Automated Lattice Perturbation Theory in the Schrödinger Functional

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WWU Münster 2010

Outline

Why Doing Lattice Perturbation Theory, why SF?

Lattice perturbation theory is needed/used for:

- Matching with perturbative results.
- Obtaining improvement coefficients.
- Renormalization.

The SF is used ...

- for *scale dependent* problems, e.g. the running coupling, or
- renormalization,
- for obtaining improvement coefficients (e.g. c_{SW}).

Automatic for the People!

Actions in LQCD

- The choice of action is not unique.
- Each possible action has advantages & disadvantages.
- There is no "best choice"!

What does it mean for LPT?

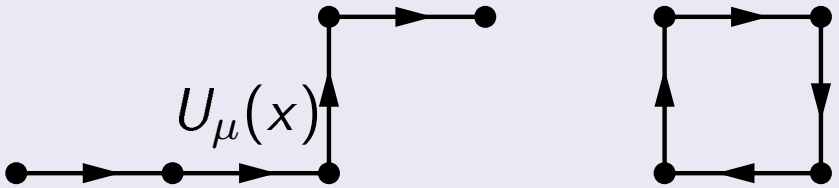
- Lots of different actions are in use.
- Each action leads to different Feynman rules!
- Calculations have to be repeated.
- Automation is suggested.

Outline

Basic Building Blocks

Curves

- Curve $\mathcal{C} = (s_1\mu_1, \ldots, s_l\mu_l)$, $\mu_i \in \{x, y, z, t\}$, $s_i = \pm 1$
- Path $U_{\mathcal{C}}$ = "Product of links along $\mathcal{C}$"



Actions

$$S_G = \sum_{\mathcal{C}_i(x,x)\in\Lambda} w_i \mathrm{ReTr}(1 - U_{\mathcal{C}_i(x,x)})$$

$$S_F = \sum_{\mathcal{C}_i(x,y)\in\Lambda} w_i \overline{\psi}(x)\Gamma U_{\mathcal{C}_i}(x,y)\psi(y)$$

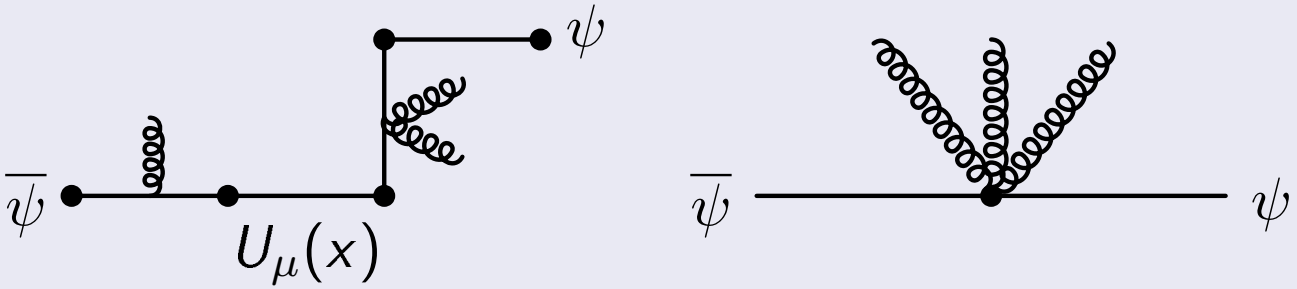
Perturbation Theory.

Expanding the action in powers of g_0,

$$S = \sum_r g_0^r S^{(r)},$$

determines the Feynman rules, i.e. propagators & vertices.

In LPT, vertices occur that have no continuum counterpart!



Paths.

A path $U_{\mathcal{C}}$ reads

$$U_{\mathcal{C}(x,y)} = U^{s_1}_{\mu_1}(\tilde{x}_1)U^{s_2}_{\mu_2}(\tilde{x}_2)\cdots U^{s_l}_{\mu_l}(\tilde{x}_l),$$

with

$$x_i + s_i\hat{\mu}_i = x_{i+1}, \quad x_0 := x, y = x_{l+1} \quad \tilde{x}_i = (x_i + x_{i+1})/2,$$

such that U lives on the mid-point between two lattice sites.

