Westfälische Wilhelms-Universität Münster

Center for Nonlinear Science

institut für theoretische physik

Statistical Description of Rayleigh–Bénard Convection yields Limit Cycle Behavior

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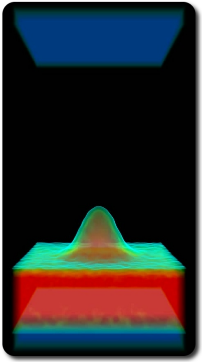

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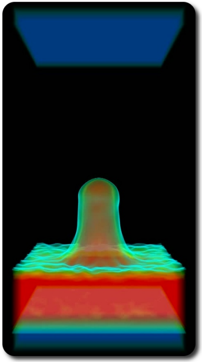
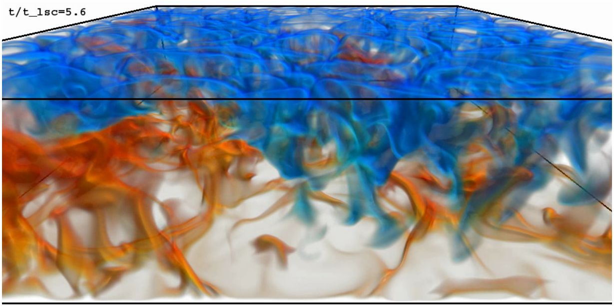


Rayleigh–Bénard Convection



- Turbulent convection common in nature and applications, yet hard to handle analytically
- Our focus: Statistical Description that connects to the Dynamics

Governing Equations

Oberbeck-Boussinesq Equations in Non-dimensional Form

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathrm{Pr}\, \Delta \mathbf{u} + \mathrm{Pr}\, \mathrm{Ra}\, T \mathbf{e}_z \quad , \quad \nabla \cdot \mathbf{u} = 0$$

Control Parameters

- Ra – Rayleigh number ($\propto$ temperature difference)
- Pr – Prandtl number ($= \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$, material parameter)
- Γ – Aspect ratio (geometry parameter)

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Statistical Description in Terms of Temperature PDF

Deriving an Evolution Equation for Temperature PDF:

- Define temperature PDF as ensemble average $\langle \cdot \rangle$ of δ-distribution: $f(T, \mathbf{x}, t) = \big\langle \delta\big(T - T(\mathbf{x}, t)\big) \big\rangle$
- Calculate and put together derivatives of PDF
- Introduce conditional averages $\langle \cdot | T, \mathbf{x}, t \rangle$
- Plug in Oberbeck-Boussinesq equations

Evolution Equation for Temperature PDF

$$\frac{\partial}{\partial t} f + \nabla \cdot \left(\left\langle \mathbf{u} \middle| T, \mathbf{x}, t \right\rangle f \right) = -\frac{\partial}{\partial T} \left(\left\langle \frac{\partial}{\partial t} T + \mathbf{u} \cdot \nabla T \middle| T, \mathbf{x}, t \right\rangle f \right)$$
$$= -\frac{\partial}{\partial T} \left(\left\langle \Delta T \middle| T, \mathbf{x}, t \right\rangle f \right)$$

Method of Characteristics yields average behaviour

Evolution Equation for Temperature PDF

$$\frac{\partial}{\partial t} f + \nabla \cdot \left(\left\langle \mathbf{u} \middle| T, \mathbf{x}, t \right\rangle f \right) = -\frac{\partial}{\partial T} \left(\left\langle \Delta T \middle| T, \mathbf{x}, t \right\rangle f \right)$$

- Apply Method of Characteristics to evolution equation of temperature PDF
- One obtains characteristic curves, i. e. trajectories along which the evolution equation transforms from a PDE into an ODE

Method of Characteristics

- Characteristics follow the vector field that is determined by the conditional averages:

Vector Field of Characteristics

$$\begin{pmatrix} \dot{T} \\ \dot{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, \mathbf{x} \rangle \\ \langle \mathbf{u} | T, \mathbf{x} \rangle \end{pmatrix}$$

