Synchronization and Pattern Formation in Coupled Nonlinear Optical Systems

Guido Krüger

gkruger@uni-muenster.de

WWU Münster
<table>
<thead>
<tr>
<th>Introduction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coupled Nonlinear Optical Systems with Pattern Formation</td>
</tr>
<tr>
<td>- Defining the System</td>
</tr>
<tr>
<td>- Theoretical Analysis</td>
</tr>
<tr>
<td>- Synchronization</td>
</tr>
<tr>
<td>- Numerical Results</td>
</tr>
<tr>
<td>Conclusion / Outro</td>
</tr>
</tbody>
</table>
Introduction
Introduction

- Analyse coupled nonlinear optical systems
- Possible information transfer device
- Interaction of coupled solitary (localized) structures
Coupled Nonlinear Optical Systems with Pattern Formation
Defining the System

- Two pattern forming single feedback mirror systems
- Unidirectional coupling from system 1 to system 2
- Separation of refraction and diffraction

[Scroggie, Firth, PRA 53,2752 (1996)] [Mitschke et al., PRA 33,3219 (1986)]
Defining the System

- Interesting points
  - Synchronization of patterns (regular and chaotic)
  - Transmission of patterns
  - Solitary structures

[Scroggie, Firth, PRA 53,2752 (1996)] [Mitschke et al., PRA 33,3219 (1986)]
Theoretical Analysis - The Generalized Bloch Type Equation

- Generalized Bloch type equation for \( \mathbf{m} = (u, v, w) \)

\[
\dot{\mathbf{m}} = \Omega \times \mathbf{m} - \gamma_{\text{eff}} \mathbf{m} + \mathbf{P}
\]  

\( \Omega = (\Omega_x, \Omega_y, \Omega_z) \): magnetic field

\( \gamma_{\text{eff}} \): effective relaxation rate

\( \mathbf{P} \): pumprate due to the injected electrical field (i.e. the laser)

- No magnetic field
- Equations uncouple
- Quasiskalar description

\[
\dot{w} = - (\gamma - \text{Diff} \nabla^2_\perp) w + P_+ (1 - w) - P_- (1 + w)
\]

\( w \): magnetization of the medium

\( P_\pm \): pumprate of the positive / negative circular polarized light

[Mitschke et al., PRA 33,3219 (1986)] [Scroggie, Firth, PRA 53,2752 (1996)]
Theoretical Analysis - Where Are The Patterns?

- blue = positive magnetization
- red = negative magnetization
- \( w = \) magnetization of the medium (Na-vapor)
- magnetization \( \Leftrightarrow \) Intensity
## Synchronization / Correlation

### Synchronization

- Transfer fields $\phi_i \in \mathbb{R} (n \times n)$ to vectors $\phi_i \in \mathbb{R} (n \times n)$
- Subtract mean values, to only calculate the synchronization of the varying fields

$$u_i = \phi_i - < \phi_i>.$$ (3)

- The function to derive the synchronization rates is chosen to

$$SyncRate(u, v) = \frac{1}{||u|| ||v||} u.v$$ (4)

### Local Correlation

- Time series of one pixel $\phi_i(r, t) \forall r$
- Subtract mean values (Correlation coefficient from Pearson)
- Take last $N$ timesteps $\rightarrow u(r) = (\phi_i(r, t_n))_{n=1..N}$

$$LocCorr(u, v)(r) = \frac{1}{||u|| ||v||} u.v$$ (5)
Numerical Results – One Cell

$\frac{\lambda}{8}$-Case
Numerical Results - Solitary Structures Region

- $\lambda/8$-Setting, Bistable system, Pitchfork-bifurcation

Homogenous state of the system (a) $\alpha = 0^\circ$, (b) $\alpha = 10^\circ$. Results of the linear stability Analysis (c) $\alpha = 0^\circ$, (d) $\alpha = 10^\circ$ (— positive branch, - - negative branch).
Numerical Results - Solitary Structures Region

(a) Grey shaded is the region where solitary structures exist. (b) Solitary object in a patterned underground ($I_{0I} = 400$ mW). (c),(d) Cut through, real image of a solitary object on a homogenous underground ($I_{0I} = 300$ mW). The parameters are $\gamma = 200$ /s, $D = 250 \text{mm}^2$ /s, $\alpha_0 = 2.2168$, $1/(2k_0) = 0.468$ mm, $L = 15$ mm, $d = 0.88$ dm, $\alpha_I = 10^\circ$, $\alpha_{II} = -10^\circ$, $\varphi_I = \varphi_{II} = \pi/4$, $R_I = R_{II} = 0.99$, $s = 0.0$, $\bar{\Delta} = 6.0$. 
Numerical Results - Different Solitary Structures

Different solitary structures stable

Discrete Family of solitary structures

[Presch et al., PRL 95,143906(2005)]
(right) Regions of existence for labyrinths and localized patterns.
(left) Hexagon, localized pattern (positive and negative Hexagon) and labyrinthine structures.
Parameters: like solitary structures, except $\alpha_I = 0^\circ$

