Stochastische Punktwirbeldynamik

-

Stochastic Point Vortex Dynamics

als Diplomarbeit vorgelegt von
Michael Rath

Januar 2010

Westfälische-Wilhelms-Universität Münster

institut für
theoretische physik

Contents

Introduction

> *"Observe the motion of the surface of the water, which resembles that of hair, which has two motions, of which one is caused by the weight of the hair, the other by the direction of the curls; thus the water has eddying motions, one part of which is due to the principal current, the other to random and reverse motion."*
> (Leonardo da Vinci)

The subject of fluid turbulence has attracted the attention of physicists over centuries. Leonardo da Vinci was probably the first one to use the term "la turbolenza" to describe the corresponding flow phenomenon. Being a derivative of 'turbare' (Latin, in English: to disorder), turbulence refers to the irregular motion of liquids and gases. Turbulence arises in any liquid or gas provided that the flow velocity is sufficiently high. In nature as well as in engineering, it is the dominating flow regime; on the one hand, it accelerates mixing processes, on the other hand it leads to considerable flow resistances of vehicles. Concerning the structure of a turbulent flow one observes that it is extremely complex: it is unsteady, random and chaotic. This complexity arises from the inherent non-linearities of the fluid motion, which are essential for the creation and maintenance of turbulence. Turbulence itself is still the most important unsolved problem of classical mechanics (cf. [28]).

Fluid turbulence as a continuum phenomenon can be described by the classical laws of fluid motion; the flow of gases and liquids is governed by a partial differential equation discovered independently by Stokes and Navier. It is the backbone of modern fluid dynamics and has to be understood as Newton's law of motion formulated for the flow of a fluid. The Navier-Stokes equation may describe both laminar and turbulent flows. Many complexities of fluid turbulence may be found in the equation in form of non-locality, viscous diffusion and dissipation. An important scientific study of turbulence dates back to 1883, when Osborne Reynolds established pipe flow experiments (cf. [25]) where he showed that the appearance of eddies depends on the value of a particular combination of parameters - the so-called Reynolds number. If it exceeds a certain limit, the fluid viscosity is no longer able to stabilize the motion.

The vital constituents of all turbulent flows are whirling eddies and vortices of a wide range of the size. In 1922, Richardson delivered a qualitative description of the so-called turbulent energy cascade:

> *"Big whorls have little whorls that feed on their velocity, and little whorls*

have lesser whorls and so on to viscosity."
(Lewis F. Richardson)

As this quotation shows he painted a picture in which the energy is passed down from the large scales of the motion to smaller scales until reaching a sufficiently small length scale such that the tiniest eddies - being still significantly larger than the intermolecular distances - eventually are viscously dissipated into heat. Adding in this context constantly energy on a large scale comparable to the size of the system, the so-called integral scale, we arrive in a flux balance or dynamic equilibrium, respectively. In his theory of 1941, Kolmogorov used these qualitative ideas to establish the first quantitative theory in turbulence research (cf. [18]), within founding the field of "Statistical Hydrodynamics". Postulating that for a very high Reynolds number the small scale turbulent motions are statistically isotropic, he made predictions claiming universal statistics on medium and small scales, in the so-called inertial range. These predictions hold for low order statistics.[1].

The Navier-Stokes equation itself was of limited use for about more than a century until the computer processing rates became sufficiently fast such that numerical solutions could be obtained for some simple two-dimensional flows in the 1960s. The special role of two-dimensional turbulence is that it is nowhere realized in nature or the laboratory in a strict sense, but only in computer simulations. Nevertheless, one can justify investigating it - it idealizes geophysical phenomena for example in the atmosphere or oceans and provides a way to model them (cf. [20]). Additionally, results may be compared to three-dimensional investigations. There were also developed various experiments in the recent time to model two-dimensional turbulence in the labaratory (cf. [6]). The probably most striking difference between two- and three-dimensional turbulence is that in two-dimensional systems one gets an inverse energy cascade going to larger scales, detaining the system to dissipate energy at small scales. (cf. [28]). Nonetheless there can be found a different direct cascade, the so-called enstrophy cascade. The quantity "enstrophy" is directly related to the vorticity of the flow.

Experiments as well as numerical calculations give the indication that so-called coherent structures characterize turbulent flow fields, whereby the concept "coherent structure" denotes a spatiotemporal collection of vorticity, which emerges for longer times. A still open question is why this local structures develop. Hence it is not astonishing that vortices as the basic elements of all turbulent flows are still in the focus of investigation and interest. Unfortunately, the Navier-equation has only few exact vortex solutions, the simplest case is a point vortex with localized vorticity in one point of the plane as a solution of the two-dimensional Navier-Stokes equation with vanishing viscosity, the so-called Euler equation. An approach to model flow

[1] Compare also [10] for an extensive overview. - We may additionally note here that after being confronted with systematic deviations of his theory both experimental as well as numerical, embodied in the so-called phenomenon intermittency, Kolmogorov refined it in 1962.

fields is to take a system consisting of a finite set of point vortices. It is possible to investigate its properties on the basis of a finite-dimensional Hamiltonian system. And since it is Hamiltonian, it is possible to perform a statistical treatment along the lines of equilibrium statistical mechanics (cf. [23]). Using this ansatz, Onsager found a possibility to explain theoretically the phenomenon of the tendency of vortices to form large structures (cf. [7]).

So coherent structures as well as fluctuating velocity fields are characteristics of turbulent flows and should be comprised in statistical considerations of hydrodynamics. In this sense, this thesis can be seen as an attempt to get familiar with the concept of taking in account both deterministic dynamics and methods of statistical physics. Therefore we investigate a simple setup containing ingredients from both: We take point vortices and investigate their stochastic dynamics under the influence of an incompressible, stochastic Kraichnan velocity field (e. g. introduced in [8]). Kraichnan velocity fields are a useful tool to model short-time correlated velocity fields and are δ-correlated in time, guaranteeing essential simplifications in the analytical calculations. It is rather a purely academic motivation, leaving the opportunity to perhaps expand and match the problem to more applied questions. So simply speaking, this thesis is about first deriving a Fokker-Planck equation for a set of point vortices in a Kraichnan velocity field, and, subsequently, solving it. For a constant diffusion coefficient and two identical vortices with the "standard vortex profile"- which qualitatively is proportional to the reciprocal distance to the vortex core, this problem has already been solved by Agullo and Verga (compare [2]). With respect to generalization, there are some open issues to the problem of two vortices:

- One question is whether and under which circumstances it is also solvable for vortices with different circulation, also with regard to limiting cases.
- A second question would be if there exist solutions for the problem in the case of a non constant diffusion matrix.
- Finally there is the question if it is solvable for different "non-standard" vortex profiles, which do not necessarily have to be solutions of the basic hydrodynamical equations. This has to be seen as an attempt of generalization which may possibly have application in modelling vortices or eddy structures differently. There might also be application in the field of screened vortices (cf. [9]).

The thesis is partitioned into two chapters.

The first chapter introduces the methods and the theoretical foundations we will work with. After a short treatment of hydrodynamical and statistical basics, we will derive a general formalism to carry over from the both deterministically and stochastically affected trajectories of particles or vortices in a fluctuating field to a stochastic partial differential equation describing the time evolution of the system in a probabilistic manner. In mathematical nomenclature this is: We have a finite set of n Langevin equations, each containing both deterministic interaction between the i^{th} constituent and the $n-1$ other ones and a fluctuating part; these equations are then conveyed by means of ensemble averaging to one Fokker-Planck equation governing the probability density function of the whole system.

The second chapter is the main part of the thesis. We first briefly consider one vortex described by one Langevin equation, lacking the deterministic interaction with other vortices. Thus we will obtain a simple Brownian motion, and after having applied the formalism derived in the previous chapter, we will see that the Fokker-Planck reduces to a simple heat equation. Then we come to the treatment of two votices. After a change to central and relative coordinates, we will apply the formalism derived before on the two Langevin equations of the vortices in order to obtain a Fokker-Planck equation describing the system. We investigate on vortices with arbitrary, different circulation, and decouple the relative and central motion of the vortex pair. It will come clear that the problem of solving the central motion is in most cases approximately and sometimes even exactly reduced to solving the heat equation. Afterwards we focus on solving the relative part for different vortex profiles and diffusion matrices. Eventually we derive solutions in the case of scaled diffusions coefficients and a particularly matching generalized vortex profile. These solutions pass over into already known solutions in limiting cases. The direct derivation of the known case can be found in the appendix to compare the results. After having solved the equation, the solutions are graphically presented and commented. In the case of a scaled diffusion matrix and a standard un-scaled vortex profile we give an outline of an asymptotic solution.
A last section of the second chapter describes an analogy of the two point vortex system to the system of two so-called anyons. The subject of anyons is a concept out of quantum field theory. Anyons are quasi particles known for example for explaining the Fractional Quantum Hall Effect (FQHE) (cf. [24]). After a short introduction, we point out the analogy; two free anyons behave statically in an analogous way to the one of two vortices in a fluctuating flow field.

1 Theoretical foundations

In this chapter the theoretical foundations of physical as well as the ones of mathematical nature concerning the subject will be presented. We will first bring in the basic hydrodynamic equations; especially we will derive the equations describing point vortex dynamics in two dimensions. Subsequently, we introduce the Kraichnan model for velocity fields and a formalism to derive a Fokker-Planck equation for a set of point vortices in a Kraichnan velocity field.

1.1 Hydrodynamics and vortices

This section deals with the introduction of the basic hydrodynamical equations and the derivation of equations describing vortex dynamics in two dimensions. Basic hydrodynamical equations are known since the 1755, when Euler established the equations for fluids without friction. About 70 years later they were generalized by Navier and Stokes by taking into account molecular friction forces, leading to the well-known Navier-Stokes equation

$$\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) = -\nabla p(\mathbf{x},t) + \nu\Delta\mathbf{u}(\mathbf{x},t) + \mathbf{f}(\mathbf{x},t). \tag{1.1}$$

Thereby we have

- the velocity field $\mathbf{u}(\mathbf{x},t)$,
- $p(\mathbf{x},t) = \frac{P(\mathbf{x},t)}{\rho}$ with $P(\mathbf{x},t)$: pressure and ρ : constant mass density
- ν denotes the kinematic viscosity and
- $\mathbf{f}(\mathbf{x},t)$ as the force per volume accumulates all external forces

In the following, we will only consider incompressible fluids with constant mass density, the incompressibility condition reads

$$\nabla\cdot\mathbf{u}(\mathbf{x},t) = 0 \tag{1.2}$$

Taking the curl of the velocity, we obtain a quantity denoted by “vorticity”

$$\boldsymbol{\omega}(\mathbf{x},t) = \nabla\times\mathbf{u}(\mathbf{x},t) \tag{1.3}$$

In two dimensions, this quantity tends to build oval structures, like for example the so-called Red Spot on Jupiter (cf. [21]). In three dimensions, thin filaments emerge.

Considering equation (1.2) and (1.3) as an analogon to magnetostatics, we may use the Biot-Savart law (cf. [5]) and derive a representation for $\mathbf{u}(\mathbf{x},t)$:

$$\mathbf{u}(\mathbf{x},t) = \int \boldsymbol{\omega}(\mathbf{x}',t) \times \mathbf{K}(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}' + \nabla\Phi$$

whereby $\mathbf{K}(\mathbf{x}-\mathbf{x}') = -\nabla G(\mathbf{x}-\mathbf{x}')$. $G(\mathbf{x}-\mathbf{x}')$ denotes the Green's function of the Laplacian, by definition $\Delta G(\mathbf{x}-\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}')$. Φ has to fulfill $\Delta\Phi = 0$ (cf. [23]). So if we knew the vorticity and corresponding boundary conditions, we could calculate the velocity field. Taking the curl of the Navier-Stokes equation, we arrive at the vorticity equation (cf. [15])

$$\left(\frac{\partial}{\partial t} + \mathbf{u}(\mathbf{x},t)\cdot\nabla\right)\boldsymbol{\omega}(\mathbf{x},t) = \boldsymbol{\omega}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) + \nu\Delta\boldsymbol{\omega}(\mathbf{x},t) + \nabla\times\mathbf{f}(\mathbf{x},t) \tag{1.4}$$

describing the time evolution of the vorticity.

Vortex dynamics in two dimensions

In two dimensions now, the vorticity has only one vorticity component perpendicular to the plane. If it is the z-component, we get

$$\boldsymbol{\omega}(\mathbf{x},t)\cdot\nabla\mathbf{u}(\mathbf{x},t) = \omega_z(x,y,t)\frac{\partial}{\partial z}\mathbf{u}(x,y,t) = 0$$

for the so-called vortex stretching term in (1.4). If we additionally neglect the external forces, our vorticity equation reduces to

$$\frac{\partial}{\partial t}\omega + \mathbf{u}\cdot\nabla\omega = \nu\Delta\omega$$

and for the nonviscous case, $\nu = 0$ this entails

$$\frac{\partial}{\partial t}\omega + \mathbf{u}\cdot\nabla\omega = 0$$

If we now consider a vorticity field generated by n point vortices,

$$\omega(\mathbf{x},t) = \sum_{i=1}^{n}\Gamma_i\delta(\mathbf{x}-\mathbf{x}_i(t)) \tag{1.5}$$

we get

$$0 = \sum_{i=1}^{n}\Gamma_i\left[-\frac{d}{dt}\mathbf{x}_i(t) + \mathbf{u}(\mathbf{x}_i,t)\right]\nabla_{\mathbf{x}}\delta(\mathbf{x}_i(t)-\mathbf{x})$$

Assuming that the n vortices are never in contact in the time of investigation, we obtain for every time in this interval

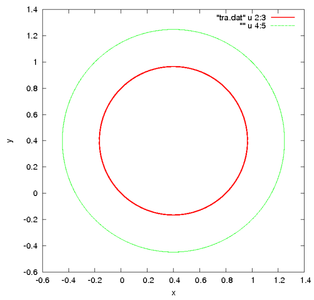
$$\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{u}(\mathbf{x}_i, t)$$

Under special conditions vortices may collide, the reason why we want to avoid this scenario is, that in that particular moment the last two equations would lose their validity.

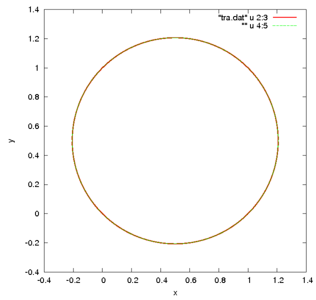
If we want to obtain the velocity field from the vorticity field now, we have to apply the Biot-Savart law, yielding in our case of two dimensions (cf. [3])

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{2\pi} \int \frac{(\mathbf{x} - \mathbf{x}')^{\perp}}{|\mathbf{x} - \mathbf{x}'|^2} \omega(\mathbf{x}', t) d\mathbf{x}' \tag{1.6}$$

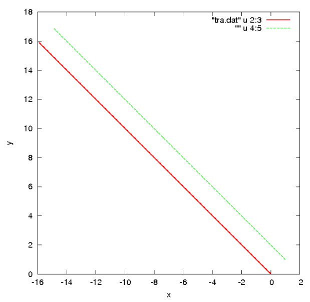
whereby $(x, y)^{\perp} = (-y, x)$ by definition is describing an azimuthal trajectory. Inserting (1.5) into (1.6) entails



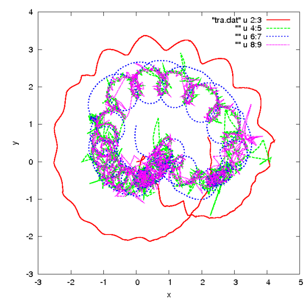
(a) Two vortices with different circulation, rotating around the center of vorticity



(b) Two vortices with the same circulation, rotating around the center of vorticity on the same trajectory



(c) Two vortices with counter-rotating circulation, their center of vorticity follows a straight line



(d) Four vortices showing chaotic behavior

Figure 1.1: Trajectories of point vortices presented qualitatively, showing purely deterministic as well as chaotic behavior. Derived using the Runge-Kutta forth order method.

$$\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{u}(\mathbf{x}_i, t) = \sum_{i=1}^{n} \Gamma_i \mathbf{K}(\mathbf{x} - \mathbf{x}_i(t))$$

whereby

$$\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi}\frac{\mathbf{x}^{\perp}}{|\mathbf{x}|^2}, \qquad |\mathbf{x}| \neq 0$$

is the derivative of the Laplacian Green function and describes the “standard” point vortex profile[1]. Combining our results, we finally arrive at a set of differential equations describing the evolution of the point vortex positions:

$$\boxed{\frac{d}{dt}\mathbf{x}_i(t) = \mathbf{u}(\mathbf{x}_i, t) = \sum_{j=1, j\neq i}^{n} \frac{\Gamma_j}{2\pi}\frac{(\mathbf{x}_j - \mathbf{x}_i)^{\perp}}{|\mathbf{x}_j - \mathbf{x}_i|^2}} \tag{1.7}$$

With a Runge-Kutta forth order method one can exemplify the behavior of such systems of point vortices. Two and three vortex systems are integrable, above four one arrives in Hamiltonian chaos.(cf. [15]) An important conserved quantity is the center of vorticity,

$$R = \frac{\sum_j \Gamma_j \mathbf{x}_j}{\sum_j \Gamma_j} \qquad , \qquad \sum_j \Gamma_j \neq 0,$$

which can be directly observed in the pictures for two point vortices, but of course also holds in the case of four vortices. In the case of two counter-rotating vortices, $\sum_j \Gamma_j \neq 0$ is not valid, therefore the center of vorticity is not conserved.

Lamb-Oseen vortex

A further vortex solution can be given by the so-called Lamb-Oseen vortex. If we have a vorticity distribution of the form $\omega(x, y, t) = \omega(r, t)\mathbf{e}_z$ with $r = \sqrt{x^2 + y^2}$, the velocity field is purely azimuthal; reducing the vorticity equation to

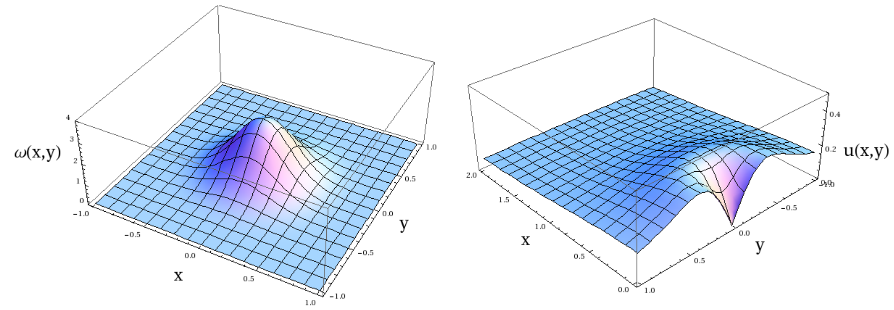
$$\frac{\partial \omega(r, t)}{\partial t} = \nu \Delta \omega(r, t),$$

in fact having the form of a heat equation for $\omega(r, t)$. In polar coordinates this is (cf. [15])

$$\frac{\partial \omega(r, t)}{\partial t} = \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right)\omega(r, t),$$

leading to the solution for the vorticity

[1] Later we will also allow vortex profiles of the form $\mathbf{K}(\mathbf{x}) = \frac{1}{2\pi}\frac{\mathbf{x}^{\perp}}{|\mathbf{x}|^{2-\xi}}$, which are of course not necessarily a solution of the basic hydrodynamical equations.



(a) Vorticity field of the Lamb-Oseen vortex, with viscosity $\nu = 0.02$ after $t = 3$ time units.

(b) Velocity field of the Lamb-Oseen vortex, in dimensionless units with viscosity $\nu = 0.02$, circulation $\Gamma = 1$ after $t = 3$ time units.

$$\omega(r,t) = \frac{\Gamma}{4\pi\nu t} \exp\left(-\frac{r^2}{4\nu t}\right)$$

For the velocity field, one can derive

$$\mathbf{u}(\mathbf{x},t) = \frac{\Gamma}{2\pi r}\left(1 - \exp\left(-\frac{r^2}{4\nu t}\right)\right)\mathbf{e}_\varphi$$

1.2 Determinism and stochastics in hydrodynamics

The Navier-Stokes equation is a deterministic, nonlinear partial differential equation. Nonlinear partial differential equation may show complex and chaotic behavior caused by local instabilities. If one takes in account that turbulent flows are characterized by many degrees of freedom which couple nonlocally, it is obvious that a description of turbulent fluid motions by a combination of the deterministic, basic hydrodynamic equations and probabilstic methods is reasonable. Like in statistical physics, the tool of choice in the theoretical treatment is ensemble averaging (cf. for example [14]). In experiments and numerical simulations the ensemble average is replaced by the time average.

As already explained in the introductory part, the main focus of this thesis is the investigation on two point vortices in an incompressible Kraichnan velocity field. In the following, we thus will introduce the concept of the Kraichnan velocity field. After bringing in the basics of the Fokker-Planck equation, we will derive a formalism to carry over from a finite set of Langevin equations to one multidimensional Fokker-Planck equation. The reason underlying is, that we want to use this formalism to investigate generally on the dynamics of n vortices (or particles) in the same stochastic field, each described by its particular Langevin equation - to eventually derive one stochastic differential equation describing the time evolution of the locations of the vortices in a probabilistic sense.

1.2.1 The Kraichnan model

In this section we introduce the concept of the Kraichnan velocity field. The model of the passive scalar advection introduced by Kraichnan in 1968 (cf. [19]) attracted a lot of attention in theoretical and numerical studies. The essence of the model is that it considers a synthetic Gaussian ensemble of velocities decorrelated in time, defining the ensemble by specifying the 1-point and the 2-point correlations of velocities:

1. The mean velocity $\langle \mathbf{u}(\mathbf{x}, t) \rangle$ has to vanish
2. The 2-point correlation has to fulfill the relation

$$\left\langle u^{\alpha}(\mathbf{x}_i, t') u^{\beta}(\mathbf{x}_j, t'') \right\rangle = d_0^{(\alpha,\beta)} - d^{(\alpha,\beta)}(\mathbf{x}_i - \mathbf{x}_j)\delta(t' - t'')$$
$$= Q^{(\alpha,\beta)}(\mathbf{x}_i - \mathbf{x}_j)\delta(t' - t'') = Q_{ij}^{(\alpha,\beta)}\delta(t' - t'')$$

whereby $d_0^{(\alpha,\beta)}$ is constant and $d^{(\alpha,\beta)}(\mathbf{x}_i - \mathbf{x}_j)$ is assumed to be $\propto r^{\xi}$, but actually only for rather short distances. The latter property models the scaling behavior of the equal-time velocity correlation functions of realistic turbulent flows in the $Re \to \infty$ limit[11]. It can be motivated by the loss of small-scale smoothness of turbulent velocities when $Re \to \infty$. The parameter ξ is taken between 0 and 2 so that the typical velocities of the ensemble are not Lip- schitz continuous[2].

[2] For more detailed examinations and further readings we refer to [11] and [8]

In this thesis, we will always claim isotropic turbulence. This expression means, that all statistical properties of the flow - especially the average velocity fluctuations - are equal all over the flow field and not depending on the direction. Hence, we have invariance of translation and rotation.

If we make the assumption of isotropy and scaling behavior on all scales for now, the velocity 2-point function of the Kraichnan ensemble can be given by (cf. [11])

$$d^{(\alpha,\beta)}(\mathbf{r}) = (A + (d + \xi - 1)B)\, \delta_{\alpha\beta} r^{\xi} + (A - B)\, \xi r^{\xi} \frac{r_\alpha r_\beta}{r^2} \tag{1.8}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta function, d the dimension, ξ the scaling exponent and $A, B \leq 0$ are two parameters. We have $A \propto \left\langle (\nabla_{\mathbf{r}} \cdot \mathbf{u}(\mathbf{r}, t'))^2 \right\rangle$, so in the incompressible case, $\nabla_{\mathbf{r}} \cdot \mathbf{u}(\mathbf{r}, t') = 0$, which we will consider, we get (cf. also [4] p. 19)

$$d^{(\alpha,\beta)}(\mathbf{r}) = r^{\xi} \left[B(d + \xi - 1)\delta_{\alpha\beta} - B\xi \frac{r_\alpha r_\beta}{r^2} \right] \tag{1.9}$$

When deriving a Fokker-Planck equation for the Kraichnan model, we will find that the diffusion coefficients correspond to the spatial correlation functions, as we will soon see in section (1.2.3).

Characteristic and of big importance is of course the time decorrelation of the velocity ensemble, which is not a very physical assumption. But although it will never be exactly fulfilled in nature, this concept is nevertheless valuable. Noting that the δ-correlation with respect to time also implies the Markovian property and therefore is a good candidate for modelling processes with no memory, it can be taken as a good approximation for turbulent velocity fields with fast-decaying correlation function. Additionally it can be seen as a long time approximation for flow fields with finite correlation function. Eventually it makes, however, analytic studies easier.

1.2.2 The Fokker-Planck equation

The Fokker-Planck equation is a second order partial differential equation for the time evolution of a probability density[3] $f(\mathbf{x}_1, ..., \mathbf{x}_n, t)$, which first was used by Fokker and Planck to model the Brownian motion of particles fluids. For one spatial variable x it reads (cf. [26])

$$\frac{\partial}{\partial t} f(x,t) = -\frac{\partial}{\partial x} D_1(x,t) f(x,t) + \frac{1}{2} \frac{\partial^2}{\partial x^2} D_2(x,t) f(x,t) \tag{1.10}$$

[3]Here, we will always talk about the "distribution function" in the mathematical sense as the integral over the "probability density function"; this denomination is unambiguous in contrast to the non-distinction common in physical literature.

whereby D_1 and $\frac{1}{2}D_2$ are the so-called drift and diffusion coefficients; the $\frac{1}{2}$ is a matter of definition, often one finds 1, too. Now more generally, for N variables we have

$$\frac{\partial}{\partial t}f(x_1,...,x_N,t) = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}D_1^{(i)}f(x_1,...,x_N,t)+\frac{1}{2}\sum_{i,j=1}^{N}\frac{\partial^2}{\partial x_i\partial x_j}D_2^{(i,j)}f(x_1,...,x_N,t) \tag{1.11}$$

with $D_1^{(i)} = D_1^{(i)}(x_1,...,x_N,t)$ and $D_2^{(i,j)} = D_2^{(i,j)}(x_1,...,x_N,t)$

Alternatively, if we think of n variables each consisting of m components, we may find

$$\frac{\partial}{\partial t}f(x_1^{(1)},...,x_1^{(m)},...,x_n^{(1)},...,x_n^{(m)},t) = \frac{\partial}{\partial t}f(\mathbf{x}_1,...,\mathbf{x}_n,t) = \frac{\partial}{\partial t}f(\underline{\mathbf{x}},t)$$

$$= -\sum_{i=1}^{n}\sum_{\alpha=1}^{m}\frac{\partial}{\partial x_i^{(\alpha)}}D_1^{(i),(\alpha)}f(\underline{\mathbf{x}},t)+\frac{1}{2}\sum_{i,j}^{n}\sum_{\alpha,\beta}^{m}\frac{\partial^2}{\partial x_i^{(\alpha)}\partial x_j^{(\beta)}}D_2^{(i,j),(\alpha,\beta)}f(\underline{\mathbf{x}},t)$$

$$= -\sum_{i=1}^{n}\frac{\partial}{\partial \mathbf{x}_i}\mathbf{D}_1^{(i)}f(\underline{\mathbf{x}},t)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial \mathbf{x}_i\partial \mathbf{x}_j}\mathbf{D}_2^{(i,j)}f(\underline{\mathbf{x}},t)$$

$$\Rightarrow$$

$$\boxed{\frac{\partial}{\partial t}f(\underline{\mathbf{x}},t) = -\sum_{i=1}^{n}\frac{\partial}{\partial \mathbf{x}_i}\mathbf{D}_1^{(i)}f(\underline{\mathbf{x}},t)+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2}{\partial \mathbf{x}_i\partial \mathbf{x}_j}\mathbf{D}_2^{(i,j)}f(\underline{\mathbf{x}},t)} \tag{1.12}$$

as a useful notion for this case. Thereby we defined

$\underline{\mathbf{x}} \stackrel{\text{def}}{=} (\mathbf{x}_1,...,\mathbf{x}_n) = (x_1^{(1)},...,x_1^{(m)},...,x_n^{(1)},...,x_n^{(m)})$ and $\frac{\partial}{\partial \mathbf{x}_i} \stackrel{\text{def}}{=} \nabla_{\mathbf{x}_i}$

Of course the two multidimensional descriptions are equivalent for $n \cdot m = N$ variables.