Expanding Paths

Expanding U_C in powers of g_0 yields

$$U_{C(x,y)} = \prod_{i=1}^{I} \sum_{r_i} s_i^{r_i} \frac{g_0^{r_i}}{r_i!} A_{\mu_i}^{r_i}(\tilde{x}_i),$$

or

$$U_{C(x,y)} = \prod_{i=1}^{I} \sum_{r_i} \frac{g_0^{r_i}}{r_i!} (I^a X_i^a)^{r_i},$$

with

$$X_i = s_i A_{\mu_i}(\tilde{x}_i).$$

Expanding Paths (2)

Picking *one* power r of g_0 ...

$$U^{(r)}_{C(x,y)} = \frac{g_0^r}{\alpha_1! \cdots \alpha_l!} I^{a_1} \cdots I^{a_r} X^{a_1}_{i_1} \cdots X^{a_r}_{i_l},$$

where $\alpha_j = \sum_k \delta(j, i_k)$ counts how many of the X belong to the jth link.

We have a *path independent* color factor

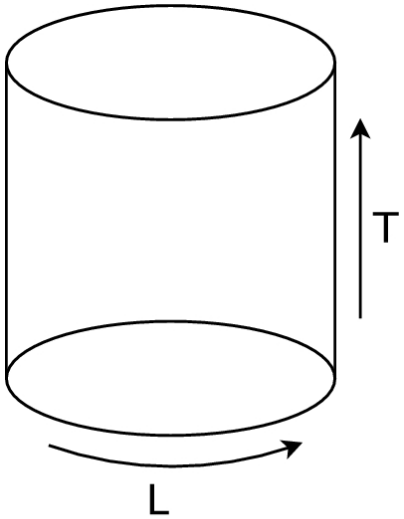
$$C^{a_1,\ldots,a_r} = I^{a_1} \cdots I^{a_r}$$

and a *path dependent* kinetic factor

$$K(C, x) = X^{a_1}_{i_1} \cdots X^{a_r}_{i_l}.$$

Outline

Meet the Schrödinger Functional.



- *M. Lüscher, R. Narayanan, P. Weisz et al.*,
'The Schrödinger functional: A Renormalizable probe for nonAbelian gauge theories',
Nucl. Phys. **B384**, 168-228 (1992).
`hep-lat/9207009`
- *S. Sint*,
'The Schrödinger functional in QCD',
Nucl. Phys. Proc. Suppl. **42**, 835-837 (1995).
`hep-lat/9411063`

- QCD on a $L^3 \times T$ cylinder.
- *Fixed temporal* boundary conditions at $t = 0, t = T$.
- *Periodic spatial* boundary conditions.

The Schrödinger Functional Boundary.

Fermions.

- Temporal: Dirichlet.
- Spatial: Periodic with phase angle θ,

$$\psi(x + L\hat{k}) = e^{i\theta_k}\psi(x).$$

Gluons.

- $U_k(x)|_{x_0=0} = W_k(\mathbf{x}) \quad U_k(x)|_{x_0=T} = W'_k(\mathbf{x})$,
- Commonly $W_k = e^C$ with C diagonal (W' analog).
- Spatial: Periodic.

The Background Field.

Gluon Boundary Conditions Induce a *Background Field*.

$$V_\mu(x) = (V[C, C'])_\mu(x)$$

- V_μ minimizes the action for given C, C'.
- Therefore, V_μ is the new expansion point.
- Decompose $U_\mu(x) = e^{g_0 q_\mu(x)} V_\mu(x_0)$.
- $q_\mu(x)$ are the d.o.f.

Expansion with a Background Field.

Expanding paths with background field will yield terms like

$$U^{(r)}_{\mathcal{C}(x,y)} = \underbrace{I^{a_1} I^{a_2} V_{\mu_1}(x_1) I^{a_3} \cdots I^{a_r} V_{\mu_l}(x_l)}_{\text{color factor C}} \times \underbrace{q^{a_1}_{\mu_1}(\tilde{x}_1) q^{a_2}_{\mu_1}(\tilde{x}_1) \cdots q^{a_r}_{\mu_l}(\tilde{x}_l)}_{\text{kinetic}} + \ldots$$

The color factor becomes path dependent as well!

Outline

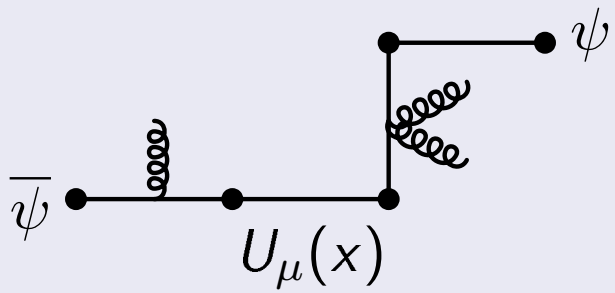
How to Automate?

Lüscher and Weisz: Expansion 'by induction'.

