The elements

They are summarized in the following:

Conjugate classes of O	φ	Conjugate classes of ^{2}O	φ
$E = \{id\}$	2π	$E = \{id\}$ $J = \{-id\}$	4π 2π
$6C_2 = \{\mathsf{C}_{2i}\,(\varphi)\}$	π	$12C_4$	$\pm\pi$
$8C_3 = \{\mathsf{C}_{3i}\,(\varphi)\}$	$\pm\frac{2\pi}{3}$	$8C_6$ $8C_6^2$	$\pm\frac{2\pi}{3}$ $\pm\frac{4\pi}{3}$
$6C_4 = \{\mathsf{C}_{4i}\,(\varphi)\}$	$\pm\frac{\pi}{2}$	$6C_8$ $6C_8'$	$\pm\frac{\pi}{2}$ $\pm\frac{3\pi}{2}$
$3C_4^2 = \{\mathsf{C}_{4i}\,(\varphi)\}$	π	$6C_8^2$	$\pm\pi$

Irreps

- ^2O has 8 classes $\Longrightarrow$ 8 irreps. $(d_{G_1}, d_{G_2}, d_H) = (2, 2, 4)$

- The irreps $\mathcal{R}^j$ of $SU(2)$ are subduced representations of ^2O and they decompose into irreps of ^2O to describe spin j particles on the lattice. The spin content are:

$SU(2)$ Spin j	Its decomposition in irreps of ^{2}O
1/2	G_1
3/2	H
5/2	$G_2 \oplus H$
7/2	$G_1 \oplus G_2 \oplus H$
9/2	$G_1 \oplus 2H$
11/2	$G_1 \oplus G_2 \oplus 2H$

The characters of G_1, G_2 and H

Using the formula of charcters of $SU(2)$ irreps,

Conj. Class. E	J	C_4	C_6^2	C_6	C_8	C_8'	C_8^2	
num. elem. 1	1	12	8	8	6	6	6	
G_1	2	-2	0	-1	1	-$\sqrt{2}$	$\sqrt{2}$	0
G_2	2	-2	0	-1	1	$\sqrt{2}$	-$\sqrt{2}$	0
H	4	-4	0	1	-1	0	0	0
θ	4π	2π	$\pm\pi$	$\pm 4\pi/3$	$\pm 2\pi/3$	$\pm 3\pi/2$	$\pm\pi/2$	$\pm\pi$

Spinors

Q: In which Representation $\tilde{\mathcal{R}}$ of ${}^2O$ transforms a spinor ?

A: First, find the characters of its rotation matrices, and then apply the formula to find the mutiplicity of irreps within $\tilde{\mathcal{R}}$.

- Rotations

$$\psi_\alpha \longmapsto \psi'_\alpha = S_{\text{Rot.}(\alpha\beta)}(n,\theta)\psi_\beta \ ,$$

where

$$S_{\text{Rot.}}(n,\theta) = \exp\left(\tfrac{i}{2}\theta\Sigma\cdot n\right) \ .$$

- Characters

Conj. Class.	E	J	C_4	C_6^2	C_6	C_8	C_8'	C_8^2
$\chi^{\tilde{\mathcal{R}}}(S_{\text{Rot.}}(\theta))$	4	-4	0	-2	2	$-2\sqrt{2}$	$2\sqrt{2}$	0

- Using the fomula for the multiplicity of irreps occuring in $\tilde{\mathcal{R}}$,

$$a_{G_1} = 2 \ ,$$
$$a_{G_2} = 0 \ ,$$
$$a_H = 0 \ .$$

- $$\tilde{\mathcal{R}} = 2\,G_1 = G_1 \oplus G_1 \ .$$

Spinors

- To include the parity transformation the double group is extended by constructing the product ${}^2O \otimes \{e, I\}$ to get the *the full double group* ${}^2O_h$, the spinor now is transforming in the *irreducible* representation $\tilde{\mathcal{R}}^{\pm} = G_1^{\pm} \oplus G_1^{\pm}$ of the ${}^2O_h$ group.