The Fokker-Planck equation as a continuity equation

There is a practical property of the solution of the Fokker-Planck equation: If $f(\underline{\mathbf{x}},t)$ is normalized for a fixed time t_0, say

$$\int\limits_{\mathbb{R}^{n\cdot m}} f(\underline{\mathbf{x}}, t_0)\mathrm{d}\underline{\mathbf{x}} = N(t_0),$$

it stays normalized for all time, $N(t) = \text{constant} \equiv N(t_0)$. This is due to

$$\frac{\partial}{\partial t}N(t) = \frac{\partial}{\partial t}\int\limits_{\mathbb{R}^{n\cdot m}} f(\underline{\mathbf{x}}, t)\mathrm{d}\underline{\mathbf{x}} = \int\limits_{\mathbb{R}^{n\cdot m}} \frac{\partial}{\partial t}f(\underline{\mathbf{x}}, t)\mathrm{d}\underline{\mathbf{x}}$$

$$= \int\limits_{\mathbb{R}^{n\cdot m}} \mathrm{d}\underline{\mathbf{x}} \left(-\sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{x}_i}\mathbf{D}_1^{(i)} f(\underline{\mathbf{x}}, t) + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j}\mathbf{D}_2^{(i,j)} f(\underline{\mathbf{x}}, t) \right)$$

$$= \int\limits_{\mathbb{R}^{n\cdot m}} \mathrm{d}\underline{\mathbf{x}} \left(-\sum_{i=1}^{n} \nabla_{\mathbf{x}_i}\cdot\mathbf{D}_1^{(i)} f(\underline{\mathbf{x}}, t) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \nabla_{\mathbf{x}_i}\cdot\left(\nabla_{\mathbf{x}_j}\cdot\mathbf{D}_2^{(i,j)}\right)^{tr} f(\underline{\mathbf{x}}, t) \right)$$

$$= -\int\limits_{\mathbb{R}^{n\cdot m}} \mathrm{d}\underline{\mathbf{x}}\ \nabla_{\underline{\mathbf{x}}}\cdot\underline{\mathbf{D}}_1 f(\underline{\mathbf{x}}, t) + \frac{1}{2}\int\limits_{\mathbb{R}^{n\cdot m}} \mathrm{d}\underline{\mathbf{x}}\ \nabla_{\underline{\mathbf{x}}}\cdot\left(\nabla_{\underline{\mathbf{x}}}\cdot\underline{\mathbf{D}}_2\right)^{tr} f(\underline{\mathbf{x}}, t)$$

whereby for the Nabla operator we have

$$\begin{aligned}\nabla_{\underline{\mathbf{x}}} &= \left(\tfrac{\partial}{\partial \mathbf{x}_1},\quad \cdots,\quad \tfrac{\partial}{\partial \mathbf{x}_n}\right) = \left(\nabla_{\mathbf{x}_1},\quad \cdots,\quad \nabla_{\mathbf{x}_n}\right)\\ &= \left(\tfrac{\partial}{\partial x_1^{(1)}},\quad \cdots,\quad \tfrac{\partial}{\partial x_1^{(m)}},\quad \cdots,\quad \tfrac{\partial}{\partial x_n^{(1)}},\quad \cdots,\quad \tfrac{\partial}{\partial x_n^{(m)}}\right)\end{aligned}$$

Now we can use the divergence theorem and obtain

$$= -\oint\limits_{\partial\mathbb{R}^{n\cdot m}=S} \mathbf{n}\mathrm{dS}\cdot\underline{\mathbf{D}}_1 f(\underline{\mathbf{x}}, t) + \frac{1}{2}\oint\limits_{\partial\mathbb{R}^{n\cdot m}=S} \mathbf{n}\mathrm{dS}\cdot\left(\nabla_{\underline{\mathbf{x}}}\cdot\underline{\mathbf{D}}_2\right)^{tr} f(\underline{\mathbf{x}}, t) \overset{!}{=} 0$$

We have the so-called natural boundary condition letting the probability density vanish in infinity. (cf. [26]) More exactly, we have to claim that $f(|\underline{\mathbf{x}}| \to \infty, t) \to 0$ faster than $\frac{1}{|x|^{nm-1}} \to 0$ (which is fulfilled in most cases) to save the normalizability of the probability density function f. We see that all terms have to vanish, since we integrate on the boundary of an $n \cdot m$-dimensional sphere in infinity. Hence, we finally obtain

$$\frac{\partial}{\partial t} N(t) = 0 \Rightarrow N(t) = \text{constant} \equiv N(t_0)$$

So in fact, the Fokker-Planck equation behaves as a continuity equation for the probability measure,

$$\frac{\partial}{\partial t} f(\underline{\mathbf{x}}, t) = -\nabla_{\underline{\mathbf{x}}} \cdot \underline{\mathbf{j}}(\underline{\mathbf{x}}, t)$$

with probability current density function

$$\underline{\mathbf{j}}(\underline{\mathbf{x}}, t) = \underline{\mathbf{D}}_1 f(\underline{\mathbf{x}}, t) - \frac{1}{2} \left(\nabla_{\underline{\mathbf{x}}} \cdot \underline{\mathbf{D}}_2 \right)^{tr} f(\underline{\mathbf{x}}, t).$$

The big advantage of this is: given a solution of the Fokker-Planck equation in dependence of t, we only have to show for one point in time that it is normalized.

Later, we will in fact always try to show that our solutions of the Fokker-Planck equation correspond to δ-functions for $t = 0$, which of course are by definition normalizable.

Another important feature, which can be derived via path integrals, is that the solution of the Fokker-Planck equation stay positive if it was initially psoitive. [26]

1.2.3 Derivation of a Fokker-Planck equation for a finite set of Langevin equations

In this section we derive a multidimensional Fokker-Planck equation of the form (1.12) for a finite, n-dimensional set of m-dimensional Langevin equations.[4]
This derivations holds especially for Kraichnan velocity fields. Later we will apply this generally established formalism on the example of two Langevin equations for $n = 2$ point vortices which live of course only in $m = 2$ dimensions, finding themselves in a stochastic Kraichnan velocity field.

Consider

$$\boxed{\dot{\mathbf{x}}_i(t) = \mathbf{K}_i(\underline{\mathbf{x}}, t) + \mathbf{u}(\mathbf{x}_i, t)}, \tag{1.13}$$

a quite general Langevin equation describing the time evolution of $\mathbf{x}_i$. Later, in our case, it will be describing the deterministically as well as stochastically affected trajectory of point vortex i. Thereby $\mathbf{x}_i$ and $\dot{\mathbf{x}}_i(t)$ are the space and velocity vectors

[4] compare [12] for the "one-dimensional" case, say: one physical issue represented in one one-dimensional Langevin equation; by the way, this formalism could easily be generalized to n Langevin equations of not necessarily the same dimension m, but in physics this case is rarely of interest.

of equation i - we have again $\underline{\mathbf{x}} = (\mathbf{x}_1, ..., \mathbf{x}_n) = (x_1^{(1)}, ..., x_1^{(m)}, ..., x_n^{(1)}, ..., x_n^{(m)})$ with e.g. $m = 2$ in $\mathbb{R}^2$.

Furthermore, $\mathbf{K}_i(\underline{\mathbf{x}}, t)$ denotes the deterministic part which contains the deterministic interaction between vortex i and the other vortices in the system; therefore it depends on all spatial coordinates of the involved vortices.

$\mathbf{u}(\mathbf{x}_i, t)$ in contrast denotes the stochastical velocity field which is the same for every Langevin equation evaluated at the particular corresponding coordinate $\mathbf{x}_i$. For the stochastic part of (1.13), we assume that it behaves like Gaussian distributed white noise [5] which implies for its average $\langle \mathbf{u}(\mathbf{x}_i, t) \rangle = 0$.
Its correlation function is assumed to fulfill

$$\left\langle \mathbf{u}(\mathbf{x}_i, t')\mathbf{u}(\mathbf{x}_j, t'') \right\rangle = \mathbf{Q}(\mathbf{x}_i - \mathbf{x}_j)\delta(t' - t''),$$

this is a second-order tensor field; in a different notion, we have for $\mathbf{u}(\mathbf{x}_i, t) \in \mathbb{R}^m$ and $\alpha,\ \beta \in \{1, \cdots, m\}$

$$\left\langle u^{(\alpha)}(\mathbf{x}_i, t')u^{(\beta)}(\mathbf{x}_j, t'') \right\rangle = Q^{(\alpha,\beta)}(\mathbf{x}_i - \mathbf{x}_j)\delta(t' - t'')$$

Note that we have the same stochastic field realized at different locations for every Langevin equation.

We now consider $P(\mathbf{x}_i, t) = \delta(\mathbf{x}_i - \mathbf{x}_i(t))$ which is a probabilistic representation of the path described by $\mathbf{x}_i(t)$ whereas $\mathbf{x}_i(t)$ has to satisfy the particular i^{th} Langevin equation (1.13). Generalizing this leads to $P(\underline{\mathbf{x}}, t) = \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t))$. Note that the δ-function has to be understood in terms of the product of δ-functions for every single component:

$$\begin{aligned}\delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) &= \delta(\mathbf{x}_1 - \mathbf{x}_1(t)) \cdots \delta(\mathbf{x}_n - \mathbf{x}_n(t)) \\ &= \delta(x_1^{(1)} - x_1^{(1)}(t)) \cdots \delta(x_1^{(m)} - x_1^{(m)}(t))\delta(x_2^{(1)} - x_2^{(1)}(t)) \cdots \delta(x_n^{(m)} - x_n^{(m)}(t))\end{aligned}$$

Now we take the ensemble average over all possible paths and introduce the function

$$f(\underline{\mathbf{x}}, t) = \langle P(\underline{\mathbf{x}}, t) \rangle = \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle \tag{1.14}$$

To derive now a differential equation for f, let us consider a change $\Delta f(\underline{\mathbf{x}}, t)$ of f in a time interval τ:

$$\begin{aligned}\Delta f(\underline{\mathbf{x}}, t) &= f(\underline{\mathbf{x}}, t + \tau) - f(\underline{\mathbf{x}}, t) \\ &= \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t + \tau)) \rangle - \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle\end{aligned}$$

[5] An exact definition may be found in [10] p 41, more details to Gaussian stochastic processes may be found in [12], p. 86

If we now put $\mathbf{x}_i(t+\tau) = \mathbf{x}_i(t) + \Delta\mathbf{x}_i(t)$ and perform a Taylor expansion up to powers quadratic in $\Delta\mathbf{x}_i$, we get[6]

$$\Delta f(\underline{\mathbf{x}}, t)$$

$$= \left\langle \left(-\sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{x}_i} \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \right) \Delta\mathbf{x}_i(t) \right\rangle + \frac{1}{2} \left\langle \sum_{i,j}^{n} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \left(\Delta\mathbf{x}_i(t)\Delta\mathbf{x}_j(t)\right) \right\rangle$$

Considering now

$$\int_t^{t+\tau} \dot{\mathbf{x}}_i(t')dt' = \mathbf{x}_i(t+\tau) - \mathbf{x}_i(t) = \Delta\mathbf{x}_i(t)$$

we may calculate $\Delta\mathbf{x}_i(t)$. To reach this goal, we have to accept that $\int_t^{t+\tau} dt'\mathbf{K}_i(\underline{\mathbf{x}}, t') = \mathbf{K}_i(\underline{\mathbf{x}}, t)\tau$. Consider this: If $\mathfrak{K}_i(\underline{\mathbf{x}}, t')$ is the primitive of $\mathbf{K}_i(\underline{\mathbf{x}}, t')$, then

$$\int_t^{t+\tau} dt'\mathbf{K}_i(\underline{\mathbf{x}}, t') = [\mathfrak{K}_i(\underline{\mathbf{x}}, t')]_t^{t+\tau} = \mathfrak{K}_i(\underline{\mathbf{x}}, t+\tau) - \mathfrak{K}_i(\underline{\mathbf{x}}, t) \overset{\star}{=} \mathbf{K}_i(\underline{\mathbf{x}}, t)\tau = \frac{d}{dt}(\mathfrak{K}_i(\underline{\mathbf{x}}, t))\tau$$

and $\star$ is obviously true if we consider

$$\lim_{\tau\to 0} \frac{\mathfrak{K}_i(\underline{\mathbf{x}}, t+\tau) - \mathfrak{K}_i(\underline{\mathbf{x}}, t)}{\tau} = \frac{d}{dt}(\mathfrak{K}_i(\underline{\mathbf{x}}, t)).$$

Now using (1.13), we get

$$\begin{aligned} \Delta\mathbf{x}_i(t) &= \int_t^{t+\tau} \dot{\mathbf{x}}_i(t')dt' \overset{(1.13)}{=} \int_t^{t+\tau} dt' \left(\mathbf{K}_i(\underline{\mathbf{x}}, t') + \mathbf{u}(\mathbf{x}_i, t')\right) \\ &= \int_t^{t+\tau} dt'\mathbf{K}_i(\underline{\mathbf{x}}, t') + \int_t^{t+\tau} dt'\mathbf{u}(\mathbf{x}_i, t') \overset{\star}{=} \mathbf{K}_i(\underline{\mathbf{x}}, t)\tau + \Delta\mathbf{u}(\mathbf{x}_i, t) \end{aligned} \quad (1.15)$$

whereas τ is small and

$$\Delta\mathbf{u}(\mathbf{x}_i, t) \equiv \int_t^{t+\tau} dt'\mathbf{u}(\mathbf{x}_i, t').$$

[6]This is due to

$$\left\langle \frac{d}{dq(t)} \delta(q - q(t)) \right\rangle = \left\langle -\frac{d}{dq} \delta(q - q(t)) \right\rangle,$$

compare also [12]

Now we can evaluate the first term of the Taylor expansion,or the drift, respectively:

$$\begin{aligned}
&\left\langle \left(-\frac{\partial}{\partial \mathbf{x}_i} \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \right) \Delta \mathbf{x}_i(t) \right\rangle \\
=& \frac{\partial}{\partial \mathbf{x}_i} \left[\langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t))(\mathbf{K}_i(\underline{\mathbf{x}}, t)\tau) \rangle + \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \Delta \mathbf{u}(\mathbf{x}_i, t) \rangle \right] \\
=& \frac{\partial}{\partial \mathbf{x}_i} \left[\langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t))(\mathbf{K}_i(\underline{\mathbf{x}}, t)\tau) \rangle + \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle \langle \Delta \mathbf{u}(\mathbf{x}_i, t) \rangle \right]
\end{aligned}$$

The first step is possible due to a property of the δ-function (details can be found in [26], pp. 69, 70 and pp. 81, 82). The second step may be performed, because $\mathbf{x}_i(t)$ is determined by times $t' < t$ and $\Delta \mathbf{u}(\mathbf{x}_i, t)$ by times $t' > t$ (compare also [12]). Now we take in account that the stochastic part of (1.13) has to vanish in the average,

$$\langle \mathbf{u}(\mathbf{x}_i, t) \rangle = 0, \text{ which also implies}$$

$$\langle \Delta \mathbf{u}(\mathbf{x}_i, t) \rangle = \left\langle \int_t^{t+\tau} dt' \mathbf{u}(\mathbf{x}_i, t') \right\rangle = \int_t^{t+\tau} dt' \left\langle \mathbf{u}(\mathbf{x}_i, t') \right\rangle = 0$$

because the integration interchanges. This reduces the drift term to

$$\frac{\partial}{\partial \mathbf{x}_i} \left(\langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t))(\mathbf{K}_i(\underline{\mathbf{x}}, t)\tau) \rangle \right)$$

The next step will be the calculation of the second term of the Taylor expansion, namely the diffusion. We have

$$\begin{aligned}
&\frac{1}{2} \left\langle \sum_{i,j} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t))(\Delta \mathbf{x}_i(t) \Delta \mathbf{x}_j(t)) \right\rangle \\
=& \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle \langle (\Delta \mathbf{x}_i(t) \Delta \mathbf{x}_j(t)) \rangle \\
\overset{(1.15)}{=}& \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle \langle [\mathbf{K}_i(\underline{\mathbf{x}}, t)\tau + \Delta \mathbf{u}(\mathbf{x}_i, t)] \, [\mathbf{K}_j(\underline{\mathbf{x}}, t)\tau + \Delta \mathbf{u}(\mathbf{x}_j, t)] \rangle \\
=& \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle \big\langle \mathbf{K}_i(\underline{\mathbf{x}}, t) \mathbf{K}_j(\underline{\mathbf{x}}, t)\tau^2 + \Delta \mathbf{u}(\mathbf{x}_i, t) \mathbf{K}_j(\underline{\mathbf{x}}, t)\tau \\
&+ \mathbf{K}_i(\underline{\mathbf{x}}, t)\tau \Delta \mathbf{u}(\mathbf{x}_j, t) + \Delta \mathbf{u}(\mathbf{x}_i, t) \Delta \mathbf{u}(\mathbf{x}_j, t) \rangle \\
=& \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle \left(\big\langle \mathbf{K}_i(\underline{\mathbf{x}}, t) \mathbf{K}_j(\underline{\mathbf{x}}, t)\tau^2 \big\rangle + \langle \Delta \mathbf{u}(\mathbf{x}_i, t) \mathbf{K}_j(\underline{\mathbf{x}}, t)\tau \rangle \right. \\
&\left. + \langle \mathbf{K}_i(\underline{\mathbf{x}}, t)\tau \Delta \mathbf{u}(\mathbf{x}_j, t) \rangle + \langle \Delta \mathbf{u}(\mathbf{x}_i, t) \Delta \mathbf{u}(\mathbf{x}_j, t) \rangle \right)
\end{aligned}$$

Note that the second step may be performed due to the same argumentation used above: $\mathbf{x}_i(t)$ is determined for times $t' < t$ and $\Delta\mathbf{x}_i$ for times $t' > t$.

When inserting now equation (1.15) for $\Delta\mathbf{x}_i$, one finds terms containing τ^2, $\tau\Delta\mathbf{u}(\mathbf{x}_i,t)$ and $\Delta\mathbf{u}(\mathbf{x}_i,t)\Delta\mathbf{u}(\mathbf{x}_j,t)$. $\langle\Delta\mathbf{u}_i(x,t)\rangle$ vanishes, so the only relevant terms left are those containing $\Delta\mathbf{u}(\mathbf{x}_i,t)\Delta\mathbf{u}(\mathbf{x}_j,t)$ and those $\propto \tau^2$:

$$
\begin{aligned}
\Rightarrow &\frac{1}{2}\left\langle \sum_{i,j} \frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\delta(\underline{\mathbf{x}}-\underline{\mathbf{x}}(t))(\Delta\mathbf{x}_i(t)\Delta\mathbf{x}_j(t))\right\rangle \\
= &\frac{1}{2}\sum_{i,j} \frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j} \langle\delta(\underline{\mathbf{x}}-\underline{\mathbf{x}})(t)\rangle \left(\langle\mathbf{K}_i(\underline{\mathbf{x}},t)\mathbf{K}_j(\underline{\mathbf{x}},t)\rangle\tau^2 + \langle\Delta\mathbf{u}(\mathbf{x}_i,t)\Delta\mathbf{u}(\mathbf{x}_j,t)\rangle\right)
\end{aligned}
$$

Note that now the derivatives act also on all other terms next to the δ-function, again this is due to the property of the δ-function already used above (details can be found in [26], pp 69, 81, 82.)

If we take into account that the correlation function of the velocities is δ-correlated in time, like a Kraichnan velocity field,

$$
\left\langle \mathbf{u}(\mathbf{x}_i,t')\mathbf{u}(\mathbf{x}_j,t'')\right\rangle = \mathbf{Q}(\mathbf{x}_i-\mathbf{x}_j)\delta(t'-t''),
$$

we are able to evaluate

$$
\begin{aligned}
\langle\Delta\mathbf{u}(\mathbf{x}_i,t)\Delta\mathbf{u}(\mathbf{x}_j,t)\rangle &= \int_t^{t+\tau}\int_t^{t+\tau} dt'dt'' \left\langle \mathbf{u}(\mathbf{x}_i,t')\mathbf{u}(\mathbf{x}_j,t'')\right\rangle \\
&= \int_t^{t+\tau}\int_t^{t+\tau} dt'dt''\mathbf{Q}_{ij}\delta(t'-t'') = \mathbf{Q}_{ij}\tau
\end{aligned}
$$

If we now combine our results, we find

$$
\begin{aligned}
\Delta f(\underline{\mathbf{x}},t) = &-\nabla_{\underline{\mathbf{x}}}\cdot\langle\delta(\underline{\mathbf{x}}-\underline{\mathbf{x}}(t))(\underline{\mathbf{K}}(\underline{\mathbf{x}},t)\tau)\rangle \\
&+\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\langle\delta(\underline{\mathbf{x}}-\underline{\mathbf{x}}(t))\rangle\left(\langle\mathbf{K}_i(\underline{\mathbf{x}},t)\mathbf{K}_j(\underline{\mathbf{x}},t)\rangle\tau^2+\mathbf{Q}_{ij}\tau\right)
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow \quad \frac{\Delta f(\underline{\mathbf{x}},t)}{\tau} = \frac{f(\underline{\mathbf{x}},t+\tau)-f(\underline{\mathbf{x}},t)}{\tau} &= -\nabla_{\underline{\mathbf{x}}}\cdot\langle\delta(\underline{\mathbf{x}}-\underline{\mathbf{x}}(t))(\underline{\mathbf{K}}(\underline{\mathbf{x}},t))\rangle \\
&+\frac{1}{2}\sum_{i,j}\frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\langle\delta(\underline{\mathbf{x}}-\underline{\mathbf{x}}(t))\rangle\left(\langle\mathbf{K}_i(\underline{\mathbf{x}},t)\mathbf{K}_j(\underline{\mathbf{x}},t)\rangle\tau+\mathbf{Q}_{ij}\right)
\end{aligned}
$$

Finally using equation (1.14)

$$f(\underline{\mathbf{x}}, t) = \langle \delta(\underline{\mathbf{x}} - \underline{\mathbf{x}}(t)) \rangle$$

and taking the limit $\tau \to 0$ we get

$$\frac{\partial}{\partial t} f(\underline{\mathbf{x}}, t) = -\nabla_{\underline{\mathbf{x}}} \cdot \left(\underline{\mathbf{K}}(\underline{\mathbf{x}}, t) f(\underline{\mathbf{x}}, t) \right) + \frac{1}{2} \sum_{i,j} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \mathbf{Q}_{ij} f(\underline{\mathbf{x}}, t) \qquad (1.16)$$

So we derived a partial differential equation - the Fokker-Planck equation - for a system of n Langevin equations. It describes the change of the probability density of the systems state during the course of time. To illustrate this, figure 1.16 shows an example.

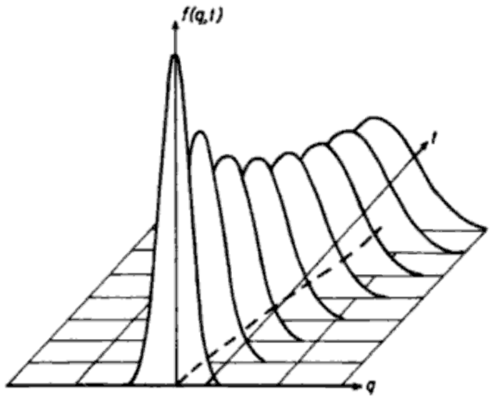


Figure 1.2: Example for the case $f(\underline{\mathbf{x}}, t) = f(q, t)$ as a function of time t and one spatial variable q. The dashed line describes the most probable path. (from [12], p. 163)

Note that we only considered derivatives up to second order in our Taylor expansion. Nevertheless, the derived Fokker-Planck equation is exact in this case, because the examined process characterized by the fluctuating Kraichnan velocity field with its particular properties is Gaussian, all higher terms vanish for $\tau \to 0$

In the formalism of a Kramers-Moyal expansion, one could alternatively verify that caused by the Pawula Theorem all moments except the first two vanish. Thus, the Kramers-Moyal expansion passes in an exact manner into the Fokker-Planck equation. We pass on here without detailed examinations and refer the reader to [12], pp. 85 ff..,164 and [26].

2 Vortices in a Kraichnan velocity field

This chapter is about the analytical treatment of the dynamics of point vortices in a incompressible, statistical isotropic, Kraichnan velocity field. We will on the one hand investigate on "standard" point vortices as solution of the Euler equation with the vortex profile

$$\mathbf{K}(\mathbf{x}_j - \mathbf{x}_i) = \frac{1}{2\pi}\frac{(\mathbf{x}_j - \mathbf{x}_i)^{\perp}}{|(\mathbf{x}_j - \mathbf{x}_i)|^2}$$

whereby by convention we have $(x, y)^{\perp} = (-y, x)$. It is describing the velocity field in point x_j generated by a point vortex located at x_i. On the other hand, we also allow vortex profiles of the form

$$\mathbf{K}(\mathbf{x}_j - \mathbf{x}_i) = \frac{1}{2\pi}\frac{(\mathbf{x}_j - \mathbf{x}_i)^{\perp}}{|(\mathbf{x}_j - \mathbf{x}_i)|^{2-\xi}}$$

with $\xi \in [0, 2]$. For $\xi = 0$, we of course get the standard point vortex profile. It is important to notice that these are not necessarily compatible with the basic hydrodynamical equations, but may be an alternative way to model systems involving vorticity. Note that the dimension changes when $\xi \neq 0$, it has to be matched for example by a reasonable choice of the dimension of the "circulation" Γ, that then also does not hold the properties it does for the standard profile.

In a first step we will describe the system in terms of two Langevin equations. Visually, we modelled the system for different circulations with a modified Runge-Kutta method including white noise. After a change of coordinates, we then apply the formalism we derived in the previous chapter on the two Langevin equations of two point vortices in order to obtain one Fokker-Planck equation describing the time evolution of the whole system in a probabilistic manner. Subsequently, we will break down the derived Fokker-Planck equation analytically and finally solve it.

2.1 One vortex in a Kraichnan velocity field

The first simple case we want to treat is one point vortex in a Kraichnan velocity field. We get the Langevin equation

$$\dot{\mathbf{x}}_0 = \mathbf{u}(\mathbf{x}_0, t)$$

whereby we treat the system of course in two spatial dimensions.

The evolution of the system is by no means dependent on the deterministic velocity field generated by the vortex. Hence, applying our formalism, we get

$$\frac{\partial}{\partial t} f(\mathbf{x}_0, t) = \frac{1}{2} \frac{\partial^2}{\partial \mathbf{x}_0^2} \mathbf{Q}_{00} f(\mathbf{x}_0, t)$$

$$= \frac{1}{2} \nabla_{\mathbf{x}_0} \cdot \mathbf{Q}(0) \cdot \nabla_{\mathbf{x}_0} \; f(\mathbf{x}_0, t)$$

whereby we used that $\mathbf{Q}_{00} = \mathbf{Q}(\mathbf{x}_0 - \mathbf{x}_0) = \mathbf{Q}(0)$

So we obtain just the well known Brownian motion.

Assuming[1] that

$$\mathbf{Q}(0) = \begin{pmatrix} \theta_0 & 0 \\ 0 & \theta_0 \end{pmatrix}$$

[1]In section (2.3.2) an equation will be derived, namely (2.23), with which can be obtained that this is exact in the case of isotropic statistics and if $\mathbf{u}(\mathbf{x}_0, t)$ is incompressible.