- Start with a single link $U_{\mathcal{C}} = U_\mu(x)$
- Find a multiplication algorithm

$$U^{(r)}_{\mathcal{C}}, U^{(s)}_{\mathcal{C}'} \mapsto U^{(r)}_{\mathcal{C}} \cdot U^{(s)}_{\mathcal{C}'} \in U^{(r+s)}_{\mathcal{C},\mathcal{C}'}$$

- Build up the desired path by multiplying links.



M. Lüscher, P. Weisz,
Nucl. Phys. **B266**, 309 (1986).

Expansion of a Link.

A single link reads

$$U_\mu(x) = \left(\sum_r \frac{g_0}{r!} I^a q_\mu^a(x) \right) V_\mu(x),$$

with a simple color factor in each order

$$C^{a_1,\ldots,a_r}(\mathcal{C}) = I^{a_1} \cdots I^{a_r} V_\mu(x).$$

What about the multiplication?

The Background Field

Useful Properties of *Abelian* Background Fields

- $V_\mu(x)^\dagger = V_\mu(x)^{-1}, \quad V_0(x) = 1, \quad V_k(x) = V(x_0)$
- $V(x_0) I^a = I^a V(x_0) e^{i\phi_a(x_0)}$
- $V(x_0 + \Delta t) = V(x_0) e^{i\mathcal{E}\Delta t}$
- $e^{i\mathcal{E}} I^a = I^a e^{i\mathcal{E}} e^{i\phi'_a}$

Color Factors Can be Cast Into a *Standard Form*.

$$C^{a_1,\ldots a_r}(x_0, A, B, \mathbf{C}, \mathbf{D}) = I^{a_1} \ldots I^{a_r} V(x_0)^A e^{i\mathcal{E}B} e^{i/2(\phi'\cdot\mathbf{C} + \phi(x_0)\cdot\mathbf{D})}$$

S. Takeda, U. Wolff,
PoS **LAT2007**, 257 (2007).

Color Multiplication.

The color factor decomposes into a path independent and path dependent part.

$$C^{a_1,\dots a_r}(x_0, A, B, \mathbf{C}, \mathbf{D}) = I^{a_1} \cdots I^{a_r} C(x_0, A, B, \mathbf{C}, \mathbf{D}).$$

One can formulate a multiplication for the path dependent bit.

$$C, C' \mapsto C'' = C \cdot C'$$
$$(x_0, A, B, \mathbf{C}, \mathbf{D}), (x_0', A', B', \mathbf{C}', \mathbf{D}') \mapsto (x_0, A'', B'', \mathbf{C}'', \mathbf{D}'').$$

S. Takeda, U. Wolff,
PoS **LAT2007**, 257 (2007).

Kinetic Multiplication.

Formulating a multiplication for the kinetic part is much simpler!
Let's remember: The kinetic part consists of lists

- of the directions $(\mu_1, \ldots, \mu_l)$,
- of the signs $(s_1, \ldots, s_l)$,
- and the sites $(\tilde{x}_1, \ldots \tilde{x}_l)$.

Multiplication of these boils down to concatenating the lists!

$$(\mu_1, \ldots, \mu_l), (\mu'_1, \ldots, \mu'_s) \mapsto (\mu_1, \ldots, \mu_l, \mu'_1, \ldots, \mu'_s)$$

etc.

Outline

Some History.

What has been done so far?

- *Lüscher & Weisz* proposed the algorithm.

  M. Lüscher, P. Weisz, Nucl. Phys. **B266**, 309 (1986).

- *Hart et al.* implemented a recursive version in `python`.

  A. Hart, G. M. von Hippel, R. R. Horgan et al., Comput. Phys. Commun. **180**, 2698-2716 (2009).

- *Takeda & Wolff* started extending Hart's code to the SF.

  S. Takeda, U. Wolff, PoS **LAT2007**, 257 (2007).

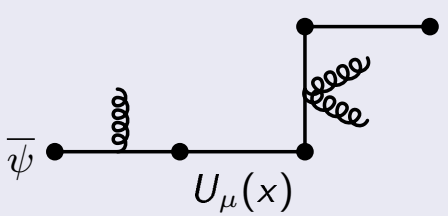
The issues.

- Code is OK for gluonic and simple fermionic Paths.
- It is not OK for general fermionic paths.
- Interplay between `python` and `FORTRAN` code.

