Westfälische Wilhelms-Universität Münster

Investigation of Temperature Statistics in Turbulent Rayleigh-Bénard Convection

Application of PDF Methods

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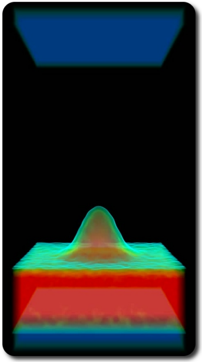

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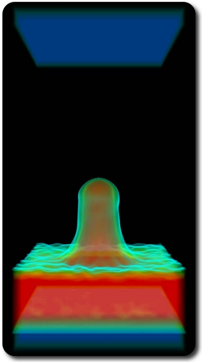
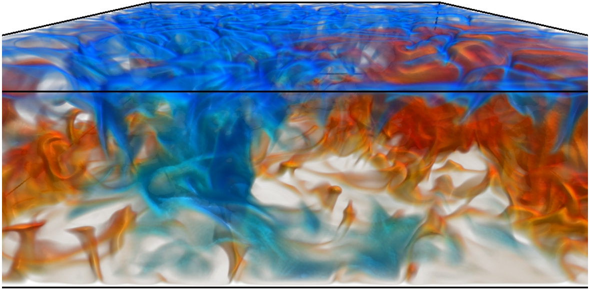


Phenomenon

Rayleigh-Bénard Convection:

- Heated from below, cooled ...
- Ubiquitous in nature: Atmosphere, ozeans, plate tectonics, ...
- Different patterns, from stable laminar to highly turbulent flows
- Hard to handle analytically



More videos at

www.youtube.com/
turbulenceteamms

Governing Equations

Oberbeck-Boussinesq Equations in Rescaled Form

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mathrm{Pr}\,\Delta \mathbf{u} + \mathrm{Pr}\,\mathrm{Ra}\,T\mathbf{e}_z \quad , \quad \nabla \cdot \mathbf{u} = 0$$

Dimensionless Parameters

Ra – Rayleigh number

Pr – Prandtl number

Boundary Conditions

$$z \in [0, 1]$$

$$T(z=0) = \tfrac{1}{2}\ , \quad T(z=1) = -\tfrac{1}{2}$$

$$\mathbf{u}(z=0) = \mathbf{u}(z=1) = \mathbf{0}$$

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A Statistical Description

Goal: Achieve a description in terms of the Temperature PDF

Deriving an Evolution Equation for the Temperature PDF

- Define temperature PDF as ensemble average of fine-grained PDF:

$$f(\tau; \mathbf{x}, t) = \Big\langle \delta\big(\tau - T(\mathbf{x}, t)\big) \Big\rangle$$

- Calculate and put together derivatives of f
- Plug in Oberbeck-Boussinesq equations
- Introduce conditional averages
- Make use of statistical symmetries

Deriving an Evolution Equation

Evolution equation for $f = \left\langle \delta(\tau - T(\mathbf{x}, t)) \right\rangle$

$$\frac{\partial}{\partial t} f + \nabla \cdot \left\langle \mathbf{u}\, \delta\left(\tau - T\right) \right\rangle = -\frac{\partial}{\partial \tau} \left\langle \left(\frac{\partial}{\partial t} T + \mathbf{u} \cdot \nabla T \right) \delta\left(\tau - T\right) \right\rangle$$
$$= -\frac{\partial}{\partial \tau} \left\langle \Delta T\, \delta\left(\tau - T\right) \right\rangle$$

Deriving an Evolution Equation

Introducing Conditional Averages

$$\left\langle \mathbf{u}\,\delta\big(\tau - T(\mathbf{x},t)\big)\right\rangle = \langle \mathbf{u}|\tau,\mathbf{x},t\rangle\, f(\tau;\mathbf{x},t)$$

$$\left\langle \Delta T\,\delta\big(\tau - T(\mathbf{x},t)\big)\right\rangle = \langle \Delta T|\tau,\mathbf{x},t\rangle\, f(\tau;\mathbf{x},t)$$

Resulting Evolution Equation

$$\frac{\partial}{\partial t} f + \nabla\cdot\Big(\langle \mathbf{u}|\tau,\mathbf{x},t\rangle\, f\Big) = -\frac{\partial}{\partial \tau}\Big(\langle \Delta T|\tau,\mathbf{x},t\rangle\, f\Big)$$

Deriving an Evolution Equation

Statistical Symmetries

- Flow is homogeneous in x, y-planes: $f(\tau; \mathbf{x}, t) = f(\tau; z, t)$
- Flow is stationary in time: $f(\tau; z, t) = f(\tau; z)$

The same applies to the conditional averages:

$$\langle \mathbf{u} | \tau, \mathbf{x}, t \rangle = \langle \mathbf{u} | \tau, z \rangle$$
$$\langle \Delta T | \tau, \mathbf{x}, t \rangle = \langle \Delta T | \tau, z \rangle$$