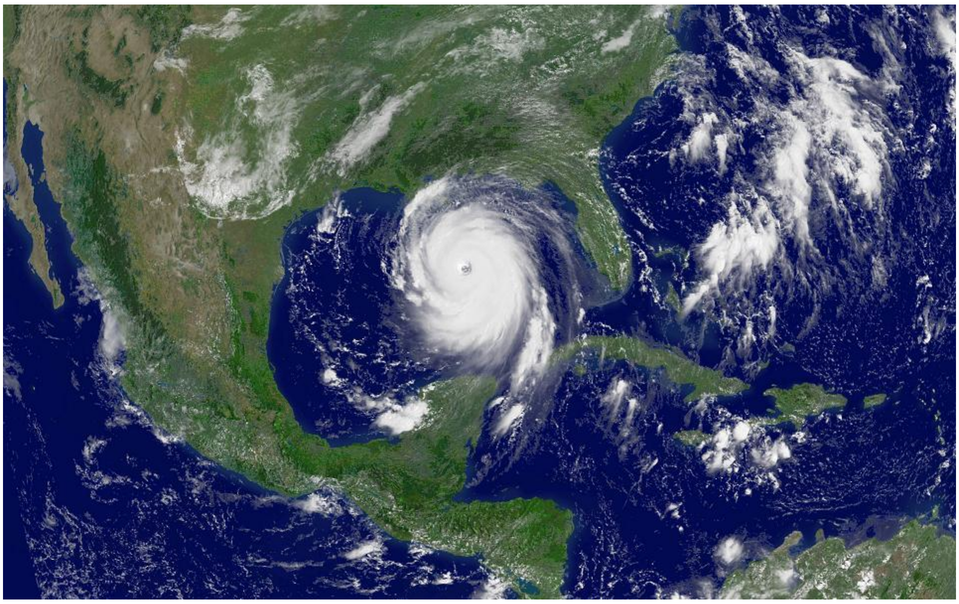
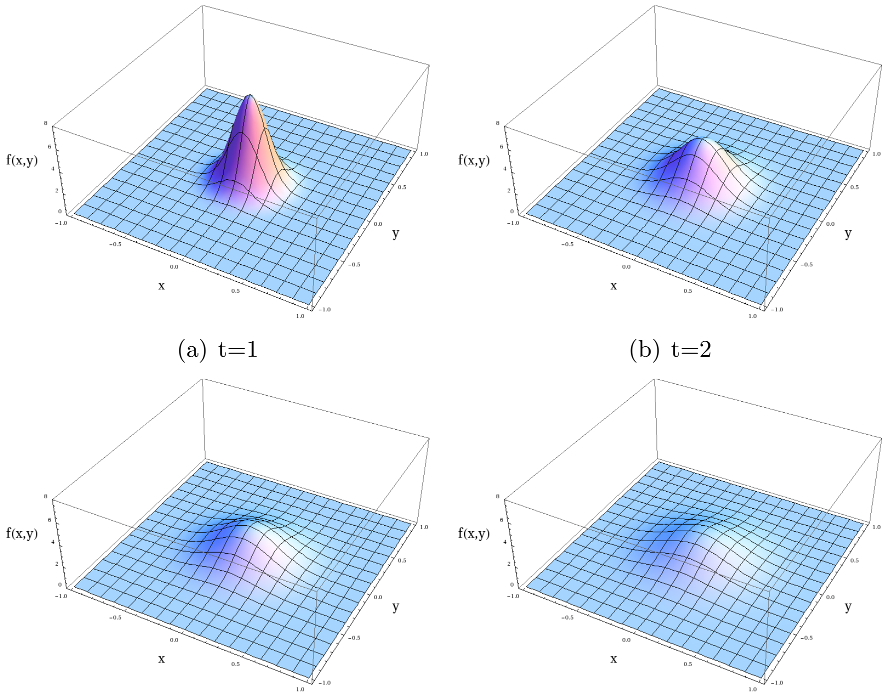


Figure 2.1: Hurricane Katrina, on August 28^{th}, 2005. In contrast to the point vortices we investigate on, hurricanes like Katrina are (fortunately) decaying eddies. Source: `http://www.noaanews.noaa.gov/stories2005/images/katrina-08-28-2005-1545z.jpg`



(a) t=1 (b) t=2

(c) t=3 (d) t=4

Figure 2.2: $f(\mathbf{x}_0, t) = f(x, y, t)$ for $\theta_0 = 0.02$ and $t \in \{1, 2, 3, 4\}$

we can also write

$$\frac{\partial}{\partial t} f(\mathbf{x}_0, t) = \frac{1}{2} \theta_0 \Delta_{\mathbf{x}_0} f(\mathbf{x}_0, t)$$

whereby

$$\Delta_{\mathbf{x}_0} = \frac{\partial^2}{\partial x_{0,1}^2} + \frac{\partial^2}{\partial x_{0,2}^2}$$

is the Laplacian. Assuming that the vortex is initially located at the origin,

$$f(\mathbf{x}_0, t = 0) = \delta(\mathbf{x}_0)$$

the solution can be given by

$$f(\mathbf{x}_0, t) = \frac{1}{2\theta_0 \pi t} \exp\left(-\frac{|\mathbf{x}_0|^2}{2\theta_0 t} \right),$$

Hence we have a formal similarity to the Lamb-Ossen vortex presented in section (1.1). In [3] the point vortex solution, itself a *probability density function* describing the time evolution of the location of one point vortex with strictly located vorticity is even identified with the Lamb-Oseen vortex, which in contrast describes a *vorticity field* of the corresponding form

$$\omega_z(r,t) = \frac{\Gamma}{4\pi\nu t} \exp\left(-\frac{r^2}{4\nu t}\right)$$

(compare for example [15]) with viscosity ν and circulation Γ. Note that this is of course not the same; we in fact only have two differential equations equivalent to the heat equation which of course the same solutions.

2.2 Two vortices in a Kraichnan velocity field

If we consider the motion of two vortices with constant in time vortex profiles under their mutual hydrodynamic interaction, perturbed by a fluctuating stochastical velocity field, the corresponding Langevin equations read

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \Gamma_2 \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{u}(\mathbf{x}_1, t) \\ \dot{\mathbf{x}}_2 &= \Gamma_1 \mathbf{K}(\mathbf{x}_2 - \mathbf{x}_1) + \mathbf{u}(\mathbf{x}_2, t)\end{aligned} \tag{2.1}$$

$\Longleftrightarrow$

$$\boxed{\begin{aligned}\dot{\mathbf{x}}_1 &= \Gamma_2 \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{u}(\mathbf{x}_1, t) \\ \dot{\mathbf{x}}_2 &= -\Gamma_1 \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{u}(\mathbf{x}_2, t)\end{aligned}} \tag{2.2}$$

whereas $\Gamma_i \mathbf{K}(\mathbf{x}_j - \mathbf{x}_i) \stackrel{!}{=} -\Gamma_i \mathbf{K}(\mathbf{x}_i - \mathbf{x}_j)$ in each case is the deterministic part, which is for now - with respect to generality - quite arbitrary.
Later, it will be chosen to

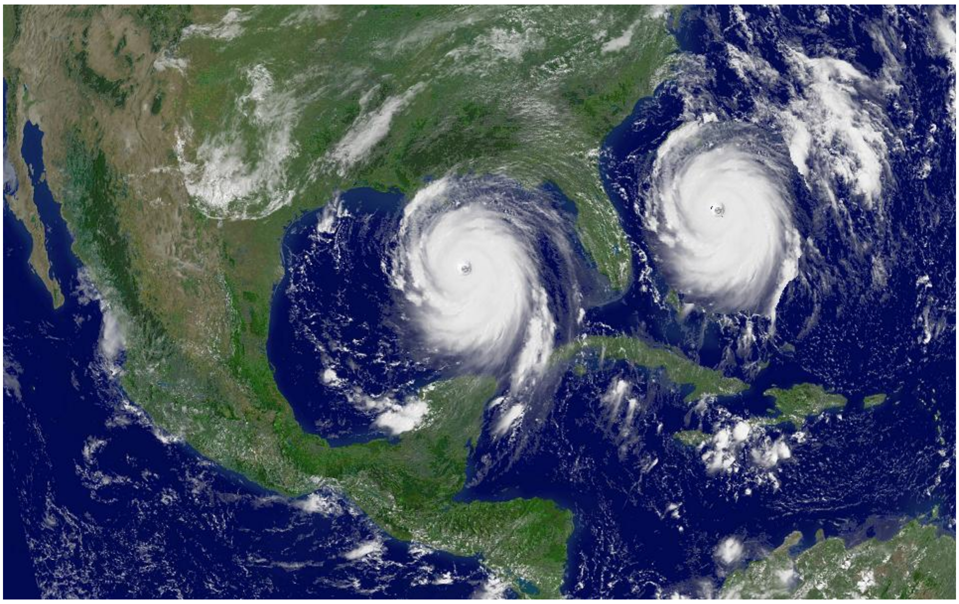


Figure 2.3: Fiction: Katrina 1 + Katrina 2 attacking Florida (Edited with MS Paint). Source: `http://www.noaanews.noaa.gov/stories2005/images/katrina-08-28-2005-1545z.jpg`

$$\Gamma_i \mathbf{K}(\mathbf{x}_j - \mathbf{x}_i) = \frac{\Gamma_i}{2\pi} \frac{(\mathbf{x}_k - \mathbf{x}_l)^{\perp}}{|(\mathbf{x}_k - \mathbf{x}_l)|^{2-\xi}}$$

whereas $\xi \in [0,2]$. Note that we have the standard vortex profile, itself a solution of the Euler equation, only in the case $\xi = 0$. In all other cases, we have a generalized model not necessarily compatible with the basic hydrodynamic equations, additionally following that we have to match the dimension of Γ_i. Although knowing that this stands in contrast to the ordinary dimension of the hydrodynamical circulation, we will nevertheless treat the denotions Γ_i and "circulation" equivalently.

The stochastical part $\mathbf{u}(\mathbf{x}_i, t)$ is assumed to an incompressible Kraichnan velocity field, thus it has to fulfill the properties

$$\langle \mathbf{u}(\mathbf{x}_i, t) \rangle = 0$$

and

$$\left\langle \mathbf{u}(\mathbf{x}_i, t)\mathbf{u}(\mathbf{x}_j, t') \right\rangle = \mathbf{Q}_{ij}\delta(t - t') = \mathbf{Q}(\mathbf{x}_i - \mathbf{x}_j) = \mathbf{Q}(\mathbf{x}_j - \mathbf{x}_i)$$

Note, that for $i \in \{1, 2\}$, we have

$$\mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_1) = \mathbf{Q}(\mathbf{x}_2 - \mathbf{x}_2) = \mathbf{Q}(0) = \mathbf{Q}_{ii}$$

Just to emphasize once again that we have to deal with a double vectorial character of the considered quantities, we write down the complete correlation matrix:

$$\begin{aligned} \underline{\mathbf{Q}}\ \delta(t-t') &= \begin{pmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{pmatrix} \delta(t-t') \\ &= \begin{pmatrix} \langle \mathbf{u}(\mathbf{x}_1, t)\mathbf{u}(\mathbf{x}_1, t') \rangle & \langle \mathbf{u}(\mathbf{x}_1, t)\mathbf{u}(\mathbf{x}_2, t') \rangle \\ \langle \mathbf{u}(\mathbf{x}_2, t)\mathbf{u}(\mathbf{x}_1, t') \rangle & \langle \mathbf{u}(\mathbf{x}_2, t)\mathbf{u}(\mathbf{x}_2, t') \rangle \end{pmatrix} \end{aligned} \tag{2.3}$$

$$= \begin{pmatrix} \langle u_1(\mathbf{x}_1,t)u_1(\mathbf{x}_1,t')\rangle & \langle u_1(\mathbf{x}_1,t)u_2(\mathbf{x}_1,t')\rangle & \langle u_1(\mathbf{x}_1,t)u_1(\mathbf{x}_2,t')\rangle & \langle u_1(\mathbf{x}_1,t)u_2(\mathbf{x}_2,t')\rangle \\ \langle u_2(\mathbf{x}_1,t)u_1(\mathbf{x}_1,t')\rangle & \langle u_2(\mathbf{x}_1,t)u_2(\mathbf{x}_1,t')\rangle & \langle u_2(\mathbf{x}_1,t)u_1(\mathbf{x}_2,t')\rangle & \langle u_2(\mathbf{x}_1,t)u_2(\mathbf{x}_2,t')\rangle \\ \langle u_1(\mathbf{x}_2,t)u_1(\mathbf{x}_1,t')\rangle & \langle u_1(\mathbf{x}_2,t)u_2(\mathbf{x}_1,t')\rangle & \langle u_1(\mathbf{x}_2,t)u_1(\mathbf{x}_2,t')\rangle & \langle u_1(\mathbf{x}_2,t)u_2(\mathbf{x}_2,t')\rangle \\ \langle u_2(\mathbf{x}_2,t)u_1(\mathbf{x}_1,t')\rangle & \langle u_2(\mathbf{x}_2,t)u_2(\mathbf{x}_1,t')\rangle & \langle u_2(\mathbf{x}_2,t)u_1(\mathbf{x}_2,t')\rangle & \langle u_2(\mathbf{x}_2,t)u_2(\mathbf{x}_2,t')\rangle \end{pmatrix}$$

Among other things, some parts of this matrix will explicitly be calculated in the following sections for different cases. After a coordinate transformation, we will primarily focus on the upper left part of the matrix, which will contain the spatial correlations for the relative motion of the vortices[2]. As a remark, we refer to the case $\mathbf{x}_1 = \mathbf{x}_2$ which describes the collision of the vortices and which is not allowed since it would result in a break down of the equations. Two vortices initially located in the same place would result of course in the case of one localized vortex with accumulated circulations, as treated in the previous section. So from now on, we assume $\mathbf{x}_1 \neq \mathbf{x}_2$ for the whole time of investigation on the system.

[2]Of course it will be the upper left part only by convention, this depends directly on the sort sequence of the Langevin equations.

2.2.1 The cases $\Gamma_1, \Gamma_2 \in \mathbb{R}$ with $\Gamma_1 \neq -\Gamma_2$

We will now set up of the corresponding Fokker-Planck equation for all cases $\Gamma_1, \Gamma_2 \in \mathbb{R}$ with $\Gamma_1 \neq -\Gamma_2$ by applying the formalism of the previous chapter. But before, we will introduce center of vorticity coordinates: if we put

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2 \text{ and } \mathbf{R} = \frac{\Gamma_1 \mathbf{x}_1 + \Gamma_2 \mathbf{x}_2}{\Gamma_1 + \Gamma_2}$$

we get

$$\begin{aligned}
\mathbf{r} &= \dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2 \\
&\overset{(2.2)}{=} \Gamma_2 \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{K}(\mathbf{x}_1, t) - (-\Gamma_1 \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{u}(\mathbf{x}_2, t)) \\
&= (\Gamma_1 + \Gamma_2)\mathbf{u}(\mathbf{r}) + \mathbf{u}(\mathbf{x}_1, t) - \mathbf{u}(\mathbf{x}_2, t))
\end{aligned}$$

and

$$\begin{aligned}
\dot{\mathbf{R}} &= \frac{\Gamma_1 \dot{\mathbf{x}}_1 + \Gamma_2 \dot{\mathbf{x}}_2}{\Gamma_1 + \Gamma_2} \\
&\overset{(2.2)}{=} \frac{1}{\Gamma_1 + \Gamma_2} \left(\Gamma_1 \Gamma_2 \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \Gamma_1 \mathbf{u}(\mathbf{x}_1, t) - \Gamma_2 \Gamma_1 \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \Gamma_2 \mathbf{u}(\mathbf{x}_2, t)\right) \\
&= \frac{1}{\Gamma_1 + \Gamma_2} \left(\Gamma_1 \mathbf{u}(\mathbf{x}_1, t) + \Gamma_2 \mathbf{u}(\mathbf{x}_2, t)\right) \\
&= \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \mathbf{u}(\mathbf{x}_1, t) + \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \mathbf{u}(\mathbf{x}_2, t)
\end{aligned}$$

and obtain

$$\begin{aligned}
\dot{\mathbf{r}} &= (\Gamma_1 + \Gamma_2)\mathbf{K}(\mathbf{r}) + \mathbf{u}_m(t) \\
\dot{\mathbf{R}} &= \mathbf{u}_p(t)
\end{aligned} \tag{2.4}$$

as the Langevin equations describing the interaction between the vortices and the Kraichnan velocity field. Thereby we define

$$\mathbf{u}_m(t) = \mathbf{u}(\mathbf{x}_1, t) - \mathbf{u}(\mathbf{x}_2, t)$$

and

$$\mathbf{u}_p(t) = \frac{\Gamma_1}{\Gamma_1 + \Gamma_2} \mathbf{u}(\mathbf{x}_1, t) + \frac{\Gamma_2}{\Gamma_1 + \Gamma_2} \mathbf{u}(\mathbf{x}_2, t).$$

If we now apply the formalism derived in the previous chapter, we get a Fokker-Planck equation of the form

$$\boxed{\frac{\partial}{\partial t}f(\mathbf{r},\mathbf{R},t) = -\nabla\cdot\begin{pmatrix}(\Gamma_1+\Gamma_2)\mathbf{K}(\mathbf{r})\\0\end{pmatrix}f(\mathbf{r},\mathbf{R},t)+\frac{1}{2}\sum_{i;j\in\{m,p\}}\frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\tilde{\mathbf{Q}}_{ij}f(\mathbf{r},\mathbf{R},t)}$$

Thereby, we have

$$\nabla=\begin{pmatrix}\nabla_{\mathbf{r}}\\\nabla_{\mathbf{R}}\end{pmatrix}=\begin{pmatrix}\frac{\partial}{\partial\mathbf{r}}\\\frac{\partial}{\partial\mathbf{R}}\end{pmatrix}\stackrel{!}{=}\begin{pmatrix}\frac{\partial}{\partial\mathbf{x}_m}\\\frac{\partial}{\partial\mathbf{x}_p}\end{pmatrix},$$

It is important to accept that $\frac{\partial}{\partial\mathbf{x}_m}$ has to be identified with $\frac{\partial}{\partial\mathbf{r}}$ as well as $\frac{\partial}{\partial\mathbf{x}_p}$ has to be identified with $\frac{\partial}{\partial\mathbf{R}}$. The reason underlies already in the Langevin equation: The first Langevin equation in (2.4) represents the component $\mathbf{r}$ and the second one $\mathbf{R}$. The associated stochastical term for $\mathbf{r}$ is $\mathbf{u}_m(t)$ and for $\mathbf{R}$ is $\mathbf{u}_p(t)$. Index renaming to achieve an identifying context would result in an easier summation in the diffusion term. This would be just over $i,j\in\{\mathbf{r},\mathbf{R}\}$ and the application of the introduced formalism could be accomplished without any further thoughts about connections between indices. But on the other side, for example concerning the renaming of $\mathbf{u}_p(t)$ into $\mathbf{u}_{\mathbf{R}}(t)$, we would get a problem in an intuitive sense - the reason is that $\langle\mathbf{u}_p(t)\mathbf{u}_p(t')\rangle$ is dependent on $\mathbf{r}$, but not on $\mathbf{R}$ as we will see below. Hence, the apparently missing connection is only an effect of a coordinate transformation and after having clarified the notation, so we will not rename any variables.

We will now first concentrate on the diffusive term. To find its correct form, we have to investigate explicitly the question what the correlation matrices of $\mathbf{u}_m$ and $\mathbf{u}_p$ do look like.

First, for $\langle\mathbf{u}_m(t)\mathbf{u}_m(t')\rangle$ we get

$$\begin{aligned}&\left\langle\mathbf{u}_m(t)\mathbf{u}_m(t')\right\rangle\\=&\left\langle(\mathbf{u}(\mathbf{x}_1,t)-\mathbf{u}(\mathbf{x}_2,t))(\mathbf{u}(\mathbf{x}_1,t')-\mathbf{u}(\mathbf{x}_2,t'))\right\rangle\\=&\left\langle\mathbf{u}(\mathbf{x}_1,t)\mathbf{u}(\mathbf{x}_1,t')\right\rangle+\left\langle\mathbf{u}(\mathbf{x}_2,t)\mathbf{u}(\mathbf{x}_2,t')\right\rangle-\left\langle\mathbf{u}(\mathbf{x}_1,t)\mathbf{u}(\mathbf{x}_2,t')\right\rangle-\left\langle\mathbf{u}(\mathbf{x}_2,t)\mathbf{u}(\mathbf{x}_1,t')\right\rangle\\=&2[\mathbf{Q}(0)-\mathbf{Q}(\mathbf{x}_1-\mathbf{x}_2)]\delta(t-t')\\\stackrel{\text{def}}{=}&\tilde{\mathbf{C}}(\mathbf{r})\delta(t-t')\end{aligned}\tag{2.5}$$

Second, for $\langle\mathbf{u}_p(t)\mathbf{u}_p(t')\rangle$, we have

$$\langle \mathbf{u}_p(t)\mathbf{u}_p(t') \rangle$$

$$= \left\langle \left(\frac{\Gamma_1}{\Gamma_1+\Gamma_2}\mathbf{u}(\mathbf{x}_1,t) + \frac{\Gamma_2}{\Gamma_1+\Gamma_2}\mathbf{u}(\mathbf{x}_2,t) \right) \left(\frac{\Gamma_1}{\Gamma_1+\Gamma_2}\mathbf{u}(\mathbf{x}_1,t') + \frac{\Gamma_2}{\Gamma_1+\Gamma_2}\mathbf{u}(\mathbf{x}_2,t') \right) \right\rangle$$

$$= \left(\frac{\Gamma_1}{\Gamma_1+\Gamma_2} \right)^2 \langle \mathbf{u}(\mathbf{x}_1,t)\mathbf{u}(\mathbf{x}_1,t') \rangle + \left(\frac{\Gamma_2}{\Gamma_1+\Gamma_2} \right)^2 \langle \mathbf{u}(\mathbf{x}_2,t)\mathbf{u}(\mathbf{x}_2,t') \rangle$$

$$+ \frac{\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2} \langle \mathbf{u}(\mathbf{x}_1,t)\mathbf{u}(\mathbf{x}_2,t') \rangle + \frac{\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2} \langle \mathbf{u}(\mathbf{x}_2,t)\mathbf{u}(\mathbf{x}_1,t') \rangle$$

$$= \left[\frac{\Gamma_1^2+\Gamma_2^2}{(\Gamma_1+\Gamma_2)^2}\mathbf{Q}(0) + \frac{2\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2}\mathbf{Q}(\mathbf{x}_1-\mathbf{x}_2) \right] \delta(t-t')$$

$$\stackrel{\text{def}}{=} \tilde{\mathbf{D}}(\mathbf{r})\delta(t-t') \tag{2.6}$$

and finally for the mixed terms, we have

$$\langle \mathbf{u}_m(t)\mathbf{u}_p(t') \rangle = \langle \mathbf{u}_p(t)\mathbf{u}_m(t') \rangle$$

$$= \left\langle (\mathbf{u}(\mathbf{x}_1,t) - \mathbf{u}(\mathbf{x}_2,t)) \left(\frac{\Gamma_1}{\Gamma_1+\Gamma_2}\mathbf{u}(\mathbf{x}_1,t) + \frac{\Gamma_2}{\Gamma_1+\Gamma_2}\mathbf{u}(\mathbf{x}_2,t) \right) \right\rangle$$

$$= \frac{\Gamma_1-\Gamma_2}{\Gamma_1+\Gamma_2} [\mathbf{Q}(0) - \mathbf{Q}(\mathbf{r})]\, \delta(t-t')$$

$$\stackrel{\text{def}}{=} \tilde{\mathbf{G}}(\mathbf{r})\delta(t-t') \tag{2.7}$$

So for the complete correlation matrix (compare (2.3)), we get

$$\underline{\tilde{\mathbf{Q}}} = \begin{pmatrix} \tilde{\mathbf{C}}(\mathbf{r}) & \tilde{\mathbf{G}}(\mathbf{r}) \\ \tilde{\mathbf{G}}(\mathbf{r}) & \tilde{\mathbf{D}}(\mathbf{r}) \end{pmatrix} \tag{2.8}$$

Inserting this in our Fokker-Planck equation, we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial t} f(\mathbf{r},\mathbf{R},t) + \nabla\cdot\begin{pmatrix}(\Gamma_1+\Gamma_2)\mathbf{K}(\mathbf{r})\\ 0\end{pmatrix} f(\mathbf{r},\mathbf{R},t)\\
=&\frac{1}{2}\sum_{i;j\in\{m,p\}}\frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\tilde{\mathbf{Q}}_{ij} f(\mathbf{r},\mathbf{R},t)\\
=&\frac{1}{2}\int_0^\infty dt'\delta(t-t')\sum_{i;j\in\{m,p\}}\frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\tilde{\mathbf{Q}}_{ij} f(\mathbf{r},\mathbf{R},t)\\
=&\frac{1}{2}\int_0^\infty dt'\sum_{i,j\in\{m;p\}}\frac{\partial^2}{\partial\mathbf{x}_i\partial\mathbf{x}_j}\left\langle\mathbf{u}_i(t)\mathbf{u}_j(t')\right\rangle f(\mathbf{r},\mathbf{R},t')\\
=&\frac{1}{2}\int_0^\infty dt'\,[\,\frac{\partial^2}{\partial\mathbf{x}_m^2}\left\langle\mathbf{u}_m(t)\mathbf{u}_m(t')\right\rangle+\frac{\partial^2}{\partial\mathbf{x}_p^2}\left\langle\mathbf{u}_p(t)\mathbf{u}_p(t')\right\rangle\\
&+2\frac{\partial^2}{\partial\mathbf{x}_p\partial\mathbf{x}_m}\left\langle\mathbf{u}_p(t)\mathbf{u}_m(t')\right\rangle\,]\, f(\mathbf{r},\mathbf{R},t')\\
\overset{(2.5),(2.6),(2.7)}{=}&\frac{1}{2}\int_0^\infty dt'\delta(t-t')\left[\frac{\partial^2}{\partial\mathbf{r}^2}\tilde{\mathbf{C}}(\mathbf{r})+\frac{\partial^2}{\partial\mathbf{R}^2}\tilde{\mathbf{D}}(\mathbf{r})+2\frac{\partial^2}{\partial\mathbf{R}\partial\mathbf{r}}\tilde{\mathbf{G}}(\mathbf{r})\right] f(\mathbf{r},\mathbf{R},t')\\
=&\int_0^\infty dt'\delta(t-t')\frac{1}{2}\left[\nabla_\mathbf{r}\cdot\tilde{\mathbf{C}}(\mathbf{r})\cdot\nabla_\mathbf{r}+\nabla_\mathbf{R}\cdot\tilde{\mathbf{D}}(\mathbf{r})\cdot\nabla_\mathbf{R}+\nabla_\mathbf{R}\cdot\tilde{\mathbf{G}}(\mathbf{r})\cdot\nabla_\mathbf{r}\right] f(\mathbf{r},\mathbf{R},t')
\end{aligned}
$$

and this finally yields[3]

$$
\begin{aligned}
&\frac{\partial}{\partial t} f(\mathbf{r},\mathbf{R},t) + \nabla\cdot\begin{pmatrix}(\Gamma_1+\Gamma_2)\mathbf{u}(\mathbf{r})\\ 0\end{pmatrix} f(\mathbf{r},\mathbf{R},t)\\
&=\frac{1}{2}\left[\nabla_\mathbf{r}\cdot\tilde{\mathbf{C}}(\mathbf{r})\cdot\nabla_\mathbf{r}+\nabla_\mathbf{R}\cdot\tilde{\mathbf{D}}(\mathbf{r})\cdot\nabla_\mathbf{R}+\nabla_\mathbf{R}\cdot\tilde{\mathbf{G}}(\mathbf{r})\cdot\nabla_\mathbf{r}\right] f(\mathbf{r},\mathbf{R},t) \qquad (2.9)
\end{aligned}
$$

[3]The last step of the calculation has to be understood as

$$\nabla_\mathbf{r}\cdot\tilde{\mathbf{C}}(\mathbf{r})\cdot\nabla_\mathbf{r}=\begin{pmatrix}\frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2}\end{pmatrix}\cdot\begin{pmatrix}\tilde{C}_{11}(\mathbf{r}) & \tilde{C}_{12}(\mathbf{r})\\ \tilde{C}_{21}(\mathbf{r}) & \tilde{C}_{22}(\mathbf{r})\end{pmatrix}\cdot\begin{pmatrix}\frac{\partial}{\partial r_1}\\ \frac{\partial}{\partial r_2}\end{pmatrix}\overset{\star}{=}\sum_{k,l=1}^{2}\frac{\partial^2}{\partial r_k\partial r_l}\tilde{C}_{kl}(\mathbf{r});$$

$\star$ holds for incompressible flow fields, it will be shown in section (2.3.1), that $\nabla_\mathbf{r}\cdot\mathbf{Q}=0$ due to incompressibility, and thus

$$\nabla_\mathbf{r}\cdot\tilde{\mathbf{C}}=\nabla_\mathbf{r}\cdot\tilde{\mathbf{D}}=\nabla_\mathbf{r}\cdot\tilde{\mathbf{G}}=0.$$

Decoupling of the central and relative motion

We will now use the divergence theorem (compare appendix, paragraph [1]) to decouple the stochastical motion into a central and relative part. Fortunately, we will see that some correlations in the separated parts do not give any contribution and vanish caused by a normalization argument.