Expansion Algorithm

Wanted: Expansion $U_{\mathcal{C}}^{(r)}$ (length$(\mathcal{C}) = l$, Λ: lattice, $\mathcal{B}$: boundary)

for $i = 1$ to l **do**
  if $\tilde{x}_i \notin \Lambda$ **then**
    return
  else
    expand links $X_i^{(j)}$, $j = 0, \ldots, r$
  end if
end for
for all $(n_1, \ldots, n_l)$, $\sum_k n_k = r$, $n_i = 0$ if $\tilde{x}_i \in \mathcal{B}$ **do**
  $U_{\mathcal{C}}^{(r)} \to U_{\mathcal{C}}^{(r)} \cup \left\{ X_1^{(n_1)} X_2^{(n_2)} \ldots X_l^{(n_l)} \right\}$
end for



$$\to (1,0,2,0) \to X_1^{(1)} V_{\mu_2}(\tilde{x}_2) X_3^{(2)} V_{\mu_4}(\tilde{x}_4)$$

Outline

Some Remarks.

Lattice Perturbation Theory in the SF.

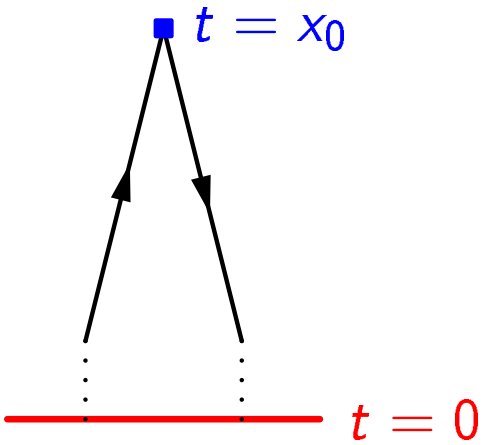
- Usually coordinate or time-momentum space.
- No integral over Brillouin zone, but sum (finite volume).

How to use the Code.

- Propagators are calculated by expanding S and inverting $(CG)^a$.
- Quark-anti quark-n-gluon vertices and n-gluon vertices also calculated from the actions.
- Diagrams are coded explicitly, like one would do it on paper.

[a] The gluon propagator needs to be *gauge-fixed* before inversion.

Correlation Functions.



$$f_A(x_0) = -\sum_{\mathbf{y},\mathbf{z}} \frac{1}{3} \langle A_0^a(x) \zeta(\mathbf{y}) \gamma_5 \tau^a / 2 \zeta(\mathbf{z}) \rangle$$

$$f_P(x_0) = -\sum_{\mathbf{y},\mathbf{z}} \frac{1}{3} \langle P^a(x) \zeta(\mathbf{y}) \gamma_5 \tau^a / 2 \zeta(\mathbf{z}) \rangle$$

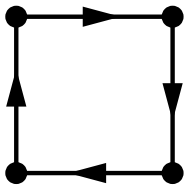
M. Lüscher, S. Sint, R. Sommer et al.,
Nucl. Phys. **B478**, 365-400 (1996).`hep-lat/9605038`

$$f_X^{(0)}(x_0) = -\frac{1}{2} \sum_{\mathbf{y},\mathbf{z}} \operatorname{tr} \{ P_+ \gamma_5 S_0(y,x) \gamma_5 \Gamma_X S_0(x,z) \} \Big|_{y_0 = z_0 = a}$$

$$\Gamma_A = -\gamma_0,\ \Gamma_P = 1,\ P_+ = (1+\gamma_0)/2$$

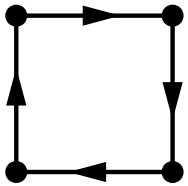
Some Action(s)!

Wilson

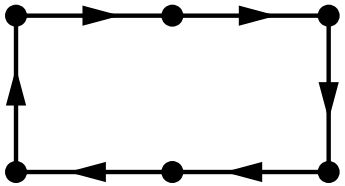


Iwasaki

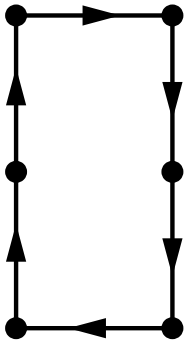
c_0



$+ c_1$

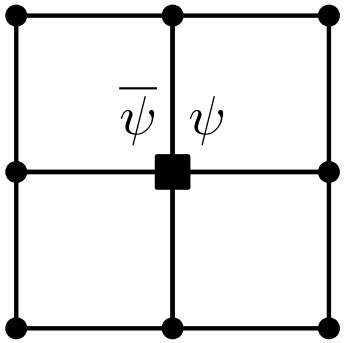


$+ c_1$



Clover term

$\sigma_{\mu\nu}$



Outline

Expanding made Easy!

