Westfälische Wilhelms-Universität Münster

Center for Nonlinear Science

institut für theoretische physik

Using PDF Equations to describe Rayleigh–Bénard Convection

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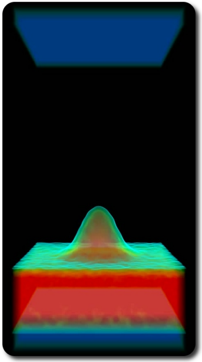
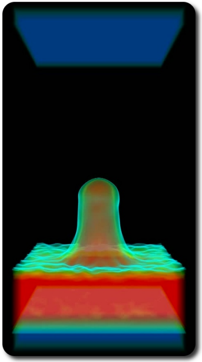
Contents



Contents



- Introduction

Temperature Statistics and PDF Equations

(a) Periodic Boundary Conditions 2D

(b) Periodic Boundary Conditions 3D

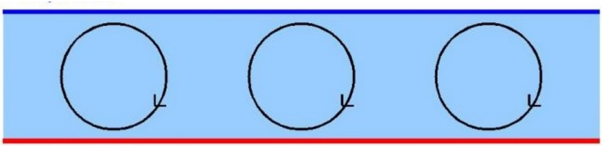
(c) Cylindrical Vessel

Summary

Phenomenon

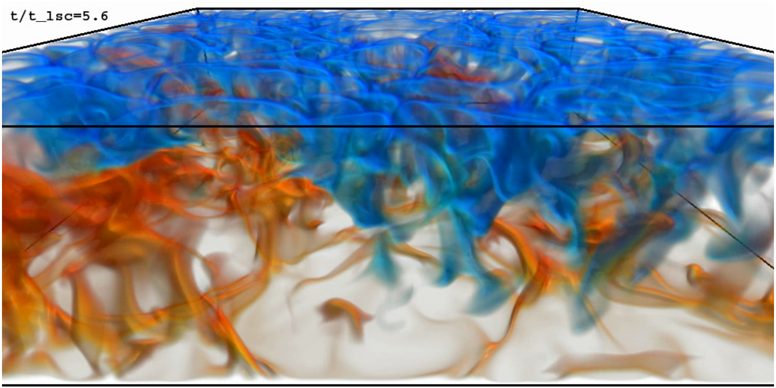
Rayleigh–Bénard Convection:

- Heated from below, cooled from above
- Ubiquitous in nature: Atmosphere, ozeans, plate tectonics, ...
- Different patterns, from stable laminar to highly turbulent flows
- Turbulent flows common in nature and applications, yet hard to handle analytically
- Our focus: Statistical Description that connects to the Dynamics



Phenomenon

Turbulent Rayleigh–Bénard Convection



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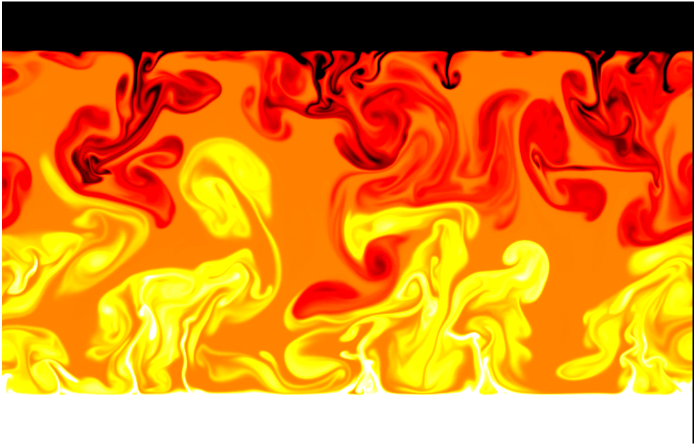
Phenomenon

Turbulent Rayleigh–Bénard Convection



Governing Equations

Oberbeck-Boussinesq Equations in Non-dimensional Form

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathrm{Pr}\, \Delta \mathbf{u} + \mathrm{Pr}\, \mathrm{Ra}\, T \mathbf{e}_z \quad , \quad \nabla \cdot \mathbf{u} = 0$$

Control Parameters

- Ra – Rayleigh number ($\propto$ temperature difference)
- Pr – Prandtl number ($= \frac{\text{kinematic viscosity}}{\text{thermal diffusivity}}$, material parameter)
- Γ – Aspect ratio (geometry parameter)

Contents

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Goal: A Statistical Description from First Principles

Laminar convection: Analytical solution of basic equations is possible

Turbulent convection: Analytical solution is (up to now) impossible

But:

- Analytical solution not necessarily needed
- Compare ideal gas: We can't predict 10^{23} particles, but we still can predict temperature, pressure, energy, ...