First, we focus on the relative part. If we want to derive an equation spatially only dependent on the relative distance $\mathbf{r}$ of the two vortices, we have to take a look at equation (2.9), and integrate over all possible values of $\mathbf{R}$.[4] We define

$$\int d\mathbf{R} f(\mathbf{R}, \mathbf{r}, t) \stackrel{\text{def}}{=} h(\mathbf{r}, t),$$

and obtain

$$\int d\mathbf{R} \frac{\partial}{\partial t} f(\mathbf{R}, \mathbf{r}, t) = \frac{\partial}{\partial t} \int d\mathbf{R} f(\mathbf{R}, \mathbf{r}, t) = \frac{\partial}{\partial t} h(\mathbf{r}, t)$$

$$= \int d\mathbf{R} \left[-\nabla \cdot \begin{pmatrix} (\Gamma_1 + \Gamma_2)\mathbf{K}(\mathbf{r}) \\ 0 \end{pmatrix} f(\mathbf{r}, \mathbf{R}, t) \right]$$
$$+ \int d\mathbf{R} \left[\frac{1}{2} \left[\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} + \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{D}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}} + \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} \right] f(\mathbf{r}, \mathbf{R}, t) \right]$$

$$= - \int d\mathbf{R}\ \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2)\mathbf{K}(\mathbf{r}) f(\mathbf{r}, \mathbf{R}, t) + \frac{1}{2} \int d\mathbf{R}\ \nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{R}, t)$$
$$+ \frac{1}{2} \int d\mathbf{R}\ \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{D}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}} f(\mathbf{r}, \mathbf{R}, t) + \frac{1}{2} \int d\mathbf{R}\ \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{R}, t)$$

$$= - \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2)\mathbf{K}(\mathbf{r}) \int d\mathbf{R} f(\mathbf{r}, \mathbf{R}, t) + \frac{1}{2} \nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} \int d\mathbf{R} f(\mathbf{r}, \mathbf{R}, t)$$
$$+ \frac{1}{2} \int d\mathbf{R}\ \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{D}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}} f(\mathbf{r}, \mathbf{R}, t) + \frac{1}{2} \int d\mathbf{R}\ \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{R}, t)$$

$$= - \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2)\mathbf{K}(\mathbf{r}) h(\mathbf{r}, t) + \frac{1}{2} \nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} h(\mathbf{r}, t)$$
$$+ \underbrace{\frac{1}{2} \int d\mathbf{R}\ \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{D}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}}\ f(\mathbf{r}, \mathbf{R}, t)}_{I} + \underbrace{\frac{1}{2} \int d\mathbf{R}\ \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{R}, t)}_{II}$$

We are now able to use the divergence theorem on I and II; exemplary for term II, the $\tilde{\mathbf{G}}(\mathbf{r})$ -term, this yields

[4] more precisely: by $\int d\mathbf{R} f(\mathbf{R}, \mathbf{r}, t)$, we of course mean $\int_0^\infty dR_1 \int_0^\infty dR_1 f(R_1, R_2, r_1, r_2, t)$

$$\int\limits_{V(\mathbf{R})} \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{R}, t) dV = \oint\limits_{\partial V(\mathbf{R})} \left(\tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{R}, t) \right) \cdot d\mathbf{S} \stackrel{!}{=} 0$$

In the limit $\mathbf{R} \to \infty$, the integration on the boundary of $V, \partial V$, is the integration over the boundary of a circle with infinite radius. But since f is a probability density, it has to vanish for $\mathbf{R} \to \infty$, and so does II. Hence for the relative part, we finally have

$$\boxed{\frac{\partial}{\partial t} h(\mathbf{r}, t) + \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2) \mathbf{K}(\mathbf{r}) h(\mathbf{r}, t) = \frac{1}{2} \left[\nabla_{\mathbf{r}} \cdot \mathbf{C}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} \right] h(\mathbf{r}, t)} \tag{2.10}$$

This will be the equation we will later concentrate on. The study of the relative position of the two vortices gives most benefit in understanding their mutual interaction.

To get an equation solely for the motion of the center of vorticity, we integrate over $\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$,

$$\int d\mathbf{r} f(\mathbf{R}, \mathbf{r}, t) \stackrel{\text{def}}{=} g(\mathbf{R}, t),$$

and obtain

$$\int d\mathbf{r} \frac{\partial}{\partial t} f(\mathbf{R}, \mathbf{r}, t) = \frac{\partial}{\partial t} \int d\mathbf{r} f(\mathbf{R}, \mathbf{r}, t) = \frac{\partial}{\partial t} g(\mathbf{R}, t)$$

$$= \int d\mathbf{r} \left[-\nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2) \mathbf{K}(\mathbf{r}) f(\mathbf{r}, \mathbf{R}, t) \right]$$
$$+ \int d\mathbf{r} \left[\frac{1}{2} \left[\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} + \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{D}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}} + \nabla_{\mathbf{r}} \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}} \right] \; f(\mathbf{r}, \mathbf{R}, t) \right]$$

$$= - \underbrace{\int d\mathbf{r} \; \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2) \mathbf{K}(\mathbf{r}) f(\mathbf{r}, \mathbf{R}, t)}_{I} + \frac{1}{2} \underbrace{\int d\mathbf{r} \nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \mathbf{R}, t)}_{II}$$
$$+ \frac{1}{2} \int d\mathbf{r} \; \nabla_{\mathbf{R}} \cdot \tilde{\mathbf{D}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}} \; f(\mathbf{r}, \mathbf{R}, t) + \underbrace{\frac{1}{2} \int d\mathbf{r} \; \nabla_{\mathbf{r}} \cdot \tilde{\mathbf{G}}(\mathbf{r}) \cdot \nabla_{\mathbf{R}} f(\mathbf{r}, \mathbf{R}, t)}_{III}$$

I, II and III vanish due to the same argumentation accomplished above, say application of the divergence theorem and vanishing probability density on the boundary of an infinite circle. We finally get

$$\boxed{\frac{\partial}{\partial t} g(\mathbf{R}, t) = \frac{1}{2} \int d\mathbf{r}\ \nabla_{\mathbf{R}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\mathbf{R}}\ f(\mathbf{r}, \mathbf{R}, t)} \tag{2.11}$$

For the case that $\mathbf{D}(\mathbf{r}) = \mathbf{D}$ is constant, we get

$$\frac{\partial}{\partial t} g(\mathbf{R}, t) = \frac{1}{2} \int d\mathbf{r}\ \nabla_{\mathbf{R}} \cdot \mathbf{D} \cdot \nabla_{\mathbf{R}}\ f(\mathbf{r}, \mathbf{R}, t) = \frac{1}{2} \nabla_{\mathbf{R}} \cdot \mathbf{D} \cdot \nabla_{\mathbf{R}} \int d\mathbf{r}\ f(\mathbf{r}, \mathbf{R}, t)$$

$$\Rightarrow$$

$$\boxed{\frac{\partial}{\partial t} g(\mathbf{R}, t) = \frac{1}{2} \nabla_{\mathbf{R}} \cdot \mathbf{D} \cdot \nabla_{\mathbf{R}}\ g(\mathbf{R}, t)} \tag{2.12}$$

thus a pure diffusive equation. The solution looks exactly like in the one vortex case in section (2.1).

Numerical simulation of two point vortices with different circulation affected by white noise

In this section we show some stochastic affected trajectories of the whole two point vortex system for vortices with different circulation. Their qualitative behavior is presented in the case of a constant correlation matrix $\underline{\mathbf{Q}}(0) = Q \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for different values of Q. The numerical method is a modification of Runge-Kutta forth order (compare appendix), in every calculation step we add a Gaussian distributed white noise term multiplied with $\sqrt{Q\mathrm{dt}}$ whereby dt is the time-step. The two trajectories start at the gridpoints (0,0) and (1,1).

The special case $\Gamma_1 = \Gamma_2 \equiv \Gamma$: Two identical vortices

We now shortly consider the case where $\Gamma_1 = \Gamma_2 \equiv \Gamma$, because it has already been investigated. If we put

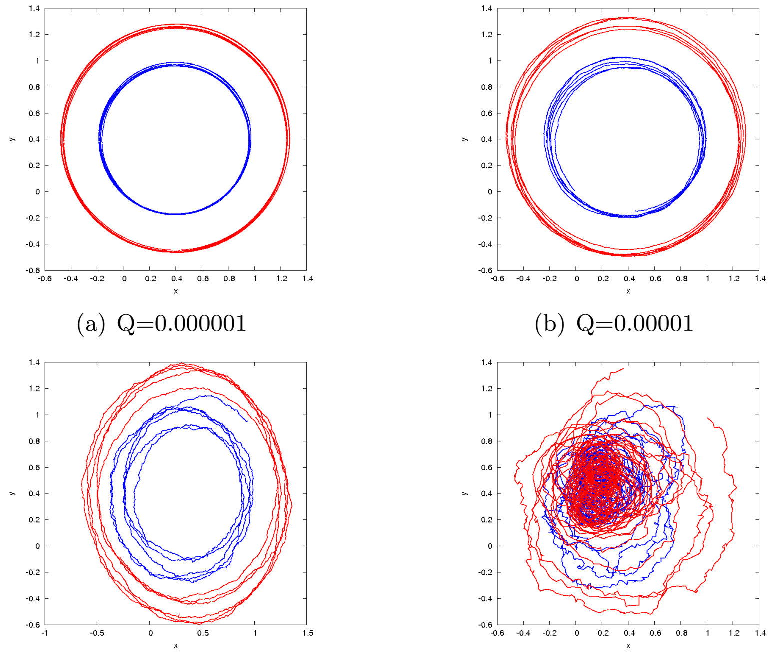
$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2 \text{ and } \mathbf{R} = \frac{\Gamma \mathbf{x}_1 + \Gamma \mathbf{x}_2}{\Gamma + \Gamma} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$$

we obtain

$$\boxed{\begin{array}{lll} \dot{\mathbf{r}} = & 2\Gamma \mathbf{K}(\mathbf{r}) + & \mathbf{u}_m(t) \\ \dot{\mathbf{R}} = & & \mathbf{u}_p(t) \end{array}} \tag{2.13}$$

whereby

$$\mathbf{u}_m(t) = \mathbf{u}(\mathbf{x}_1, t) - \mathbf{u}(\mathbf{x}_2, t) \text{ and } \mathbf{u}_p(t) = \frac{1}{2}(\mathbf{u}(\mathbf{x}_1, t) + \mathbf{u}(\mathbf{x}_2, t)).$$



(a) Q=0.000001 (b) Q=0.00001

(c) Q=0.0001 (d) Q=0.001

Figure 2.4: Stochastically affected trajectories of the whole two point vortex system for vortices with different circulation: Qualitative behavior for different values of Q

This has the consequence, that the $\tilde{\mathbf{G}}(\mathbf{r})$-term does not appear at all. This becomes clear with

$$\left\langle \mathbf{u}_m(t)\mathbf{u}_p(t')\right\rangle = \left\langle \mathbf{u}_p(t)\mathbf{u}_m(t')\right\rangle$$

$$= \left\langle (\mathbf{u}(\mathbf{x}_1,t) - \mathbf{u}(\mathbf{x}_2,t))\frac{1}{2}(\mathbf{u}(\mathbf{x}_1,t') + \mathbf{u}(\mathbf{x}_2,t'))\right\rangle = \frac{1}{2}(\langle \mathbf{u}(\mathbf{x}_1,t)\mathbf{u}(\mathbf{x}_1,t')\rangle - \langle \mathbf{u}(\mathbf{x}_2,t)\mathbf{u}(\mathbf{x}_2,t')\rangle)$$

$$= \frac{1}{2}[\mathbf{Q}(0) - \mathbf{Q}(0)]\delta(t-t') = 0 \qquad (2.14)$$

or when inserting $\Gamma_1 = \Gamma_2 \equiv \Gamma$ in (2.7), we get $\tilde{\mathbf{G}}(\mathbf{r}) = 0$, respectively.

Hence for the complete correlation matrix (compare (2.3)), we get

$$\underline{\tilde{\mathbf{Q}}} = \begin{pmatrix} \tilde{\mathbf{C}}(\mathbf{r}) & 0 \\ 0 & \tilde{\mathbf{D}}(\mathbf{r}) \end{pmatrix} \qquad (2.15)$$

When setting $\tilde{\mathbf{G}}(\mathbf{r}) = 0$ in (2.9), we get the same two equations describing the dynamic interaction between the vortices in a probabilistic manner:

$$\frac{\partial}{\partial t}h(\mathbf{r},t) + \nabla_{\mathbf{r}} \cdot 2\Gamma\mathbf{K}(\mathbf{r})h(\mathbf{r},t) = \frac{1}{2}\left[\nabla_{\mathbf{r}} \cdot \mathbf{C}(\mathbf{r}) \cdot \nabla_{\mathbf{r}}\right] h(\mathbf{r},t)$$

and

$$\frac{\partial}{\partial t}g(\mathbf{R},t) = \frac{1}{2}\int d\mathbf{r}\ \nabla_{\mathbf{R}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\mathbf{R}}\ f(\mathbf{r},\mathbf{R},t)$$

Interestingly, the structure of $\tilde{\mathbf{G}}(\mathbf{r})$ has no effect on the stochastic motion of the two vortices when we analyze the decoupled equations apart of each other. But it obviously has an effect, if we take a look at the equation for the joint-probability (2.9). The reason underlying is the stochastic independency of the decoupled equations in the case $\Gamma_1 = \Gamma_2$, as shown above, the correlation of the two terms vanishes and thus they are stochastically independent. Hence, $f(\mathbf{r},\mathbf{R},t)$ turns out to be the product of the probability density functions of the center of vorticity and the relative vortex distance

$$f(\mathbf{r},\mathbf{R},t) = h(\mathbf{r},t)\ g(\mathbf{R},t)$$

So for $\Gamma_1 \neq \Gamma_2$, we have thus a slight generalization of the case $\Gamma_1 = \Gamma_2 \equiv \Gamma$ treated in [2] and [3], emphasizing that we must not use the simple product $f(\mathbf{r},\mathbf{R},t) = h(\mathbf{r},t)\ g(\mathbf{R},t)$ if we treat the general case $\Gamma_1 \neq \Gamma_2$.

Written in terms of conditional probability we exactly have

$$f(\mathbf{r},\mathbf{R},t) = h(\mathbf{r},t)\ g(\mathbf{R}|\mathbf{r},t)$$

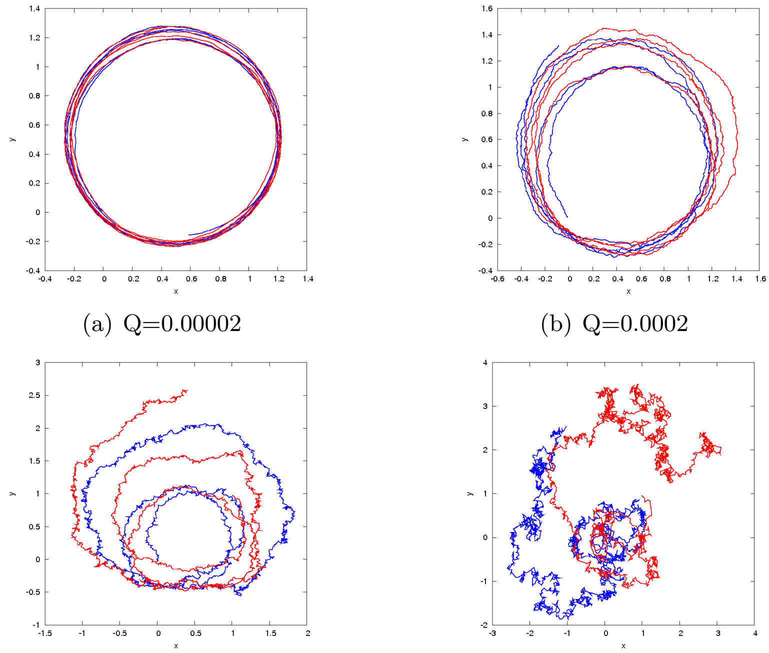
or

$$f(\mathbf{r},\mathbf{R},t) = h(\mathbf{r}|\mathbf{R},t)\ g(\mathbf{R},t),$$

respectively. In the sense of a Born-Oppenheim approximation, we will treat the motions as stochastically independent, noting that this approximation becomes more and more exact, if $\Gamma_1 \rightarrow \Gamma_2$.

Numerical simulation of two point vortices with the same circulation affected by white noise

This section deals with the stochastic affected trajectories of the whole two point vortex system for vortices with the same circulation. Their qualitative behavior is presented again in the case of a constant correlation matrix $\underline{\mathbf{Q}}(0) = Q\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ for different values of Q, the numerical method is again the modification of Runge-Kutta forth order used above and the two trajectories start again at the gridpoints (0,0) and (1,1).



(a) Q=0.00002 (b) Q=0.0002

(c) Q=0.002 (d) Q=0.04

Figure 2.5: Stochastically affected trajectories of the whole two point vortex system for vortices with the same circulation: Qualitative behavior for different values of Q

2.2.2 The case $\Gamma_1 = -\Gamma_2 \equiv \Gamma$: Two according to amount equipollent, counter-rotating vortices

In this case, the Langevin equations read

$$\dot{\mathbf{x}}_1 = -\Gamma \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{u}(\mathbf{x}_1, t) \tag{2.16}$$

$$\dot{\mathbf{x}}_2 = -\Gamma \mathbf{K}(\mathbf{x}_1 - \mathbf{x}_2) + \mathbf{u}(\mathbf{x}_2, t)$$

We have to use other coordinates this time[5]:

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2 \text{ and } \tilde{\mathbf{R}} = \frac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2)$$

We are used to, e. g., the center-of-mass or the center-of-vorticity transformation. But this transformation is also possible and leads to reasonable results. Now the transformation of our system of linear differential equations entails

[5]If we tried to apply the equations derived in the general case $|\Gamma_1| \neq |\Gamma_2|$ with center-of-vorticity coordinates, singularities would emerge in some of the correlation matrices we calculated in the previous section.

$$\boxed{\begin{array}{lll} \dot{\mathbf{r}} = & & \mathbf{u}_m(t) \\ \dot{\tilde{\mathbf{R}}} = & -\Gamma\mathbf{K}(\mathbf{r}) + & \mathbf{u}_p(t) \end{array}} \tag{2.17}$$

whereas again

$$\mathbf{u}_m(t) = \mathbf{u}(\mathbf{x}_1, t) - \mathbf{u}(\mathbf{x}_2, t) \text{ but this time } \mathbf{u}_p(t) = \frac{1}{2}(\mathbf{u}(\mathbf{x}_1, t) + \mathbf{u}(\mathbf{x}_2, t)).$$

The Fokker-Planck equation now reads

$$\frac{\partial}{\partial t} f(\mathbf{r}, \tilde{\mathbf{R}}, t) = -\nabla \cdot \begin{pmatrix} 0 \\ -\Gamma\mathbf{K}(\mathbf{r}) \end{pmatrix} f(\mathbf{r}, \tilde{\mathbf{R}}, t) + \frac{1}{2} \sum_{i;j \in \{m,p\}} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \tilde{\mathbf{Q}}_{ij} f(\mathbf{r}, \tilde{\mathbf{R}}, t)$$

with

$$\nabla = \begin{pmatrix} \nabla_{\mathbf{r}} \\ \nabla_{\tilde{\mathbf{R}}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \mathbf{r}} \\ \frac{\partial}{\partial \tilde{\mathbf{R}}} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial \mathbf{x}_m} \\ \frac{\partial}{\partial \mathbf{x}_p} \end{pmatrix}$$

we have

$$\boxed{\frac{\partial}{\partial t} f(\mathbf{r}, \tilde{\mathbf{R}}, t) = +\nabla_{\tilde{\mathbf{R}}} \cdot \Gamma\mathbf{K}(\mathbf{r}) f(\mathbf{r}, \tilde{\mathbf{R}}, t) + \frac{1}{2} \sum_{i;j \in \{m,p\}} \frac{\partial^2}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \tilde{\mathbf{Q}}_{ij} f(\mathbf{r}, \tilde{\mathbf{R}}, t)}$$

Now we again have to investigate the question what $\tilde{\mathbf{Q}}_{ij}$ does look like. $\langle \mathbf{u}_m(t)\mathbf{u}_m(t')\rangle$ has already been calculated (compare the calculation to (2.6)), it is

$$\langle \mathbf{u}_m(t)\mathbf{u}_m(t')\rangle = \tilde{\mathbf{C}}(\mathbf{r})\delta(t - t')$$

Analogous, we find

$$\begin{aligned}
&\langle \mathbf{u}_p(t)\mathbf{u}_p(t')\rangle \\
=&\left\langle \frac{1}{2}(\mathbf{u}(\mathbf{x}_1, t) + \mathbf{u}(\mathbf{x}_2, t))\frac{1}{2}(\mathbf{u}(\mathbf{x}_1, t') + \mathbf{u}(\mathbf{x}_2, t'))\right\rangle \\
=&\frac{1}{4}(\langle \mathbf{u}(\mathbf{x}_1, t)\mathbf{u}(\mathbf{x}_1, t')\rangle + \langle \mathbf{u}(\mathbf{x}_2, t)\mathbf{u}(\mathbf{x}_2, t')\rangle + \langle \mathbf{u}(\mathbf{x}_1, t)\mathbf{u}(\mathbf{x}_2, t')\rangle + \langle \mathbf{u}(\mathbf{x}_2, t)\mathbf{u}(\mathbf{x}_1, t')\rangle) \\
=&\frac{1}{2}[\mathbf{Q}(0) + \mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_2)]\delta(t - t') \\
\equiv&\mathbf{D}(\mathbf{r})\delta(t - t')
\end{aligned} \tag{2.18}$$

and

$$\langle \mathbf{u}_m(t)\mathbf{u}_p(t')\rangle = \langle \mathbf{u}_p(t)\mathbf{u}_m(t')\rangle$$

$$= \left\langle (\mathbf{u}(\mathbf{x}_1,t) - \mathbf{u}(\mathbf{x}_2,t))\frac{1}{2}(\mathbf{u}(\mathbf{x}_1,t') + \mathbf{u}(\mathbf{x}_2,t'))\right\rangle = \frac{1}{2}(\langle \mathbf{u}(\mathbf{x}_1,t)\mathbf{u}(\mathbf{x}_1,t')\rangle - \langle \mathbf{u}(\mathbf{x}_2,t)\mathbf{u}(\mathbf{x}_2,t')\rangle)$$

$$= \frac{1}{2}[\mathbf{Q}(0) - \mathbf{Q}(0)]\delta(t-t') = 0 \tag{2.19}$$

Finally we obtain

$$\boxed{\frac{\partial}{\partial t} f(\mathbf{r},\tilde{\mathbf{R}},t) = \nabla_{\mathbf{R}} \cdot \Gamma \mathbf{K}(\mathbf{r}) f(\mathbf{r},\tilde{\mathbf{R}},t) + \frac{1}{2}\left[\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} + \nabla_{\tilde{\mathbf{R}}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\tilde{\mathbf{R}}}\right] f(\mathbf{r},\tilde{\mathbf{R}},t')}$$

Note that again the stochastic terms are independent and so $f(\mathbf{r},\tilde{\mathbf{R}},t)$ turns out to be the product of the probability density functions of the center of vorticity and the relative vortex distance in an exact manner:

$$f(\mathbf{r},\tilde{\mathbf{R}},t) \overset{\text{exact}}{=} h(\mathbf{r},t)\ g(\tilde{\mathbf{R}},t)$$

Decoupling of the central and relative motion

If we again perform an integration over $\tilde{\mathbf{R}}$ to decouple the equations, we obtain

$$\int d\tilde{\mathbf{R}} \frac{\partial}{\partial t} f(\tilde{\mathbf{R}}, \mathbf{r}, t) = \frac{\partial}{\partial t} \int d\tilde{\mathbf{R}} f(\tilde{\mathbf{R}}, \mathbf{r}, t) = \frac{\partial}{\partial t} h(\mathbf{r}, t)$$

$$= \int d\tilde{\mathbf{R}} \left[\nabla \cdot \begin{pmatrix} 0 \\ \Gamma \mathbf{K}(\mathbf{r}) \end{pmatrix} f(\mathbf{r}, \tilde{\mathbf{R}}, t) + \frac{1}{2} \left[\nabla_{\mathbf{r}} \cdot \mathbf{C}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} + \nabla_{\tilde{\mathbf{R}}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\tilde{\mathbf{R}}} \right] f(\mathbf{r}, \tilde{\mathbf{R}}, t) \right]$$

$$= \int d\tilde{\mathbf{R}} \ \left(\nabla_{\tilde{\mathbf{R}}} \cdot \Gamma \mathbf{K}(\mathbf{r}) \right) f(\mathbf{r}, \tilde{\mathbf{R}}, t)$$
$$+ \frac{1}{2} \int d\tilde{\mathbf{R}} \ \nabla_{\mathbf{r}} \cdot \mathbf{C}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} f(\mathbf{r}, \tilde{\mathbf{R}}, t) + \frac{1}{2} \int d\tilde{\mathbf{R}} \ \nabla_{\tilde{\mathbf{R}}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\tilde{\mathbf{R}}} f(\mathbf{r}, \tilde{\mathbf{R}}, t)$$

$$= \int d\tilde{\mathbf{R}} \ \nabla_{\tilde{\mathbf{R}}} \cdot \Gamma \mathbf{K}(\mathbf{r})$$
$$+ \frac{1}{2} \nabla_{\mathbf{r}} \cdot \mathbf{C}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} \int d\tilde{\mathbf{R}} f(\mathbf{r}, \tilde{\mathbf{R}}, t) + \frac{1}{2} \int d\tilde{\mathbf{R}} \ \nabla_{\tilde{\mathbf{R}}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\tilde{\mathbf{R}}} f(\mathbf{r}, \tilde{\mathbf{R}}, t)$$

$$= \underbrace{\int d\tilde{\mathbf{R}} \ \nabla_{\tilde{\mathbf{R}}} \cdot \Gamma \mathbf{K}(\mathbf{r})}_{I}$$
$$+ \frac{1}{2} \nabla_{\mathbf{r}} \cdot \mathbf{C}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} \ h(\mathbf{r}, t) + \underbrace{\frac{1}{2} \int d\tilde{\mathbf{R}} \ \nabla_{\tilde{\mathbf{R}}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\tilde{\mathbf{R}}} \ f(\mathbf{r}, \tilde{\mathbf{R}}, t)}_{II}$$

We again are able to use the divergence theorem on I and II to see that these terms vanish in the limit $\tilde{\mathbf{R}} \to \infty$ because of normalization. So this time, we finally get a pure diffusive equation for the relative part:

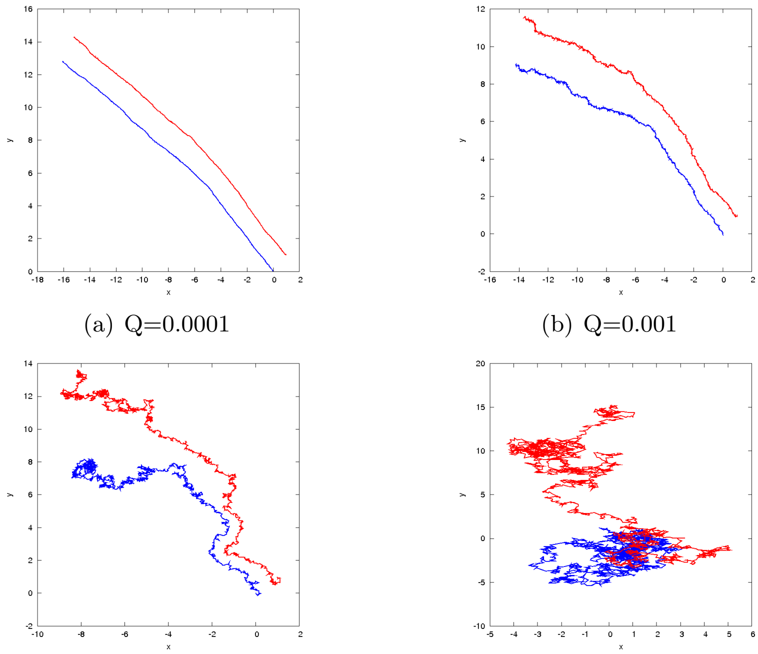
$$\frac{\partial}{\partial t} h(\mathbf{r}, t) = \frac{1}{2} \left[\nabla_{\mathbf{r}} \cdot \mathbf{C}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} \right] h(\mathbf{r}, t)$$

but for the center-of-mass part, we get a drift-diffusion equation:

$$\frac{\partial}{\partial t} g(\tilde{\mathbf{R}}, t) = \nabla_{\tilde{\mathbf{R}}} \cdot \Gamma \mathbf{K}(\mathbf{r}) g(\tilde{\mathbf{R}}, t) + \frac{1}{2} \left[\nabla_{\tilde{\mathbf{R}}} \cdot \mathbf{D}(\mathbf{r}) \cdot \nabla_{\tilde{\mathbf{R}}} \right] g(\tilde{\mathbf{R}}, t)$$

Numerical simulation of two point vortices with counter-rotating circulation affected by white noise

Here we present the qualitative behavior of two point vortices with counter-rotating circulation affected by white noise. Again we only investigate the case of a constant correlation matrix $\underline{\mathbf{Q}}(0) = Q \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for different values of Q with the same numerical method as used before; the two trajectories also starting again at the gridpoints (0,0) and (1,1).