Reprentation of ${}^2O_h$ on some Mixed Majorana fermion and Links operators

- **Example of N=1 SU(2) SYM operators**
  - Goal: Spectrum of gauge theory on the lattice, Gluballs, Hadrons,..
  - Generate field configuations of the theory.
  - $\Longrightarrow$ Construction of **gauge invariant lattice operators (Observables)** with given spin j and PC content, ONB.
  - Monte Carlo simulation of lattice operators $\rightarrow$ Time slice correlations.
  - Mass estimates in lattice unit $\rightarrow$ Fit, estimating methods,...

Majorana-Majorana operators

- Majorana spinor has 2 degrees of freedom occuring positive charge conjugation C.

$$\lambda = \begin{pmatrix} \lambda^R \\ \lambda^L \end{pmatrix},$$
$$\bar{\lambda} = \lambda^t C$$
$$\lambda^C = \lambda$$

- From two Majorana fermion ϕ and χ one can form bilinear covariants

$$\bar{\phi} M \chi = \begin{cases} +(\bar{\chi} M \phi) & M = 1,\ \gamma_5\gamma_\mu,\ \gamma_5 \\ -(\bar{\chi} M \phi) & M = \gamma_\mu,\ [\gamma_\mu, \gamma_\mu] \end{cases}$$

in particular, if from single Majorana, one can form only

$$\bar{\lambda}\lambda, \quad \bar{\lambda}\gamma_5\gamma_\mu\lambda, \quad \bar{\lambda}\gamma_5\lambda.$$

1- Adjoint scalar-like mesons

- First, the Majorana-Majorana operator is created on a lattice site by

$$\bar{\lambda}\lambda = \bar{\lambda}(x)\lambda(x) \; ,$$

  and it transforms in the continum as a Lorentz scalar ! (true scalar)

$$\bar{\lambda}'\lambda' = \bar{\lambda}S^{-1}S\lambda = \bar{\lambda}\lambda \; .$$

- Under the action of elements of ${}^2O$ which are also elements of the Lorentz group, this remains also a trivial scalar occuring $P = +1$ in the same it transforms in the irrep A_1^{++} of ${}^2O_h$. In the continum it transforms in $(0,0)$ irrep of the Lorentz group with parity, it has the quantum number

$$J^{PC} = 0^{++} \Longrightarrow \qquad \text{adjoint meson a-}f_0.$$

- In the same way one can create a non trivial scalar,

$$\bar{\lambda}\gamma_5\lambda$$

  This is a Lorentz pseudoscalar with $J^{PC} = 0^{-+}$ transforming in the A_1^{-+} irrep of ${}^2O_h$.

2- Adjoint vector-like mesons

- The vector supermultiplet of N=1 SYM contains also a vector particle (Boson / triplet), one candidate can be the **vector** $A_\mu = \bar{\lambda}\gamma_5\gamma_\mu\lambda$, the Lorentz transformation of Dirac matrices are

$$\begin{aligned} S^{-1}(\Lambda)\gamma_5(\Lambda)S &= \gamma_5 \,, \\ S^{-1}(\Lambda)\gamma^\mu(\Lambda)S &= \Lambda^\mu_\nu\gamma^\nu \,. \end{aligned}$$

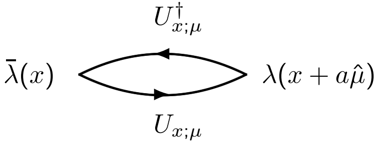
If we restrict the rotations Λ to the elements of the cubic group, this operator transforms in the T_1^{++} of ${}^2O_h$, and will dscribe a spin $J^{PC} = 1^{++}$ particle.

