Introduction to the Standard Model
Exercises 2

Deadline: Monday 25 April 2016 (12 am)
at Dr. Giudice's office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Euler-Lagrange equations, local gauge symmetry, scalar QED, photon propagator, spinor identities.

1. (3.5 P) Using the Euler-Lagrange equations for fields, derive the field equations for the following Lagrange densities:

   (a) $\mathscr{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{g}{4}\phi^4, \quad \phi(x) \in \mathbb{R}$

   (b) $\mathscr{L} = \partial_\mu\phi^*\,\partial^\mu\phi - m^2\phi^*\phi\ , \quad \phi(x) \in \mathbb{C}$;
   Hint: consider ϕ and ϕ^* as independent components

   (c) $\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

   (d) $\mathscr{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{m^2}{2}B_\mu B^\mu$,
   where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad B_\mu(x) \in \mathbb{R}$

   (e) Show that the field equations obtained in (d) are equivalent to
   $(\Box - m^2)B_\mu = 0, \quad \partial_\mu B^\mu = 0.$

   (f) Is the Lagrange density given in (d) invariant under local gauge transformations $B_\mu(x) \longrightarrow B'_\mu(x) = B_\mu(x) + \partial_\mu\alpha(x)$? Is it invariant in any special case ?

2. (1.0 P) Show that the Lagrangian of QED

$$\mathscr{L} = \bar{\psi}(\mathrm{i}\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\ ,$$

   where the covariant derivative is $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$, is locally gauge invariant. Moreover, show that the additional term $\frac{1}{2}m^2A_\mu A^\mu$ is prohibited by gauge invariance, i.e. the photon must be massless. Remember that the local gauge transformations for the fields are given by:

$$\begin{aligned} A_\mu &\rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha(x)\ , \\ \psi &\rightarrow \mathrm{e}^{\mathrm{i}\alpha(x)}\psi\ . \end{aligned}$$

3. (4.5 P) Scalar QED:

   (a) For a free complex scalar field $\phi(x)$ there is a global U(1) symmetry given by
   $$\phi(x) \longrightarrow \phi' = \mathrm{e}^{-\mathrm{i}q\alpha}\phi(x).$$
   Calculate the corresponding Noether current.

   (b) Write down the Lagrangian $\mathscr{L}$ for $\phi(x)$ interacting with the Maxwell field $A_\mu(x)$ (without gauge fixing term). Derive the field equation for $\phi(x)$. Write the field equation by using the covariant derivative D_μ.

   (c) Calculate the Noether current corresponding to the global U(1) symmetry of $\mathscr{L}$, and calculate its divergence $\partial_\mu j^\mu$ using the field equations.

   (d) Check whether the interaction term in (b) is proportional to $j^\mu(x)A_\mu(x)$, where j^μ is the Noether current from (c).

   (e) Draw the interaction vertices belonging to $\mathscr{L}$. Use lines with arrows to distinguish ϕ from ϕ^*.

4. (3.0 P) Consider the Lagrangian density for the Abelian field A_μ:
$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 \ .$$

   (a) Show that it can be written in the form: $\mathscr{L} = \frac{1}{2}A^\mu M_{\mu\nu} A^\nu$.

   (b) Calculate the photon propagator: $\mathrm{i}\Pi_{\mu\nu}(k) = \mathrm{i}\left(\tilde{M}^{-1}\right)_{\mu\nu}(k)$.

5. (2.0 P) Show that:
$$\sum_s u_s(p)\ \bar{u}_s = \not{p} + m \ ,$$
$$\sum_s v_s(p)\ \bar{v}_s = \not{p} - m \ .$$