Introduction to the Standard Model
Exercises 6

Deadline: Monday 30 May 2016 (12 am)
at Dr. Giudice's office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Gluon diagrams, running coupling.

1. (2.0 P) Prove that the following quantity:

$$\begin{aligned} N^{\mu\nu} &= \left[g^{\mu\alpha}(p+k)^\rho + g^{\alpha\rho}(p-2k)^\mu + g^{\rho\mu}(k-2p)^\alpha\right] g^{\alpha\beta}g^{\rho\sigma} \\ &\times \left[g^{\nu\beta}(p+k)^\sigma - g^{\beta\sigma}(2k-p)^\nu - g^{\sigma\nu}(2p-k)^\beta\right] , \end{aligned}$$

after the shift $k \to k + xp$, is equal to (keep in mind that $g^{\mu\nu}g_{\mu\nu} = d$):

$$\begin{aligned} \tilde{N}^{\mu\nu} &= 2k^2 g^{\mu\nu} - (6-4d)k^\mu k^\nu - \left[6(x^2-x+1) - d(1-2x)^2\right] p^\mu p^\nu \\ &+ (2x^2-2x+5)p^2 g^{\mu\nu} - (2-4x)g^{\mu\nu}(kp) + (2d-3)(2x-1)(k^\mu p^\nu + k^\nu p^\mu) . \end{aligned}$$

2. (3.0 P) Draw the gluon bubble and show that its expression, using the appropriate QCD Feyman rules, is equal to:

$$\mathrm{i}\mathcal{M}_3^{ab\mu\nu} = \frac{g^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{-\mathrm{i}}{k^2}\frac{-\mathrm{i}}{(k-p)^2} f^{ace}f^{bdf}\delta^{cf}\delta^{ed}N^{\mu\nu} .$$

Using the Feynman parameters, completing the square and shifting conveniently the momentum k, derive the following expression:

$$\begin{aligned} \mathcal{M}_3^{ab\mu\nu} &= -\frac{g^2}{2}\frac{\mu^{4-d}}{(4\pi)^{d/2}}\delta^{ab}C_A\int_0^1 dx\, \Delta^{\frac{d}{2}-2} \times \left\{ g^{\mu\nu}3(d-1)\Gamma\left(1-\frac{d}{2}\right)\Delta \right. \\ &+ p^\mu p^\nu \left[6(x^2-x+1) - d(1-2x)^2\right]\Gamma\left(2-\frac{d}{2}\right) \\ &+ \left. g^{\mu\nu}p^2\left[(-2x^2+2x-5)\Gamma\left(2-\frac{d}{2}\right)\right]\right\} . \end{aligned}$$

3. (2.0 P) Calculate the ghost bubble and show that its expression, using the appropriate QCD Feyman rules, is equal to:

$$\mathrm{i}\mathcal{M}_{gh}^{ab\mu\nu} = -g^2\int \frac{d^4k}{(2\pi)^4}\frac{\mathrm{i}}{(k-p)^2}\frac{\mathrm{i}}{k^2} f^{cad}f^{dbc}k^\mu(k-p)^\nu .$$

Using the Feynman parameters, completing the square and shifting conveniently the momentum k, derive the following expression:

$$\begin{aligned}\mathcal{M}_{gh}^{ab\mu\nu} &= g^2 \frac{\mu^{4-d}}{(4\pi)^{d/2}} \delta^{ab} C_A \int_0^1 dx\ \Delta^{\frac{d}{2}-2} \times \left\{ g^{\mu\nu} \frac{1}{2} \Gamma\left(1-\frac{d}{2}\right)\Delta \right. \\ &+ \left. p^\mu p^\nu \left[x(1-x)\Gamma\left(2-\frac{d}{2}\right)\right]\right\} .\end{aligned}$$

4. Let $\beta(g) = -\beta_0 g^3 - \beta_1 g^5 - \beta_2 g^7 - \ldots$ be the renormalisation group β-function of QCD in a particular scheme.

   (a) (1.0 P) From
   $$Q \frac{d}{dQ} g_R(Q) = \beta(g_R(Q))$$
   show that
   $$g_R^2(Q) = \frac{g_R^2(\mu)}{1 + \beta_0\ g_R^2(\mu)\ \ln(Q^2/\mu^2)} ,$$
   if the β-function is approximated by its lowest order term.

   (b) (1.0 P) Taking into account higher orders, show that:
   $$\ln(Q^2/\mu^2) = F(g_R(Q)) - F(g_R(\mu)) ,$$
   where
   $$F(g) = \frac{1}{\beta_0 g^2} + \frac{2\beta_1}{\beta_0^2} \ln g + \mathcal{O}(g^2) .$$

   (c) (1.0 P) Let $\tilde{g}$ be a different renormalised coupling, defined in another scheme, such that
   $$\tilde{g} = g + cg^3 + \mathcal{O}(g^5) .$$
   The corresponding β-function, given by
   $$Q \frac{d}{dQ} \tilde{g}(Q) = \tilde{\beta}(\tilde{g}(Q))$$
   is of the form
   $$\tilde{\beta}(\tilde{g}) = -\tilde{\beta}_0 \tilde{g}^3 - \tilde{\beta}_1 \tilde{g}^5 - \tilde{\beta}_2 \tilde{g}^7 - \ldots$$
   Show that
   $$\tilde{\beta}_0 = \beta_0, \quad \tilde{\beta}_1 = \beta_1 ,$$
   and that $\tilde{\beta}_2 \neq \beta_2$ in general.