Topological Representations of the Quantum Group $U_q\left(\mathfrak{sl}_2(\mathbb{C})\right)$ in Two Dimensional Conformal Field Theory

Als Habilitationsschrift dem Fachbereich Physik der Westfälischen Wilhelms–Universität Münster vorgelegt von

Christian Wieczerkowski

Münster 1996

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Contents

1 Introduction

Conformal invariance has a long history in physics. Maxwell's theory of electrodynamics and also Einstein's general relativity are examples of conformal invariant classical field theories. Cunningham [Cu09] and Bateman [Ba10] were the first to observe that Maxwell's equations are covariant not only under the Lorentz group but also under the larger conformal group. Conformal transformations are coordinate transformations under which the metric tensor $g_{\mu\nu}(x)$ is multiplied by a space–time dependent scalar factor $\sigma(x)$. They include in particular Poincaré transformations (homogeneous and inhomogeneous Lorentz transformations) where the conformal factor is one. They also include dilatations where the conformal factor is a constant different from one, and so called special conformal transformations. As a brief introduction to conformal invariance in classical physics we mention the article [FRW62] by Fulton, Rohrlich, and Witten. In two dimensions conformal invariance is a particularly powerful concept. If one considers two real coordinates (x, y) as one complex number $z = x + iy$, then conformal transformations consist of substitutions $z \mapsto f(z)$ where f is a holomorphic function. It is not surprising that holomorphic mappings therefore play an important role in the electrodynamics of two dimensional systems.

The standard model of particle physics is conformal invariant as a classical field theory if all particle masses are put to zero. In quantum field theory an early hope was that conformal invariance might be an approximate symmetry at high energies. The reason behind this hope was that conformal invariance should take place at energies where the particle masses become negligible. It was argued that theories should not have an intrinsic scale at high energies and therefore be scale invariant. Unfortunately, renormalization tends to break scale invariance. The renormalization of a quantum field theory amounts to the subtractions of singularities due to vacuum fluctuations on all length scales. Their subtraction requires the introduction of a renormalization scale, which breaks the conformal invariance. At low energies scale invariance is furthermore broken explicitely in the standard model due to non–vanishing particle masses. Scale transformations were subsequently refined to the renormalization group, an important structure underlying quantum field theory. An introduction to this subject can be found in [Co71] by Coleman.

The construction of conformal invariant solutions to quantum field equations has remained a promising enterprise. The idea is to use the constraints imposed by conformal invariance to obtain truly non–perturbative information about quantized field theories. In conjunction with the idea of operator product expansions, this program is known in particle physics as conformal bootstrap. It still awaits its fruitful completion in higher dimensions than two. A status of the work preceding the present boost in two dimensions, together with references to the original articles, can be found in the report [DMPPT77] by Dobrev, Mack, Petkova, Petrova, and Todorov.

In two dimensions the conformal bootstrap is a story of astounding success. The seminal work of Belavin, Polyakov, and Zamolodchikov [BPZ84] started an avalanche of activity which is now an entire branch of theoretical physics. The wealth of results obtained in its development has had deep impact on various branches of physics and mathematics, and we may well expect more surprising results to come.

The motivation in physics to study two dimensional conformal invariant field theories is at least threefold. One motivation is string theory. String theory, as lectured for instance by Green, Schwarz, and Witten [GSW87], is a candidate for a quantum theory of all interactions including gravity. Conformal invariance comes in string theory as reparametrization invariance of world sheets. Two dimensional conformal field theories are the classical solutions of string theory. A particular conformal field theory determines the string vacuum and encodes information about the number of space–time dimensions and the gauge group of its low energy limit. Not every conformal field theory however is an acceptable string vacuum. One has constraints both from the requirement of internal consistency, for example the vanishing of the conformal anomaly and modular invariance, and also from the requirement that it should reproduce the four dimensional standard model as its low energy limit. The investigations of these constraints is an ambitious ongoing program.

The second motivation is the study of two dimensional statistical systems at the critical point [Ca89]. Two dimensional systems come in statistical physics and in solid state physics in the form two dimensional films in three dimensions, layered systems in three dimensions which are effectively two dimensional, and as surface effects of three dimensional bulk systems. Examples of which are the quantum Hall effect and layered models of high T_c super–conductivity. From a theoretical point of view, two dimensional systems are highly valuable as models of critical behavior because of their solvability. They also serve as a laboratory to test approximations with exact results.