(a) Q=0.0001 (b) Q=0.001

(c) Q=0.01 (d) Q=0.1

Figure 2.6: Stochastic affected trajectories of the whole two point vortex system for vortices with antipodal circulation: Qualitative behavior for different values of Q

2.3 Derivation and implementation of additional constraints

The derived additional constraints in this section - namely some consequences of the postulation of incompressibility and isotropic statistics - have general application. Hence they could have already been established in the basic theoretical foundations. Nevertheless, it may seem more reasonable to adopt them but now here to emphasize their practical application and to show their direct effect on the treated problem.

2.3.1 Consequences of the postulation of incompressibility

If we assume an imcompressible flow, we have

$$\nabla_{\mathbf{x}_i} \cdot \mathbf{u}(\mathbf{x}_i, t) = \sum_\alpha \frac{\partial}{\partial x_i^\alpha} u^{(\alpha)}(\mathbf{x}_i, t) = 0 \tag{2.20}$$

Now we consider the expression

$$\langle (\nabla_{\mathbf{x}_1} \cdot \mathbf{u}(\mathbf{x}_1, t))\mathbf{u}(\mathbf{x}_2, t)\rangle$$

which is

$$= \langle (\nabla_{\mathbf{x}_1} \cdot \mathbf{u}(\mathbf{x}_1, t))\mathbf{u}(\mathbf{x}_2, t)\rangle = \langle 0 \; \mathbf{u}(\mathbf{x}_2, t)\rangle = 0$$

due to (2.20), but also

$$= \langle (\nabla_{\mathbf{x}_1} \cdot \mathbf{u}(\mathbf{x}_1, t))\mathbf{u}(\mathbf{x}_2, t)\rangle = \left\langle (\sum_\alpha \frac{\partial}{\partial x_1^{(\alpha)}} u^{(\alpha)}(\mathbf{x}_1, t))\mathbf{u}(\mathbf{x}_2, t) \right\rangle$$

$$= \sum_\alpha \left\langle \frac{\partial}{\partial x_1^{(\alpha)}} u^{(\alpha)}(\mathbf{x}_1, t)\mathbf{u}(\mathbf{x}_2, t) \right\rangle = \sum_\alpha \frac{\partial}{\partial x_1^{(\alpha)}} \left\langle u^{(\alpha)}(\mathbf{x}_1, t)\mathbf{u}(\mathbf{x}_2, t) \right\rangle$$

$$= \nabla_{\mathbf{x}_1} \cdot \langle (\mathbf{u}(\mathbf{x}_1, t))\mathbf{u}(\mathbf{x}_2, t)\rangle = \nabla_{\mathbf{x}_1} \cdot \mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_2)\delta(t - t') = 0$$

Hence we get

$$\nabla_{\mathbf{x}_1} \cdot \mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_2) = 0 \vee \delta(t - t') = 0$$

And finally because of

$$\nabla_{\mathbf{x}_1} \cdot (\mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_2)) = \nabla_{\mathbf{r}} \cdot \mathbf{Q}(\mathbf{x}_1 - \mathbf{x}_2) = \nabla_{\mathbf{r}} \cdot \mathbf{Q}(\mathbf{r})$$

we have

$$\boxed{\nabla_{\mathbf{r}} \cdot \mathbf{Q}(\mathbf{r}) = 0.} \tag{2.21}$$

With (2.21), we of course also have the identity

$$\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) = \nabla_{\mathbf{r}} \cdot 2[\mathbf{Q}(0) - \mathbf{Q}(\mathbf{r})] = 0 \tag{2.22}$$

which will be used in the following calculations.

2.3.2 Consequences of the postulation of isotropic statistics

For isotropic turbulence, all statistical properties of the flow - especially the average velocity fluctuations - are equal all over the flow field and not depending on the direction. Hence, we have invariance of translation and rotation. The translational invariance by the way became already visible within the fact, that the correlation matrix does not depend on R. Claiming isotropic statistics, we may in mathematical terms assume that the correlation matrix[6] for the relative part $\tilde{\mathbf{C}}(\mathbf{r})$ is only dependent on r (cf. [15]) and that it is symmetric:

$$\tilde{C}_{\alpha\beta}(\mathbf{r}) = \theta(r)\delta_{\alpha\beta} + \psi(r)\frac{r_\alpha}{r}\frac{r_\beta}{r}$$

with $\theta(r)$, $\psi(r)$ only dependent on $|\mathbf{r}|$

With (2.22), we will now show that

$$\boxed{\frac{\partial}{\partial r}\theta(r) + \frac{1}{r}\frac{\partial}{\partial r}r\psi(r) = 0.} \tag{2.23}$$

This is an important identity connecting incompressibility of the flow and the isotropy of the statistics and has particular application in the following. To calculate it, we make use of the identity[7]

$$\frac{\partial}{\partial r_i} = \frac{r_i}{r}\frac{\partial}{\partial r} \tag{2.24}$$

With $\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) = 0$ on the one hand, we get

[6] The complete correlation matrix for the whole system was given by

$$\underline{\tilde{\mathbf{Q}}} = \begin{pmatrix} \tilde{\mathbf{C}}(\mathbf{r}) & \tilde{\mathbf{G}}(\mathbf{r}) \\ \tilde{\mathbf{G}}(\mathbf{r}) & \tilde{\mathbf{D}}(\mathbf{r}) \end{pmatrix}$$

[7] In the most general case with $r = |\mathbf{r}| = \sqrt{r_1^2 + \cdots + r_i^2 + \cdots + r_n^2}$ for $\mathbf{r} = (r_1, \ldots, r_i \ldots r_n)$, it is

$$\frac{\partial}{\partial r_i}f(r) = \frac{\partial}{\partial r_i}f(\sqrt{r_1^2 + \cdots + r_i^2 + \cdots + r_n^2}) = 2r_i \cdot \frac{1}{2}(r_1^2 + \cdots + r_i^2 + \cdots + r_n^2)^{-\frac{1}{2}}\frac{\partial}{\partial r}f(r) = \frac{r_1}{r}\frac{\partial}{\partial r}f(r)$$

or

$$\frac{\partial}{\partial r_i} = \frac{r_i}{r}\frac{\partial}{\partial r},$$

respectively.

$$0 = \begin{pmatrix} 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} \end{pmatrix} \cdot \begin{pmatrix} \theta(r) + \psi(r)\frac{r_1^2}{r^2} & \psi(r)\frac{r_1 r_2}{r^2} \\ \psi(r)\frac{r_1 r_2}{r^2} & \theta(r) + \psi(r)\frac{r_2^2}{r^2} \end{pmatrix}$$

$$\overset{i,j\in\{1;2\},i\neq j}{\Longleftrightarrow} 0 = \frac{\partial}{\partial r_i}\theta(r) + \frac{\partial}{\partial r_i}\psi(r)\frac{r_i^2}{r^2} + \frac{\partial}{\partial r_j}\psi(r)\frac{r_i r_j}{r^2} \tag{2.25}$$

$$= \frac{\partial}{\partial r_i}\theta(r) + \psi(r)\frac{\partial}{\partial r_i}\frac{r_i^2}{r^2} + \frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}\psi(r) + \psi(r)\frac{\partial}{\partial r_j}\frac{r_i r_j}{r^2} + \frac{r_i r_j}{r^2}\frac{\partial}{\partial r_j}\psi(r)$$

$$= \frac{\partial}{\partial r_i}\theta(r) + \psi(r)(\frac{2r_i}{r^2} - \frac{2r_i^3}{r^4}) + \frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}\psi(r) + \psi(r)(\frac{r_i}{r^2} - \frac{2r_i r_j^2}{r^4}) + \frac{r_i r_j}{r^2}\frac{\partial}{\partial r_j}\psi(r)$$

$$= \frac{\partial}{\partial r_i}\theta(r) + \psi(r)\frac{3r_i}{r^2} - \psi(r)\frac{2r_i^3}{r^4} - \psi(r)\frac{2r_i r_j^2}{r^4} + \frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}\psi(r) + \frac{r_i r_j}{r^2}\frac{\partial}{\partial r_j}\psi(r)$$

$$= \frac{\partial}{\partial r_i}\theta(r) + \psi(r)\frac{3r_i}{r^2} - \psi(r)\frac{2r_i}{r^4}(r_i^2 + r_j^2) + \frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}\psi(r) + \frac{r_i r_j}{r^2}\frac{\partial}{\partial r_j}\psi(r)$$

$$= \frac{\partial}{\partial r_i}\theta(r) + \frac{1}{r^2}\left[3r_i\psi(r) - 2r_i\psi(r) + r_i^2\frac{\partial}{\partial r_i}\psi(r) + r_i r_j\frac{\partial}{\partial r_j}\psi(r)\right]$$

$$\overset{(2.24)}{=} \frac{r_i}{r}\frac{\partial}{\partial r_r}\theta(r) + \frac{1}{r^2}\left[r_i\psi(r) + \frac{r_i^3}{r}\frac{\partial}{\partial r}\psi(r) + r_i r_j\frac{r_j}{r}\frac{\partial}{\partial r}\psi(r)\right] = 0$$

Since both components vanish identically, we multiply the i^{th} component by $\frac{r}{r_i}$ and obtain

$$\Leftrightarrow 0 = \frac{\partial}{\partial r}\theta(r) + \frac{1}{r}\left[\psi(r) + \frac{r_i^2}{r}\frac{\partial}{\partial r}\psi(r) + \frac{r_j^2}{r}\frac{\partial}{\partial r}\psi(r)\right]$$

and because $r_i^2 + r_j^2 = r_1^2 + r_2^2 = r^2$ for $i, j \in \{1; 2\}, i \neq j$, we finally have

$$= \frac{\partial}{\partial r}\theta(r) + \frac{1}{r}\left[\psi(r) + \frac{r^2}{r}\frac{\partial}{\partial r}\psi(r)\right]$$

$$= \frac{\partial}{\partial r}\theta(r) + \frac{1}{r}\left[\psi(r) + r\frac{\partial}{\partial r}\psi(r)\right]$$

$$= \frac{\partial}{\partial r}\theta(r) + \frac{1}{r}\frac{\partial}{\partial r}r\psi(r) = 0, \ \square$$

2.4 The relative part of the motion

From now on there will be only a consideration of the Fokker-Planck equation for the relative part in the cases where $\Gamma_1 \neq -\Gamma_2$ and $\Gamma_1 = \Gamma_2$ which we derived in the penultimate section:

$$\frac{\partial}{\partial t} h(\mathbf{r}, t) + \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2)\mathbf{u}(\mathbf{r}) h(\mathbf{r}, t) = \frac{1}{2}[\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}}] h(\mathbf{r}, t).$$

Again, we emphasize that to describe the whole system we need the product of the probability density of the relative part and the conditional probability density of the center part, which becomes more and more unconditional for $\Gamma_1 \to \Gamma_2$. With the result (2.23) from the previous section, we may now further reduce the diffusion term of this Fokker-Planck-equation. We consider

$$[\nabla_{\mathbf{r}} \cdot \tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}}] h(\mathbf{r}, t)$$

which entails

$$= \begin{pmatrix} \frac{\partial}{\partial r_1} & \frac{\partial}{\partial r_2} \end{pmatrix} \cdot \begin{pmatrix} \theta(r) + \psi(r)\frac{r_1^2}{r^2} & \psi(r)\frac{r_1 r_2}{r^2} \\ \psi(r)\frac{r_1 r_2}{r^2} & \theta(r) + \psi(r)\frac{r_2^2}{r^2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r_1} \\ \frac{\partial}{\partial r_2} \end{pmatrix} h(\mathbf{r}, t)$$

We first take a look at the right part of it

$$[\tilde{\mathbf{C}}(\mathbf{r}) \cdot \nabla_{\mathbf{r}}] h(\mathbf{r}, t)$$

$$= \begin{pmatrix} \theta(r) + \psi(r)\frac{r_1^2}{r^2} & \psi(r)\frac{r_1 r_2}{r^2} \\ \psi(r)\frac{r_1 r_2}{r^2} & \theta(r) + \psi(r)\frac{r_2^2}{r^2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r_1} \\ \frac{\partial}{\partial r_2} \end{pmatrix} h(\mathbf{r}, t)$$

$$\overset{i,j\in\{1;2\},i\neq j}{\Longrightarrow} h(\mathbf{r},t)\frac{\partial}{\partial r_i}\theta(r) + h(\mathbf{r},t)\frac{\partial}{\partial r_i}\psi(r)\frac{r_i^2}{r^2} + h(\mathbf{r},t)\frac{\partial}{\partial r_j}\psi(r)\frac{r_ir_j}{r^2}$$

$$+\theta(r)\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_ir_j}{r^2}\frac{\partial}{\partial r_j}h(\mathbf{r},t)$$

$$=h(\mathbf{r},t)\underbrace{\left(\frac{\partial}{\partial r_i}\theta(r) + \frac{\partial}{\partial r_i}\psi(r)\frac{r_i^2}{r^2} + \frac{\partial}{\partial r_j}\psi(r)\frac{r_ir_j}{r^2}\right)}_{\overset{(2.25)}{=}0}$$

$$+\theta(r)\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_ir_j}{r^2}\frac{\partial}{\partial r_j}h(\mathbf{r},t)$$

$$=\theta(r)\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_ir_j}{r^2}\frac{\partial}{\partial r_j}h(\mathbf{r},t)$$

$$\Rightarrow$$

$$[\tilde{\mathbf{C}}(\mathbf{r})\cdot\nabla_{\mathbf{r}}]h(\mathbf{r},t) = \begin{pmatrix}\theta(r)\frac{\partial}{\partial r_1}h(\mathbf{r},t) + \psi(r)\frac{r_1^2}{r^2}\frac{\partial}{\partial r_1}h(\mathbf{r},t) + \psi(r)\frac{r_1r_2}{r^2}\frac{\partial}{\partial r_2}h(\mathbf{r},t)\\ \theta(r)\frac{\partial}{\partial r_2}h(\mathbf{r},t) + \psi(r)\frac{r_2^2}{r^2}\frac{\partial}{\partial r_2}h(\mathbf{r},t) + \psi(r)\frac{r_2r_1}{r^2}\frac{\partial}{\partial r_1}h(\mathbf{r},t)\end{pmatrix}$$

If we now also take the left differential operator into account, we have

$$[\nabla_{\mathbf{r}}\cdot\tilde{\mathbf{C}}(\mathbf{r})\cdot\nabla_{\mathbf{r}}]h(\mathbf{r},t)$$

$$= \sum_{i,j\in\{1;2\},i\neq j}\frac{\partial}{\partial r_i}\left[\theta(r)\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_i^2}{r^2}\frac{\partial}{\partial r_i}h(\mathbf{r},t) + \psi(r)\frac{r_ir_j}{r^2}\frac{\partial}{\partial r_j}h(\mathbf{r},t)\right]$$

$$= \frac{\partial}{\partial r_1}\left[\theta(r)\frac{\partial}{\partial r_1}h(\mathbf{r},t) + \psi(r)\frac{r_1^2}{r^2}\frac{\partial}{\partial r_1}h(\mathbf{r},t) + \psi(r)\frac{r_1r_2}{r^2}\frac{\partial}{\partial r_2}h(\mathbf{r},t)\right]$$

$$+ \frac{\partial}{\partial r_2}\left[\theta(r)\frac{\partial}{\partial r_2}h(\mathbf{r},t) + \psi(r)\frac{r_2^2}{r^2}\frac{\partial}{\partial r_2}h(\mathbf{r},t) + \psi(r)\frac{r_1r_2}{r^2}\frac{\partial}{\partial r_1}h(\mathbf{r},t)\right]$$

$$= (\frac{\partial}{\partial r_1}\theta(r))(\frac{\partial}{\partial r_1}h(\mathbf{r},t)) + (\frac{\partial}{\partial r_1}\psi(r)\frac{r_1^2}{r^2})(\frac{\partial}{\partial r_1}h(\mathbf{r},t)) + (\frac{\partial}{\partial r_1}\psi(r)\frac{r_1 r_2}{r^2})(\frac{\partial}{\partial r_2}h(\mathbf{r},t))$$

$$+ \theta(r)\frac{\partial^2}{\partial r_1^2}h(\mathbf{r},t) + \psi(r)\frac{r_1^2}{r^2}\frac{\partial^2}{\partial r_1^2}h(\mathbf{r},t) + \psi(r)\frac{r_1 r_2}{r^2}\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}h(\mathbf{r},t)$$

$$+ (\frac{\partial}{\partial r_2}\theta(r))(\frac{\partial}{\partial r_2}h(\mathbf{r},t)) + (\frac{\partial}{\partial r_2}\psi(r)\frac{r_2^2}{r^2})(\frac{\partial}{\partial r_2}h(\mathbf{r},t)) + (\frac{\partial}{\partial r_2}\psi(r)\frac{r_1 r_2}{r^2})(\frac{\partial}{\partial r_1}h(\mathbf{r},t))$$

$$+ \theta(r)\frac{\partial^2}{\partial r_2^2}h(\mathbf{r},t) + \psi(r)\frac{r_2^2}{r^2}\frac{\partial^2}{\partial r_2^2}h(\mathbf{r},t) + \psi(r)\frac{r_1 r_2}{r^2}\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}h(\mathbf{r},t)$$

$$\overset{2.25}{=} \underbrace{\left[(\frac{\partial}{\partial r_1}\theta(r)) + (\frac{\partial}{\partial r_1}\psi(r)\frac{r_1^2}{r^2}) + (\frac{\partial}{\partial r_1}\psi(r)\frac{r_1 r_2}{r^2})\right]}_{=0}(\frac{\partial}{\partial r_1}h(\mathbf{r},t))$$

$$+ \theta(r)\frac{\partial^2}{\partial r_1^2}h(\mathbf{r},t) + \psi(r)\frac{r_1^2}{r^2}\frac{\partial^2}{\partial r_1^2}h(\mathbf{r},t) + \psi(r)\frac{r_1 r_2}{r^2}\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}h(\mathbf{r},t)$$

$$+ \underbrace{\left[(\frac{\partial}{\partial r_2}\theta(r)) + (\frac{\partial}{\partial r_2}\psi(r)\frac{r_2^2}{r^2}) + (\frac{\partial}{\partial r_2}\psi(r)\frac{r_1 r_2}{r^2})\right]}_{=0}(\frac{\partial}{\partial r_2}h(\mathbf{r},t))$$

$$+ \theta(r)\frac{\partial^2}{\partial r_2^2}h(\mathbf{r},t) + \psi(r)\frac{r_2^2}{r^2}\frac{\partial^2}{\partial r_2^2}h(\mathbf{r},t) + \psi(r)\frac{r_1 r_2}{r^2}\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}h(\mathbf{r},t)$$

$$= \theta(r)\frac{\partial^2}{\partial r_1^2}h(\mathbf{r},t) + \psi(r)\frac{r_1^2}{r^2}\frac{\partial^2}{\partial r_1^2}h(\mathbf{r},t) + \psi(r)\frac{r_1 r_2}{r^2}\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}h(\mathbf{r},t)$$

$$+ \theta(r)\frac{\partial^2}{\partial r_2^2}h(\mathbf{r},t) + \psi(r)\frac{r_2^2}{r^2}\frac{\partial^2}{\partial r_2^2}h(\mathbf{r},t) + \psi(r)\frac{r_1 r_2}{r^2}\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}h(\mathbf{r},t)$$

$$= \theta(r)(\frac{\partial^2}{\partial r_1^2}+\frac{\partial^2}{\partial r_2^2})h(\mathbf{r},t)+\psi(r)\frac{r_1^2\frac{\partial^2}{\partial r_1^2}h(\mathbf{r},t)+r_2^2\frac{\partial^2}{\partial r_2^2}h(\mathbf{r},t)}{r^2}+2\psi(r)\frac{r_1r_2}{r^2}\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}h(\mathbf{r},t)$$

$$\overset{(2.24)}{=} \theta(r)\Delta_{\mathbf{r}}h(\mathbf{r},t)+\psi(r)\frac{r_1^4+r_2^4}{r^4}\frac{\partial^2}{\partial r^2}h(\mathbf{r},t)+2\psi(r)\frac{r_1^2r_2^2}{r^4}\frac{\partial^2}{\partial r^2}h(\mathbf{r},t)$$

$$= \theta(r)\Delta_{\mathbf{r}}h(\mathbf{r},t)+\psi(r)\frac{(r_1^2+r_2^2)^2}{r^4}\frac{\partial^2}{\partial r^2}h(\mathbf{r},t)$$

$$= \theta(r)\Delta_{\mathbf{r}}h(\mathbf{r},t)+\psi(r)\frac{\partial^2}{\partial r^2}h(\mathbf{r},t) \tag{2.26}$$

we thus have

$$\Rightarrow$$

$$\boxed{[\nabla_{\mathbf{r}}\cdot\tilde{\mathbf{C}}(\mathbf{r})\cdot\nabla_{\mathbf{r}}]h(\mathbf{r},t)=\theta(r)\Delta_{\mathbf{r}}h(\mathbf{r},t)+\psi(r)\frac{\partial^2}{\partial r^2}h(\mathbf{r},t)}$$

and finally get

$$\frac{\partial}{\partial t}h(\mathbf{r},t)+\nabla_{\mathbf{r}}\cdot(\Gamma_1+\Gamma_2)\mathbf{K}(\mathbf{r})h(\mathbf{r},t)=\frac{1}{2}[\nabla_{\mathbf{r}}\cdot\tilde{\mathbf{C}}(\mathbf{r})\cdot\nabla_{\mathbf{r}}]h(\mathbf{r},t).$$

$$\overset{(2.26)}{\Longleftrightarrow}$$

$$\boxed{\frac{\partial}{\partial t}h(\mathbf{r},t)+\nabla_{\mathbf{r}}\cdot(\Gamma_1+\Gamma_2)\mathbf{K}(\mathbf{r})h(\mathbf{r},t)=\frac{1}{2}\left[\theta(r)\Delta_{\mathbf{r}}+\psi(r)\frac{\partial^2}{\partial r^2}\right]h(\mathbf{r},t) \quad (2.27)}$$

For $\psi(r) = 0$ and $\theta(r) = 2\nu$ = constant this equation has already been solved (cf. [2], compare also appendix, paragraph A.2.).

2.4.1 Change to polar coordinates

From now on we only want to focus on vortices with pure azimuthal velocity field, in polar coordinates this means

$$\mathbf{K}(\mathbf{r}) = \mathbf{K}(r) = \mathbf{e}_\varphi\, v(r).$$

Hence we take equation (2.27),

$$\frac{\partial}{\partial t}h(\mathbf{r},t) + \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2)\mathbf{K}(\mathbf{r})h(\mathbf{r},t) = \frac{1}{2}\left[\theta(r)\Delta_{\mathbf{r}} + \psi(r)\frac{\partial^2}{\partial r^2}\right]h(\mathbf{r},t)$$

and make a coordinate transformation to polar coordinates,

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \rightarrow \begin{pmatrix} r \\ \varphi \end{pmatrix}$$

Thereby we have the relations

$$\nabla_{\mathbf{r}} h(r,\varphi,t) = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \varphi} \end{pmatrix} h(r,\varphi,t),$$

$$\nabla_{\mathbf{r}} \cdot \mathbf{K}(\mathbf{r}) = \frac{1}{r}\frac{\partial}{\partial r}(rK_r(\mathbf{r})) + \frac{1}{r}\frac{\partial}{\partial \varphi}K_\varphi(\mathbf{r}),$$

$$\Delta_{\mathbf{r}} = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

We obtain

$$\frac{\partial}{\partial t}h(r,\varphi,t) + \nabla_{\mathbf{r}} \cdot (\Gamma_1 + \Gamma_2)\mathbf{K}(\mathbf{r})h(r,\varphi,t)$$

$$=\frac{1}{2}\left[\theta(r)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}) + \psi(r)\frac{\partial^2}{\partial r^2}\right]h(r,\varphi,t)$$

$$=\frac{1}{2}\left[[\theta(r) + \psi(r)]\frac{\partial^2}{\partial r^2} + \theta(r)\frac{1}{r}\frac{\partial}{\partial r} + \frac{\theta(r)}{r^2}\frac{\partial^2}{\partial \varphi^2}\right]h(r,\varphi,t) \tag{2.28}$$

As we said before, we from now on consider the case where

$$\mathbf{K}(\mathbf{r}) = \mathbf{K}(r) = \mathbf{e}_\varphi \; v(r). \tag{2.29}$$

Taking a look at the drift term we obtain

$$\nabla_{\mathbf{r}} \cdot (h(r,\varphi,t)\mathbf{K}(\mathbf{r})) = \mathbf{u}(\mathbf{r}) \cdot \nabla_{\mathbf{r}} h(r,\varphi,t) + h(r,\varphi,t)\nabla_{\mathbf{r}} \cdot \mathbf{K}(\mathbf{r})$$

$$\overset{(2.4.1)}{=} v(r)\frac{1}{r}\frac{\partial}{\partial \varphi}h(r,\varphi,t) + h(r,\varphi,t)\frac{1}{r}\frac{\partial}{\partial \varphi}v(r)$$

$$=v(r)\frac{1}{r}\frac{\partial}{\partial \varphi}h(\mathbf{r},t)$$

and hence

$$\frac{\partial}{\partial t}h(r,\varphi,t)+v(r)\frac{(\Gamma_1+\Gamma_2)}{r}\frac{\partial}{\partial\varphi}h(r,\varphi,t)=\frac{1}{2}\left[[\theta(r)+\psi(r)]\frac{\partial^2}{\partial r^2}+\theta(r)\frac{1}{r}\frac{\partial}{\partial r}+\frac{\theta(r)}{r^2}\frac{\partial^2}{\partial\varphi^2}\right]h(r,\varphi,t) \tag{2.30}$$

2.4.2 Mode ansatz

Until now, we have derived this quite general stochastic partial equation describing the time evolution of the relative distance and position of the two vortices in a probabilistic way with quite few assumptions and simplifications. The only assumptions we have made so far are having an incompressible, Kraichnan velocity field, isotropic statistics, pure azimuthal vortex profiles and that the two vortices are of the same type. Hence, at this point, we theoretically may still insert arbitrary functions for $v(r), \theta(r)$ and $\psi(r)$ and try to derive a solution for the resulting equation. But as one can easily think of, this equation is not solvable for every set of desired functions, it is only solvable for very few cases, requiring additional investigation: after having eventually solved the equation, we still have a *stochastic* differential equation, which demands from us to test and verify that the obtained "solution" is a solution - meaning that it is a positive, normalized probability density.