Goal:

- Achieve statistical description for turbulent Rayleigh–Bénard convection in terms of probability density function (PDF) of temperature!

Statistical Description in Terms of Temperature PDF

Deriving an Evolution Equation for Temperature PDF:

- Define temperature PDF as ensemble average $\langle\cdot\rangle$ of δ-distribution: $f(T, \mathbf{x}, t) = \langle \delta(T - T(\mathbf{x}, t)) \rangle$
- Calculate and put together derivatives of PDF
- Introduce conditional averages $\langle \cdot | T, \mathbf{x}, t \rangle$
- Plug in Oberbeck-Boussinesq equations

Evolution Equation for Temperature PDF

$$\frac{\partial}{\partial t} f + \nabla \cdot \left(\left\langle \mathbf{u} \middle| T, \mathbf{x}, t \right\rangle f \right) = -\frac{\partial}{\partial T} \left(\left\langle \frac{\partial}{\partial t} T + \mathbf{u} \cdot \nabla T \middle| T, \mathbf{x}, t \right\rangle f \right)$$
$$= -\frac{\partial}{\partial T} \left(\left\langle \Delta T \middle| T, \mathbf{x}, t \right\rangle f \right)$$

Method of Characteristics yields average behaviour

Evolution Equation for Temperature PDF

$$\frac{\partial}{\partial t} f + \nabla \cdot \left(\left\langle \mathbf{u} \middle| T, \mathbf{x}, t \right\rangle f \right) = -\frac{\partial}{\partial T} \left(\left\langle \Delta T \middle| T, \mathbf{x}, t \right\rangle f \right)$$

- Apply Method of Characteristics to evolution equation of temperature PDF
- One obtains characteristic curves, i. e. trajectories along which the evolution equation transforms from a PDE into an ODE

Method of Characteristics

- Characteristics follow the vector field that is determined by the conditional averages:

Vector Field of Characteristics

$$\begin{pmatrix} \dot{T} \\ \dot{\mathbf{x}} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, \mathbf{x} \rangle \\ \langle \mathbf{u} | T, \mathbf{x} \rangle \end{pmatrix}$$

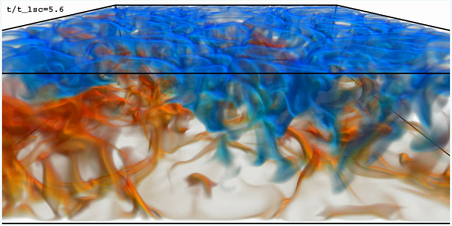
- Solutions / characteristics $\begin{pmatrix} T(t) \\ \mathbf{x}(t) \end{pmatrix}$ show average behavior of a fluid particle in T-$\mathbf{x}$-phase space
- This general framework is now applied to 3 different RB settings with different symmetries;
  $\langle \cdot | T, \mathbf{x} \rangle$ are estimated from 3 different DNS

Contents

Periodic Boundary Conditions 3D

Rayleigh–Bénard System

- 3D convection, homogenous in horizontal direction
- No-slip bottom and top plates $\mathbf{u} = \mathbf{0}$
- Parameters: $\mathrm{Ra} = 2 \cdot 10^7$, $\mathrm{Pr} = 1$, $\Gamma = 4$
- Numerics: Pseudospectral and Volume Penalization



Periodic Boundary Conditions 3D

Symmetries

- Statistics do not depend on horizontal coordinates:

$$f(T, \mathbf{x}) = f(T, z)$$
$$\langle \cdot | T, \mathbf{x} \rangle = \langle \cdot | T, z \rangle$$

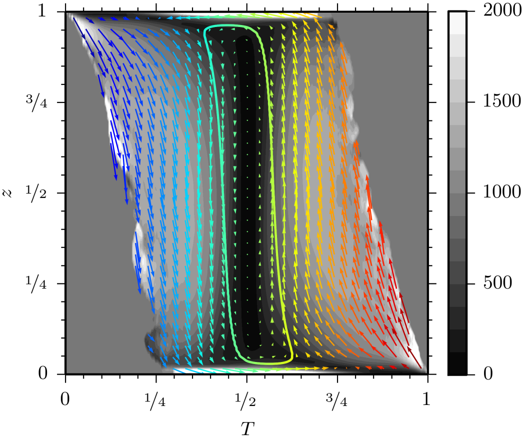
- Phase space becomes 2D, spanned by T and z
- Characteristics simplify to:

$$\begin{pmatrix} \dot{T} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, z \rangle \\ \langle u_z | T, z \rangle \end{pmatrix}$$

Periodic Boundary Conditions 3D

Integrating characteristics for arb. initial positions: Convergence towards Limit Cycle!