In the statistical physics of critical systems one addresses the following inter–related questions. What is the classification of universality classes of critical behavior? What are the criteria to decide to which universality class a given system belongs? What are the values of critical exponents and universal amplitudes? What are the finite size effects when a critical system is put into a finite volume? What is the effect of perturbations which drive a system away from criticality? Conformal invariance is a powerful tool in their investigation. The strategy is to identify universality classes with conformal field theories. The critical exponents of a universality class are encoded in the spectrum of anomalous dimensions of the scaling operators in the conformal field theory. Conformal invariance puts restrictive constraints on their possible values. Finite size effects are studied as the response of a conformal field theories to a variation of the size and shape of the Euclidean space–time region. Last not least, the effect of perturbations is studied by conformal perturbation theory. A basic concept in this strategy is to view the critical system from the point of a Euclidean field theory.

Critical behavior is a collective phenomenon due to fluctuations of a statistical system on all length scales. An illustrative example in statistical physics is the two dimensional Ising model. Its critical properties can be investigated by means of the renormalization group [BGZ76, WK74]. The link to conformal field theory is the concept of scale invariance. In Wilson's renormalization group [WK74] it comes about as follows. Applying the renormalization group in the infra–red direction, one looks at a system on larger and larger scales. Technically one averages out short distance fluctuations and studies the resulting effective theory of long distance fluctuations. After having integrated out all fluctuations between two length scales, say L_1 and L_2 ($L_1 < L_2$), one compares the result with that of a pure scale transformation by L_2/L_1. Suppose first that your system is non–critical in the sense that fluctuations on a particular scale are dominating. Such a model is solved in a single step by averaging out precisely these fluctuations. The analysis of critical systems require an infinite number of steps, when in each step a finite portion of fluctuations is averaged. The idea is then to study the flow of effective theories, always rescaling to a unit scale where the effective distributions can be compared. Universality classes come out as domains of attraction. Two different systems in the same universality class tend to converge to the same effective distribution. Critical systems come out as renormalization group fixed points. Applying the scale analysis to a critical system, one flows into a fixed point which characterizes the universality class. The two dimensional (and also the three dimensional) Ising model has such a non–trivial infrared fixed point. This fixed point encodes the critical properties of the Ising model, for instance its critical exponents. Their investigation is a truly non–perturbative problem. The toolbox of the renormalization group for models in any number of dimensions contains the ϵ-expansion, the $1/N$-expansion, and numerical techniques. All of them make approximations which are difficult to control. In two dimensions conformal invariance provides a method which is truly non–perturbative and exact. Critical properties are therefore advantageously studied in terms of conformal field theory. For instance, the critical exponents can be inferred from the spectrum of anomalous dimensions. Conversely, if the critical indices are known from other calculations they can be used to identify the conformal field theory. Other prominent examples besides the Ising model are the tri–critical Ising model and the three states Potts model [Do84]. They belong to a series of conformal field theories called minimal models [BPZ84], whose symmetry properties form a subject of this thesis.

A critical system is scale invariant in the sense that looking upon it with a poorer magnifying glass reproduces the same picture up to a trivial scale transformation. Belavin, Polyakov, and Zamolodchikov [BPZ84] argue that scale invariance implies invariance under special conformal transformations. A system which is Euclidean invariant to begin with is consequently invariant under the full conformal group. The argument relies on a field theoretic description of the critical system. For the Ising model this can be done quite explicitely, see for instance [ID89]. For a general lattice field theory it requires to perform a scaling limit of the lattice correlation functions. Roughly speaking, the continuum field theory is contained in their long distance behavior. The argument concludes that the scaling limit of a critical system is a massless Euclidean field theory. It encloses a traceless stress–energy tensor, which generates the space–time symmetries. Provided that the field theory satisfies general axioms it follows that tracelessness implies conformal invariance. In two dimensions the technical conclusion is that the stress–energy tensor is a Lie field, and that its charges satisfy the commutation relations of two copies of the Virasoro algebra. This fact is known as

Lüscher–Mack theorem [LM76] and is the starting point of [BPZ84].

Two dimensional conformal field theories exhibit a wealth of interesting structures. A basic structure is the factorization into holomorphic and anti–holomorphic chiral field theories. The observation is that, if one changes from real Euclidean coordinates x^1 and x^2 to the complex ones $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$, two dimensional conformal field theory on the plane presents itself in terms of quantities which depend either on z or on $\bar{z}$. Such quantities are called chiral. In this thesis we will restrict our attention to the holomorphic chiral sector. A more involved structure, which is a main issue in this thesis, is the multi–valuedness of chiral conformal correlators as functions of the insertion points. In mathematical terms, chiral conformal correlators have a non–trivial monodromy. In two dimensions the exchange of two insertion points in a chiral correlator depends on the path along which they are exchanged. In dimensions higher than two any two such paths can be continuously deformed into one another. In two dimensions this is not the case. As a consequence, the monodromy of chiral correlators is not governed by the permutation group but rather by the braid group. In a nutshell, two dimensional field theories allow for more than the Bose–Fermi alternative. We mention that braid group statistics is a fascinating topic in itself, whose consequences are far from completely explored. An illuminating reference to braid group statistics are the lecture notes [Fr88] by Fröhlich.

