Scale Dependent Renormalization and the Schrödinger Functional

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Outline

1. Scale Dependent Renormalization
2. The Schrödinger Functional
3. Renormalizing Quark Masses
Some Conventions

We Are Lattice People

We do Monte Carlo (MC) simulations with
- Lattice size $L$
- Lattice spacing $a$
- $(L/a)^4$ lattice points (in 4 dimensions)
- The lattice introduces a momentum cutoff $a^{-1}$

First we will consider pure Yang-Mills-Theory, later switch to QCD.
### The Running Coupling

**QCD**

Bare coupling constant

\[ g_0 \]

has to be **renormalized**.

**The Real World**

Physical, scale dependant coupling \( \alpha(\mu) \), e.g.

\[
\alpha(\mu) \propto \frac{\sigma(e^+e^- \rightarrow q\bar{q}g)}{\sigma(e^+e^- \rightarrow q\bar{q})}
\]
A Picture

Stolen from S. Bethke: $\alpha_s$ 2002
For high energies $\mu$, one can use perturbation theory (PT) to make predictions. The renormalization group (RG) tells us that

$$\mu \frac{\partial g}{\partial \mu} = \beta(g),$$

PT then yields

$$\beta(g) \to 0 - g^3(b_0 + g^2 b_1 + \ldots)$$

But QCD should also describe low energy phenomena ...
### What Do We Want To Do?

#### A Test For QCD

- Determine $\alpha(\mu)$ non-perturbatively on the lattice
- Make connection to PT in the high energy sector
- i.e. connect low- and high-energy regimes of QCD, predict e.g. $\Lambda/F_\pi$ or simply $\Lambda$ trough hadronic input
- Use PT (or some other effective theory) for 'real world predictions'
- Compare with experiments
Define a physical coupling, e.g.

\[ \alpha_{\bar{q}q}(\mu) := \left. \frac{1}{C_F} r^2 F(r) \right|_{\mu=1/r} \]

and measure it on the lattice!

Simple?
How To Do This?

Define a physical coupling, e.g.

\[ \alpha_{\bar{q}q}(\mu) := \frac{1}{C_F} \left. r^2 F(r) \right|_{\mu=1/r} \]

and measure it on the lattice!

This doesn’t work!
Why Doesn’t It Work?

We have to satisfy constraints:

- $\mu \geq 10 \text{ GeV}$ for PT matching
- $\mu \ll a^{-1}$ to control discretization errors
- $L \gg \frac{1}{m_\pi}$, $r_0$ to control finite size effects

This leads to

$$L \gg r_0, \frac{1}{m_\pi} \sim \frac{1}{0.14 \text{ GeV}} \gg \frac{1}{\mu} \sim \frac{1}{10 \text{ GeV}} \gg a$$

$\Rightarrow$ Simulate $L/a \gg 70$ lattice points in MC simulation

$\rightarrow$ (today) not possible
Besides $a^{-1}$, another energy scale is accessible in MC simulations, namely $L$

**The $L$ Trick**

- Identify $\mu = \frac{1}{L}$, i.e. choose finite size effects as observable
- Find a clever definition for $\alpha(L)$
- Split up the
  - Renormalization of $\alpha(L)$ for fixed $L$ and
  - Computation of the scale dependence of $\alpha$
To investigate the scale evolution of $\alpha$, define the step scaling function $\sigma$

**The Step Scaling Function**

- Choose starting point $u_0 = \bar{g}^2(L)$
- Choose a scaling factor $s$
- Define $\sigma(s, u_0) = \bar{g}^2(sL)$

This is a discrete integrated $\beta$-function
The SSF on a Lattice

REMEMBER: We Are Lattice People

- Obtained on a lattice, \( \sigma \) will carry a dependence on \( a/L \)
- So define

\[
\Sigma(s, u, a/L) = \left. \frac{\bar{g}^2}{g^2} \right|_{g^2(L)=u,g_0 \text{ fixed},a/L \text{ fixed}}
\]

- Calculate \( \Sigma(s, u, a/L) \) for several lattice resolutions and take the limit

\[
\sigma(s, u) = \lim_{a/L \to 0} \Sigma(s, u, a/L)
\]
σ in Three $n$ Steps

**How To Obtain σ?**

1. Choose initial $(L/a)^4$ lattice
2. Tune $\beta$ such that $\bar{g}(L) = u$ is where you want to start
3. Compute $\bar{g}(2L)$ with the same bare parameters and get $\Sigma(2, u, a/L)$

Repeat for several resolutions $a/L$ and extrapolate $a/L \to 0$

**Note:**

- Step 2) takes care of renormalization
- Step 3) computes the scale-evolution of the renormalized coupling
σ: A Comic Approach

Stolen from ALPHA Collaboration
Does It Work?