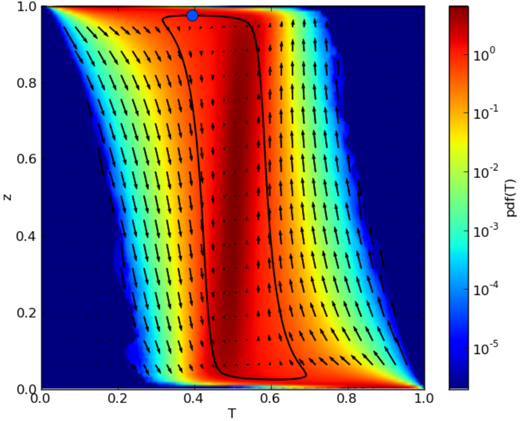
- Blue→red: Temperature
- Arrows: vector field of characteristics, i.e. $\begin{pmatrix} \langle \Delta T | T, z \rangle \\ \langle u_z | T, z \rangle \end{pmatrix}$
- Black→white: Norm of vector field (i.e., phase space speed)



Periodic Boundary Conditions 3D

Integrating characteristics for arb. initial positions: Convergence towards Limit Cycle!

- Color: pdf of T
- Arrows: Vector field
- Colored Circle: Solution $\binom{T}{z}$ following limit cycle; Temperature color coded



Contents

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Summary

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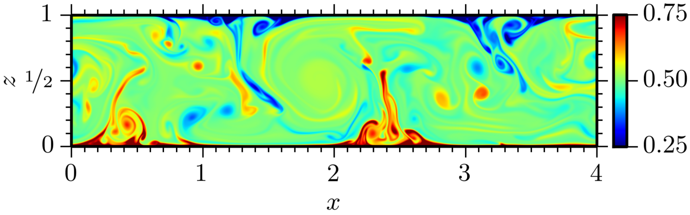
Periodic Boundary Conditions 2D

RB System

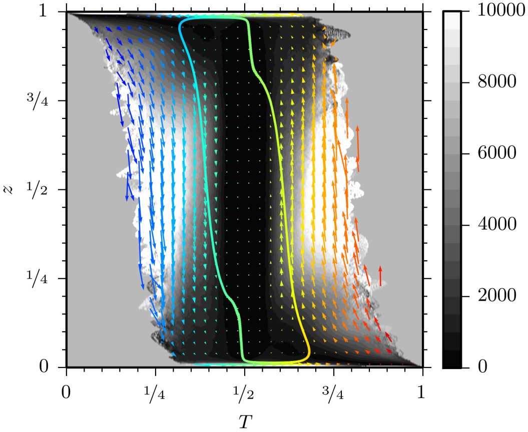
- 2D convection, homogenous in horizontal direction
- No-slip bottom and top plates $\mathbf{u} = \mathbf{0}$
- Parameters: $\mathrm{Ra} = 5 \cdot 10^8$, $\mathrm{Pr} = 1$, $\Gamma = 4$
- Symmetries: Identical to 3D periodic case



Periodic Boundary Conditions 2D

Integrating characteristics for arb. initial positions: Convergence towards Limit Cycle!

- Blue→red: Temperature
- Arrows: vector field of characteristics
- Black→white: Norm of vector field (i.e., phase space velocity)



Periodic Boundary Conditions 2D

Integrating characteristics for arb. initial positions: Convergence towards Limit Cycle!

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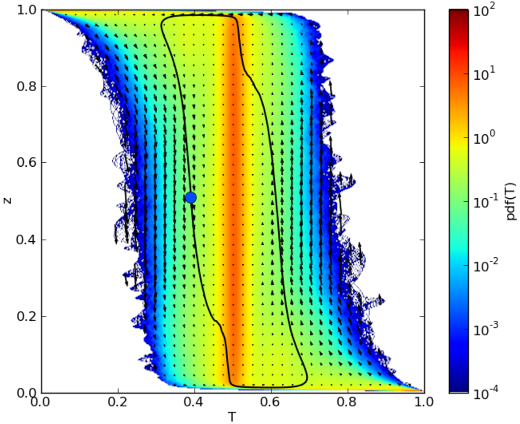

Contents

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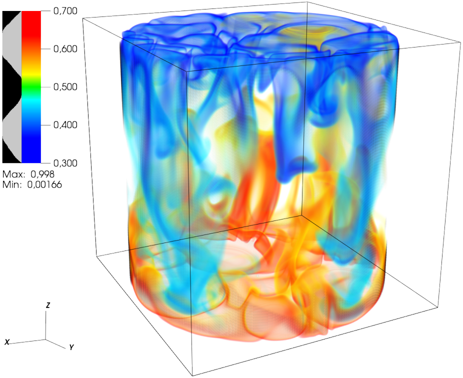
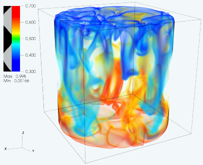
Cylindrical Vessel