The conformal field theories which we will look upon below therefore furnish indeed monodromy representations of braid groups. The complete representation theory of braid groups is a difficult mathematical problem. We will restrict our attention to those representation which arise in the conformal field theory of the minimal models due to [BPZ84] and the Wess–Zumino–Witten models [KZ84] based on $SU(2)$. They turn out to be so called R–matrix representations. The particular representation of the braid group, which is realized in a given conformal field theory is an extremely important structural datum. The investigations of this thesis were initiated by the observation that the R–matrix representations from minimal models and Wess–Zumino–Witten models based on $SU(2)$ coincide with R–matrix representations of the quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$. Abstractly speaking, one observed that conformal field theory comes together with data from the representation category of quantum groups. A natural question to ask is whether this coincidence can be traced back to a quantum group symmetry, or at least to a quantum group action on auxiliary quantities entering the construction of chiral correlators. This question will receive a positive answer in form of topological representations of quantum groups connected to integral representations of chiral correlators.

Quantum groups emerged from the solution of two dimensional integrable models by means of the algebraic Bethe ansatz. Their foundation as a mathematical object was layed by Drinfeld [D86]. Quantum groups have undergone an intense inspection since then. For our purposes, the notion of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ as a quantum deformation of the universal envelopping algebra of sl_2 will suffice. The deformation is such that the resulting object remains a Hopf algebra. In practice, one deformes both the relations of the Lie algebra and, for instance, the coproduct. The coproduct is necessary in order to preserve the notion of a tensor product of two representations. The result in the case of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is a Hopf algebra which is coassociative but non–cocommutative. The non–cocommutativity comes in form of a universal R–matrix which satisfies a Yang–Baxter equation. The Yang–Baxter equation again is the link to the algebraic Bethe ansatz [B82]. Quantum groups therefore provide a tool to solve integrable models based on the algebraic Bethe ansatz. The representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ turns out to degenerate in the case when the complex deformation parameter q becomes a root of one. In conformal field theory precisely this degenerate case turns out to be realized. An account of the representation theory of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ in the degenerate case has been given by Fröhlich and Kerler [FK93]. We mention that the decomposition of its adjoint representation in this case has been accomplished only very recently by Ostrik [O96].

That such a structure plays a role in quantum theory is an exciting surprise. Since the advent of quantum mechanics it is taught that symmetries take place in the form of representations of symmetry groups. In the presense of a symmetry quantum states can be organized into multiplets, which form representations of the symmetry group, and the symmetry implies selection rules for transitions. A question which was neglected before the appearance of quantum groups is whether one can also quantize the notion of symmetry. The most general notion of a symmetry compatible with the framework of quantum theory known today is that of a weak quasi–Hopf algebra. The quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$ falls into this category. An explanation of the appearance of quantum group data in conformal field theory is to implement the quantum group as a global symmetry algebra. The idea there is to decompose the Hilbert space of states into superselection sectors carrying quantum group quantum numbers, and to construct field operators making transitions between superselection sectors, subject to braid relations. This picture has been worked out by Mack and Schomerus [MS90]. A source of troubles are unwanted and unphysical non–decomposable finite dimensional representations.

In this thesis we proceed in a different direction. The quantum group data, braiding and fusion rules, comes encoded in chiral conformal correlators. They form the building blocks of a conformal field theory. The most powerful method for their investigation are free field representations. We will address the question why quantum group data appears in free field representations. In free field representations one works with an enlarged theory. The quantum group turns out to act on this enlarged theory. Moreover, this action can be described explicitely, the representations can be explicitely constructed, identified, and analyzed. The result is a topological representation.

The third motivation to study two dimensional conformal field theory and its symmetries is its mathematical beauty, which has attracted a lot of workers to the field. Two dimensional conformal field theory is to a considerable extent the representation theory of infinite dimensional Lie algebras, the Virasoro algebra, and current algebras. As a consequence, it yields beautiful representation theoretic interpretations of critical exponents in two dimensions. When formulated on higher topologies, intricate connections with complex geometry, for instance modular properties of higher genus surfaces, appear. The main motivation to study higher genus surfaces comes from string theory. But the lowest higher topology, the torus, plays also an important role in statistical physics. About half this thesis is devoted to the modifications that come about in the theory of topological representations in the transition from genus zero to genus one. Last not least topological representations are a theory of a large class of special functions, hypergeometric functions and generalizations thereof. As a mathematical subject they have been established by the work of Schechtman and Varchenko [SV90] and by Felder and the author [FW91]. The subject is still in motion. In particular topological representations on higher genus surfaces are still to be properly understood.