The Wilson Gauge Action

```
Curve C;
C += mu, nu, -mu, -nu; // mu, nu are keywords!
Action Wilson;
Wilson.PushBack(C);    // actions can contain
                       // many paths
```

- Any vertex of order r: `Wilson.Expand(r)`!
- Summation over $\mu \neq \nu$ implied!

Really Easy!

The Iwasaki Gauge Action

```
Curve P(c0), R1(c1), R2(c1);
P += mu, nu, -mu, -nu;                    // plaquette
R1 += mu, mu, -nu,-mu, -mu, -nu;          // 2x1 rect.
R2 += mu, nu, nu, -mu, -nu, -nu;          // 1x2 rect.
Action Iwasaki;
Iwasaki.PushBack(P);
Iwasaki.PushBack(R1);
Iwasaki.PushBack(R2);
```

Templates so far:

- Gauge: *Wilson, Iwasaki*
- Fermion: *Wilson, Clover – (**Dirac**, Pauli, Scalar)*

Tree Level.

f_A at Tree Level.

```
// get the propagator
FermionPropagator P(WilsonFermion(0));
Storage S = P.Get(0,FermionMomentum());
// get fA
fA = -0.5*(Pp * g5 * S(1, x0) *
           g5 * g0 * S(x0, 1)  ).tr();
// output
cout << "fA(param().T()/2) = "
     << fA.real() << endl;
```

Output

fA(8) = -0.5113075898406796 (-0.5113075898406484)

Outline

Z_P to one loop.

First Test: Calculate the one loop contribution to

$$Z_P(g_0, L/a) = c\frac{\sqrt{f_1}}{f_P(T/2)}.$$

S. Capitani, M. Guagnelli, M. Luescher et al.,
Nucl. Phys. Proc. Suppl. **63**, 153-158 (1998). `hep-lat/9709125`

Expectation:

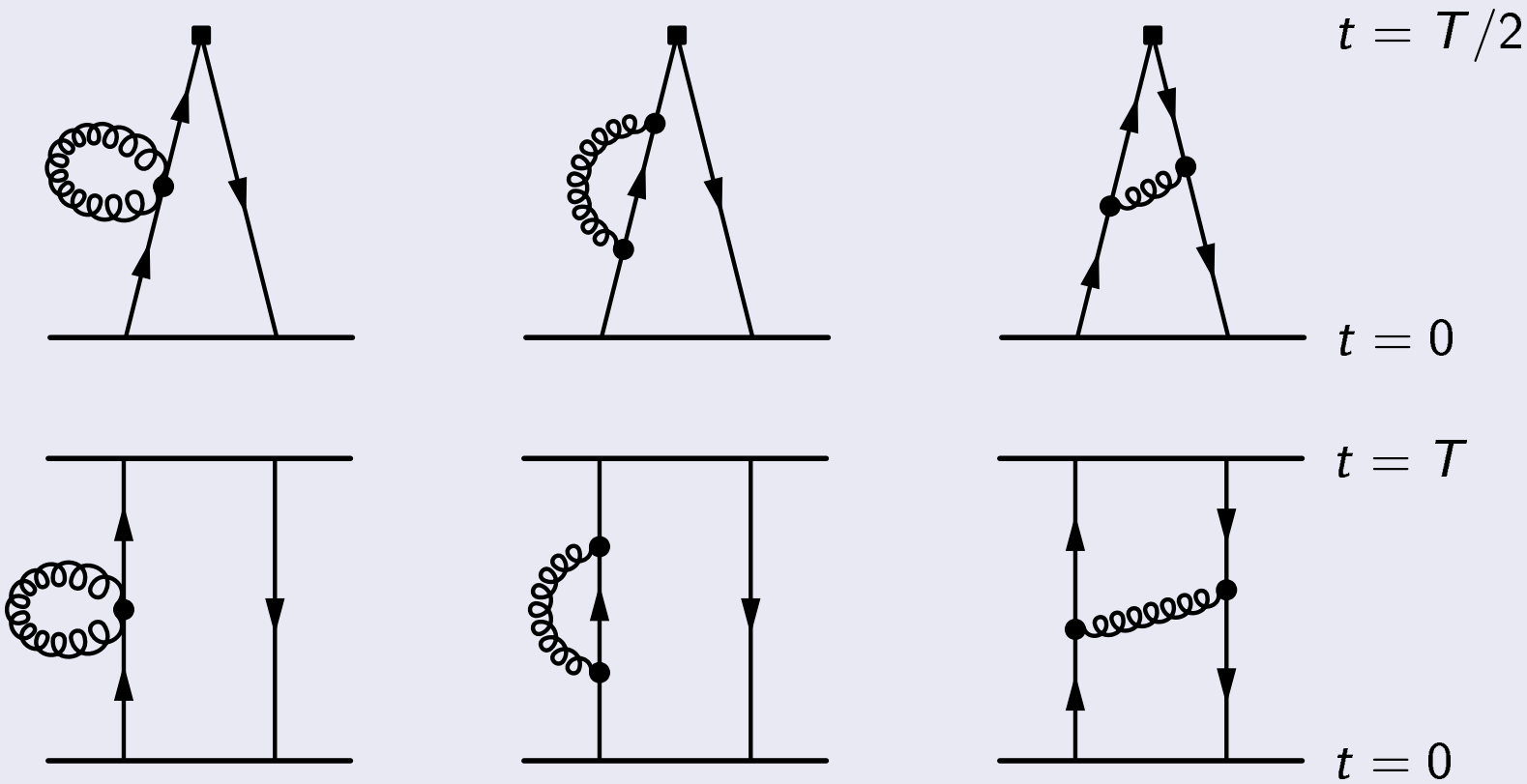
$$Z_P(g_0, L/a) = 1 + \sum_{r=0}^{\infty} g_0^{2k} Z_P^{(r)}(L/a),$$