[Diss. Schüttler, 2006]
Numerical Results – Coupling solitons
Numerical Results - Solitary Structures on Random State

(a) Synchronisation rate against increasing coupling with $\lambda/8$-plate in both systems, initial condition is a pos. solitary object in system 1 and a random state in system 2. Intensities $I_{0I} = 300\text{mW}$, Angles $: \alpha_I = \alpha_{II} = 10^\circ$. Lines drawn to guide the eye. (b)(i) Weak coupling leads to neg. hexagonal structures with weak imprint of sol. obj. in cell $II$. (ii) Medium coupling leads to imprint of sol. obj. in cell $II$ on negative background. (iii) Initial positive solitary object of cell $I$ and transmitted sol. obj. for very strong coupling ($I_{0II} = 300\text{mW}$).
numerical results - transmission of solitary objects

(a) Synchronisation rate against increasing coupling with $\lambda/8$-plate in both systems, initial condition is a pos. solitary object in system 1 and a homogenous state in system 2. Intensities $I_{0I} = 300\text{mW}$, Angles $\alpha_I = \alpha_{II} = 10^\circ$. Lines drawn to guide the eye. (b)(i) Initial positive solitary object of cell $I$ and transmitted sol. obj. for very strong coupling ($I_{0II} = 300\text{mW}$) or medium coupling ($I_{0II} = I_{II\text{eff}}$). (ii) Weak coupling leads to weak imprint of sol. obj. in cell $II$. (iii) Medium coupling leads to formation of hexagonal structures with weak imprint of sol. obj.. (iv) Strong coupling leads to hexagonal structures with transmitted sol. obj..
Numerical Results - Transmission of Solitary Objects

(a) Color coded 2D-plot of cell $II$ with $s = 0.90$, (b) inverted corresponding 3D-plot (for better visibility).
(c) Color coded 2D-plot of cell $II$ with $s = 1.00$, (d) inverted corresponding 3D-plot.

(a) Synchronisation rate with increasing coupling with $\frac{\lambda}{8}$-plate in both systems, initial condition is a pos. solitary object in system 1 and a negative one in system 2. Intensities $I_{0I} = I_{0II} = 300 \text{mW}$, Angles: $\alpha_I = -\alpha_{II} = 10^\circ$. Lines drawn to guide the eye. (b) (i) Initial negative solitary object of cell 2, (ii) “kicked” solitary objects with coupling strength $s = 0.03$, (iii) final field with $s = 0.90$ and (iv) final field with $s = 1.00$. 

Guido Krüger — gkruger@uni-muenster.de — Workshop Ameland — 2007
Numerical Results - Coupling identical solitons $s = 0.02$

Overview of stable distances

Initial fields: (top) Cell I (bottom) Cell II

1D-Cuts of stable distances along the connection line (green = Cell I, blue = Cell II)
Numerical Results - Coupling inverse solitons $s = 0.03$

Overview of stable distances

Initial fields: (top) Cell I (bottom) Cell II

1D-Cuts of stable distances along the connection line (green = Cell I, blue = Cell II)
Coupling of solitons yields three basic mechanisms:

- **Destroying** - Soliton in the second cell is destroyed.
  Resulting pattern:
  - homogenous state
  - generic hexagon

- **Moving** - Soliton in the second cell is moved.
  Resulting pattern:
  - shape of soliton in the second cell unchanged, position changed
  - shape of soliton in the second cell slightly changed, position changed

- **Morphing** - Soliton in the second cell is morphed.
  Resulting pattern:
  - soliton in the second cell synchronizes with first soliton
Numerical Results - Different Solitons - Destroying

- solitons of different sizes
- solitons with different direction

Coupling of different solitons with different direction. (left) LS4 positive - LS2 negative, (right) LS2 positive - LS4 negative.

Parameters: $\Delta = 6.0$, $\alpha_I = -\alpha_{II} = 5^\circ$, $I_I = I_{II} = 300\text{mW}$. (green = Cell I, blue = Cell II)
Numerical Results - Different Solitons - Moving

- LS1 ± - LS1 ±
- different solitons, same direction
- small couplings

(left) Stable distances and max synchronization with initial solitons in the second cell. (right) 1D-Cuts at the connection line (stable distance, max deformation). Parameters: $\Delta = 6.0$, $\alpha_I = -\alpha_{II} = 5^\circ$, $I_I = I_{II} = 300\,\text{mW}$. (green = Cell I, blue = Cell II)
Numerical Results - Different Solitons - Moving

- LS1 ± - LS1 ±
- different solitons, same direction
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(left) Stable distances and max synchronization with initial solitons in the second cell. (right) 1D-Cuts at the connection line (stable distance, max deformation). Parameters: $\Delta = 6.0$, $\alpha_I = -\alpha_{II} = 5^\circ$, $I_I = I_{II} = 300\text{mW}$. (green = Cell I, blue = Cell II)
different solitons

same direction

Morphing of solitons. (left) LS2 pos. - LS3 pos. $s = 0.010$, (right) LS2 pos. - LS4 pos. $s = 0.030$. Parameters:\n$\bar{\Delta} = 6.0$, $\alpha_I = -\alpha_{II} = 5^\circ$, $I_I = I_{II} = 300 \text{mW}$. (green = Cell I, blue = Cell II)
Numerical Results - Different Solitons - Morphing