The Majorana-link-majorana

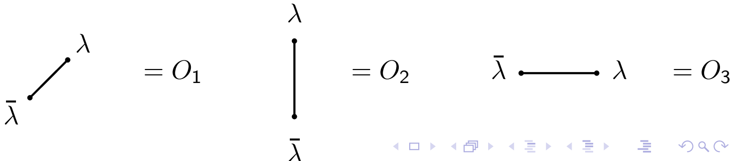
- This split-point operator gauge invariant and is formed of two links and two Majoran on two distant lattice sites,

$$O_\mu = \mathrm{Tr}\{\bar{\lambda}(x) U^\dagger_\mu(x) \Gamma \lambda(x + a\hat{\mu}) U_\mu(x)\} \; ; \quad \mu = 1, 2, 3$$
$$\Gamma = 1, \gamma_5 \; .$$

- diagramatically represented by



- We distinguish three independent basic operators,



PC

For $\Gamma = 1$ this operator is $PC = +1$ invariant. The content of this operator has two sectors:

- Majorana sector:
  The bilinear covariant $\bar{\lambda}(x)\lambda(x + a\hat{\mu})$ is Lorentz trivial

$$\Downarrow$$
$$\mathcal{R}_{MS} = A_1^{++}$$

- Links sector:
  The transformation under the elements of O_h of the three basis O_1, O_2 and O_3 (three dimensions) shows that the representation matrices of $\mathcal{R}_{LS}$ are the same one as the proper rotations in three dimensions.

$$\Downarrow$$
$$\mathcal{R}_{LS} = T_1^{++}$$

- $\mathcal{R}$?
  We need now to combine the O_h representation of the Majorana bilinear with that of spatial links-loop. The combined operator O_μ will lie in a representation given by the Glebsch-Gordon decomposition of representations.

Representations Product

Theorem

For two given unitary irreps $\mathcal{R}^\mu$ and $\mathcal{R}^\nu$ of a group G with dimensions d_μ and d_ν respectively, represented by their representation matrices $D^\mu(g)$ and $D^\nu(g)$, then the matrices

$$D^{\mu\otimes\nu}(g) := D^\mu(g) \otimes D^\nu(g)$$

for all $g \in G$ are fixed in a unitary representation $\mathcal{R}^\mu \otimes \mathcal{R}^\nu$ of dimension $d_\mu \cdot d_\nu$.

For the character of the representations tensor product, one has

$$\chi^{\mathcal{R}^\mu \otimes \mathcal{R}^\nu}(g) = \chi^{\mathcal{R}^\mu}(g)\, \chi^{\mathcal{R}^\nu}(g) \, .$$

The Majorana-link-Majorana operator transforms in

$$\mathcal{R}^{PC} = \mathcal{R}_{MS} \otimes \mathcal{R}_{LS} = A_1^{\pm +} \otimes T_1^{++} = T_1^{\pm +}.$$

irrep of O_h.

The wave function of irreducible M-L-M operator:

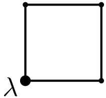
Loop-op	O_μ^P	$O_1^\pm$	$O_2^\pm$	$O_3^\pm$
$\tilde{\mathcal{R}}^{PC}$	$T_1^{\pm +}$	1		
			1	
				1

- spin content:
  The lowest spin content of Majorana-link-Majorana is $J = 1$ or also $J = 3, 4, 5, \ldots$.

Majorana-plaquette operator

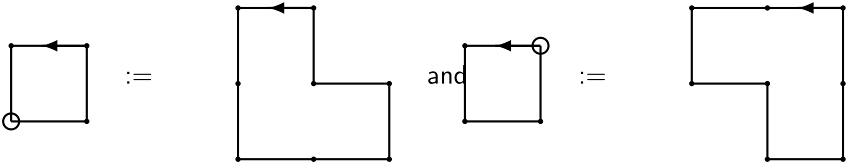
- It is given by

$$\mathrm{Tr}\left\{U_{\mu\nu}(x)\lambda_\alpha\right\} = U^{rs}_{\mu\nu}(x)\lambda^a_\alpha(T^a)^{sr}$$