2 Prologue

Two dimensional conformal field theory has its origin in the statistical physics of two dimensional systems at a critical point. It has furthermore attracted a lot of interest because of its applications to string theory, and also because of its mathematical beauty. We mention the seminal paper [BPZ84], the collection [ISZ88] of reprints, and the lectures [Gi89, Ca89, ID89] as a first guide to the literature.

2.1 Scale Invariance

The scaling limit of a critical model defines a two dimensional Euclidean quantum field theory. This field theory is not only invariant under the group of Euclidean motions but also under dilatations. One can argue [BPZ84, Ca89] that it is then invariant under the conformal group. An objective of conformal field is the classification of all two dimensional critical phenomena. The intense work in conformal field theory since [BPZ84] has changed the objective to a considerably broader nature. The connection of two dimensional conformal field theory with quantum groups is a structural question which has been posed in the course of this developement.

2.2 Energy–Momentum Tensor

We assume space–time to be two dimensional Euclidean space. Complex coordinates are introduced by $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. It is common strategy to consider z and $\bar{z}$ as independent variables, and to set them complex conjugate in the very end of the reasoning. The infinitesimal conformal transformations are given by $z \mapsto z - \epsilon z^{n+1}$. The corresponding generators on functions are $l_n = -z^{n+1}\partial_z$. They span an infinite dimensional Lie algebra with relations

$$[l_n, l_m] = (n-m)l_{n+m}. \tag{1}$$

In field theory, co–ordinate transformations are generated by the charges constructed from an energy–momentum tensor. In conformal field theory the energy–momentum tensor is traceless and decomposes in a holomorphic zz–component $T(z)$ and an anti–holomorphic $\bar{z}\bar{z}$–component $\bar{T}(\bar{z})$. It has an expansion

$$T(z) = \sum_{n\in\mathbb{Z}} L_n \, z^{-n-2}, \tag{2}$$

where the coefficients satisfy the relations of the Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{C}{12}(n^3-n)\delta_{n,-m}, \quad [L_n, C] = 0. \tag{3}$$

Thus in conformal field theory we are not dealing with the Witt algebra (1) but rather its central extension (3). This conformal anomaly was derived in [LM76] and is the starting point of [BPZ84]. Analogous equations hold for the anti–holomorphic component of the energy–momentum tensor. Eq. (3) plays the role of an infinite symmetry algebra in conformal field theory.

2.3 Minimal Models

In a given conformal field theory, the central charge C takes a definite value c, and is the most important single datum. In the Ising model $c = 1/2$. Special values of c have the important property that they yield a finite set of irreducible highest weight representations of (3). Models with this property are called rational. First examples are the minimal models of [BPZ84]. They have the property of being solvable in a strong sense. Integral representations are known for their correlation functions. The integrals have been computed explicitely in many cases. Let us briefly discuss the setup of minimal models of [BPZ84] following [F89]. The Hilbert space of a minimal model is a direct sum

$$\mathcal{H} = \bigoplus_{n',n} \mathcal{H}_{n',n} \otimes \mathcal{H}_{n',n} \tag{4}$$

of irreducible highest weight representations for two copies $Vir \oplus Vir$ of (3). The generators are L_n and $\bar{L}_n$, acting as $L_n \otimes 1$ and $1 \otimes L_n$ respectively. The central charge is

$$c = 1 - \frac{6(p'-p)^2}{p'p}, \tag{5}$$

where p' and p are two positive integers without common divisor. They label the series of minimal models. The highest weight module $\mathcal{H}_{n',n}$ is constructed as a quotient of a Verma module $V(h_{n',n}, c)$ with highest weight

$$h_{n',n} = \frac{(n'\,p - n\,p')^2 - (p-p')^2}{4p\,p'} \tag{6}$$

by its maximal proper submodule $LV(h_{n',n}, c)$. The representations are labelled by pairs n', n of integers subject to $1 \le n' < p'$, $1 \le n < p$ and $pn' > p'n$. A result of [FQS86] shows that a model is unitary if and only if $p = p' \pm 1$. For $p' = p - 1$ we obtain the series of unitary minimal models with central charge

$$c = 1 - \frac{6}{p(p+1)}, \tag{7}$$

where $p = 3, 4, 5, \ldots$. The Ising model is the first of which with $p = 3$. Let us also have a brief look at the field content of minimal models. It is convenient to introduce the abbreviations

$$N = (n', n), \quad M = (m', m), \quad L = (l', l) \tag{8}$$

for double indices. With each representation is associated a conformal primary field

$$\Phi_N(z, \bar{z}) = \sum_{M,L} C^L_{N,M} \; \phi^L_{N,M}(z) \otimes \phi^L_{N,M}(\bar{z}). \tag{9}$$

