
Exercise 1: The Korteweg-de-Vries equation

Solve the initial value problem corresponding to the 1D KdV equation:

$$\frac{\partial u}{\partial t} + 6\,u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \qquad (1)$$

using a pseudospectral method for $x \in [-\pi,\, \pi]$ with 256 spatial discretisation points. Use the "integrating factor" method and a Runge Kutta 4 algorithm for the time-stepping. The initial condition is

$$u(x,0) = \frac{c_1^2}{2}\mathrm{sech}^2\left(\frac{c_1\,(x+2)}{2}\right) + \frac{c_2^2}{2}\mathrm{sech}^2\left(\frac{c_2\,(x+1)}{2}\right)$$

with $c_1 = 5$ und $c_2 = 4$.

Exercise 2: Derivaties using the 2D FFT

In order to calculate the derivative of a field in one dimension, it is sufficient to multiply the Fourier coefficients by the imaginary unit and the corresponding wave number. For the case of a two dimensional field $f(\boldsymbol{x})$, a two dimensional field of wavevectors $\boldsymbol{k} = (k_x, k_y)$ has to be considered. The Fourier coefficients of a field $\tilde{f}$ are ordered in such a way, that the corresponding wavevectors have to be initialised according to

$$k_x(i,j) = \begin{cases} \frac{2\pi}{L} i & \text{falls} \quad i = 0, \ldots, \frac{N}{2} \\ \frac{2\pi}{L}(-N+i) & \text{falls} \quad i = \frac{N}{2}+1, \ldots, N-1 \end{cases}$$

$$k_y(i,j) = \begin{cases} \frac{2\pi}{L} j & \text{falls} \quad j = 0, \ldots, \frac{N}{2} \\ \frac{2\pi}{L}(-N+j) & \text{falls} \quad j = \frac{N}{2}+1, \ldots, N-1. \end{cases}$$

Here, L is the physical length of the system. If f is a real field, analogously to the 1D case, the array for the wavevectors only has to be half as large and the index i can take the values $i = 0, \ldots, \frac{N}{2}$. In order to calculate the derivative in x direction, the Fourier coefficients are multiplied by the imaginary unit and the corresponding k_x.

a) Initilialize a two dimensional field and make sure by applying the FFT as well as the inverse FFT that the algorithm works correctly.

b) Now take the derivatives of the fields with respect to x and y. Apply also the Laplace operator on the field. Compare your results with analytical predictions (e.g. using gnuplot).