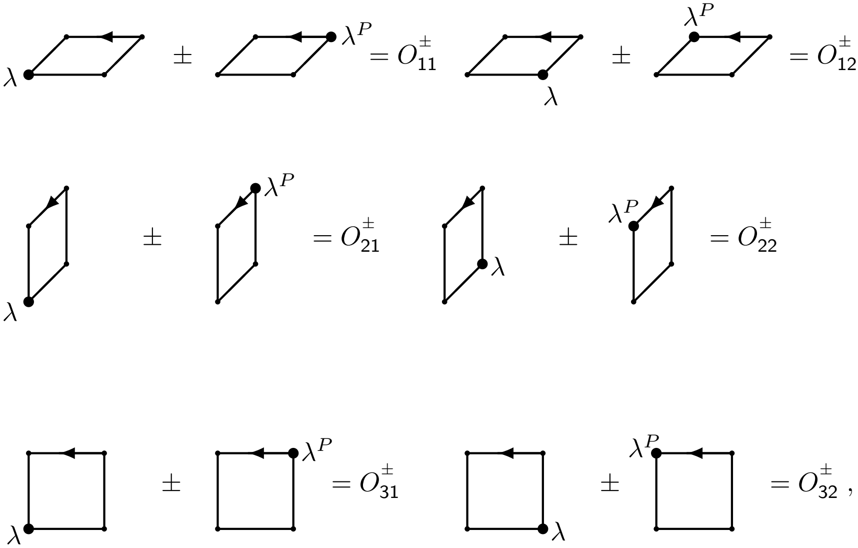


- There are $12 = 3.4$ independent operators of this form which can be built. The treatment of such operator is closely analog to the Wilson loop of length 8 **L-plaquette**

L-plaquette



Consrtuctions: First P-invariant



Constructions: Second C-invariant

$$O^P_{ij\,\pm} \equiv O^P_{ij} \pm C O^P_{ij}$$

Representations

- In Plaquette sector:
  This is encode in the following

$$
\begin{aligned}
\tilde{\mathcal{R}}_{\mathsf{PS}}^{++} &= A_1^{++} \oplus E^{++} \oplus T_2^{++} \ , \\
\tilde{\mathcal{R}}_{\mathsf{PS}}^{+-} &= A_2^{+-} \oplus E^{+-} \oplus T_1^{+-} \ , \\
\tilde{\mathcal{R}}_{\mathsf{PS}}^{-+} &= T_1^{-+} \oplus T_2^{-+} \ , \\
\tilde{\mathcal{R}}_{\mathsf{PS}}^{--} &= T_1^{--} \oplus T_2^{--} \ .
\end{aligned}
$$

- In Majorana sector:
  Majorana fermion are self antiparticle. The parity commutes with Lorentz tranformations then the combination

$$
\lambda \pm \lambda^P
$$

is Parity invariant under Lorentz transformations. The representation of Maj-Plaq-op in Majorana sector is a 4-dimensional irrep of ${}^2O_h$

$$
\tilde{\mathcal{R}}_{\mathsf{MS}}^{\pm} = G_1^{\pm} \oplus G_1^{\pm} \ .
$$

$$
\begin{aligned}
\tilde{\mathcal{R}}_1^{++} &= A_1^{++} \otimes \{G_1^{++} \oplus G_1^{++}\}\,, \\
\tilde{\mathcal{R}}_2^{++} &= E^{++} \otimes \{G_1^{++} \oplus G_1^{++}\}\,, \\
\tilde{\mathcal{R}}_3^{++} &= T_2^{++} \otimes \{G_1^{++} \oplus G_1^{++}\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{R}}_1^{+-} &= A_2^{+-} \otimes \{G_1^{+-} \oplus G_1^{+-}\}\,, \\
\tilde{\mathcal{R}}_2^{+-} &= E^{+-} \otimes \{G_1^{+-} \oplus G_1^{+-}\}\,, \\
\tilde{\mathcal{R}}_3^{+-} &= T_1^{+-} \otimes \{G_1^{+-} \oplus G_1^{+-}\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{R}}_1^{-+} &= T_1^{-+} \otimes \{G_1^{-+} \oplus G_1^{-+}\}\,, \\
\tilde{\mathcal{R}}_2^{-+} &= T_2^{-+} \otimes \{G_1^{-+} \oplus G_1^{-+}\}\,,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{R}}_1^{--} &= T_1^{--} \otimes \{G_1^{--} \oplus G_1^{--}\}\,, \\
\tilde{\mathcal{R}}_2^{--} &= T_2^{--} \otimes \{G_1^{--} \oplus G_1^{--}\}\,.
\end{aligned}
$$

