

Chapter 3

Burgers Equation

One of the major challenges in the field of complex systems is a thorough understanding of the phenomenon of turbulence. Direct numerical simulations (DNS) have substantially contributed to our understanding of the disordered flow phenomena inevitably arising at high Reynolds numbers. However, a successful theory of turbulence is still lacking which would allow to predict features of technologically important phenomena like turbulent mixing, turbulent convection, and turbulent combustion on the basis of the fundamental fluid dynamical equations. This is due to the fact that already the evolution equation for the simplest fluids, which are the so-called Newtonian incompressible fluids, have to take into account nonlinear as well as nonlocal properties:

$$\begin{aligned}\frac{\partial}{\partial t}\mathbf{u}(\mathbf{x},t) + \mathbf{u}(\mathbf{x},t) \cdot \nabla\mathbf{u}(\mathbf{x},t) &= -\nabla p(\mathbf{x},t) + \nu\Delta\mathbf{u}(\mathbf{x},t), \\ \nabla \cdot \mathbf{u}(\mathbf{x},t) &= 0.\end{aligned}\tag{3.1}$$

Nonlinearity stems from the convective term and the pressure term, whereas non-locality enters due to the pressure term. Due to incompressibility, the pressure is defined by a Poisson equation

$$\Delta p(\mathbf{x},t) = -\nabla \cdot \mathbf{u}(\mathbf{x},t) \cdot \nabla\mathbf{u}(\mathbf{x},t).\tag{3.2}$$

In 1939 the dutch scientist J.M. Burgers [1] simplified the Navier-Stokes equation (3.1) by just dropping the pressure term. In contrast to Eq. (3.1), this equation can be investigated in one spatial dimension (Physicists like to denote this as 1+1 dimensional problem in order to stress that there is one spatial and one temporal coordinate):

$$\frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t) = \nu\frac{\partial^2}{\partial x^2}u(x,t) + F(x,t)\tag{3.3}$$

Note that usually the Burgers equation is considered without external force $F(x,t)$. However, we shall include this external force field.

The Burgers equation 3.3 is nonlinear and one expects to find phenomena similar to turbulence. However, as it has been shown by Hopf [8] and Cole [3], the homogeneous Burgers equation lacks the most important property attributed to turbulence: The solutions do not exhibit chaotic features like sensitivity with respect to initial conditions. This can explicitly shown using the *Hopf-Cole transformation* which transforms Burgers equation into a linear parabolic equation. From the numerical point of view, however, this is of importance since it allows one to compare numerically obtained solutions of the nonlinear equation with the exact one. This comparison is important to investigate the quality of the applied numerical schemes. Furthermore, the equation has still interesting applications in physics and astrophysics. We will briefly mention some of them.

Growth of interfaces: Deposition models

The Burgers equation (3.3) is equivalent to the so-called *Kardar-Parisi-Zhang (KPZ-) equation* which is a model for a solid surface growing by vapor deposition, or, the opposite case, erosion of material from a solid surface. The location of the surface is described in terms of a height function $h(\mathbf{x}, t)$. This height evolves in time according to the KPZ-equation

$$\frac{\partial}{\partial t} h(\mathbf{x}, t) - \frac{1}{2} (\nabla h(\mathbf{x}, t))^2 = \nu \frac{\partial^2}{\partial x^2} h(x, t) + F(x, t). \quad (3.4)$$

This equation is obtained from the simple advection equation for a surface at $z = h(\mathbf{x}, t)$ moving with velocity $\mathbf{U}(\mathbf{x}, t)$

$$\frac{\partial}{\partial t} h(\mathbf{x}, t) + \mathbf{U} \cdot \nabla h(\mathbf{x}, t) = 0. \quad (3.5)$$

The velocity is assumed to be proportional to the gradient of $h(\mathbf{x}, t)$, i.e. the surface evolves in the direction of its gradient. Surface diffusion is described by the diffusion term.

