Introduction to the Standard Model
Exercises 5

Deadline: Monday 16 May 2016 (12 am)
at Dr. Giudice's office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Lie groups, Yang-Mills theories, gauge invariance.

1. (1.0 P) Prove that, for on-shell spinors, the Gordon identity is satisfied:

$$\bar{u}(q)\gamma^\mu u(p) = \bar{u}(q)\left[\frac{q^\mu + p^\mu}{2m} + \mathrm{i}\frac{\sigma^{\mu\nu}(q_\nu - p_\nu)}{2m}\right] u(p)\ .$$

2. The standard basis for the generators of SU(3) in its fundamental representation is

$$T_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad T_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$T_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad T_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

   (a) (1.0 P) Knowing that $[T^a, T^b] = if^{abc}T^c$, evaluate the following structure constants of SU(3): f_{123}, f_{126}, f_{147}.

   (b) (0.5 P) Check the orthogonality condition $\mathrm{Tr}\,(T_a T_b) = \lambda_F\,\delta_{ab}$ for 2 cases $a \neq b$, and evaluate the constant λ_F for this representation.

   (c) (1.0 P) Show that f_{abc} is totally antisymmetric.
   Hint: First derive $f_{abc} = -2\mathrm{i}\,\mathrm{Tr}\,([T_a, T_b]T_c)$, then use the cyclic property of the trace in a clever way.

3. (0.5 P) The color triplet $q = \begin{pmatrix} r \\ g \\ b \end{pmatrix}$ transforms under colour SU(3) as

   $q \to q' = Uq, \quad U \in \mathrm{SU}(3)$.

   Let $\bar{q}$ be another triplet transforming as $\bar{q} \to \bar{q}' = U^*\,\bar{q}$. Define the generators $\overline{T}_a$ by $U^* = \exp(-\mathrm{i}\,\alpha_a \overline{T}_a)$, where $U = \exp(-\mathrm{i}\,\alpha_a T_a)$. These are the generators of the anti-fundamental (or conjugate) representation. Show that $\overline{T}_a = -T_a^*$ and check that the $\overline{T}_a$ fulfil the SU(3) Lie algebra.

4. (1.0 P) Prove the identity $[D_\mu, D_\nu] = -\mathrm{i}gT^a F^a_{\mu\nu}$, where $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative, g is the coupling constant and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ is the field strength tensor.

5. (3.0 P) Prove that the kinetic term of a Yang-Mills theory

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} \ ,$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$, is gauge invariant. Remember that for a non-Abelian case:

$$A^a_\mu(x) \to A^a_\mu(x) + \frac{1}{g}\partial_\mu \alpha^a(x) - f^{abc}\alpha^b(x) A^c_\mu(x) \ .$$

Hint: you may need the Jacoby identity: $f^{abc}f^{ckl} + f^{acl}f^{ckb} = f^{ack}f^{bcl}$.

6. (3.0 P) The locally SU(N) gauge invariant lagrangian is given by:

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \sum_{i,j=1}^{N} \bar{\psi}_i (\delta_{ij} \mathrm{i}\partial\!\!\!/ + g A\!\!\!/^a T^a_{ij} - m\, \delta_{ij})\psi_j \ .$$

Determine the equation of motion for the gauge and spinor fields:

$$\begin{aligned} \partial_\mu F_a^{\mu\nu} + gf^{abc} A^b_\mu F_c^{\mu\nu} &= -g\bar{\psi}_i \gamma^\nu T^a_{ij} \psi_j \ , \\ (\mathrm{i}\partial\!\!\!/ - m)\psi_i &= -g A\!\!\!/^a T^a_{ij} \psi_j \ , \end{aligned}$$

and the Noether current corresponding to the global gauge symmetry:

$$J^\mu_a = -\bar{\psi}_i \gamma^\mu T^a_{ij} \psi_j + f^{abc} A^b_\nu F_c^{\mu\nu} \ .$$