One finds that

\[
\frac{\Sigma(2, u, a/L) - \sigma(2, u)}{\sigma(2, u)} = \delta_1(a/L)u + \delta_2(a/L)u^2 + \ldots
\]

where

\[
\delta_n = O(a/L).
\]

This looks good, the continuum limit is reached with errors of \(O(a/L)\).
What About Universality?

Question
Does $\sigma$ depend on the choice of the action?

Answer
It seems not ...

Strategy
Improve $\Sigma$

$$
\Sigma^{(k)}(2, u, a/L) = \frac{\Sigma(2, u, a/L)}{1 + \sum_{i=1}^{k} \delta_i(a/L)u}
$$

and calculate $\sigma$ for different actions.
Some Numerical Results

\( \bar{g}^2(2L) \)

\( \bar{g}^2(L) = 2.4484 \)

Stolen from CP-CACS Collaboration
## Putting It Together

### What We’ve Got So Far

Assume, one has

- Calculated $\sigma(u_i)$ for several $u_i$
- Interpolated a polynomial $\sigma(u)$

### The Final Step

Then one can construct the running coupling $\bar{g}^2(2^{-i}L_0) = u_i$ via the recursion

$$u_0 = \bar{g}^2(L_0), \sigma(u_{i+1}) = u_i$$
Some Results

- Scale Dependent Renormalization
- The Schrödinger Functional
- Renormalizing Quark Masses

**Done by ALPHA**
What Still Has To Be Done

The Definition of $\alpha(L)$

We have to define $\alpha(L)$ such that it has

- An easy expansion in PT
- A small Monte Carlo variance
- Small discretization errors

Which leads us to ...
Introducing: The Schrödinger Functional

The SF ...

- Was first used by Symanzik for renormalization of the Schrödinger Picture in QFT
- Then by Lüscher and Narayanan, Weisz, Wolff for finite size scaling technique
- Is the propagation kernel of some field configuration $C$ to another in euclidean time $T$
Our theory lives on a $L^3$-space-box with periodic boundary and finite time $T$, like this.
The Players I: Gauge Fields

- On our space-time live $SU(N)$ gauge fields $A_k(\vec{x})$ on $LS^3$.
- We want for a $SU(N)$ gauge transformation $\Lambda$

$$A_k^\Lambda(\vec{x}) = \Lambda(\vec{x})A_k(\vec{x})\Lambda(\vec{x})^{-1} + \Lambda(\vec{x})\partial_k\Lambda(\vec{x})^{-1}$$

to be another gauge field.
- We only admit periodic gauge transformations $\Lambda$
The Winding Number Thing

The Operators $A : S^3 \to SU(N)$ fall in disconnected topological classes, labelled by their **winding number** $n$. A simple example:

$$f : S^2 \to U(1) \simeq S^2$$

![Diagram](image_url)
The Players II: The States

A state is a wave functional $\psi[A]$. On the set of all states, a scalar product is given by

$$\langle \psi | \chi \rangle = \int \mathcal{D}[A] \psi[A]^* \chi[A]$$

with

$$\mathcal{D}[A] = \prod_{\vec{x},k,a} A_k^a(\vec{x})$$

Physical states satisfy $\psi[A^\Lambda] = \psi[A]$. We introduce the projector on the set of physical states through

$$\mathbb{P} \psi[A] = \int \mathcal{D}[\Lambda] \psi[A^\Lambda]$$
The Players III: The Boundary

How To Make Up a State ...

- Take a smooth classical gauge field $C_k(\vec{x})$
- Introduce a state $|C\rangle$ via

$$\langle C|\psi\rangle = \psi[C] \quad \forall \text{ states } \psi$$

- $C$ can be made gauge invariant by applying $\mathbb{P}$
Putting It Together

Defining the Schrödinger Functional

Let

\[ Z[C', C] = \langle C' | e^{-H_T} P | C \rangle \]

Invariant under gauge transformations due to \( P \)
Putting It Together

Defining the Schrödinger Functional

Let

\[ Z[C', C] = \langle C'| e^{-HT} P | C \rangle \]

\[ = \sum_{n=0}^{\infty} e^{E_n T} \psi_n[C'] \psi_n[C]^* \]