Burgers equation (3.3) is obtained from the KPZ-equation just by forming the gradient of $h(\mathbf{x}, t)$:

$$\mathbf{u}(\mathbf{x}, t) = -\nabla h(\mathbf{x}, t). \quad (3.6)$$

3.1 Hopf-Cole Transformation

The Hopf-Cole transformation is a transformation, which maps the solution of the Burgers equation (3.3) to the heat equation

$$\frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \nu \Delta \psi(\mathbf{x}, t). \quad (3.7)$$

We perform the ansatz

$$\psi(\mathbf{x}, t) = e^{h(\mathbf{x}, t)/2\nu} \quad (3.8)$$

and determine

$$\Delta\psi = \frac{1}{2\nu} \left[\Delta h + \frac{1}{2\nu} (\nabla h)^2 \right] e^{h/2\nu} \quad (3.9)$$

leading to

$$\frac{\partial}{\partial t} h - \frac{1}{2} (\nabla h)^2 = \nu \Delta h. \quad (3.10)$$

However, this is exactly the Kardar-Parisi-Zhang equation (3.4). The complete transformation is then obtained by combining

$$\mathbf{u}(\mathbf{x}, t) = -\frac{1}{2\nu} \nabla \ln \psi(\mathbf{x}, t). \quad (3.11)$$

We explicitly see that the Hopf-Cole transformation turns the nonlinear Burgers equation into the linear heat conduction equation. Since the heat conduction equation is explicitly solvable in terms of the so-called heat kernel we obtain a general solution of the Burgers equation. Before we construct this general solution, we want to emphasize that the Hopf-Cole transformation applied to the multi-dimensional Burgers equation only leads to the general solution provided the initial condition $\mathbf{u}(\mathbf{x}, 0)$ is a gradient field. For general initial conditions, especially for initial fields with $\nabla \times \mathbf{u}(\mathbf{x}, t)$, the solution can not be constructed using the Hopf-Cole transformation and, consequently, is not known in analytical terms. In one dimension spatial dimension it is not necessary to distinguish between these two cases.

3.2 General Solution of the 1D Burgers Equation

We are now in the position to formulate the general solution of the Burgers equation (3.3) in one spatial dimension with initial condition

$$u(x, 0), \quad \psi(x, 0) = e^{-\frac{1}{2\nu} \int^x dx' u(x', 0)}. \quad (3.12)$$

The solution of the 1D heat equation can be expressed by the heat-kernel

$$\psi(x, t) = \int dx' G(x - x', t) \psi(x', 0) \quad (3.13)$$

with the kernel

$$G(x - x', t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-x')^2}{4\nu t}} \quad (3.14)$$

In terms of the initial condition (3.12) the solution explicitly reads

$$\psi(x, t) = \frac{1}{\sqrt{4\pi t}} \int dx' e^{-\frac{(x-x')^2}{4\nu t} - \frac{1}{2\nu} \int^{x'} dx'' u(x'', 0)}. \quad (3.15)$$

The n -dimensional solution of the Burgers equation (3.3) for initial fields, which are gradient fields, are obtained analogously:

$$\psi(x, t) = \frac{1}{(4\pi t)^{d/2}} \int d\mathbf{x}' e^{-\frac{(x-x')^2}{4vt}} - \frac{1}{2v} \int^{\mathbf{x}'} d\mathbf{x}'' \cdot \mathbf{u}(\mathbf{x}'', 0). \quad (3.16)$$

Again, we see that the solution exist provided the integral is independent of the integration contour:

$$\int^{\mathbf{x}'} d\mathbf{x}'' \cdot \mathbf{u}(\mathbf{x}'', 0) = h(\mathbf{x}', t). \quad (3.17)$$

We can investigate the limiting case of vanishing viscosity, $v \rightarrow 0$. In the expression for $\psi(x, t)$, eq. (3.16), the integral is dominated by the minimum of the exponential function,

$$\min_{x'} \left[-\frac{(x-x')^2}{4vt} - \frac{1}{2v} \int^{\mathbf{x}'} d\mathbf{x}'' u(\mathbf{x}'', 0) \right]. \quad (3.18)$$

This leads to the so-called characteristics (see App. (B))

$$x = x' - tu(x', 0), \quad (3.19)$$

which we have already met in the discussion of the advection equation (2.1) (see Chapter 2). A special solution for the viscid Burgers equation is

$$u(x, t) = 1 - \tanh\left(\frac{x - x_c - t}{2v}\right). \quad (3.20)$$

3.3 Forced Burgers Equation

The Hopf-Cole transformation can be applied to the forced Burgers equation. It is straightforward to show that this leads to the parabolic differential equation

$$\frac{\partial}{\partial t} \psi(x, t) = v \Delta \psi(\mathbf{x}, t) - U(\mathbf{x}, t) \psi(\mathbf{x}, t), \quad (3.21)$$

where the potential is related to the force

$$\mathbf{F}(\mathbf{x}, t) = -\frac{1}{2v} \nabla U(\mathbf{x}, t). \quad (3.22)$$

The relationship with the Schrödinger equation for a particle moving in the potential $U(\mathbf{x}, t)$ is obvious. Recently, the Burgers equation with a fluctuating force has been investigated [12]. Interestingly, Burgers equation with a linear force, i.e. a quadratic potential

$$U(x, t) = a(t)x^2 \quad (3.23)$$

for an arbitrary time dependent coefficient $a(t)$ could be solved analytically [7].

