Introduction to the Standard Model
Exercises 11

Deadline: Monday 4 July 2016 (12 am)
at Dr. Giudice's office (KP 301) and Ms Sonja Esch (KP 310)

Topics covered: Parity operator, Goldstone bosons, symmetry breaking.

1. (1.5 P) Prove that the parity operator which acts on the Dirac spinors is given by $P = \gamma_0$. Hint: start from the Dirac equation, use the fact that $\psi'(x', y', z't') = P\psi(x, y, z, t)$, that the effect of P on the coordinate (x, y, z, t) is to send them in $(-x, -y, -z, t)$ and that $P^2 = 1$.

2. (1.0 P) Given the states, the Goldstone bosons,

$$|\pi(\vec{p})\rangle = \frac{-2\mathrm{i}}{F} \int d^3x \; e^{-\mathrm{i}\vec{p}\vec{x}} J_0(x)|\Omega\rangle \ ,$$

prove that

$$\langle\pi(\vec{q})|J_0(\vec{y})|\Omega\rangle = \mathrm{i}F\omega_q e^{\mathrm{i}\vec{q}\vec{y}} \ , \tag{1}$$

where the normalisation of the one-particle state $\langle\pi(\vec{q})|\pi(\vec{p})\rangle = 2\omega_p(2\pi)^3\delta^3(\vec{p}-\vec{q})$ has been used.

3. (1.0 P) The non linear sigma model

$$\mathcal{L} = \frac{1}{2}\partial_\mu\pi\partial^\mu\pi \ ,$$

can be derived from a linear sigma model characterised by the following invariance:

$$\pi(x) \to \pi(x) + F_\pi\theta \ .$$

Defining $|\pi\rangle$ as the state which can be created or annihilated by the field π, prove that $\langle\Omega|J^\mu(x)|\pi(q)\rangle$ is equal to $\mathrm{i}F_\pi e^{\mathrm{i}qx}q^\mu$, i.e. according to Eq. (1) the state $|\pi\rangle$ is a Goldstone boson. Hint: start calculating the current $J^\mu(x)$ in this specific case and use the normalisation $\langle\pi(p)|\pi(q)\rangle = (2\pi)^4\delta^4(p-q)$.

4. (3.0 P) Consider a theory with n real scalar fields and Lagrangian

$$\mathcal{L} = \frac{1}{2}\sum_n \partial_\mu\phi_n\partial^\mu\phi_n + \frac{1}{2}\mu^2\sum_n \phi_n\phi_n - \frac{\lambda}{4}\left(\sum_n \phi_n\phi_n\right)^2 \ .$$

   (a) What are the global symmetries of this theory ?
   (b) What are all the possible vacua of this theory? Are all the vacua equivalent?
   (c) Using simply group considerations, how many Goldstone bosons are there?
   (d) Write down the Lagrangian for small excitations around one of the vacua. How many Goldstone bosons are there?

(e) If instead of real fields we had complex fields and the Lagrangian was characterised by $SU(N)$ symmetry, how many Goldstone bosons we should expect in this case ? (again use only group considerations).

5. (1.5 P) Consider the case of one hermitian scalar field ϕ with scalar potential

$$V_0(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \ .$$

Show that $V_0(\phi)$ has degenerate minimum at $\phi = \pm v$, with $v = \sqrt{\mu^2/\lambda}$. Suppose now we add a cubic term to $V_0(\phi)$

$$\tilde{V}_0(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{2}{3}\xi\phi^3 + \frac{\lambda}{4}\phi^4 \ .$$

Show that the degeneracy in the minimum of $V_0(\phi)$ is now removed. Find the true minimum of $\tilde{V}_0(\phi)$. Also, show that, as a function of the parameter ξ, the vacuum expectation value (VEV) $\langle\phi\rangle_0$ changes discontinuously from $\langle\phi\rangle_0 = -v$ to $\langle\phi\rangle_0 = +v$ as ξ changes from positive to negative values going through 0. Hint: do all the calculations in the hypothesis of very small ξ.