where $Z_P^{(r)}(L/a)$ is a polynomial of degree r in $ln(L/a)$. We are interested in the finite part of $Z_P^{(1)}$,

$$Z_P^{(1)} = C_F z_p(\theta, \rho) - d_0 \ln(L/a) + O(a/L).$$

Diagrams.

Main Contributions to $f_P^{(1)}$, $f_1^{(1)}$.



More Details.

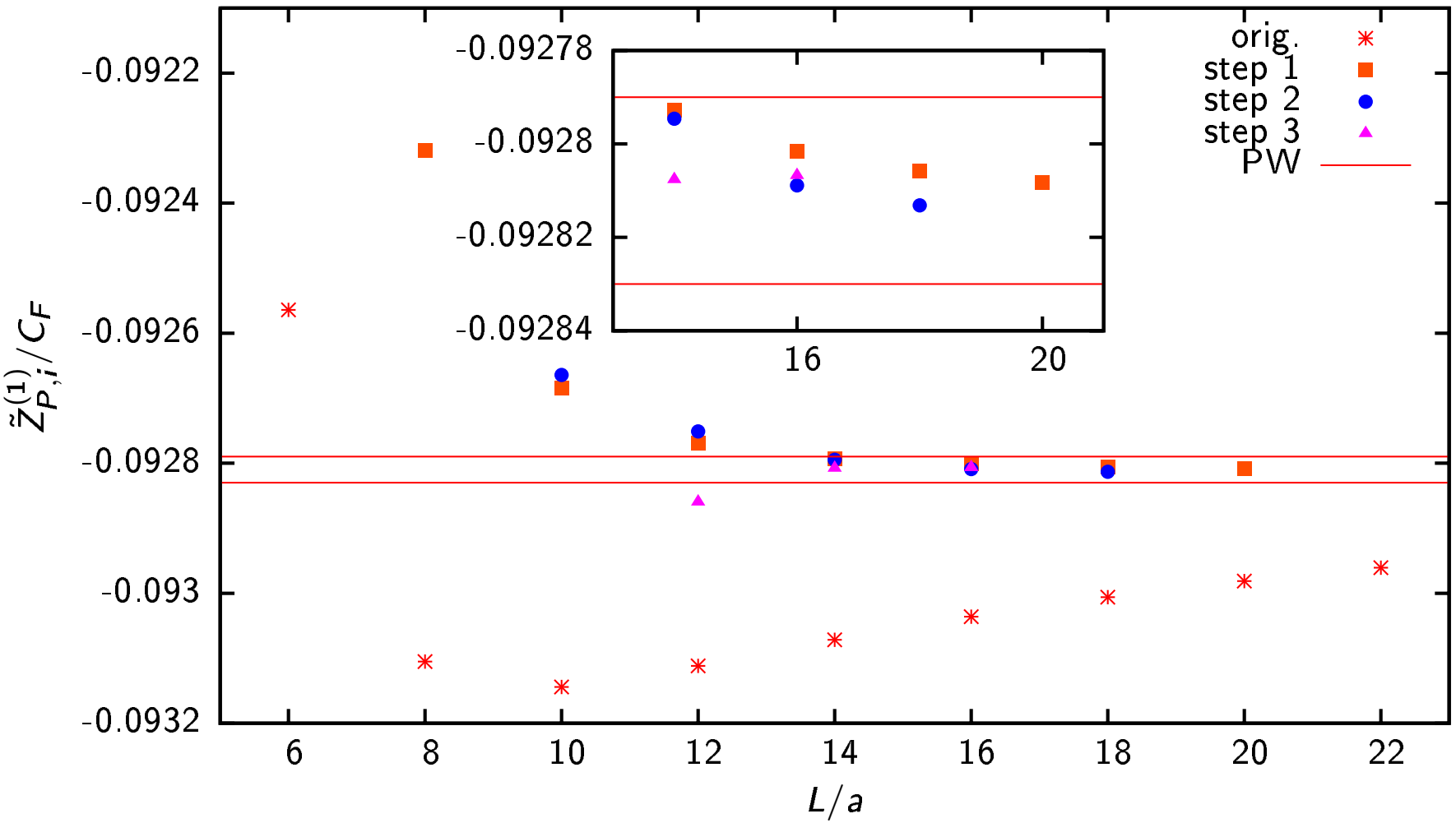
Diagrams were evaluated ...