3.4 Numerical Treatment

Let us consider a one-dimensional Burgers equation (3.3) without forcing.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}.$$

When $\nu = 0$, Burgers equation becomes *the inviscid Burgers equation*:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0, \quad (3.24)$$

which is a prototype for equations for which the solution can develop discontinuities (*shock waves*). As was mentioned above, as the solution of the advection equation (2.1), the solution of Eq. (3.24) can be constructed by the method of characteristics (see App. B). Suppose we have an initial value problem, i.e., a smooth function $u(x, 0) = u_0(x)$, $x \in \mathbb{R}$ is given. In this case the coefficients A , B and C are

$$A = u, \quad B = 1, \quad C = 0.$$

Equations (B.2-B.3) read

$$\begin{aligned} \frac{dt}{ds} &= 1 \Leftrightarrow |t(0) = 0| \Leftrightarrow t = s, \\ \frac{du}{ds} &= 0 \Leftrightarrow |u(0) = u_0(x_0)| \Leftrightarrow u(s, x_0) = u_0(x_0), \\ \frac{dx}{ds} &= u \Leftrightarrow |x(0) = x_0| \Leftrightarrow x = u_0(x_0)t + x_0. \end{aligned}$$

Hence the general solution of (3.24) takes the form

$$u(x, t) = u_0(x - u_0(x_0)t, t). \quad (3.25)$$

Eq. (3.25) is an implicit relation that determines the solution of the inviscid Burgers' equation. Note that the characteristics are straight lines, but not all the lines have the same slope. It will be possible for the characteristics to intersect. If we write the characteristics as

$$t = \frac{u}{u_0(x_0)} - \frac{x_0}{u_0(x_0)},$$

one can see, that the slope $1/u_0(x_0)$ of the characteristics depends on the point x_0 and on the initial function u_0 . For inviscid Burgers' equation (3.24), the time T_c at which the characteristics cross and a shock forms, the "breaking" time, can be determined exactly as

$$T_c = \frac{-1}{\min\{u_x(x, 0)\}}$$

This relation can be used if Eq. (3.24) has smooth initial data (so that it is differentiable). From the formula for T_c , we can see that the solution will break and a shock

will form if $u_x(x, 0)$ is negative at some point. From numerical point of view it is convenient to rewrite the Burgers' equation as

$$\frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x}(u^2) = 0 \quad (3.26)$$

Equation (3.26) describes a one-dimensional conservation law (2.13) with $F = \frac{1}{2}u^2$ and can be solve, e.g., with the upwind method (2.4) or with the Lax-Wendroff method (2.14).

Space interval	$L=10$
Initial condition	$u_0(x) = \exp(-(x-3)^2)$
Space discretization step	$\Delta x = 0.05$
Time discretization step	$\Delta t = 0.05$
Amount of time steps	$T = 36$

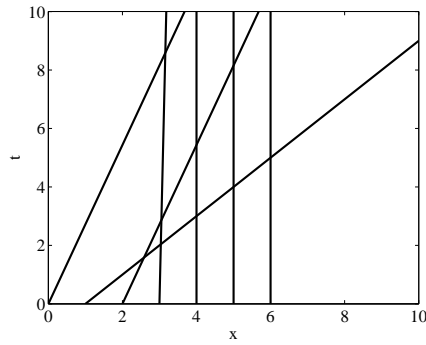


Fig. 3.1 Characteristics curves for the inviscid Burgers' equation (3.24)

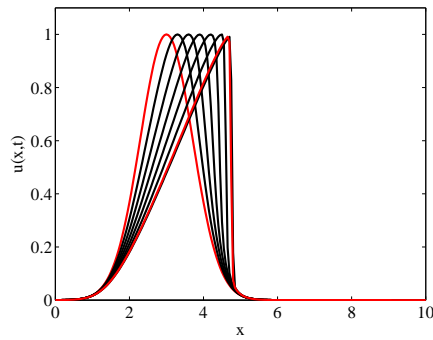


Fig. 3.2 Numerical solution of the inviscid Burgers' equation (3.24)

3.4.0.1 The Riemann Problem

A Riemann problem, named after *Bernhard Riemann*, consists of a conservation law, e.g., Eq. (3.24) together with a piecewise constant data having a single discontinuity, i.e.,

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < a; \\ u_r, & x \geq a. \end{cases} \quad (3.27)$$

The form of the solution depends on the relation between u_l and u_r .