- Cylindrical Vessel with insulating sidewalls
- Cylindrical coordinates: $\mathbf{x} = (r, \varphi, z)$
- All surfaces are no-slip $\mathbf{u} = \mathbf{0}$
- Parameters: $\mathrm{Ra} = 2 \cdot 10^8$, $\mathrm{Pr} = 1$, $\Gamma = 1$
- Numerics: Finite differences w/ gridpoint clustering

Cylindrical Vessel

Symmetries



- Homogeneous in azimuthal (φ-) direction:

$$f(T, \mathbf{x}) = f(T, r, z)$$
$$\langle \cdot | T, \mathbf{x} \rangle = \langle \cdot | T, r, z \rangle$$

Vector field of Characteristics

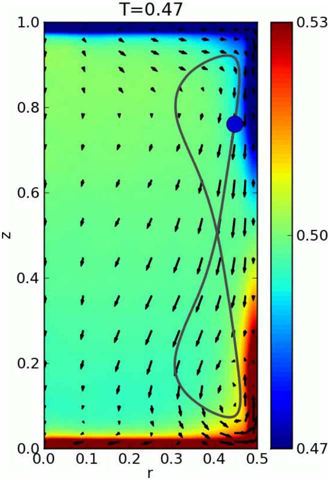
$$\begin{pmatrix} \dot{T} \\ \dot{r} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \langle \Delta T | T, r, z \rangle \\ \langle u_r | T, r, z \rangle \\ \langle u_z | T, r, z \rangle \end{pmatrix}$$

- Additional phase space dimension (radial movement)

Cylindrical Vessel

Again, Limit Cycle is found!

- RB Cycle:
  - Heating up / moving outwards at the bottom
  - Moving up / inwards in the bulk
  - Cooling down / moving outwards at the top, . . .
- Cornerflows w/o need for a LSC
- Limit Cycle lies in outer regions, $0.3 < r < 0.5$, and around the mean temperature, $0.47 < T < 0.53$



Contents

Summary

- We derived an evolution equation for the PDF of temperature from first principles
- Unclosed terms are expressed via conditional averages, which are estimated from DNS
- The Method of Characteristics is used to link statistics and dynamics of the system
- The framework allows to identify a limit cycle, which shows the average transport processes in Rayleigh–Bénard convection for three different cases ($2\times$ 2D / $1\times$ 3D phase space)
- Outlook: Further investigation of limit cycle

Summary

Methods

- PDF methods, conditionally averaged fields
- Method of Characteristics

Advantages

- from first principles
- description of average temperature-resolved behavior in different parts of cell
- results intuitive and insightful

Summary

Disadvantages / Problems

- full information $\mathbf{u}(\mathbf{x})$, $\Delta T(\mathbf{x})$ needed
- no *direct* desription of heat transport, i.e., in terms of $\mathrm{Nu}(\mathrm{Ra}, \mathrm{Pr})$
- results could be expected *a priori* ?
- limit cycle?!

Literature

- PDF Methods in general:
  - > S. B. Pope, AIAA Journal 22, 896 (1984)
  - > S. B. Pope, Turbulent Flows (Cambridge Univ. Press, 2000)
  - > M. Wilczek and R. Friedrich, Phys. Rev. E 80, 016316 (2009)
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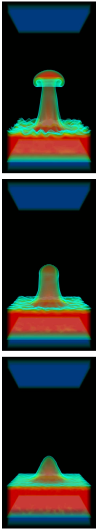
  - > R. Friedrich et al., Compt. Rend. Phys. 13, 929 (2012)
- PDF Methods in Rayleigh–Bénard convection:
  - > J. Lülff, M. Wilczek, and R. Friedrich, New J. Phys. 13, 015002 (2011)
  - > J. Lülff, M. Wilczek, R. Stevens, R. Friedrich and D. Lohse (*to be published...*)
- Volume Penalization:
  - > K. Schneider, Comput. Fluids 34, 1223 (2005)
  - > G. H. Keetels et al., J. Comput. Phys. 227, 919 (2007)



Thank you for your attention!