Decomposition

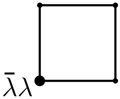
- $$a_\nu = \frac{1}{48}\sum_i n_i\, \chi^\nu(C_i)\left[\chi^{\tilde{\mathcal{R}}^\mu_{\mathsf{PS}}}(C_i)\cdot\left(\chi^{G_1}(C_i)+\chi^{G_1}(C_i)\right)\right] .$$
- The irreducible characters of ${}^2O$ are encoded in the characters of O, and after explicit calculation for different PC one obtains
  - $P = +1$, $C = +1$
  $$\begin{aligned} \tilde{\mathcal{R}}_1^{++} = A_1^{++}\otimes\{G_1^{++}\oplus G_1^{++}\} &= G_1^{++}\oplus G_1^{++} = 2G_1^{++} , \\ \tilde{\mathcal{R}}_2^{++} = E^{++}\otimes\{G_1^{++}\oplus G_1^{++}\} &= 2H^{++} \\ \tilde{\mathcal{R}}_3^{++} = T_2^{++}\otimes\{G_1^{++}\oplus G_1^{++}\} &= 2G_2^{++}\oplus 2H^{++} . \end{aligned}$$
  - $P = +1$, $C = -1$
  $$\begin{aligned} \tilde{\mathcal{R}}_1^{+-} = A_2^{+-}\otimes\{G_1^{+-}\oplus G_1^{+-}\} &= 2G_2^{+-} \\ \tilde{\mathcal{R}}_2^{+-} = E^{+-}\otimes\{G_1^{+-}\oplus G_1^{+-}\} &= 2H^{+-} \\ \tilde{\mathcal{R}}_3^{+-} = T_1^{+-}\otimes\{G_1^{+-}\oplus G_1^{+-}\} &= 2G_1^{+-}\oplus 2H^{+-} , \end{aligned}$$
  - $P = -1$, $C = \pm 1$
  $$\begin{aligned} \tilde{\mathcal{R}}_1^{-\pm} = T_1^{-\pm}\otimes\{G_1^{-\pm}\oplus G_1^{-\pm}\} &= 2G_1^{-\pm}\oplus 2H^{-\pm} \\ \tilde{\mathcal{R}}_2^{-\pm} = T_2^{-\pm}\otimes\{G_1^{-\pm}\oplus G_1^{-\pm}\} &= 2G_2^{-\pm}\oplus 2H^{-\pm} . \end{aligned}$$

The irreducible wave function (ONB): $\mathcal{R}_1^{++}$

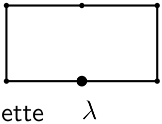
Representation $\tilde{\mathcal{R}}$ of ^{2}O$_h$	irr. operator of Majorana-Plaquette
$2G_1^{++}$ $dim = 4$	$\overset{+}{O}_{11+} + \overset{+}{O}_{12+} + \overset{+}{O}_{21+} + \overset{+}{O}_{22+} + \overset{+}{O}_{31+} + \overset{+}{O}_{32+}$

And many other examples

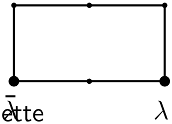
- Maj-Maj-plaquette



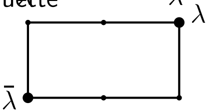
- Maj-double plaquette



- split-Maj-Maj-double plaquette



- Dig-Maj-Maj-double plaquette



Summary

How to Construct irreducibe lattice operators and determine their spin j^{PC} content ?

- Create gauge invariant product of links and fermions (Tupel)
- Determine the basis of $\mathcal{R}$: the set in different directions
- Combine to construct PC invariant basis
- Perform all possible rotations separately for each sector (link, fermion)
- Compute the characters of $\mathcal{R}$
- Decomposition of $\mathcal{R}$ into irreps
- Diagonalize and find the wave functions (ONB)

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Best wishes