- for $L = 4, 6, \ldots, 22,\ T = L,\ \theta = 0.5$
- with clover improved Wilson fermions,
- and Wilson gauge action,
- in O(hours).

S. Sint & P. Weisz in `hep-lat/9808013`

$$z_P(\theta = 0.5, T/L = 1) = -0.09281(2)$$

Z_P, a Picture.


-0.0922
-0.0924
-0.0926
-0.0928
-0.093
-0.0932
$\tilde{Z}^{(1)}_{P,i}/C_F$
6
8
10
12
14
16
18
20
22
L/a
-0.09278
-0.0928
-0.09282
-0.09284
16
20
orig.
step 1
step 2
step 3
PW

Outline

The PCAC Mass.

We define an *unrenormalized* quark mass,

$$m(m_0) = \frac{1}{2}\frac{\tilde{\partial} f_A^I(T/2)}{f_P(T/2)}.$$

K. Jansen, C. Liu, M. Luscher et al.,
Phys. Lett. **B372**, 275-282 (1996). `hep-lat/9512009`

The critical *bare* m_c mass is defined by

$$m(m_c) \stackrel{!}{=} 0,$$
$$m_c = m_c^{(0)} + g_0^2 m_c^{(1)} + \dots .$$

$\tilde{\partial} = 1/2(\partial_0^* + \partial_0)$

The Critical mass.

We expand the quark mass,

$$m(m_0) = m^{(0)}(m_0) + g_0^2 m^{(1)}(m_0) + \dots .$$

The tree and one loop parts of m_c can be obtained:

Require

$$m^{(0)}(m_c^{(0)}) + g_0^2 \left\{ m^{(1)}(m_c^{(0)}) + m_c^{(1)} \partial_{m_0} m^{(0)}(m_c^{(0)}) \right\} \stackrel{!}{=} 0.$$

At both orders!

One Loop Part.

The one loop contribution to m reads

$$m^{(1)} = \frac{1}{2}\frac{\tilde{\partial} f_A^{(1)}(T/2)}{f_P^{(0)}(T/2)} - f_P^{(1)}(T/2)\frac{\tilde{\partial} f_A^{(0)}(T/2)}{2[f_P^{(0)}(T/2)]^2} + c_A^{(1)}\frac{\partial_0^* \partial_0 f_P^{(0)}(T/2)}{2 f_P^{(0)}(T/2)}$$

Need to calculate only $\tilde{\partial} f_A^{(1)}$ at one loop!
Same diagrams (in principle) as for f_P!

Some Numbers.