To reduce the problem of solving a partial differential equation to the problem of solving an ordinary differential equation, we apply the mode ansatz

$$h(r,\varphi,t)=e^{in\varphi}e^{-\lambda t}h_{n,\lambda}(r) \tag{2.31}$$

This ansatz leads to the linear eigenvalue problem

$$-\lambda h_{n,\lambda}(r)+v(r)in\frac{\Gamma_1+\Gamma_2}{r}h_n(r)=\frac{1}{2}\left[[\theta(r)+\psi(r)]\frac{\partial^2}{\partial r^2}+\theta(r)\frac{1}{r}\frac{\partial}{\partial r}-n^2\frac{\theta(r)}{r^2}\right]h_n(r), \tag{2.32}$$

superposition leads to the general solution

$$h(r,\varphi,t)=\int d\lambda\sum_n a(n,\lambda)e^{in\varphi}e^{-\lambda t}h_{n,\lambda}(r),$$

still of course leaving the problem of determining $h_{n,\lambda}(r)$. By using this ansatz, we now have obtained a second-order linear ordinary differential equation with formal similarity to the stationary Schrödinger equation. Now the left task is an appropriate selection of $v(r), \theta(r)$ and $\psi(r)$ to find analytical solvable versions of (2.32).

One could now use the Sturm Lioville theory as presented in [13], if the equation was not complex. There are solutions in the class of Fuchs' differential equations (compare [27]), which make use of hypergeometric polynomials, but prohibit functions $v(r) \propto \frac{1}{r}$ that are of essential interest for us according to the velocity field of point vortices. We will see that all solutions to the equation that are relevant in the context of point vortices are related to Bessel's differential equation. They may be found for example in [1] or [16], and a special case we need in [22].[8]

2.4.3 Choice for $\theta(r)$ and $\psi(r)$

If we now once again consider the Kraichnan model

$$
\begin{aligned}
Q_{ij}^{(\alpha,\beta)} &= Q^{(\alpha,\beta)}(\mathbf{x}_i - \mathbf{x}_j) \\
=& d_0^{(\alpha,\beta)} - d^{(\alpha,\beta)}(\mathbf{x}_i - \mathbf{x}_j)\delta(t' - t'')
\end{aligned}
$$

where in the incompressible case we had the relation (1.9),

$$
d^{(\alpha,\beta)}(\mathbf{r}) = d^{(\alpha,\beta)}(\mathbf{r}) = r^{\xi}\left[B(d+\xi-1)\delta_{\alpha\beta} - B\xi\frac{r_\alpha r_\beta}{r^2}\right]
$$

and take (2.5)

$$
\tilde{\mathbf{C}}(\mathbf{r}) = 2\left[\mathbf{Q}(0) - \mathbf{Q}(\mathbf{r})\right], \tag{2.33}
$$

we obtain for $\alpha, \beta \in \{1, 2\}$

$$
\begin{aligned}
\tilde{C}_{\alpha\beta}(\mathbf{r}) = 2&\left[d_0^{(\alpha,\beta)} - \underbrace{d^{(\alpha,\beta)}(0)}_{=0} - (d_0^{(\alpha,\beta)} - d^{(\alpha,\beta)}(\mathbf{x}_i - \mathbf{x}_j))\right] \\
=& 2d^{(\alpha,\beta)}(\mathbf{r}) = r^{\xi}\left[\underbrace{2B(d+\xi-1)}_{\theta_0}\delta_{\alpha\beta} + \underbrace{-2B\xi}_{\psi_0}\frac{r_\alpha r_\beta}{r^2}\right] \\
=& \theta_0 r^{\xi}\delta_{\alpha\beta} + \psi_0 r^{\xi}\frac{r_\alpha r_\beta}{r^2} \\
=& \theta(r)\delta_{\alpha\beta} + \psi(r)\frac{r_\alpha r_\beta}{r^2}
\end{aligned}
$$

Hence, we have

[8] After three months of research on methods of solving differential equations, I am quite sure of having found all relevant and known analytical solutions to that class of equations.

$$\theta(r) = \theta_0 r^{\xi} \qquad , \qquad \psi(r) = \psi_0 r^{\xi} \tag{2.34}$$

Thereby θ_0 and ψ_0 are related due to incompressibility; with (2.23),

$$\frac{\partial}{\partial r}\theta(r) + \frac{1}{r}\frac{\partial}{\partial r} r\psi(r) = 0$$

it is

$$0 = \xi\theta_0 r^{\xi-1} + \frac{1}{r}(\psi_0 r^{\xi} + r\xi\psi_0 r^{\xi-1})$$

$$\Leftrightarrow \xi\theta_0 + \psi_0 + \xi\psi_0 = 0$$

$$\Leftrightarrow \xi = \frac{-\psi_0}{\theta_0 + \psi_0} \tag{2.35}$$

If we insert now our "definitions" of θ_0 and ψ_0 in (2.35), we see that

$$\xi = \frac{-\psi_0}{\theta_0 + \psi_0} = \frac{B\xi}{B(d + \xi - 1 - \xi)} = \frac{\xi}{d-1} = \xi$$

for $d = 2$ dimensions. $\Box$
So we just verified that we calculated properly.

Inserting this in our Fokker-Planck equation (2.32) results in a linear eigenvalue problem of the form

$$-\lambda h_{n,\lambda}(r) + v(r) in \frac{\Gamma_1 + \Gamma_2}{r} h_{n,\lambda} = \frac{1}{2}\left[[\theta_0 + \psi_0] r^{\xi} \frac{\partial^2}{\partial r^2} + \theta_0 r^{\xi} \frac{1}{r}\frac{\partial}{\partial r} - n^2 \frac{\theta_0 r^{\xi}}{r^2} \right] h_{n,\lambda} \tag{2.36}$$

with $h_{n,\lambda}(r) =: h$, $\frac{\partial}{\partial r} h_{n,\lambda}(r) =: h'$ and $\frac{\partial^2}{\partial r^2} h_{n,\lambda}(r) =: h''$to simplify writing, and arranging in orders of derivation, we get

$$-\lambda h + v(r) in \frac{\Gamma_1+\Gamma_2}{r} h = \frac{1}{2}[\theta_0+\psi_0] r^\xi h'' + \theta_0 r^{\xi-1} h' - \frac{n^2}{2}\frac{\theta_0 r^\xi}{r^2} h$$

$$\Leftrightarrow \; -\lambda h + in \frac{(\Gamma_1+\Gamma_2)\ v(r)}{r} h = \frac{1}{2}\theta_0 r^\xi \frac{1}{r} h' + \frac{1}{2}[\theta_0+\psi_0] r^\xi h'' - \frac{n^2}{2}\frac{\theta_0 r^\xi}{r^2} h$$

$$\Leftrightarrow \frac{-2r^2\lambda}{\theta_0+\psi_0} r^{-\xi} h + \frac{2in(\Gamma_1+\Gamma_2)\ v(r)\ r}{\theta_0+\psi_0} r^{-\xi} h = \frac{\theta_0}{\theta_0+\psi_0} r h' + r^2 h'' - \frac{n^2\theta_0}{\theta_0+\psi_0} h$$

$$\Leftrightarrow r^2 h'' + \frac{\theta_0}{\theta_0+\psi_0} r h' - \frac{n^2\theta_0}{\theta_0+\psi_0} h - r^{-\xi}\left[-\frac{2r^2\lambda}{\theta_0+\psi_0} + \frac{2in(\Gamma_1+\Gamma_2)\ v(r)\ r}{\theta_0+\psi_0}\right] h = 0$$

2.5 Solution for vortex profiles $v(r) = \frac{1}{2\pi r^{1-\xi}}$

We will now investigate cases in which the azimuthal velocity is

$$v(r) = \frac{1}{2\pi r^{1-\xi}} \tag{2.37}$$

with a parameter $\xi \in [0,2]$

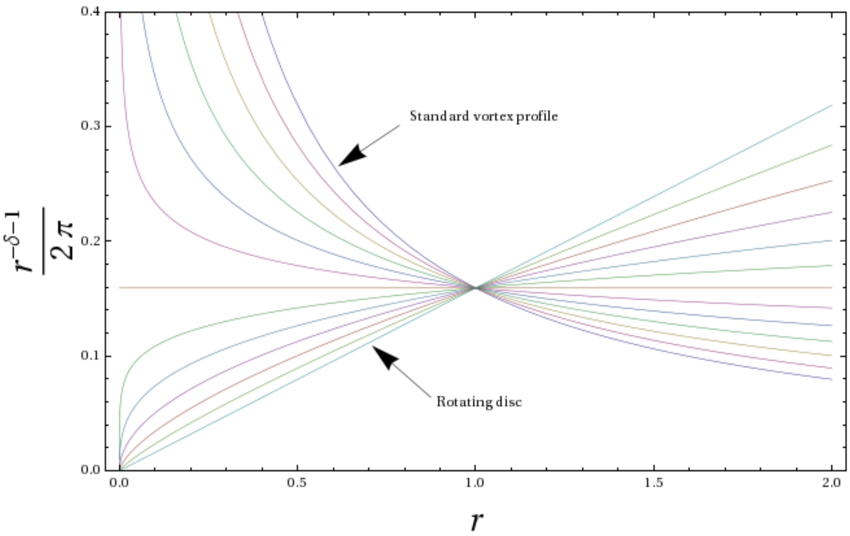


Figure 2.7: Vortex profile: r versus $v(r) = \frac{1}{2\pi r^{1-\xi}}$ for $\xi \in [0,2]$

Note that for $\xi \in [0,2]$ we have a spectrum of velocities reaching from the "standard" point vortex profile to the one of a rotating disc, represented by the monotonic increasing straight line in the plot. A rotating disc has of course a total lack of differential rotation and is therefore not an appropriate candidate to model vortices in fluids. Nevertheless, it will be included in our calculations, so it may be seen as a theoretical limiting case.

For the correlations we take a corresponding behavior, namely

$$\theta(r) = \theta_0 r^{\xi}, \qquad \psi(r) = \psi_0 r^{\xi},$$

with the same ξ. In this case it is possible to derive exact analyitical solutions. Inserting this in equation (2.36),

$$r^2 h'' + \frac{\theta_0}{\theta_0+\psi_0} r h' - \frac{n^2\theta_0}{\theta_0+\psi_0} h - r^{-\xi}\left[-\frac{2r^2\lambda}{\theta_0+\psi_0} + \frac{2in(\Gamma_1+\Gamma_2)\ v(r)\ r}{\theta_0+\psi_0}\right] h = 0$$

we get

$$r^2 h'' + \frac{\theta_0}{\theta_0+\psi_0} r h' + \left[r^{2-\xi}\frac{2\lambda}{\theta_0+\psi_0} - \frac{in(\Gamma_1+\Gamma_2)}{\pi(\theta_0+\psi_0)} - \frac{n^2\theta_0}{\theta_0+\psi_0}\right] h = 0 \qquad (2.38)$$

This equation is solvable for $2-\xi \neq 0$ (compare [22]), its particular solution is

$$h = h_{\lambda,n}(r) = r^{\frac{1}{2}-\frac{\theta_0}{2(\theta_0+\psi_0)}}\left[C_1 J_\nu\left(\frac{2}{2-\xi}\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\right) + C_2 Y_\nu\left(\frac{2}{2-\xi}\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\right)\right]$$
$$= h_{\lambda,n}(r) = r^{-\frac{\xi}{2}}\left[C_1 J_\nu\left(\frac{2}{2-\xi}\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\right) + C_2 Y_\nu\left(\frac{2}{2-\xi}\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\right)\right]$$

whereby

$$r^{\frac{1}{2}-\frac{\theta_0}{2(\theta_0+\psi_0)}} = r^{\frac{\theta_0+\psi_0}{2(\theta_0+\psi_0)}-\frac{\theta_0}{2(\theta_0+\psi_0)}} = r^{\frac{\psi_0}{2(\theta_0+\psi_0)}} \overset{(2.35)}{=} r^{-\frac{\xi}{2}}$$

C_1 and C_2 are arbitrary constants to be determined by implementation of initial conditions; $J_\nu(z)$ and $Y_\nu(z)$ are the Bessel functions of the first and second kind (cf. [5])

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{2k+\nu}}{\Gamma(\nu+k+1)k!}$$

and

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$$

with

$$\nu = \frac{1}{\xi}\sqrt{-4\left(-\frac{(\Gamma_1+\Gamma_2)in}{\pi(\theta_0+\psi_0)} - \frac{\theta_0 n^2}{\theta_0+\psi_0}\right)} = \frac{2}{\xi}\sqrt{\frac{(\Gamma_1+\Gamma_2)in}{\pi(\theta_0+\psi_0)} + \frac{\theta_0 n^2}{\theta_0+\psi_0}}$$

So our general solution finally reads

$$h(r,\varphi,t) = \sum_{n\in\mathbb{Z}} \int_0^\infty d\lambda e^{-\lambda t} e^{in\varphi} \left(r^{-\frac{\xi}{2}} \left[C_1 J_\nu \left(\frac{2r^{\frac{2-\xi}{2}}}{2-\xi} \sqrt{\frac{2\lambda}{\theta_0+\psi_0}} \right) + C_2 Y_\nu \left(\frac{2r^{\frac{2-\xi}{2}}}{2-\xi} \sqrt{\frac{2\lambda}{\theta_0+\psi_0}} \right) \right] \right) \tag{2.39}$$

We remark, that the solvability condition of this equation, $2 - \xi \neq 0$, excludes exactly the limiting case of the vortex profile for a rotating disc, which was realized for $\xi = 2$. But as we said before, this case would not have been a good candidate for modeling vortices in a turbulent fluid anyway.

2.5.1 Implementation of initial conditions

This section deals with the implementation of reasonable initial conditions for the eigenvalue problem - also with regard to a graphic presentation. It is obvious that a probabilisticly sharp localization of the relative distance of the two point vortices is such a choice, expressed by means of a Dirac delta:

$$h(r,\varphi,0) = \delta(r - r_0, \varphi) \tag{2.40}$$

So the initial distance between the two vortices is r_0, and their initial inclination is 0. This choice is possible without loss of generality, because the Fokker-Planck equation is invariant by rotation. If we manage to express the solution by a normalized function for a fixed point in time, say $t = 0$, then we have a normalized solution for all times. (cf. section (1.2.2))

To match the initial condition to our solution from the previous section at time $t = 0$, we first have to express the Dirac delta in terms of Bessel functions. Therefore we make use of the so-called Jacobi-Anger expansion (to be found in [1] p. 361)

$$\exp(iz\cos(\alpha)) = \sum_{\nu=-\infty}^{\infty} i^\nu J_\nu(z) \exp(i\nu\alpha) \tag{2.41}$$

Taking α now as the angle between $\mathbf{k}$ and $(\mathbf{r} - \mathbf{r}_0)$ and $z = |\mathbf{k}|\,|\mathbf{r} - \mathbf{r}_0| = k\,|\mathbf{r} - \mathbf{r}_0|$, we obtain with the Fourier transform of the Dirac delta

$$
\begin{aligned}
\delta(\mathbf{r}-\mathbf{r}_0) &= \frac{1}{4\pi^2}\int d\mathbf{k}\exp(i\mathbf{k}\cdot[\mathbf{r}-\mathbf{r}_0]) \\
&= \frac{1}{4\pi^2}\int_0^\infty dk\ k\int_0^{2\pi} d\alpha\exp(ik\,|\mathbf{r}-\mathbf{r}_0|\cos(\alpha)) \\
&\overset{(2.41)}{=} \frac{1}{4\pi^2}\int_0^\infty dk\ k\int_0^{2\pi} d\alpha\sum_{\nu=-\infty}^{\infty} i^\nu J_\nu(k\,|\mathbf{r}-\mathbf{r}_0|)\exp(i\nu\alpha) \\
&= \frac{1}{4\pi^2}\int_0^\infty dk\ k\left[\int_0^{2\pi} d\alpha J_0(k\,|\mathbf{r}-\mathbf{r}_0|)\exp(0) + \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} i^\nu J_\nu(k\,|\mathbf{r}-\mathbf{r}_0|)\int_0^{2\pi} d\alpha\exp(i\nu\alpha)\right] \\
&= \frac{1}{4\pi^2}\int_0^\infty dk\ k\left[2\pi J_0(k\,|\mathbf{r}-\mathbf{r}_0|) + \sum_{\substack{\nu=-\infty\\ \nu\neq 0}}^{\infty} i^\nu J_\nu(k\,|\mathbf{r}-\mathbf{r}_0|)\left(\frac{1}{i\nu}(1-1)\right)\right] \\
&= \frac{1}{2\pi}\int_0^\infty dk\ kJ_0(k\,|\mathbf{r}-\mathbf{r}_0|)
\end{aligned}
$$

$$
\Rightarrow \qquad \delta(\mathbf{r}-\mathbf{r}_0) = \frac{1}{2\pi}\int_0^\infty dk\ kJ_0(k\,|\mathbf{r}-\mathbf{r}_0|) \tag{2.42}
$$

This expression of the Dirac delta is elementary. We now use Gegenbauer's addition theorem for Bessel functions (to be found in [1] p. 363) to develop $J_0(k\,|\mathbf{r}-\mathbf{r}_0|)$:

$$
J_0(k\,|\mathbf{r}-\mathbf{r}_0|) = J_0(kr_0)J_0(kr) + 2\sum_{\nu=1}^{\infty} J_\nu(kr_0)J_\nu(kr)\cos(\nu\phi)
$$

With this and $J_{-n}(z) = (-1)^n J_n(z)$, we get

$$
\begin{aligned}
\delta(\mathbf{r}-\mathbf{r}_0) &= \frac{1}{2\pi}\int_0^\infty dk\ k\left[J_0(kr_0)J_0(kr) + 2\sum_{\nu=1}^{\infty} J_\nu(kr_0)J_\nu(kr)\cos(\nu\varphi)\right] \\
&= \frac{1}{2\pi}\int_0^\infty dk\ k\sum_{\nu\in\mathbb{Z}} J_\nu(kr_0)J_\nu(kr)e^{i\nu\varphi}
\end{aligned}
$$

There are now Bessel functions with argument $\propto r^{\frac{2-\xi}{2}}$ in our solution (2.39) of the Fokker-Planck equation and Bessel functions with argument $\propto r$ in the expression for the Dirac delta which have to be matched, following the tedious task of searching for multiplication theorems to transform the more complicated terms into simpler ones.[9]

Fortunately, there is another possibility. First, we rename $\mathbf{r}$ and $\mathbf{r}_0$ by $\tilde{\mathbf{r}}$ and $\tilde{\mathbf{r}}_0$

$$\delta(\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_0) = \frac{1}{2\pi} \int_0^\infty dk\ k \sum_{\nu \in \mathbb{Z}} J_\nu(k\tilde{r}_0) J_\nu(k\tilde{r}) e^{i\nu\varphi} \tag{2.43}$$

Considering the substitution

$$\tilde{r} = r^{\frac{2-\xi}{2}} \qquad \tilde{r_0} = r_0^{\frac{2-\xi}{2}}$$

and inserting in (2.43), we get

$$\delta(r^{\frac{2-\xi}{2}} - r_0^{\frac{2-\xi}{2}}, \varphi) = \frac{1}{2\pi} \int_0^\infty dk\ k \sum_{\nu \in \mathbb{Z}} J_\nu(k r_0^{\frac{2-\xi}{2}}) J_\nu(k r^{\frac{2-\xi}{2}}) e^{i\nu\varphi}$$

The Dirac delta has a useful property now - it is (cf. [5])

$$\delta(g(r_0)) = \sum_{i=1}^{n} \frac{\delta(r_0 - r_i)}{|g'(r_i)|}$$

whereby r_i are the simple roots of $g(r_0)$. Now we consider

$$g(r_0) = r^{\frac{2-\xi}{2}} - r_0^{\frac{2-\xi}{2}}$$

with the only simple root at $r = r_0$. Hence, with $g'(r_0) = \frac{2-\xi}{2} r^{\frac{2-\xi}{2}-1} = \frac{2-\xi}{2} r^{1-\frac{\xi}{2}-1} = \frac{2-\xi}{2} r^{-\frac{\xi}{2}}$, we get

$$\delta(r^{\frac{2-\xi}{2}} - r_0^{\frac{2-\xi}{2}}, \varphi) = \delta(r - r_0, \varphi) \frac{1}{\frac{2-\xi}{2} r^{-\frac{\xi}{2}}}$$

With (2.43), one obtains

$$\delta(r - r_0, \varphi) = \frac{1}{2\pi} \frac{2-\xi}{2} r^{-\frac{\xi}{2}} \int_0^\infty dk\ k \sum_{\nu \in \mathbb{Z}} J_\nu(k r_0^{\frac{2-\xi}{2}}) J_\nu(k r^{\frac{2-\xi}{2}}) e^{i\nu\varphi} \tag{2.44}$$

an expression for the Dirac delta with the desired properties to match the Bessel functions. Comparing this to equation (2.39),

$$h(r,\varphi,t) = \sum_{n \in \mathbb{Z}} \int_0^\infty d\lambda e^{-\lambda t} e^{in\varphi} \left(r^{-\frac{\xi}{2}} \left[C_1 J_\nu \left(\frac{2r^{\frac{2-\xi}{2}}}{2-\xi} \sqrt{\frac{2\lambda}{\theta_0 + \psi_0}} \right) + C_2 Y_\nu \left(\frac{2r^{\frac{2-\xi}{2}}}{2-\xi} \sqrt{\frac{2\lambda}{\theta_0 + \psi_0}} \right) \right] \right)$$

[9] After quite a while in hopeful search of such theorems, I myself by now believe there are none.

for $t = 0$ and $\phi = 0$, leads to $C_2 = 0$.

Now we substitute $\frac{2}{2-\xi}\sqrt{\frac{2\lambda}{\theta_0+\psi_0}}$ by k, or λ by $\frac{(2-\xi)^2 k^2(\theta_0+\psi_0)}{8}$, respectively.

We have

$$\frac{d\lambda}{dk} = k\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)$$

which leads to

$$h(r,\varphi,t) = r^{-\frac{\xi}{2}}\sum_{n\in\mathbb{Z}} e^{in\varphi}\int dk\, k\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)\,\tilde{a}(n,k)\exp\left(-\frac{(2-\xi)^2(\theta_0+\psi_0)}{8}k^2 t\right) C_1 J_\nu(kr^{\frac{2-\xi}{2}})$$

and finally we identify $\tilde{a}(n,k)$ with

$$\tilde{a}(n,k) = \frac{4}{C_1(2-\xi)^2(\theta_0+\psi_0)}\frac{1}{2\pi}\frac{2-\xi}{2}J_\nu(kr_0^{\frac{\xi}{2}}),$$

which entails[10]

$$h(r,\varphi,t) = \frac{1}{2\pi}\frac{2-\xi}{2}r^{-\frac{\xi}{2}}\sum_{n\in\mathbb{Z}} e^{in\varphi}\int dk\, k\exp\left(-\frac{(2-\xi)^2(\theta_0+\psi_0)}{8}k^2 t\right) J_\nu(kr^{\frac{2-\xi}{2}})J_\nu(kr_0^{\frac{2-\xi}{2}}) \tag{2.45}$$

With the identity

$$\int_0^\infty e^{-\rho^2 k^2} J_\nu(\tilde{r}k)J_\nu(\tilde{r}_0 k)k\; dk = \frac{1}{2\rho^2}\exp\left(-\frac{\tilde{r}^2+\tilde{r}_0^2}{4\rho^2}\right) I_\nu\left(\frac{\tilde{r}\tilde{r}_0}{2\rho^2}\right)$$

and $\rho^2 = \frac{(2-\xi)^2(\theta_0+\psi_0)}{8}t$, the modified Bessel function $I_\nu(z) = i^{-\nu}J_\nu(iz)$, $\tilde{r} = r^{\frac{2-\xi}{2}}$ and $\tilde{r_0} = r_0^{\frac{2-\xi}{2}}$, we can finally write down the normalized solution of our Fokker-Planck equation:

$$h(r,\varphi,t) = \frac{1}{2\pi}\frac{2-\xi}{2}r^{-\frac{\xi}{2}}\frac{1}{2\frac{(2-\xi)^2(\theta_0+\psi_0)t}{8}}\sum_{n\in\mathbb{Z}} e^{in\varphi}\exp\left(-\frac{r^{2-\xi}+r_0^{2-\xi}}{4\frac{(2-\xi)^2(\theta_0+\psi_0)}{8}t}\right) I_\nu\left(\frac{(rr_0)^{\frac{2-\xi}{2}}}{2\frac{(2-\xi)^2(\theta_0+\psi_0)}{8}t}\right) \tag{2.46}$$

[10] Note again that

$$\nu = \frac{2}{\xi}\sqrt{\frac{(\Gamma_1+\Gamma_2)in}{\pi(\theta_0+\psi_0)} + \frac{\theta_0 n^2}{\theta_0+\psi_0}} \propto n$$

$$\Longleftrightarrow \boxed{h(r,\varphi,t) = \frac{1}{\pi(2-\xi)(\theta_0+\psi_0)t}\; r^{-\frac{\xi}{2}} \sum_{n\in\mathbb{Z}} e^{in\varphi} \exp\left(-\frac{r^{2-\xi}+r_0^{2-\xi}}{\frac{1}{2}(2-\xi)^2(\theta_0+\psi_0)t}\right) I_\nu\left(\frac{(rr_0)^{\frac{(2-\xi)}{2}}}{\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)t}\right)}$$

$$= H(r,t)\left[1+2\Re\left(\sum_{n=1}^{\infty} e^{in\phi} I_0^{-1}\left(\frac{(rr_0)^{\frac{2-\xi}{2}}}{\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)t}\right) I_\nu\left(\frac{(rr_0)^{\frac{2-\xi}{2}}}{\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)t}\right)\right)\right] \tag{2.47}$$

whereas $H(r,t)$ is defined by

$$H(r,t) = \frac{1}{\pi(2-\xi)(\theta_0+\psi_0)t}\; r^{-\frac{\xi}{2}} \exp\left(-\frac{r^{2-\xi}+r_0^{2-\xi}}{\frac{1}{2}(2-\xi)^2(\theta_0+\psi_0)t}\right) I_0\left(\frac{(rr_0)^{\frac{\xi}{2}}}{\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)t}\right) \tag{2.48}$$

Note that we have to multiply by the factor r to have a complete probability density, r is the Jacobian of the transformation to polar coordinates. If we compare this to the result of the standard vortex profile derived in the appendix, we see that we got equivalence for $\xi = 0$: It follows from $\xi = 0$ and $\psi_0 = 0$ due to incompressiblity (compare previous chapter), that

$$h(r,\varphi,t) = H(r,t)\left[1+2\Re\left(\sum_{n=1}^{\infty} e^{in\varphi} I_0^{-1}\left(\frac{rr_0}{\theta_0 t}\right) I_\nu\left(\frac{rr_0}{\theta_0+t}\right)\right)\right]$$

whereby

$$H(r,t) = \frac{1}{2\pi\theta_0 t}\exp\left(-\frac{r^2+r_0^2}{2\theta_0 t}\right) I_0\left(\frac{rr_0}{\theta_0 t}\right)$$

2.5.2 Graphical presentation of the results

Now we come to the graphical presentation. We have to
We plot $rh(r, \varphi, t)$ for different values of t with mathematica, with respect to calculation time we only sum up the first 20-100 terms of the infinite progression. It is quite difficult to write a code for infinite sums of bessel functions (cf. citebesselnumerics); the numerics of mathematica have sometimes already problems handling a large finite sum of modified Bessel functions. But one do not has to desperate - we may take into account that the infinite progression of the modified Bessel functions converges: For

$$n \to \infty,$$

we have

$$|\nu| \sim n$$

,
and hence

$$|I_\nu(cr)| \sim I_n(cr)$$

Considering

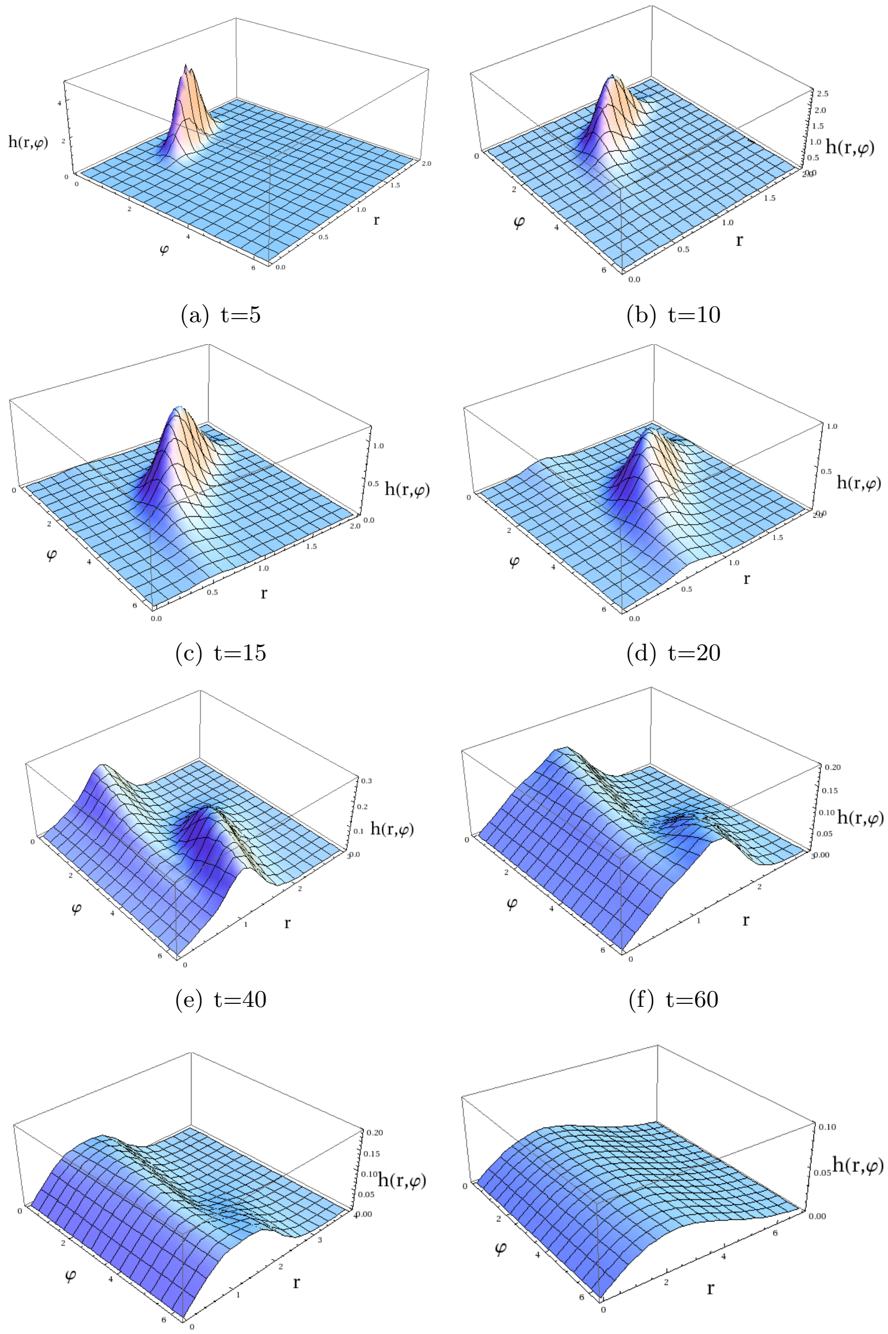
$$|I_0(cr) + 2\Re \sum_{n=1}^{\infty} I_\nu(cr) e^{in\varphi}| \leq |I_0(cr)| + 2 \sum_{n=1}^{\infty} |I_\nu(cr)|,$$

the normal convergence is a consequence of the convergence of the positive series

$$I_0(cr) + 2 \sum_{n=1}^{\infty} I_n(cr) = \exp(cr) \qquad \text{for } cr > 0$$

(cf. also [3]). Assuming that the higher terms give less contribution, we only take a finite number of terms to establish plots for the probability density. Note that the larger the value of t, the less is the contribution of the modified Bessel functions and the less the angular part is destining the evolution of the probability density.