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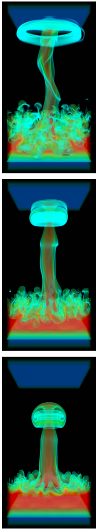
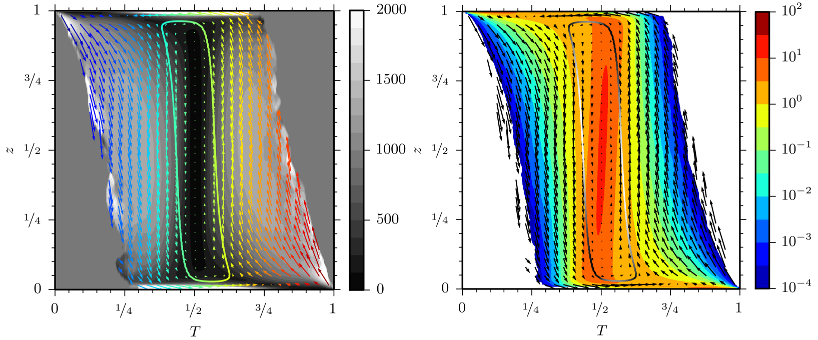
Rudolf Friedrich
† August 2012



Supplementaries

Limit Cycle 3D



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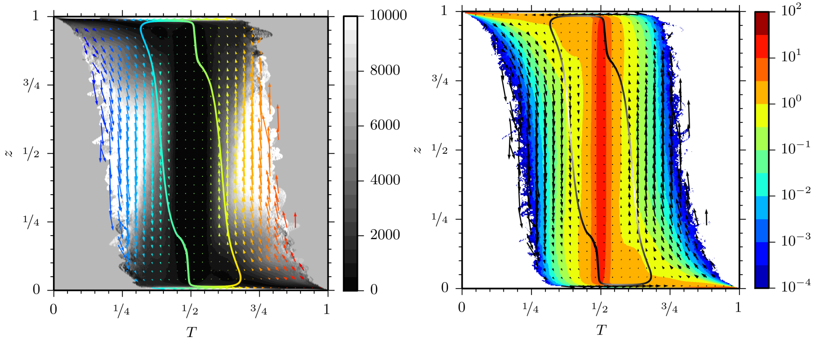
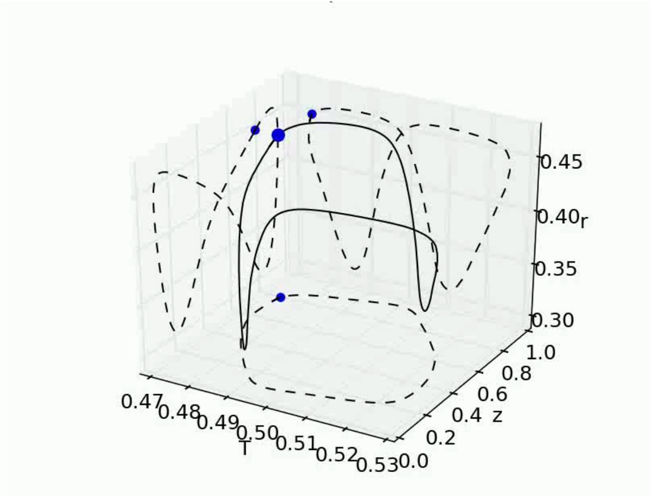
Limit Cycle 2D



Limit Cycle Cylinder



Method of Characteristics

- Along the characteristics, the PDE transforms into an ODE:

ODE along the characteristic curves

$$\frac{\mathrm{d}}{\mathrm{d}s} f(s) = -\underbrace{\Big(\frac{1}{r}\frac{\partial}{\partial r}\left(r\left\langle u_r \middle| T, r, z\right\rangle\right) + \frac{\partial}{\partial z}\left\langle u_z \middle| T, r, z\right\rangle + \frac{\partial}{\partial T}\left\langle \Delta T \middle| T, r, z\right\rangle\Big)}_{d(T,r,z)} f(s)$$

- The divergence $d(T, r, z)$ of the vectorfield describes the change of the PDF along a characteristic

Method of Characteristics

The ODE can be integrated along the characteristic curves, giving the following solution:

Integrated ODE along the characteristics

$$f(s) = f(s_0) \exp\left[-\int_{s_0}^{s} \mathrm{d}s \left(\frac{\partial}{\partial T} \langle \Delta T | T, z \rangle + \frac{\partial}{\partial z} \langle u_z | T, z \rangle \right) \Big|_{\substack{T=T(s) \\ z=z(s)}} \right]$$

- This relation describes the change of the PDF along a certain trajectory $\begin{pmatrix} T(s) \\ z(s) \end{pmatrix}$ in T, z-phase space, starting at $\begin{pmatrix} T(s_0) \\ z(s_0) \end{pmatrix}$