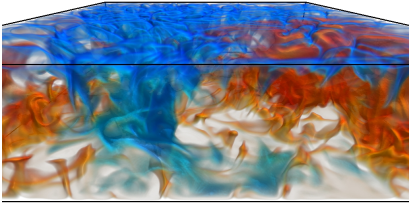
- Solutions / characteristics $\begin{pmatrix} T(t) \\ \mathbf{x}(t) \end{pmatrix}$ show average behavior of a fluid particle in T-$\mathbf{x}$-phase space
- This general framework is now applied to 3 different RB settings with different symmetries; $\langle \cdot | T, \mathbf{x} \rangle$ are estimated from 3 different DNS

Contents

Periodic Boundary Conditions 3D

RB System

- 3D convection, homogenous in horizontal direction
- No-slip bottom and top plates $\mathbf{u} = \mathbf{0}$
- Parameters: $\mathrm{Ra} = 2 \cdot 10^7$, $\mathrm{Pr} = 1$, $\Gamma = 4$
- Numerics: Pseudospectral and volume penalization methods



Periodic Boundary Conditions 3D

Symmetries

- Statistics do not depend on horizontal coordinates:

$$f(T, \mathbf{x}) = f(T, z)$$
$$\langle \cdot | T, \mathbf{x} \rangle = \langle \cdot | T, z \rangle$$

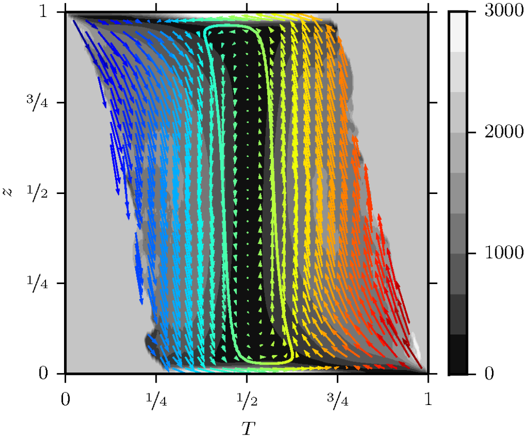
- Phase space becomes 2D, spanned by T and z
- Characteristics simplify to:

$$\begin{pmatrix} \dot{T} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, z \rangle \\ \langle u_z | T, z \rangle \end{pmatrix}$$

Periodic Boundary Conditions 3D

Integrating characteristics for arb. initial positions: Convergence towards Limit Cycle!

- Blue→red: Temperature
- Arrows: vector field of characteristics
- Black→white: Norm of vector field (i.e., phase space velocity)



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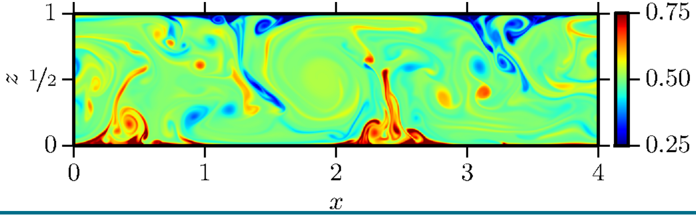
Periodic Boundary Conditions 2D

RB System

- 2D convection, homogenous in horizontal direction
- No-slip bottom and top plates $\mathbf{u} = \mathbf{0}$
- Parameters: $Ra = 5 \cdot 10^8$, $Pr = 1$, $\Gamma = 4$
- Symmetries: Identical to 3D periodic case



Periodic Boundary Conditions 2D

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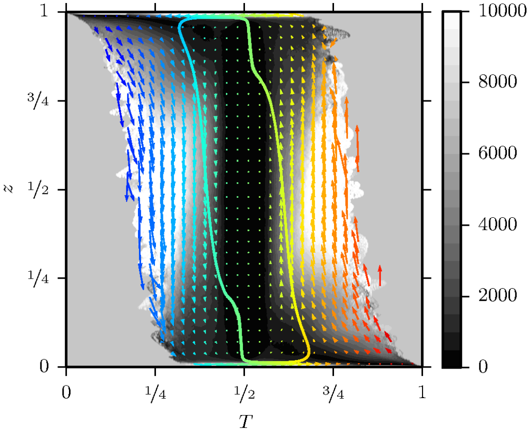