- $u_l > u_r$: The unique weak solution (see Fig. 3.2 (a)) is

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x < a + ct; \\ u_r, & x \geq a + ct \end{cases} \quad (3.28)$$

with the *shock velocity*

$$c = \frac{1}{2}(u_l + u_r).$$

Note, that in this case the characteristics in each of the region where u is constant go into the shock as time advances (see Fig. 3.3 (b)).

Space interval	$L=10$
Initial condition	$u_l = 0.8, u_r = 0.2$
Space discretization step	$\Delta x = 0.05$
Time discretization step	$\Delta t = 0.05$
Amount of time steps	$T = 100$

The initial condition is:

$$u(x,0) = u_0(x) = \begin{cases} 0.8, & x < 5; \\ 0.2, & x \geq 5. \end{cases} \quad (3.29)$$

- $u_l < u_r$: In this case there are infinitely many weak solutions. One of them is again (3.28) with the same velocity (see Fig. 3.4 (a)). Note that in this case the characteristics go out of the shock (Fig. 3.4 (b)) and the solution is not stable to perturbations.

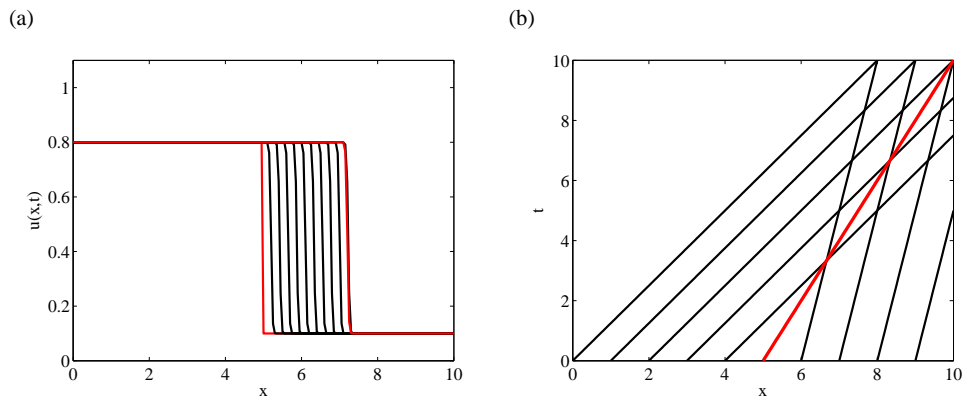


Fig. 3.3 a) Numerical solution of the inviscid Burgers' equation (3.24) for the Riemann problem for $u_l < u_r$. b) Characteristics of Eq. (3.24) with initial conditions (3.29). The red line indicates the curve $x = a + ct$.

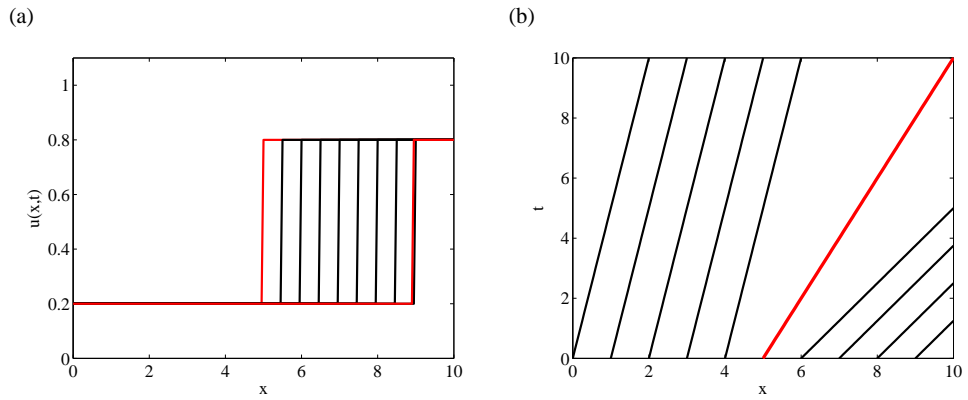


Fig. 3.4 a) Numerical solution of the inviscid Burgers' equation (??) for the Riemann problem for $u_l < u_r$. b) Characteristics of the inviscid Burgers' equation with initial conditions (??). The red line indicates the curve $x = a + ct$.