For our graphical presentation of $r\ h(r, \varphi, t)$, we first chose $\xi = 0$, $\theta_0 = 0.02$, and $r_0 = 1$.



(a) t=5 (b) t=10

(c) t=15 (d) t=20

(e) t=40 (f) t=60

(g) t=80 (h) t=800

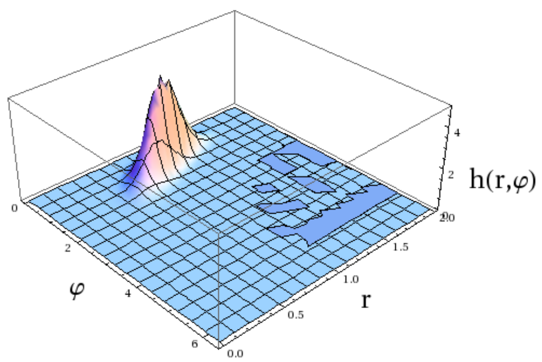
Figure 2.8: $r\ h(r, \varphi, t)$ for different values of t. Parameters: $\xi = 0$, $\theta_0 = 0.02$, and $r_0 = 1$

For $t = 5$ we still have a quite sharp located probability density function, caused by the initial condition $\delta(r-1, \varphi)$. Going further in time, we can observe, that a smaller angle and a bigger distance r, and a bigger angle and a smaller distance r correspond to each other in terms of the most probable events to a fixed time t. If we regard this in a differential manner, considering changes with time, this shows a behavior $\frac{\Delta\varphi}{\Delta t} \sim f\left(\frac{1}{\frac{\Delta r}{\Delta t}}\right)$, scaled by a certain function f. So in fact we may interpreted this observation as a description of an acceleration process if the vortices get nearer each other and a deceleration process if the move away from each other. This seems to be reasonable in the context of phenomenological fluid oberservations.
The more we progress in time, the less is the characterizing influence of the angular part of the probability density. What we eventually see is a change of the topology of the probability density function - for 800 time steps we see that for large times we arrive in an isoptropic limes concering the angular influence of the probability density.
We could have already observed this reviewing the formulas: in the asymptotic regime of $t \to \infty$, we have

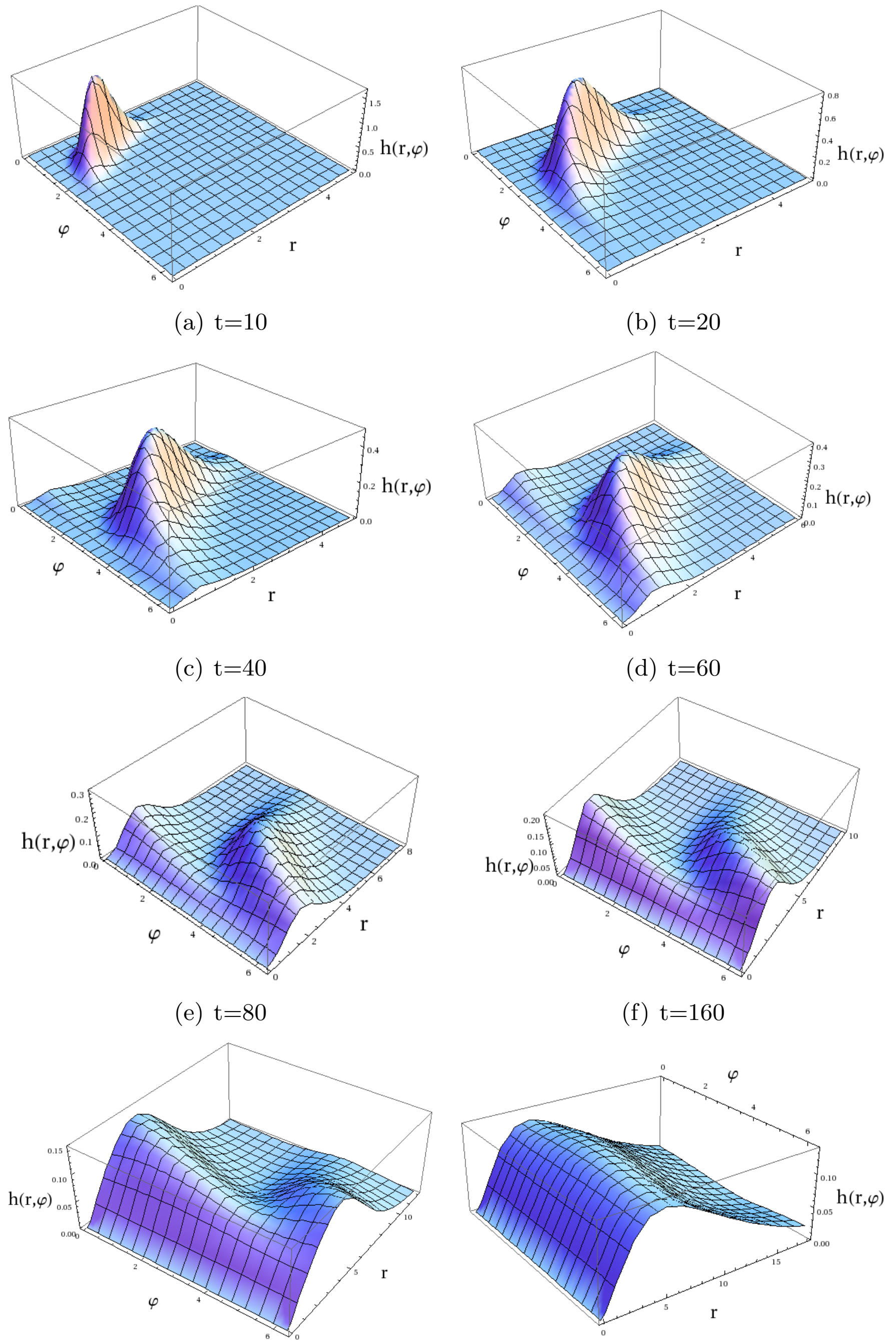
$$h(r, \varphi, t) \sim \frac{1}{2\pi\theta_0 t} \exp\left(\frac{r^2 + r_0^2}{2\theta_0 t}\right) I_0(\frac{rr_0}{\theta_0 t})$$

which corresponds to a initially concentrated probability density on a ring of radius r_0.

If we take a closer look at the graph for $t = 5$ by adjusting the plot range to stricly $r \geq 0$, we observe a feature of the graph which shows us the limits of our approximation of taking only a finite number of the terms of the progression:



The darker blue parts denote locations where the graph gets slightly negative, although taking into account the first $n = 500$ terms of the progression and a calculation time of about half an hour. For $t = 10$ time units already $n = 150$ terms are enough to have graphs without the probability density getting negative, for $t = 40$ there even is no visible difference between $n = 20$ and $n = 40$ or higher. Now in succession, we choose $\xi = 1$ for the same parameters as before $\theta_0 = 0.02$, and $r_0 = 1$



(a) t=10 (b) t=20

(c) t=40 (d) t=60

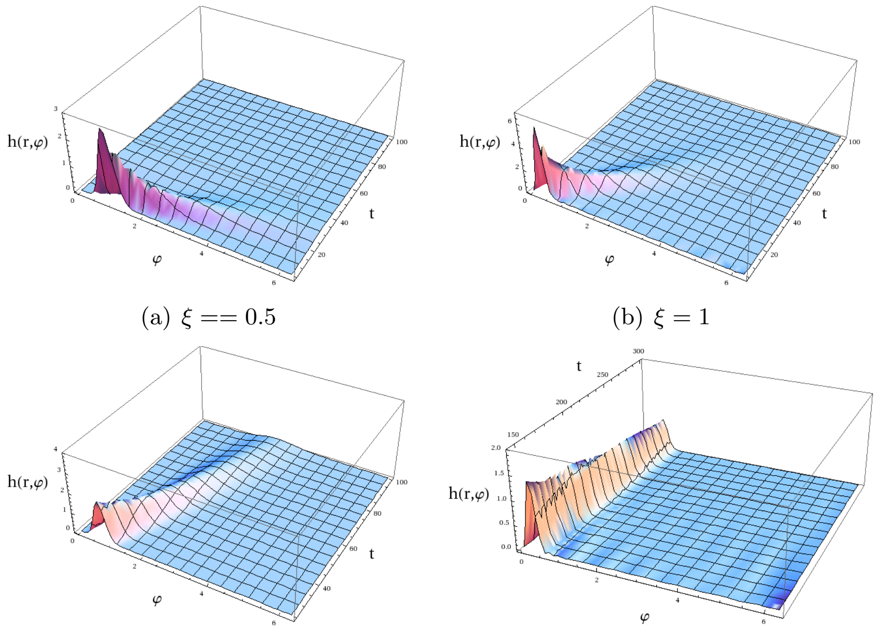
(e) t=80 (f) t=160

(g) t=240 (h) t=500

Figure 2.9: $r\ h(r, \varphi, t)$ for different values of t. Parameters: $\xi = 1$, $\theta_0 = 0.02$, and $r_0 = 1$

We see that the qualitative behavior is very similar in the two cases, a hint that the ansatz of the modified vortex profile is not a bad candidate for modelling, since the case $\xi = 0$ corresponds to the standard vortex profile as we saw before.

The one thing one can distinguish is the time in which the vortices rotate around each other and the time in which the systems finally reaches an isotropic-looking state. $\xi = 0$ is faster than $\xi = 1$. $\xi = 1.9$ does not lead to acceptable results in an appropriate amount of calculating time. One has to sum up much more terms to form a reasonable graph.



(a) $\xi == 0.5$

(b) $\xi = 1$

(c) $\xi = 1.5$

(d) $\xi = 1.8$

Figure 2.10: $r\ h(1, \varphi, t)$ for different values of ξ. Parameters: $r = 1$, $\theta_0 = 0.02$, and $r_0 = 1$

Here we see the angle in dependence of time - for smaller ξs corresponding to the standard vortex profile the angle changes faster, for larger ξ it changes slower. As one can observe, the plot for ξ is of worse quality, it although we took more calculation steps and more to compute it. In the area of since the equation is not solvable for $\xi = 2$, it is understandable that the terms show strange behavior if too few terms are added up.

2.6 Solution for vortex profiles $v(r) = \frac{1}{2\pi r}$

We will now investigate cases where the azimuthal velocity is given by

$$v(r) = \frac{1}{2\pi r}, \tag{2.49}$$

independently of the scaling exponent in the correlation matrix. With equation (2.30), the evolution equation takes the form

$$r^2 h'' + \frac{\theta_0}{\theta_0 + \psi_0} r h' - \frac{n^2\theta_0}{\theta_0 + \psi_0} h - r^{-\xi}\left[\frac{-2r^2\lambda}{\theta_0 + \psi_0} + \frac{in(\Gamma_1 + \Gamma_2)}{\pi(\theta_0 + \psi_0}) \right] h = 0 \tag{2.50}$$

This equation is not analytically solvable for $\xi \neq 0$, but at least we can try to treat an asymptotic case for large values of r before we solve it for the special case of constant spatial correlations.

2.6.1 Asymptotic behavior

To investigate on the asymptotic behavior of

$$r^2 h'' + \frac{\theta_0}{\theta_0 + \psi_0} r h' - \frac{n^2\theta_0}{\theta_0 + \psi_0} h - r^{-\xi}\left[\frac{-2r^2\lambda}{\theta_0 + \psi_0} + \frac{in(\Gamma_1 + \Gamma_2)}{\pi(\theta_0 + \psi_0}) \right] h = 0$$

for large values of r, we first make an ansatz $u(r)w(r) = uw$ for h, resulting in the relations

$$h = uw \qquad h' = u'w + uw' \qquad h'' = u''w + 2u'w' + uw'' \tag{2.51}$$

As a consequence, we get

$$r^2(u''w + 2u'w' + uw'') + \frac{\theta_0}{\theta_0 + \psi_0} r(u'w + uw') - \frac{n^2\theta_0}{\theta_0 + \psi_0} uw - r^{-\xi}\left[\frac{-2r^2\lambda}{\theta_0 + \psi_0} + \frac{in(\Gamma_1 + \Gamma_2)}{\pi(\theta_0 + \psi_0)}\right] uw = 0$$

Now we want to eliminate the terms proportional to u'. This entails

$$2r^2 w' + \zeta r w = 0$$

whereby we substituted

$$\zeta = \frac{\theta_0}{\theta_0 + \psi_0}$$

leading to the solution

$$w = r^{-\frac{1}{2}\zeta}$$

Hence we obtain

$$r^2(u'' + u\frac{w''}{w}) + \zeta r u \frac{w'}{w} - \frac{n^2\theta_0}{\theta_0 + \psi_0} u - r^{-\xi}\left[\frac{-2r^2\lambda}{\theta_0 + \psi_0} + \frac{in(\Gamma_1 + \Gamma_2)}{\pi(\theta_0 + \psi_0)}\right] u = 0,$$

For $w = r^{-\frac{1}{2}\zeta}$, we have $w' = -\frac{1}{2}\zeta r^{-\frac{1}{2}\zeta-1}$ and $w'' = (-\frac{1}{2}\zeta)(-\frac{1}{2}\zeta-1)r^{-\frac{1}{2}\zeta-2}$, following

$$\frac{w''}{w} = r^{-2}(-\frac{1}{2}\zeta)(-\frac{1}{2}\zeta - 1)$$

and

$$\frac{w'}{w} = r^{-1}(-\frac{1}{2}\zeta)$$

Inserting this yields

$$r^2u'' + \frac{\zeta^2}{4}u - \frac{n^2\theta_0}{\theta_0+\psi_0}u - r^{-\xi}\left[\frac{-2r^2\lambda}{\theta_0+\psi_0} + \frac{in(\Gamma_1+\Gamma_2)}{\pi(\theta_0+\psi_0)}\right]u = 0$$

This leads to the Schrödinger-type equation

$$u'' + \left[\left(\frac{\zeta^2}{4} - \frac{n^2\theta_0}{\theta_0+\psi_0}\right)\frac{1}{r^2} - r^{-\xi}\left(\frac{-2\lambda}{\theta_0+\psi_0} + \frac{in(\Gamma_1+\Gamma_2)}{\pi(\theta_0+\psi_0)}\frac{1}{r^2}\right)\right]u = 0 \qquad (2.52)$$

Considering now the asymptotic behavior for large r, we get

$$u'' + r^{-\xi}\frac{2\lambda}{\theta_0+\psi_0}u = 0 \qquad (2.53)$$

Equation (2.53) is again solvable using the Bessel functions (cf. [5]):

$$J_\nu(z) = \sum_{k=0}^{\infty}\frac{(-1)^k(\frac{z}{2})^{2k+\nu}}{\Gamma(\nu+k+1)k!}$$

and

$$Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}.$$

Then

$$u = C_1\sqrt{r}J_{\frac{1}{2q}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}}\frac{r^q}{q}\right) + C_2\sqrt{r}Y_{\frac{1}{2q}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}}\frac{r^q}{q}\right) \qquad (2.54)$$

is a solution of (2.53), whereas $q = \frac{1}{2}(-\xi+2)$ and C_1 and C_2 are arbitrary constants to be derived from given initial conditions (cf. [22]). So we have Bessel functions independent of n of the fixed order $\frac{1}{2-\xi}$ again.

Resubstituting, we get

$$h = r^{-\frac{1}{2}\zeta}u$$

and the general asymptotic solution of the relative part is finally the superposition

$$
\begin{aligned}
h(r,\varphi,t) &= \int d\lambda \sum_n a(n,\lambda) e^{in\varphi} e^{-\lambda t} r^{-\frac{1}{2}\zeta} u \\
&= \int d\lambda \sum_n a(n,\lambda) e^{in\varphi} e^{-\lambda t} r^{\zeta} \cdot \\
&\cdot \left(C_1 J_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right) + C_2 Y_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right) \right) \\
&= \int d\lambda e^{-\lambda t} r^{\zeta} \left(C_1 J_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right) \right. \\
&\left. + C_2 Y_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right) \right) \sum_n a(n,\lambda) e^{in\varphi}
\end{aligned}
$$

Attempt to implement initial conditions

The „arbitrary“ parameters of the equation have to be chosen in a way, so that $h(r,\varphi,t)$ is real and positive for all values of r,φ and t, as it must be a probability density function.

The problem now when trying to implement reasonable initial conditions expressed in terms of the in (2.44) derived version of the δ-function

$$
\delta(r-r_0,\varphi) = \frac{1}{2\pi} \frac{2-\xi}{2} r_0^{\frac{2-\xi}{2}-1} \int_0^\infty dk\, k \sum_{\nu\in\mathbb{Z}} J_\nu(k r_0^{\frac{2-\xi}{2}}) J_\nu(k r^{\frac{2-\xi}{2}}) e^{i\nu\varphi} \tag{2.55}
$$

is the constant order of the Bessel function independently of n in every summand in our solution for the differential equation, which detains us from simply comparing and matching it to the expression of the δ-function. To foreclose, we will not be able to implement a proper initial condition

However, the one thing we can try to do here, is to go forward in an analogous way to the already accomplished calculation, hoping to find that the solution is normalizable. Adding the multiplicative term

$$
J_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right)
$$

by taking

$$
\tilde{a}(n) J_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right) = a(n,\lambda)
$$

leads to

$$h(r,\varphi,t) = \int d\lambda e^{-\lambda t}\left(\tilde{C}_1 J_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right) J_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right)\right.$$

$$\left. + \tilde{C}_2 J_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right) Y_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right)\right)\sum_n \tilde{a}(n)e^{in\varphi}$$

$$= \int d\lambda e^{-\lambda t} C_1 J_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right)\cdot$$

$$\cdot J_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right)\sum_n \tilde{a}(n)e^{in\varphi} \tag{2.56}$$

$$+ \int d\lambda e^{-\lambda t} C_2 J_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right)\cdot$$

$$\cdot Y_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right)\sum_n \tilde{a}(n)e^{in\varphi} \tag{2.57}$$

We treat the two integrals (2.56) and (2.57) separately and substitute

$$\frac{2}{2-\xi}\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} \text{ by } k, \quad \lambda \text{ by } \frac{(2-\xi)^2 k^2(\theta_0+\psi_0)}{8}$$

and get

$$\frac{d\lambda}{dk} = k\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)$$

First, we take a look at the second integral from above (2.57):

$$\int d\lambda e^{-\lambda t} C_2 J_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right) Y_{\frac{1}{2-\xi}}\left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}}\frac{2}{2-\xi}\right)\sum_n \tilde{a}(n)e^{in\varphi}$$

leading to

$$\int dk\ k\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)\exp\left(-\frac{(2-\xi)^2(\theta_0+\psi_0)}{2}k^2 t\right)\cdot$$

$$\cdot\tilde{C}_1 Y_{\frac{1}{(2-\xi)}}(kr^{\frac{(2-\xi)}{2}}) J_{\frac{1}{(2-\xi)}}(kr_0^{\frac{(2-\xi)}{2}})\sum_{n\in\mathbb{Z}} e^{in\varphi}\ \tilde{a}(n) = N(r,\varphi,t)$$

If we want to prove, if $N(r,\varphi,t)$ is normalizable, we have to perform the integration

$$\int_{r_0}^{\infty} rdr \int_0^{2\pi} d\varphi N(r,\varphi,t)$$

and finding a finite value for all t. Note that $I_{\frac{1}{\xi}}$ has still constant order. Angular integration in the intervall $[0,2\pi]$ of this gives a multiplicative 2π. But even when not finding this term in an integral table, integrating it with mathematica shows that it diverges. So we have to choose $\tilde{C}_2 = 0$ to save the possibility of normalization in the case of keeping the ansatz we chose.

Taking a look now at (2.57),

$$\int d\lambda e^{-\lambda t} \tilde{C}_1 J_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r_0^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right) J_{\frac{1}{2-\xi}} \left(\sqrt{\frac{2\lambda}{\theta_0+\psi_0}} r^{\frac{2-\xi}{2}} \frac{2}{2-\xi} \right) \sum_n \tilde{a}(n) e^{in\varphi}$$

leads to

$$h(r,\varphi,t) = \int dk\ k \frac{1}{4} (2-\xi)^2 (\theta_0+\psi_0) \exp\left(-\frac{(2-\xi)^2(\theta_0+\psi_0)}{2} k^2 t \right) \cdot$$
$$\cdot \tilde{C}_1 J_{\frac{1}{(2-\xi)}} (k r^{\frac{2-\xi}{2}}) J_{\frac{1}{2-\xi}} (k r_0^{\frac{2-\xi}{2}}) \sum_{n\in\mathbb{Z}} e^{in\varphi}\ \tilde{a}(n)$$

With the identity

$$\int_0^{\infty} e^{-\rho^2 k^2} J_\nu(\tilde{r}k) J_\nu(\tilde{r}_0 k) k\ dk = \frac{1}{2\rho^2} \exp\left(-\frac{\tilde{r}^2+\tilde{r}_0^2}{4\rho^2} \right) I_\nu \left(\frac{\tilde{r}\tilde{r}_0}{2\rho^2} \right)$$

and $\rho^2 = \frac{(2-\xi)^2(\theta_0+\psi_0)}{8} t$, the modified Bessel function $I_\nu(z) = i^{-\nu} J_\nu(iz)$, $\tilde{r} = r^{\frac{2-\xi}{2}}$ and $\tilde{r_0} = r_0^{\frac{2-\xi}{2}}$ we obtain

$$h(r,\varphi,t) = \frac{1}{2\frac{(2-\xi)^2(\theta_0+\psi_0)t}{8}} \sum_{n\in\mathbb{Z}} \tilde{a}(n) e^{in\varphi} \exp\left(-\frac{r^{2-\xi}+r_0^{2-\xi}}{4\frac{(2-\xi)^2(\theta_0+\psi_0)}{8}t} \right) I_{\frac{1}{2-\xi}} \left(\frac{(rr_0)^{\frac{2-\xi}{2}}}{2\frac{(2-\xi)^2(\theta_0+\psi_0)}{8}t} \right)$$

$$\Leftrightarrow h(r,\varphi,t) = \frac{4}{(2-\xi)^2(\theta_0+\psi_0)t}\ \exp\left(-\frac{r^{2-\xi}+r_0^{2-\xi}}{\frac{1}{2}(2-\xi)^2(\theta_0+\psi_0)t} \right) \cdot$$
$$\cdot I_{\frac{1}{2-\xi}} \left(\frac{(rr_0)^{\frac{2-\xi}{2}}}{\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)t} \right) \sum_{n\in\mathbb{Z}} \tilde{a}(n) e^{in\varphi}$$

Angular integration in the interval $[0,2\pi]$ again gives a multiplicative 2π:

$$\int_0^{2\pi} d\varphi h(r,\varphi,t) = \tilde{N}(r,t) =$$

$$\frac{8\pi}{(2-\xi)^2(\theta_0+\psi_0)t} \exp\left(-\frac{r^{2-\xi}+r_0^{2-\xi}}{\frac{1}{2}(2-\xi)^2(\theta_0+\psi_0)t}\right) I_{\frac{1}{2-\xi}}\left(\frac{(rr_0)^{\frac{2-\xi}{2}}}{\frac{1}{4}(2-\xi)^2(\theta_0+\psi_0)t}\right)\tilde{a}(0)$$

The r-integration $\int_{r=0}^{\infty} r\tilde{N}(r,t)dr$ cannot be accomplished analytically. But at least, numerically integrating with mathematica gives different positive finite values for different ξ, and since one still may choose $\tilde{a}(0)$ in an appropriate way, it should be normalizable. Also the integrand looks quite integrable,

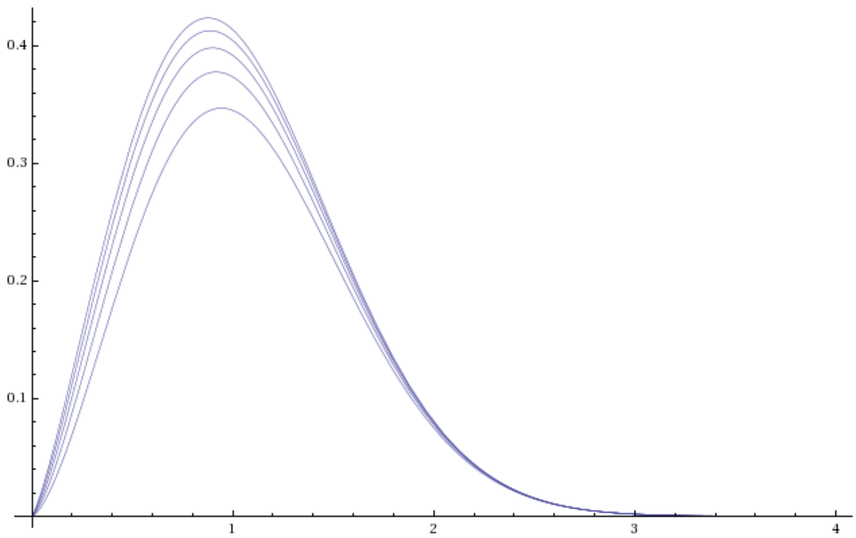


Figure 2.11: Integrand for $\delta \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$

but we are not able to give an analytical solution in this case - by the way, the calculations last quite long. Bessel functions of fractional order seem to cause some problems in the numerics of mathematica.