Resulting Evolution Equation

$$\frac{\partial}{\partial z}\Big(\langle u_z | \tau, z \rangle f \Big) = -\frac{\partial}{\partial \tau}\Big(\langle \Delta T | \tau, z \rangle f \Big)$$

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Estimating Conditional Averages from the Numerics

Overview of Numerical Scheme

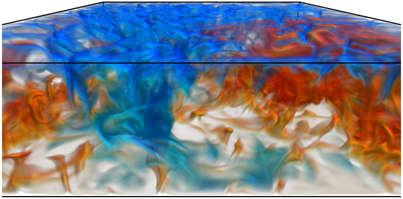
- Solves the full Oberbeck-Boussinesq equations
- 3D pseudospectral code with tri-periodic boundaries and dealiasing[1]
- Timestepping by RK3-method[2] with integrating factor
- No-slip bottom/top plates of constant temperature are enforced by volume penalization[3]
- Parallelization via MPI slab decomposition and FFTW2

[1]Thomas Y. Hou and Ruo Li. *J. Comp. Phys.*, 226:379–397, 2007
[2]Chi-Wang Shu and Stanley Osher. *J. Comp. Phys.*, 77(2):439–471, 1988
[3]Kai Schneider. *Comput. Fluids*, 34(10):1223–1238, 2005

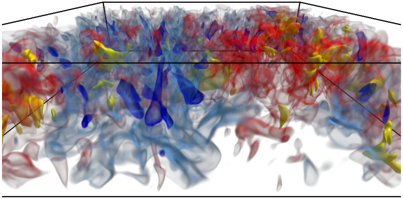
Estimating Conditional Averages from the Numerics

Simulation Details

- Parameters: $Ra = 2.44 \times 10^7$, $Pr = 1$, aspect ratio $\Gamma = 4$
- Resolution: $N_x \times N_y \times N_z = 512 \times 512 \times 192$ gridpoints

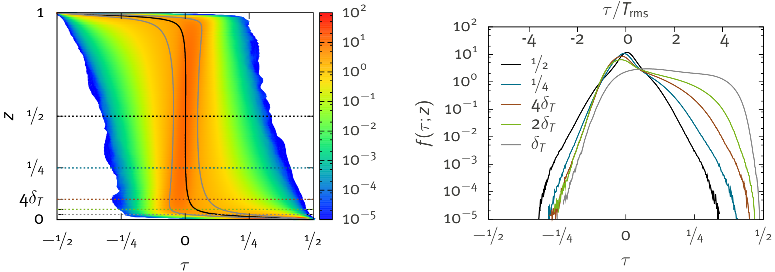


$T(\mathbf{x})$



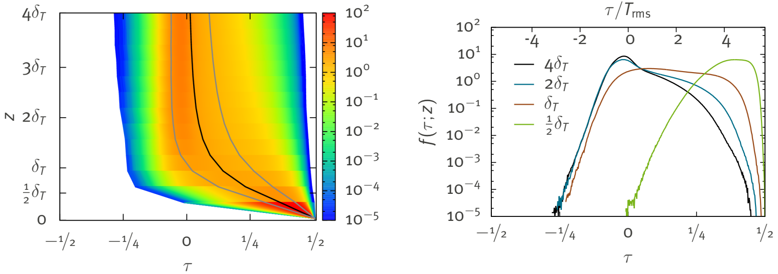
$u_z(\mathbf{x})$

Temperature PDF $f(\tau; z)$



- Rapid change from a δ-like PDF at the boundaries to a broad PDF across the boundary layer (BL)
- Average temperature (black line) is constant in the bulk
- Standard deviation (gray line) increases towards the BL and drops inside the BL

Temperature PDF $f(\tau; z)$



- Rapid change from a δ-like PDF at the boundaries to a broad PDF across the boundary layer (BL)
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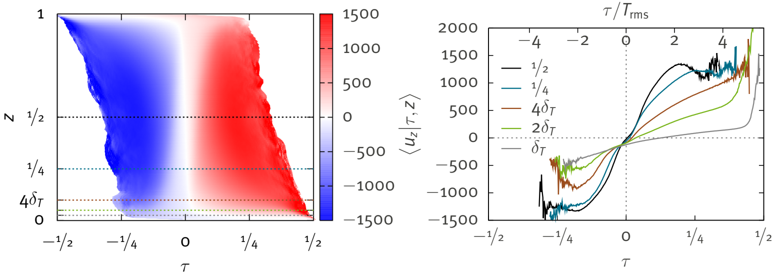
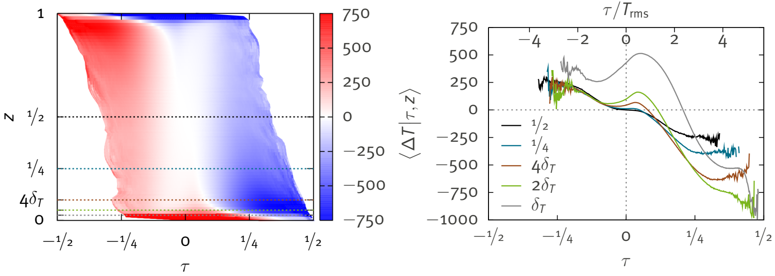
Conditional velocity $\langle u_z | \tau, z \rangle$