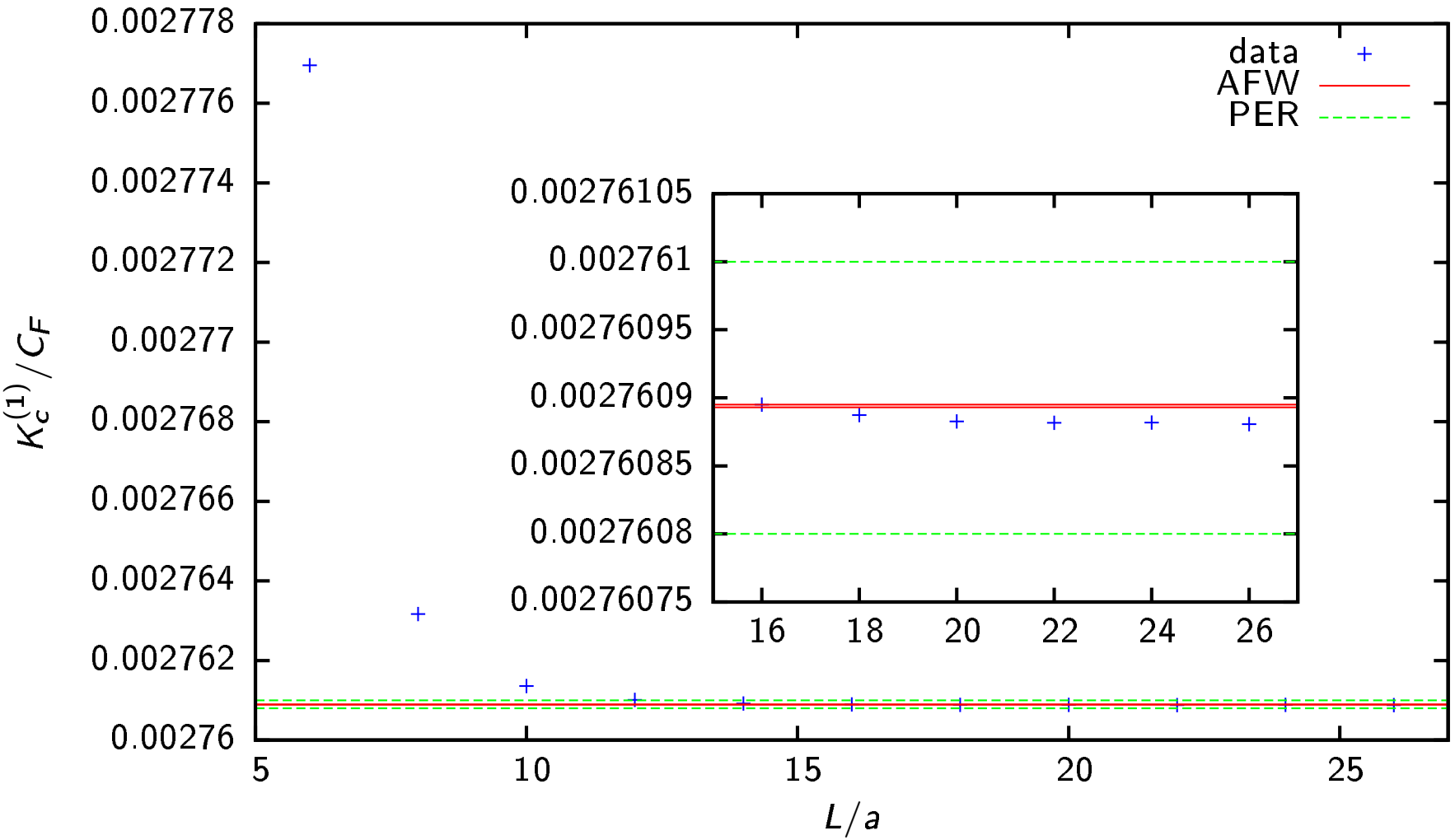
Calculations were performed...

- with Iwasaki gauge action and
- clover improved Wilson fermions ($\theta = 0$).
- For $T = L = 6, 8, \ldots$.

Aoki, Frezzotti, Weisz in `hep-lat/9808007`

$$K_C^{(1)} = 2.760894(1) \times 10^{-3} C_F$$

m_C, a Picture.


0.002778
0.002776
0.002774
0.002772
0.00277
0.002768
0.002766
0.002764
0.002762
0.00276
5
10
15
20
25
L/a
$K_c^{(1)}/C_F$
data
AFW
PER
0.00276105
0.002761
0.00276095
0.0027609
0.00276085
0.0027608
0.00276075
16
18
20
22
24
26

Summary

- Perturbative calculation must be done.
- You should and can automate them, even in the Schrödinger functional!

- Outlook
  - The code is still beta!
  - More applications to come (e.g. HQET)!

Thank you!

For Further Reading


M. Lüscher and P. Weisz,
Efficient Numerical Techniques For Perturbative Lattice Gauge Theory Computations.
Nucl. Phys. B **266**, *309 (1986).*


A. Hart, G. M. von Hippel, R. R. Horgan and E. H. Müller,
Automated generation of lattice QCD Feynman rules.
Comput. Phys. Commun. **180**, *2698 (2009) [arXiv:0904.0375 [hep-lat]].*


S. Takeda and U. Wolff,
Automatic generation of vertices for the Schrödinger functional.
PoS **LAT2007**, *257 (2007) [arXiv:0709.4167 [hep-lat]].*