- different solitons
- same direction

Morphing of solitons. (left) LS3 pos. - LS2 pos. $s = 0.035$ and (right) LS4 pos. - LS3 pos. $s = 0.045$. Parameters:

$\Delta = 6.0, \alpha_I = -\alpha_{II} = 5^\circ, I_I = I_{II} = 300$ mW. (green = Cell I, blue = Cell II)
Numerical Results – Labyrinth coupling
Numerical Results - Labyrinth coupling

Entropy = \(- \sum_k |c_k^{(2)}| \log \left( \frac{|c_k^{(1)}|}{|c_k^{(2)}|} \right) \)  \( \sum_k |c_k^{(i)}| = 1 \)
Numerical Results – Domains
\( \frac{\lambda}{4}, \frac{\lambda}{8} \)
Numerical Results - Domains

- L4.L8 case
- second cell bistable

(Left) Region of existance of domain structures (right) Developing of domains with increasing coupling (top left)

$8f(II) + 8f(I), s = 2.2$, (top right) domains and fronts $D, s = 2.5$ (bottom left) labyrinthine structure $s = 3.0$.

(Parameter : $\bar{\Delta} = 6.0$, $I_{0I} = 300 \text{mW}$ and $I_{0II} = 290 \text{mW}$). (bottom right) pure circular domains (Parameter : $\bar{\Delta} = 5.5$, $I_{0I} = 355 \text{mW}$ und $I_{0II} = 345 \text{mW}$).
Conclusions / Outro
Conclusions / Outro

- Coupled two transverse pattern forming nonlinear optical devices
- Transmission of solitary objects
- Interaction of pos. and neg. solitary objects
- Moving of solitary objects
- Destroying of solitary objects
- Morphing of solitary objects
- Coupling of labyrinthine structurse → Synchronisation
- Coupling of two different regular patterns yields domain structures
Conclusions / Outro

- Coupled two transverse pattern forming nonlinear optical devices
- Transmission of solitary objects
- Interaction of pos. and neg. solitary objects
- Moving of solitary objects
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Thank you for listening!
Theoretical Analysis - The Pumprates final

Free propagation

\[ P_{FP}(x) = \exp[-ix\nabla_\perp^2/2k_0] \]

Propagation in the medium

\[ P_{\pm,M}(x, w) = \exp[i\alpha_0 \Delta x(1 \mp w)] \]

Matrixoperator \( \lambda/x \)-plate

\[ M(\phi, \alpha) \]

- Matrixoperator for the medium given by

\[ P_M(x, w) = \begin{pmatrix} P_{+,M}(x, w) & 0 \\ 0 & P_{-,M}(x, w) \end{pmatrix} \]

- Combining the operators easily gives the pumprates at certain points

\[ P_{\pm} \sim |\mathcal{E}_{\pm,f}(0, t)|^2 + |\mathcal{E}_{\pm,b}(0, t)|^2 \]

\[ P_{\pm} \sim |\mathcal{E}_{\pm}^0|^2 + R|M(\phi, \alpha)P_{FP}(2d)P_M(L, w)|^2 \]

Theoretical Analysis - Inserting the Coupling

- Uncoupled systems gives two differential equations

\[ \dot{w}_{I,II} = - (\gamma - \text{Diff} \nabla^2_\perp) w_{I,II} \]
\[ + P_{+I,II}(1 - w_{I,II}) - P_{-I,II}(1 + w_{I,II}) \]

(8) \[ = \text{NL}_{I,II}(w_{I,II}) \]

- Coupling the systems with coupling strength \( k \):

- Input in cell two is

\[ \mathcal{E}^0_{\pm,II} + \mathcal{E}_{\pm,I,b}(L, t) \]

(9)

- Neglecting interference terms

\[ \mathcal{E}^0_{\pm,II} \mathcal{E}_{\pm,I,b}(L, t) \rightarrow 0, \mathcal{E}^0_{\pm,II} \mathcal{E}_{\pm,I,b}(L, t)^* \rightarrow 0 \]
Theoretical Analysis - Inserting the Coupling

- Coupled System gives rise to differential eq. system

\[\dot{w}_I = NL_I(w_I)\]
\[\dot{w}_{II} = NL_{II}(w_{II}) + kNL_{III}(w_I, w_{II})\]

- \(NL_{II}(w_{II})\): Terms due to laser 2
- \(NL_{III}(w_I, w_{II})\): Terms due to the coupling

- “Coupling” nonlinearity is

\[NL_{III}(w_I, w_{II}) \sim |\mathcal{E}_{\pm,I,b}(L, t)|^2 + R|M(\phi_{II}, \alpha_{II})\mathcal{P}_{FP}(2d)P_M(L, w_{II})\mathcal{E}_{\pm,I,b}(L, t)|^2\]