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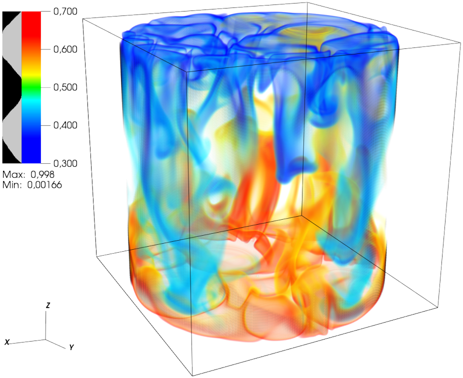
Cylindrical Vessel



- Cylindrical Vessel with insulating sidewalls
- Cylindrical coordinates: $\mathbf{x} = (r, \varphi, z)$
- All surfaces are no-slip $\mathbf{u} = \mathbf{0}$
- Parameters: $\mathrm{Ra} = 2 \cdot 10^8$, $\mathrm{Pr} = 1$, $\Gamma = 1$
- Numerics: Finite differences w/ gridpoint clustering

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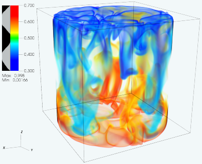
Cylindrical Vessel

Symmetries



- Homogeneous in azimuthal (φ-) direction:

$$f(T,\mathbf{x}) = f(T,r,z)$$
$$\langle\cdot|T,\mathbf{x}\rangle = \langle\cdot|T,r,z\rangle$$

Vector field of Characteristics

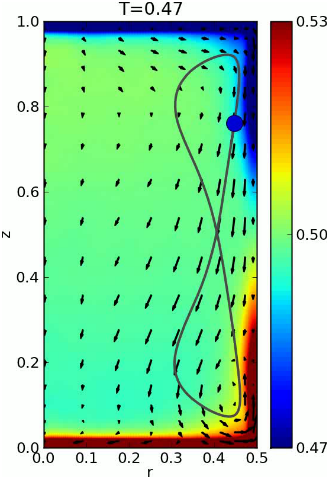
$$\begin{pmatrix} \dot{T} \\ \dot{r} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \langle \Delta T|T,r,z\rangle \\ \langle u_r|T,r,z\rangle \\ \langle u_z|T,r,z\rangle \end{pmatrix}$$

- Additional phase space dimension (radial movement)

Cylindrical Vessel

Again, Limit Cycle is found!

- RB Cycle:
  - Heating up / moving outwards at the bottom
  - Moving up / inwards in the bulk
  - Cooling down / moving outwards at the top, ...
- Cornerflows w/o need for a LSC
- Limit Cycle lies in outer regions, $0.3 < r < 0.5$, and around the mean temperature, $0.47 < T < 0.53$



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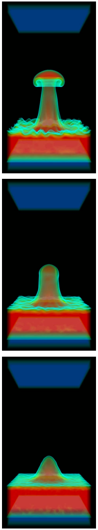
Summary

- We derived an evolution equation for the PDF of temperature from first principles
- Unclosed terms are expressed via conditional averages, which are estimated from DNS
- The Method of Characteristics is used to link statistics and dynamics of the system
- The framework allows to identify a limit cycle, which shows the average transport processes in Rayleigh–Bénard convection for three different cases ($2\times$ 2D / $1\times$ 3D phase space)
- Outlook: Further investigation of limit cycle



Thank you for your attention!

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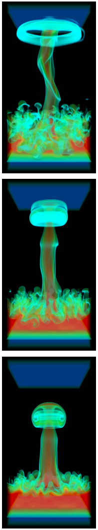
Rudolf Friedrich
† August 2012



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