Introduction to the Standard Model
Exercises 8

Deadline: Monday 13 June 2016 (12 am)
at Dr. Giudice's office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Parton model, color factors, β-function.

1. (2.0 P) In the parton model we defined the two hadronic structure functions $W_1(x,Q)$ and $W_2(x,Q)$:

$$\begin{aligned} W_1(x,Q) &= 2\pi \sum_q Q_q^2 f_q(x,Q) \ , \\ W_2(x,Q) &= 8\pi \frac{x^2}{Q^2} \sum_q Q_q^2 f_q(x,Q) \ , \end{aligned}$$

where Q_q is the electric charge of the parton, Q is the momentum transfer and $f_q(x,Q)$ are the parton distribution functions (PDFs). Let us define now the dimensionless structure functions:

$$F_1(x) = \frac{1}{4\pi} W_1(x), \quad F_2(x) = \frac{Q^2}{8\pi x} W_2(x) \ .$$

Determine the Callan-Gross relation using both the dimension and the dimensionless structure functions.

Let us now call the parton distribution functions of the up, anti-up, down and anti-down quark of the proton $u^p(x), \bar{u}^p(x), d^p(x), \bar{d}^p(x)$ and of the neutron $u^n(x), \bar{u}^n(x), d^n(x), \bar{d}^n(x)$.

If we assume that the proton quark sea has the same number of anti-up and anti-down pairs, i.e. in terms of the antiquark density $\bar{u}^p(x) = \bar{d}^p(x)$, show the Gottfried sum rule:

$$\int_0^1 \frac{dx}{x} [F_2^p(x) - F_2^n(x)] = 1/3 \ ,$$

where F_2^p is the structure function of the proton and F_2^n of the neutron.
Hint: note that, for example, the probability to find a quark up in a proton is equal to that of finding a quark down in a neutron.

2. (2.0 P) Consider the following two processes where an electron and a positron are annihilated: $e^+e^- \to \mu^+\mu^-\gamma$ and $e^+e^- \to \bar{q}qg$ (in the first case producing two muons and a photon, in the second case producing two quarks and a gluon). Draw the two diagrams which contribute to each process and write down the corresponding amplitudes. Comparing them (you do not need to do explicit calculations), show that you can obtain the cross section of one from the other one just multipling for g_s^2/e^2 and for the "color factor" $3C_F \sum_f Q_f^2$, where C_F is the quadratic Casimir and Q_f is the charge corresponding to the f flavour (in unit of e).

3. (2.0 P) As we have seen in the previous problem one can obtain a QCD amplitude from a similar QED process by multiplying his amplitude for a colour factor. Let us consider the parton process $q_i q_j \to q_k q_l$ mediated by a gluon. We label the incoming and the outgoing quark colours by i, j, k, l; they can take three values labelled by r, g, b. We label the exchanged gluon by the letter a. This process is similar to the QED process $e^-\mu^- \to e^-\mu^-$; the QCD amplitude $i\mathcal{M}$ can be obtained from it by the replecement $e^2 \to -g_s^2$ and multiplying by the colour factor $C(ij \to jl)$, which takes into account the sum over the eight possible exchanged gluons:

$$C(ik \to jl) = \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{ji} \lambda^a_{lk} \ .$$

Consider the following cases: all colours are the same, e.g. $rr \to rr$; the two initial-state quarks have different colours but do not change colour, e.g $rb \to rb$; the two initial-state quarks have different colours and exchange colour, e.g. $rb \to br$; and all three colours are involved, e.g. $rb \to bg$. Draw the different cases (Feynman diagrams), calculate the colour factor, and say which gluon has been exchanged. Remember that λ_1 and λ_2 correspond to the exchange of $r\bar{g}$ and $g\bar{r}$ gluons, λ_4 and λ_5 represent $r\bar{b}$ and $b\bar{r}$ gluons, λ_6 and λ_7 correspond to the exchange of $g\bar{b}$ and $b\bar{g}$ gluons, and λ_3 and λ_8 represent the exchange of $\frac{1}{2}(r\bar{r} + g\bar{g})$ and $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$ gluons.

4. (1.5 P) The β-function of QED is of the form $\beta(e) = \beta_0 e^3 + \mathcal{O}(e^5)$, where $\beta_0 = 1/(12\pi^2)$, and e is the electric charge.

   (a) Show that:

   $$e_R^2(Q) = \frac{e_R^2(\mu)}{1 - \beta_0 \ e_R^2(\mu) \ \ln\left(Q^2/\mu^2\right)} \ ,$$

   if the β-function is approximated by its lowest order term.

   (b) Compare this result with that of problem sheet 6, exercise n.4(a). What is the origin of the difference? Which are the consequences of this difference?

   (c) Estimate the value of Q, where $e_R^2(Q)$ would diverge in this approximation, if μ is taken to be the electron mass m_e, and $e_R^2(m_e)/4\pi \approx 137.036$.

5. (1.5 P) The divergence which appears in the problem 4(c) is called the Landau pole problem. It can be avoided if an ultraviolet (UV) fixed point appears in the theory. We explore this possibility in this problem. Consider the following β-function $\beta(g) = a_1 g - a_2 g^3$, with $a_1, a_2 > 0$.

   (a) Draw a sketch of $\beta(g)$. On the graph of $\beta(g)$ indicate the flow of g for increasing momentum scale Q.

   (b) Find the limiting value g_c of g for $Q \to \infty$. Show that:

   $$g_R(Q) - g_c \propto \left(\frac{Q^2}{\mu^2}\right)^{-\gamma} \quad \text{as} \quad Q \to \infty \ ,$$

   and calculate the exponent γ.