Where \( \psi_n \) is the \( n \)-th (physical) energy eigenstate

Invariant under gauge transformations due to \( P \)
Going Functional

We Are Lattice People

We want a functional integral:

$$Z[C', C] = \int \mathcal{D}[\Lambda] \mathcal{D}[A] e^{S[A]}$$

(modulo renormalization factor) where

$$A_k(x) = \begin{cases} C^\Lambda_k(\vec{x}) & \text{at } x^0 = 0 \\ C'_k(\vec{x}) & \text{at } x^0 = T \end{cases}$$

and

$$S[A] = -\frac{1}{2g_0^2} \int d^4x \, \text{tr}(F_{\mu\nu}F_{\mu\nu})$$
The Topology Trick

After the $\mathcal{D}[A]$ integration, $Z$ reads

$$Z[C', C] = \int \mathcal{D}[\Lambda] F[\Lambda]$$

and actually, $F$ only depends on the winding number $n$. So we find that

$$Z[C', C] = \sum_{n=-\infty}^{\infty} \int \mathcal{D}[A] e^{S[A]}$$

where

$$A_k(x) = \begin{cases} C^\Lambda_n(\vec{x}) & \text{at } x^0 = 0 \\ C'_k(\vec{x}) & \text{at } x^0 = T \end{cases}$$
The Action And the Winding Number

A Boundary for the Action

- One finds that $S[A]$ is bounded by

$$S[A] \geq \frac{1}{2g_0^2} |S_{CS}[C] - S_{CS}[C'] + n|$$

- Where $S_{CS}$ is the Chern-Simons action
- And $n$ the winding number of $A$
- Only have to check a few topological sectors for minimal action gauge fields, which dominate the integral
The Action And the Winding Number

A Boundary for the Action

One finds that $S[A]$ is bounded by

$$S[A] \geq \frac{1}{2g_0^2} \left| S_{CS}[C] - S_{CS}[C'] + n \right|$$

$$= \frac{1}{2g_0^2} |\text{some number} + n|$$

- Where $S_{CS}$ is the Chern-Simons action
- And $n$ the winding number of $A$
- Only have to check a few topological sectors for minimal action gauge fields, which dominate the integral
Finding The Minimum

How To Obtain a Minimal Action Configuration $B$?

- Generally difficult
- Easy if we
  - Take a known solution $B$ of the field eqns. and
  - Define $C, C'$ as
    \[ C_k(x) |_{x^0 = 0} = B_k(x) |_{x^0 = 0} \quad C'_k(x) = B_k(x) |_{x^0 = T} \]
- If
  - $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$ is self dual and
  - $S_{SC}[C'] - S_{CS}[C] < 1/2$ and
  - $n(B) = 0$
- Then $B$ is the unique (up to gauge transformations) minimal action configuration
A Simple Example

A One-Parameter Family of Background Fields

Consider the BG-field

\[ B_0(x) = 0 \quad B_k(x) = b(x^0) l_k \quad [l_k, l_l] = \epsilon_{klj} l_j. \]

Self-duality condition reduces to

\[ \partial_0 b = b^2 \quad \Rightarrow \quad b(x^0) = (\tau - x^0)^{-1}. \]

We just found a family of globally stable background fields!
A Simple Example

A One-Parameter Family of Background Fields

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We just found a family of globally stable background fields!

We will need this for $\alpha$!
What About Renormalization?

**Question:** Is the SF Renormalizable?

In the weak coupling domain, expand the SF around the induced background field and obtain for the effective action:

\[
\Gamma[B] = -\ln Z[C', C] = g_0^{-2}\Gamma_0[B] + \Gamma_1[B] + g_0^2\Gamma_2[B] + \ldots
\]

With \(\Gamma_0[B] = g_0^2 S[B]\), divergent in each power of \(g_0\)

**Answer:** Most Probably ... Yes

- Of course, one has to renormalize \(g_0, (m)\)
- In general, one has to add boundary counter-terms
- This should be sufficient (checked up to 2-loop order in QCD)
- In Yang-Mills theory, no such counter-terms are needed
The Running Coupling (Finally)

A Running Coupling Recipe

- Choose a background field $B$ depending on a dimensionless parameter $\eta$
- Then $\Gamma'[B] = -\frac{\partial}{\partial \eta} \Gamma[B]$ is a renormalization group invariant.
- Set $T = L$ and define a physical coupling via
  \[
  \bar{g}^2(L) := \frac{\Gamma'[B]}{\Gamma_0'[B]}
  \]
- This is a Casimir force between the boundary fields
- If the chosen field depends on parameters with dimension $\neq 1$, scale them proportional to $L$, e.g. in our example set $\tau = -L/\eta$
The Result

From ALPHA again
Let’s Measure a Mass

Fermions

The next interesting quantities which needs scale dependent Renormalization are the quark Masses.