Hence, we were not able to implement a δ-function as initial condition, but at least it seems that the solution is normalizable for all ξ and that the analogous way to the already performed one did kind of work out here.

2.7 A remark on the analogy to anyon statistics

Because of being a partial differential equation, the Fokker–Planck equation can be solved analytically only in special cases. The formal analogy of the Fokker–Planck equation with the Schrödinger equation allows the use of advanced operator techniques known from quantum mechanics for its solution in a number of cases. An interesting fact one may observe is that the Fokker-Planck equation we derived for the two point vortices can be treated in analogy of the so-called two anyon system - in a consideration of our Fokker-Planck equation in imaginary time results in a Schrödinger equation, which corresponds to the two anyon problem. The timedependence can be separated both in the vortex and the anyon calculation, the rest of the calculation in cylindrical coordinates exhibits the same differential equations.

An anyon can be described as an object consisting of a spinless charge orbiting around a flux. If the charge is proportional to the flux, this object has a fractional spin of

$$s = \frac{q\phi}{4\pi}$$

whereby ϕ is the flux and q is the charge. Due to the fractional spin, anyons obey fractional statistics. In the appendix this feature is presented in the context of two anyons.(A good introduction to the subject of anyons is found in [24]).

2.7.1 The two anyon system

The Hamiltonian of a two anyon system can be given by

$$\hat{H} = \frac{(\mathbf{p}_1 - q\mathbf{a}_1)^2}{2m} + \frac{(\mathbf{p}_2 - q\mathbf{a}_2)^2}{2m}$$

whereby we have the vector potentials

$$\mathbf{a}_i = \frac{\phi}{2\pi} \frac{\hat{z} \times (\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2}$$

for $i, j \in \{1, 2\}$,$i \neq j$. If we work in center of mass and relative coordinates similar like we did before, the Hamilitonian reads

$$\hat{H} = \frac{\mathbf{P}^2}{4m} + \frac{(\mathbf{p} - q\mathbf{a}_{\text{rel}})^2}{m}$$

whereby

$$\mathbf{R} = \frac{\mathbf{r}_1 + \mathbf{r}_2}{2} \Rightarrow \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2$$
$$\mathbf{r} = \frac{\mathbf{r}_1 - \mathbf{r}_2}{2} \Rightarrow \mathbf{p} = \frac{\mathbf{p}_1 - \mathbf{p}_2}{2}$$
$$\mathbf{a}_{\text{rel}} = \frac{\phi}{2\pi} \frac{\hat{z} \times \mathbf{r}}{|\mathbf{r}|^2} = \begin{pmatrix} a_{\text{rel}}^{(r)} \\ a_{\text{rel}}^{(\theta)} \end{pmatrix}$$

in cylindrical coordinates. Solving this quantum mechanical problem now for the center of mass motion (CM), we obtain $E_{\text{CM}} = \frac{\mathbf{P}}{4m}$ and $\psi_{\text{CM}} = e^{i\mathbf{P}\cdot\mathbf{R}}$.

We now consider a bosonic charge orbiting around a bosonic flux, the wavefunction of the system is symmetric under exchange, meaning

$$\psi_{rel}(r, \theta + \pi) = \psi_{rel}(r, \theta) \tag{2.58}$$

For the relative motion we go over to cylindrical coordinates in the Hamiltonian, too, and for $\hbar = 1$ we obtain

$$\hat{H}_{\text{rel}}\psi_{\text{rel}}(r, \theta) = \frac{(\mathbf{p} - q\mathbf{a}_{\text{rel}})^2}{m}\psi_{\text{rel}}(r, \theta) \tag{2.59}$$
$$= \left[-\frac{1}{m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right) + \frac{1}{mr^2}\left(i\frac{\partial}{\partial \theta} - \frac{q\phi}{2\pi}\right)^2 \right] \psi_{\text{rel}}(r, \theta) = E_{\text{rel}}\psi_{\text{rel}}(r, \theta) \tag{2.60}$$

The Hamiltonian is separable in r and θ, so we have

$$\psi_{\text{rel}}(r, \theta) = h(r)Y_l(\theta)$$

The angular solution can be obtained by solving

$$\left(i\frac{\partial}{\partial \theta} - \frac{q\phi}{2\pi}\right)^2 Y_l(\theta) = \eta Y_l(\theta),$$

leading to

$$Y_l(\theta) = e^{il\theta}$$

with l even to match the boundary condition for bosons (2.58). This, of course, leads to

$$\eta = (l + \frac{q\phi}{2\pi})^2$$

with l even. Now inserting η in the remaining radial equation we obtain an differential equation totally alike to the one we solved for two vortices in an incompressible, Kraichnan velocity field for $\delta = 0$, the standard vortex profile:

$$\left[-\frac{1}{m}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)+\frac{1}{mr^2}\left(l-\frac{q\phi}{2\pi}\right)^2\right]h(r)=E_{\mathrm{rel}}h(r)$$

Multiplying by $-mr^2$ and abbreviating $\frac{\partial}{\partial r}h(r)=h'$ we get

$$r^2h''+rh'+(E_{\mathrm{rel}}mr^2-(l+\frac{q\phi}{2\pi})^2)h=0$$

chapter*Conclusio Comparing this for example to equation (A.1) in the appendix,

$$r^2h''+rh'+(\frac{2\lambda}{\theta_0}r^2-(n^2+\frac{in(\Gamma_1+\Gamma_2}{\pi\theta_0}))h=0$$

we can identify

$$E_{\mathrm{rel}}m \leftrightarrow \frac{2\lambda}{\theta_0}$$
$$(l+\frac{q\phi}{2\pi})^2 \leftrightarrow (n^2+\frac{in(\Gamma_1+\Gamma_2}{\pi\theta_0})$$

and find the predicted analogy between the two vortex system we investigated on and the two anyon system. The solutions we derived just would have to be modified by renaming, if one wanted to present the solutions for the two anyon system.

Conclusion

The central issue of this thesis was the investigation on the dynamics of point vortices in an incompressible and statistical isotropic Kraichnan velocity field. We combined both deterministic dynamics and methods of statistical physics to model a simple system containing coherent structures as well as fluctuating flow fields, which are characteristics of turbulent fluid motion. To obtain an appropriate description of the system, we derived a Fokker-Planck equation describing the stochastical dynamics and solved it subsequently.

The thesis was partitioned into two chapters. The first chapter introduced the methods and the theoretical foundations we needed as a basis for our investigations. After a brief introduction to hydrodynamics and especially vortex dynamics, we presented the Kraichnan model. Despite the fact that Kraichnan velocity fields are a useful tool to model short-time correlated velocity fields, they deliver important simplifications in our analytical calculations due to their decorrelation in time. Then we derived a general formalism to carry over from a finite set of Langevin equations to one Fokker-Planck equation.

The second chapter was the main part of the thesis, first, we briefly considered one vortex described by one Langevin equation. The equation was lacking the deterministic interaction and reduced to a simple Brownian motion.
In the following treatment of two vortices, we first made a change to central and relative coordinates in the two Langevin equations and subsequently we applied the formalism derived in the theoretical foundations.
We decoupled the obtained Fokker-Planck equation into a central and relative part for vortices with arbitrary, different circulation. Solving the central motion was in many cases approximately and sometimes even exactly reduced to solving the heat equation. Separating the motion required a consideration in terms of conditional probability. For vortices with the same circulation and with antipodal circulation the solutions of the central and relative motion decouple were statistically independent and the complete joint probability density function reduced to a simple product of the relative and central part. For different circulations, one can only treat the decoupling with conditioned probability densities, for example one can consider an exact relative motion and a conditioned central motion. In the sense of a Born-Oppenheimer approximation, we assumed now that the relative motion and the central motion do not affect each other that much and limited ourselves to the consideration of a Brownian motion of the center of vorticity and an exact solution of the relative part. Visually, we modelled the different cases with a modified Runge-

Kutta method including white noise.
Now exactly solving the relative part was the more complicated part of the work. We first derived and implemented some consecutions of incompressibility and isotropic statistics. Then we made a change to polar coordinates and applied a mode ansatz on the obtained Fokker-Planck equation to reduce the problem of solving ordinary differential equations. We eventually derived solutions in the case of scaled diffusions coefficients and a particularly matching generalized vortex profile, solving the ordinary differential equations and implementing a δ-function as a proper initial condition. As we have shown, these solutions pass over into already known solutions in limiting cases; the known solutions are derived in the appendix. After having solved the equations, the solutions were presented graphically and commented. The case of a scaled diffusion matrix and a standard (not scaled) vortex profile was investigated asymptotically, but unsatisfyingly we were not able to implement proper initial conditions. At least, it seems that the solutions are normalizable. Finally we described the analogy of the two point vortex system to the system of two free anyons, which behave statistically in an analogous way.

To find out more about stochastic vortex dynamics in future, one could concern the analogy to the anyons, for example there exist some classes of solutions in the n-anyon problem [17], which perhaps may be carried over to solve similar n-vortex systems.

A Appendix

A.1 The special case $\xi = 0$ - direct derivation

We investigate equation(2.50), but this time in the case $\xi = 0$. This has already been done in a similar way in ([3]), the calculation is often analogous. The benefit of performing it anyway, it is that it is a possibility to control, if the generalized results for a not constant correlation matrix and the modified vortex profile $v(r) = \frac{1}{2\pi r^{1-\xi}}$ match to the simple case in the limit, say for $\delta \to 0$

$\delta 00$ implies also $\psi_0 = 0$ (compare equation (2.35), leading to

$$\begin{aligned} r^2 h'' + rh' - n^2 h - \left[-\frac{2r^2\lambda}{\theta_0} + \frac{in(\Gamma_1 + \Gamma_2)}{\pi\theta_0} \right] h = 0 \\ \Leftrightarrow r^2 h'' + rh' + \left[\frac{2r^2\lambda}{\theta_0} - (n^2 + \frac{in(\Gamma_1 + \Gamma_2)}{\pi\theta_0}) \right] h = 0 \end{aligned} \tag{A.1}$$

This again can be solved in terms of the Bessel functions of the first and second kind

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{2k+\nu}}{\Gamma(\nu + k + 1)k!} \quad , \quad Y_\nu(z) = \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}$$

We get (cf.[22])

$$h(r) = C_1 J_\nu \left(\sqrt{\frac{2\lambda}{\theta_0}} r \right) + C_2 Y_\nu \left(\sqrt{\frac{2\lambda}{\theta_0}} r \right) \tag{A.2}$$

as a solution of $(A.1)$, whereas C_1 and C_2 are arbitrary constants to be derived from any given initial condition and $\nu = \sqrt{n^2 + \frac{in(\Gamma_1+\Gamma_2)}{\pi\theta_0}}$.

So the complete and generalized solution of the relative part reads (cf. [22])

$$\begin{aligned} h(r,\varphi,t) &= \sum_{n\in\mathbb{Z}} \int d\lambda\ a(n,\lambda) e^{in\varphi} e^{-\lambda t} h_{n,\lambda}(r) \\ &= \sum_{n\in\mathbb{Z}} \int d\lambda\ a(n,\lambda) e^{in\varphi} e^{-\lambda t} \left[C_1 J_\nu \left(\sqrt{\frac{2\lambda}{\theta_0}} r \right) + C_2 Y_\nu \left(\sqrt{\frac{2\lambda}{\theta_0}} r \right) \right] \end{aligned} \tag{A.3}$$

Implementation of initial conditions

Now we again want to implement an initial condition in terms of a Dirac delta

$$h(r,\phi,0) = \delta(r - r_0, \varphi) \tag{A.4}$$

and match it to our general solution (2.39) from the previous subsection. Exactly like in the previous section we need an expression for the Dirac delta in terms of Bessel functions. We already accomplished this task using the Jacobi-Anger expansion (cf. [1] p. 361) on the Fourier transform of the Dirac delta, and then applying Gegenbauer's additions theorem to obtain the expression

$$\begin{aligned}\delta(\mathbf{r}-\mathbf{r}_0) &= \frac{1}{2\pi}\int_0^\infty dk\ k\left[J_0(kr_0)J_0(kr) + 2\sum_{\nu=1}^{\infty} J_\nu(kr_0)J_\nu(kr)\cos(\nu\varphi)\right]\\ &= \frac{1}{2\pi}\int_0^\infty dk\ k\sum_{\nu\in\mathbb{Z}} J_\nu(kr_0)J_\nu(kr)e^{i\nu\varphi}\end{aligned}$$

Now comparing this to equation (A.3)

$$h(r,\varphi,t) = \sum_{n\in\mathbb{Z}}\int d\lambda\ a(n,\lambda)e^{in\varphi}\exp\left(-\lambda t\right)\left[C_1 J_\nu\left(\sqrt{\frac{2\lambda}{\theta_0}}r\right) + C_2 Y_\nu\left(\sqrt{\frac{2\lambda}{\theta_0}}r\right)\right]$$

for $t = 0$ and $\phi = 0$, we immediately see that we have to choose $C_2 = 0$. Further, we have to substitute $\sqrt{\frac{2\lambda}{\theta_0}}$ by k, or λ by $\frac{k^2\theta_0}{2}$, respectively. This leads to

$$h(r,\varphi,t) = \sum_{n\in\mathbb{Z}} e^{in\varphi}\int dk\ k\theta_0\ \tilde{a}(n,k)\exp\left(-\frac{\theta_0}{2}k^2 t\right) C_1 J_\nu(kr)$$

The last thing to do for now is to identify $\tilde{a}(n,k)$ with $\frac{1}{C_1\theta_0}\frac{1}{2\pi}J_\nu(kr_0)$, so we finally obtain

$$h(r,\varphi,t) = \frac{1}{2\pi}\sum_{n\in\mathbb{Z}} e^{in\varphi}\int dk\ k\exp\left(-\frac{\theta_0}{2}k^2 t\right) J_\nu(kr)J_\nu(kr_0) \tag{A.5}$$

Note that $\nu = \sqrt{n^2 + \frac{in(\Gamma_1+\Gamma_2)}{\pi\theta_0}}$ is $\propto n$ - and becomes exactly equal to n for $(\Gamma_1 + \Gamma_2) \to 0$. We now additionally show that the initial condition is also verified for $(\Gamma_1 + \Gamma_2) \neq 0$: With the identity

$$\int_0^\infty e^{-\rho^2 k^2} J_\nu(rk)J_\nu(r_0 k)k\ dk = \frac{1}{2\rho^2}\exp\left(-\frac{r^2+r_0^2}{4\rho^2}\right) I_\nu\left(\frac{rr_0}{2\rho^2}\right)$$

we obtain for $\rho^2 = \frac{\theta_0}{2}t$ and the modified Bessel function $I_\nu(z) = i^{-\nu}J_\nu(iz)$ the solution for the relative part(compare [3], p.7)

$$h(r,\varphi,t) = \frac{1}{2\pi\theta_0 t}\exp\left(-\frac{r^2+r_0^2}{2\theta_0 t}\right)\sum_{n\in\mathbb{Z}} e^{in\varphi} I_\nu\left(\frac{rr_0}{\theta_0 t}\right) \tag{A.6}$$

$$= H(r,t)\left[1 + 2\Re\left(\sum_{n=1}^{\infty} I_0^{-1}\left(\frac{rr_0}{\theta_0 t}\right) I_\nu\left(\frac{rr_0}{\theta_0 t}\right)e^{in\varphi}\right)\right] \tag{A.7}$$

$$\tag{A.8}$$

whereas $H(r,t)$ reads

$$H(r,t) = \frac{1}{2\pi\theta_0 t}\exp\left(-\frac{r^2+r_0^2}{2\theta_0 t}\right) I_0\left(\frac{rr_0}{\theta_0 t}\right) \tag{A.9}$$

which matches exactly the generalized case as shown above.

Remark In [3], the Fokker-Planck equation was given by

$$\frac{\partial}{\partial t}P(r,\theta,t) = -\frac{1}{r^2}\frac{\partial}{\partial\theta}P(r,\theta,t) + \nu\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)P(r,\theta,t)$$

and its solution by

$$P(r,\theta,t) = G(r,t)\left[1 + 2\Re\left(\sum_{p=1}^{\infty} I_0^{-1}\left(\frac{r}{2\nu t}\right) I_{\mu_p}\left(\frac{r}{2\nu t}\right)e^{ip\theta}\right)\right]$$

with

$$G(r,t) = \frac{1}{4\pi\nu t}\exp\left(-\frac{r^2+1}{4\nu t}\right) I_0\left(\frac{r}{2\nu t}\right) \tag{A.10}$$

whereby the choice was $r_0 = 1$

So the results are compatible.

A.2 Fractional statistics of anyons

We consider

$$\hat{H} = \frac{\mathbf{P}^2}{4m} + \frac{(\mathbf{p} - q\mathbf{a}_{\text{rel}})^2}{m}$$

whereby

$$\mathbf{a}_{\text{rel}} = \frac{\phi}{2\pi} \frac{\hat{z} \times \mathbf{r}}{|\mathbf{r}|^2}$$

The center of mass motion thus follows a free translation. However the relative one depends on whether the orbiting charge is bosonic, fermionic or corresponds to an anyon. It has reduced to a system of a single charge of mass $\frac{m}{2}$ orbiting around a flux at a distance $\mathbf{r}$, proposing cylindrical coordinates $\mathbf{r} = (r, \theta)$. Thus for the vector potential

$$\mathbf{a}_{\text{rel}} = \begin{pmatrix} a_{\text{rel}}^{(r)} \\ a_{\text{rel}}^{(\theta)} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\phi}{2\pi r} \end{pmatrix}$$

If we take a bosonic charge orbiting around a bosonic flux, the wavefunction of the system is symmetric under exchange, meaning $\psi_{rel}(r, \theta + \pi) = \psi_{rel}(r, \theta)$.
If we perform a gauge transformation now

$$\mathbf{a}_{\text{rel}} \rightarrow \mathbf{a'}_{\text{rel}} = \mathbf{a}_{\text{rel}} - \nabla\Lambda(r\theta) = \mathbf{a}_{\text{rel}} - \nabla(\frac{\phi}{2\pi}\theta$$

we have

$$\begin{pmatrix} a'^{(r)}_{\text{rel}} \\ a'^{(\theta)}_{\text{rel}} \end{pmatrix} = \begin{pmatrix} a_{\text{rel}}^{(r)} - \frac{1}{r}\frac{\partial}{\partial\theta}(\frac{\phi\theta}{2\pi}) \\ a_{\text{rel}}^{(\theta)} \end{pmatrix} = \begin{pmatrix} \frac{\phi}{2\pi r} - \frac{\phi}{2\pi r} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and hence the Hamiltonian can be given by

$$\hat{H} = \frac{\mathbf{P}^2}{4m} + \frac{\mathbf{p}^2}{m},$$

the Hamiltonian of two free particles. Now we have to take into account that the wavefunction of the relative Hamiltonian has changed, we have

$$\psi'_{\text{rel}}(r, \theta) = e^{-iq\Lambda}\psi_{rel}(r, \theta) = \exp(-i\frac{q\phi}{2\pi}\theta)\psi_{rel}(r, \theta)$$

Note that this wavefunction is no longer symmetric under $\mathbf{r} \rightarrow -\mathbf{r}$, because

$$\psi'_{\text{rel}}(r, \theta + \pi) = \exp(-i\frac{q\phi}{2})\psi'_{\text{rel}}(r, \theta) = e^{-i\alpha}\psi'_{rel}(r, \theta)$$

So the initially assumed two objects with bosonic charge orbiting around a bosonic flux behave like two particles developing a phase $e^{-i\alpha}$ under exchange, meaning that they follow fractional statistics.

A.3 Modified Runge-Kutta method - source code

```
PROGRAM punktwirbel

IMPLICIT NONE

INTEGER :: n_iter=2000, i
REAL :: dt=0.1,Q
REAL, DIMENSION(2) :: x1,x2,k1_1,k1_2,k1_3,k1_4,k2_1,k2_2,
    k2_3,k2_4,bm1,bm2

!Startpunkte
x1=(/0,0/)
x2=(/1,1/)

Q=0.000001 !Rauschparameter

!Iteration
OPEN(9,FILE="tra.dat")

DO i=1,n_iter

   CALL calc_rhs(x1,x2,k1_1,k2_1)
   CALL calc_rhs(x1+dt/2.0*k1_1,x2+dt/2.0*k2_1,k1_2,k2_2)
   CALL calc_rhs(x1+dt/2.0*k1_2,x2+dt/2.0*k2_2,k1_3,k2_3)
   CALL calc_rhs(x1+dt*k1_3,x2+dt*k2_3,k1_4,k2_4)
   CALL gausZufallBM(bm1)
   CALL gausZufallBM(bm2)

   x1=x1+dt/6.0*(k1_1+2.0*k1_2+2.0*k1_3+k1_4) + sqrt(2*Q*dt)*
       bm1
   x2=x2+dt/6.0*(k2_1+2.0*k2_2+2.0*k2_3+k2_4) + sqrt(2*Q*dt)*
       bm2

   PRINT*,"i",i
   WRITE(9,*) dt*i,x1(1:2),x2(1:2)

END DO
CLOSE(9)
!
```

```
    <<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<<

END PROGRAM

!
    ************************************************************
!
    ************************************************************

SUBROUTINE calc_rhs(x1_in,x2_in,x1_out,x2_out) !Subroutine
    Runge Kutta

IMPLICIT NONE

REAL :: gamma1=1.0, gamma2=-1.5
REAL :: r, pi
REAL, DIMENSION(2), INTENT(IN) :: x1_in, x2_in
REAL, DIMENSION(2), INTENT(OUT) :: x1_out, x2_out

pi=ACOS(-1.0)

r=((x2_in(1)-x1_in(1))**2+(x2_in(2)-x1_in(2))**2)**0.5

x1_out=((/-(x2_in(2)-x1_in(2)),x2_in(1)-x1_in(1)/)*gamma1)
    /(2.0*pi*r**2)

x2_out=((/-(x1_in(2)-x2_in(2)),x1_in(1)-x2_in(1)/)*gamma2)
    /(2.0*pi*r**2)

END SUBROUTINE

SUBROUTINE gausZufallBM(box_mueller) !Subroutine gaussche
    Zufallszahl

IMPLICIT NONE

REAL :: pi
REAL, DIMENSION(2), INTENT(INOUT) :: box_mueller
REAL, DIMENSION(2) :: random
CALL random_number(random(1))
```

```
CALL random_number(random(2))
pi=ACOS(-1.0)

box_mueller(1) = sqrt(-2*log(1-random(1)))*cos(2*pi*random
    (2));
box_mueller(2) = sqrt(-2*log(1-random(1)))*sin(2*pi*random
    (2));

END SUBROUTINE
```

Bibliography

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions.* U.S. Dept. of Commerce, National Institute of Standards and Technology, 1972.

[2] O. Agullo and A. Verga. Exact two vortices solution of navier-stokes equations. *Physical Review Letters*, 78:2361, 1997.

[3] O. Agullo and A. Verga. Effect of viscosity in the dynamics of two point vortices: Exact results. *Phys. Rev. E*, 63(5):056304, Apr 2001.

[4] D. Bernard. Turbulence for (and by) amateurs, 2000.

[5] S. Bronstein. *Teubner-Taschenbuch der Mathematik.* B. G. Teubner Stuttgart Leipzig, 1996.

[6] H. J. H. Clercx and G. J. F. van Heijst. Two-dimensional navier–stokes turbulence in bounded domains. *Applied Mechanics Reviews*, 62:020802 (25 pages), 2009.

[7] G. L. Eyink and K. R. Sreenivasan. Onsager and the theory of hydrodynamic turbulence. *Rev. Mod. Phys.*, 78(1):87, Jan 2006.

[8] G. Falkovich, K. Gawedzki, and M. Vergassola. Particles and fields in fluid turbulence. *Reviews of Modern Physics*, 73:913, 2001.

[9] R. Friedrich. Vortrag am 18.1.2010. none, 2010.

[10] U. Frisch. *Turbulence - The Legacy of A. N. Kolmogorov.* Cambridge University Press, 1995.

[11] K. Gawedzki and M. Vergassola. Phase transition in the passive scalar advection. *Physica D: Nonlinear Phenomena*, 138(1-2):63 – 90, 2000.

[12] H. Haken. *Synergetics - An Introduction.* Springer-Verlag Berlin Heidelberg New York, 1977.

[13] C. Harper. *Analytical Mehtods in Physics.* Wiley, 1999.

[14] W. Hoeber. *Repititorium Theoretische Physik, 2. Auflage.* Springerverlag Berlin Heidelberg New York, 2005.

[15] M. H. R. F. John Argyris, Gunter Faust. *Die Erforschung des Chaos.* Springer-Verlag Berlin Heidelberg, 2010.

[16] E. Kamke. *Differentialgleichungungen.* Akademische Verlagsgesellschaft Geest & Portig K.-G., 1969.

[17] A. Khare. A class of exact solutions for n anyons in a n-body potential. *Physics Letters A*, 221(6):365 – 369, 1996.

[18] A. N. Kolmogorov. The local structure of turbulence in incompressible viscous fluid for very large reynolds numbers. *Proceedings: Mathematical and Physical Sciences*, 434(1890):9–13, 1991.

[19] R. H. Kraichnan. Small-scale structure of a scalar field converted by turbulence. *Phys. Fluids*, 11:945, 1968.

[20] R. H. Kraichnan and D. Montgomery. Two-dimensional turbulence. *Reports on Progress in Physics*, 43(5):547–619, 1980.

[21] J. Miller, P. B. Weichman, and M. C. Cross. Statistical mechanics, euler's equation, and jupiter's red spot. *Phys. Rev. A*, 45(4):2328–2359, Feb 1992.

[22] A. D. Polyanin and V. F. Zaitsev. *Handbook of Exact Solutions for Ordinary Differential Equations, 2nd Edition.* Chapman & Hall/CRC, Boca Raton, 2003.

[23] J. P. R. Friedrich. Fluid dynamics: Turbulence. Institute for Theoretical Physics, University of Muenster, Institute of Physics, Carl-von-Ossietzky University Oldenburg.

[24] S. Rao. An anyon primer, 1992.

[25] O. Reynolds. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. *Philosophical Transactions of the Royal Society of London*, 174:935–982, 1883.

[26] H. Risken. *The Fokker-Planck equation. Methods of Solution and Applications.* Springer Verlag Berlin Heidelberg New York Tokyo, 1984.

[27] N. Svartholm. Die lösung der fuchsschen differentialgleichung zweiter ordnung durch hypergeometrische polynome. *Math. Ann.*, 116:413–421, 1938.

[28] P. Tabeling. Two-dimensional turbulence: a physicist approach. *Physics Reports*, 362(1):1 – 62, 2002.

Danksagung

Abschließend möchte ich allen danken, die zum Gelingen dieser Arbeit beigetragen haben, insbesondere danke ich Prof. Dr. R. Friedrich für das interessante Thema der Diplomarbeit und die freundliche Betreuung zu jedem Zeitpunkt der Bearbeitungszeit. Außerdem danke ich Michael Wilczek, Oliver Kamps, Robert Hahn, Christoph Honisch, Johannes Greber, Michel Voskuhle, Michael Köpf und Anton Daitche für Hilfestellungen bei verschiedensten Problemen und für inspirierende Diskussionen.
Meiner Familie, und ganz besonders meinen Eltern Norbert und Monika Rath, danke ich für deren Unterstützung, ohne die mir mein Studium wahrscheinlich nicht möglich gewesen wäre.
Bei allen Mitarbeiterinnen und Mitarbeitern des Instituts für Theoretische Physik bedanke ich mich für ihre Hilfsbereitschaft und die angenehme Arbeitsatmosphäre.

Hiermit versichere ich, die vorliegende Arbeit selbstständig angefertigt und mich keiner anderen als der angegebenen Quellen und Hilfsmittel bedient zu haben.

....	....
Datum	Unterschrift