- Very hot (cold) fluid rises up (sinks down) fast
- Fluid of mean temperature (i.e. at $\tau = 0$) is at rest
- No-slip boundaries for $z \approx 0$ and $z \approx 1$

Conditional heat diffusion $\left\langle \Delta T \middle| \tau, z \right\rangle$



- Highest in the boundary layers
- High (low) for cold (hot) fluid
- Slices in τ-direction show over- and undershoot (more pronounced at the boundaries)

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Method of Characteristics

Evolution Equation

$$\frac{\partial}{\partial z}\Big(\langle u_z|\tau,z\rangle f\Big)=-\frac{\partial}{\partial\tau}\Big(\langle\Delta T|\tau,z\rangle f\Big)$$

- Apply Method of Characteristics to evolution equation of temperature PDF
- One obtains characteristic curves $\big(\tau(s),z(s)\big)$ in τ,z-phase space, parametrized by s

Method of Characteristics

Equations determining the characteristic curves

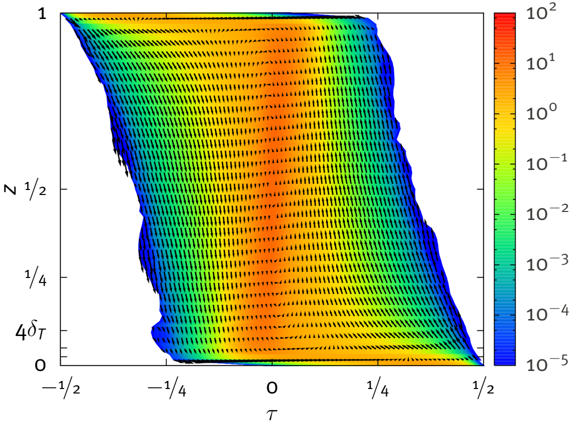
$$\frac{\mathrm{d}}{\mathrm{d}s}\tau(s) = \left\langle \Delta T \middle| \tau, z \right\rangle \Big|_{\substack{\tau=\tau(s)\\ z=z(s)}}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}z(s) = \left\langle u_z \middle| \tau, z \right\rangle \Big|_{\substack{\tau=\tau(s)\\ z=z(s)}}$$

- Remember: $\left\langle \Delta T \middle| \tau, z \right\rangle$ and $\left\langle u_z \middle| \tau, z \right\rangle$ are “known” quantities!

Method of Characteristics

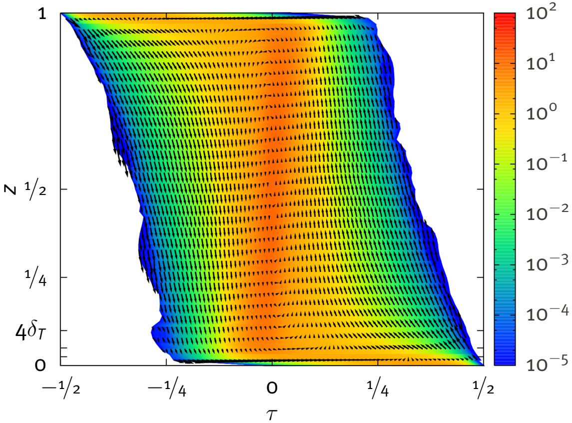
Vector field plot of the characteristics $\begin{pmatrix} \tau(s) \\ z(s) \end{pmatrix}$ in τ, z-phase space



- Vector field shows typical “Rayleigh-Bénard cycle”
- Vector field is strongest near boundaries and for high temperatures

Method of Characteristics

Vector field plot of the characteristics $\begin{pmatrix} \tau(s) \\ z(s) \end{pmatrix}$ in τ, z-phase space



- Connection between dynamics and statistics of RB convection!

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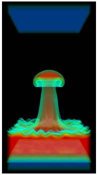
Conclusion

Goal: A statistical description of Rayleigh-Bénard convection

- We derived an evolution equation for the PDF of temperature
- Unclosed terms are expressed via conditional averages
- DNS using pseudospectral and volume penalization methods is used to estimate the conditional averages
- The Method of Characteristics is used to examine the evolution equation
- The statistical quantities in the form of Characteristics give insight into the RB dynamics
- To-do: z-resolved conditionally averaged dissipation rates appear in variations of our equations!



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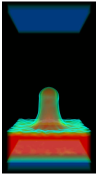

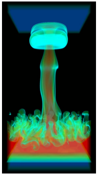

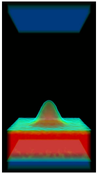

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