- Define $N_f$ fermion fields $\psi_s$ on our periodic space time
- Define boundary fields $\zeta, \zeta'$ for quark fields
- Add counter terms for $\psi$ at the boundary for renormalization
Definition for $\bar{m}$

Defining a Running Quark Mass

- Use the PCAC relation to define $\bar{m}$

$$\partial_\mu A_\mu^R(x) = (\bar{m}_s + \bar{m}_s') P^R(x)$$

with

$$A_\mu^R(x) = Z_A A_\mu(x) = Z_A \bar{\psi}_s(x) \gamma_\mu \gamma_5 \psi_s'(x)$$

$$P^R(x) = Z_P P(x) = Z_P \bar{\psi}_s(x) \gamma_5 \psi_s'(x)$$

- $A_\mu(x)$ is renormalized through current algebra relations
- Scale- & scheme-dependence arises through renormalization of $P(x)$, $Z_P = Z_P(\mu)$
- the corresponding RG function reads $\tau(\bar{g}) \bar{m}_s = \mu \frac{\partial \bar{m}_s}{\partial \mu}$
A Definition for $Z_P(L)$

We drop $s$ and define

$$Z_P(L) = \frac{\sqrt{3f_1}}{f_P(L/2)}$$

where $\sqrt{3f_1}$ is only a normalization factor, defined as

$$f_P(x) = -\frac{1}{3} \int d^3y \, d^3z \, \langle \overline{\psi}(x) \gamma_5 \frac{1}{2} \tau^a \psi(x) \overline{\zeta}(y) \gamma_5 \frac{1}{2} \tau^a \zeta(z) \rangle$$

$$f_1 = -\frac{1}{3L^6} \int d^3u \, d^3v \, d^3y \, d^3z \, \langle \overline{\zeta'}(u) \gamma_5 \frac{1}{2} \tau^a \zeta'(v) \overline{\zeta}(y) \gamma_5 \frac{1}{2} \tau^a \zeta(z) \rangle$$

which look complicated, but...
$f_p$ and $f_1$, an Illustration

... can be illustrated like this:
Calculating $M(m, \mu)$, Pt. 1

Yet Another Step Scaling Function

So far, we have

$$\overline{m}(\mu)_s = \frac{Z_A}{Z_P(L)} m_s$$

Define the step scaling function $\sigma_P$ as

$$Z_P(2L) = \sigma_P(u) Z_P(L)$$

and compute $\sigma(L_0), \ldots, \sigma(2^k L_0)$. Use these for

$$\frac{M}{m(2^k L_0)} = \frac{M}{m(L_0)} \underbrace{\frac{\overline{m}(L_0)}{m(2L_0)} \frac{\overline{m}(2L_0)}{m(2^2 L_0)} \cdots \frac{\overline{m}(2^{k-1} L_0)}{m(2^k L_0)}}_{\text{accessible in PT}} \sim \text{SSF}^{-1}$$
Calculating $M(m, \mu)$, Pt. 2

The Final Step

Finally, we can compute

$$M = \frac{M}{\bar{m}(2^k L_0)} \bar{m}(2^k L_0)$$

$$= \frac{M}{\bar{m}(2^k L_0)} Z_A \frac{1}{Z_P (\mu = (2^k L_0)^{-1})} m$$

(known from Pt. 1) (from simulations)

$$= Z(\mu) m$$

We found the overall renormalization factor!
Scale Dependent Renormalization

The Schrödinger Functional

Renormalizing Quark Masses

This Talk’s Last Picture

\[ \frac{\bar{m}(\mu)}{M} \]

\[ \text{SF scheme, } N_f=2 \]

- 2/3-loop
- 1/2-loop

\[ \mu/\Lambda \]

ALPHA once more
Conclusions

- Important physical quantities like $\alpha$ and $m$ require scale dependent renormalization.
- Scale dependent renormalization is a difficult task, because a large variety of energy scales has to be covered.
- This problem can be fixed by using a finite scaling technique.
- The Schrödinger Functional provides a good framework for the definition of scale dependent quantities.

Thank you!

Some literature:

- R. Sommer: Non-perturbative QCD [...], hep-lat/0611020
- Capitani, Lüscher, Sommer, Wittig: Non-perturbative quark mass renormalization in quenched lattice QCD, hep-lat/9810063