In the renormalization group terminology it is a scaling field. Here $C^L_{N,M}$ are $\mathbb{C}$-valued structure constants. The field $\phi^L_{N,M}(z)$ is a holomorphic chiral primary fields. It maps $\mathcal{H}_M$ to $\mathcal{H}_L$, is zero on the other $\mathcal{H}_K$, and is a conformal field of weight h_N:

$$[L_n, \phi^L_{N,M}(z)] = \left\{ z^{k+1} \frac{\mathrm{d}}{\mathrm{d}z} + (k+1)\; h_N \; z^k \right\} \phi^L_{N,M}(z). \tag{10}$$

We remark that (10) defines an action of Vir on a space of operators generated from the primary field. The operators obtained from the primary field by acting with products of L_{-n}, $n > 0$, are called descendants. The family of field operators forms a conformal multiplet [BPZ84]. In principle all matrix elements of chiral primary fields can be computed up to normalization constants by means of (10). The normalization constants can be fixed by

$$(v_L, \phi^L_{N,M}(1) v_M) = 1, \tag{11}$$

where v_M denotes the highest weight vector in $\mathcal{H}_M$. It follows that the theory is completely determined by the structure constants. Their evaluation is the subject of the conformal bootstrap [BPZ84]. Correlation functions of primary fields (9) factorize into

$$\langle \Phi_{N_1}(z_1, \bar{z}_1) \cdots \Phi_{N_k}(z_1, \bar{z}_k) \rangle = \sum_{\mu} \lambda_\mu F_\mu(z_1, \ldots, z_k) F_\mu(\bar{z}_1, \ldots, \bar{z}_k). \tag{12}$$

The index μ stands for a sequence of intermediate representations $(M_1, \ldots, M_{k-1})$. The constants λ_μ are products of structure constants. The factors in (12) are correlation functions

$$F_\mu(z_1, \ldots, z_k) = \left(v_{M_0}, \phi^{M_0}_{N_1,M_1}(z_1) \cdots \phi^{M_{k-1}}_{N_k,M_k}(z_k)\; v_{M_k} \right). \tag{13}$$

Here $M_0 = M_k = (1,1)$ denotes the vacuum representation. The correlation functions (13) are called chiral conformal blocks. They are the main objects of this investigation.

In principle they can be computed by doing the sum over intermediate representations. Again there exists a more powerful tool, the method of integral representations. In a nut shell, the program is to deduce integral representations for the solutions of differential equations for the conformal blocks which follow from (10). The result is generally a multiple contour integral. Boundary data is then encoded in the choice of integration contours. At this point the quantum group enters the scene. The quantum group organizes which contours yield the physical conformal blocks. By locality (12) is required to be single valued. This is not the case for the conformal blocks (13). They are generally multi–valued functions of the insertion points. Upon analytic continuation they define a representation of the braid group on the configuration space of insertion points. This representation is one of the interesting features of two dimensional conformal field theory. More on this will be said in the bulk of this thesis.

2.4 $U(1)$–Current Algebra

Integral representations for the conformal blocks of minimal models are efficiently deduced from free field representations. See [DF84, F89] and references therein. Since they form the point of departure of this thesis, we give a brief derivation for the case of minimal models. We do not follow the standard free field computations but rather choose an alternative geometrical construction. We use an adaptation of the approach in [F92] for Wess–Zumino–Witten models to the case of $U(1)$–current algebra. We find it amusing by itself. It also underlies the more involved constructions of the last chapters.

We begin with an introduction of pre–requisites, simultaneously fixing the notation. Let $g = \mathbb{C}$ be the complexified Lie algebra of $U(1)$. The associated loop algebra is $Lg = \mathbb{C}((t))$, the space of complex valued formal Laurent series in t. The loop algebra is again a Lie algebra, and is abelian in this case. It has a central extension $\widehat{Lg} = Lg \oplus \mathbb{C}k$. This central extension is a non–abelian Lie algebra with bracket

$$[f \oplus \zeta k, g \oplus \zeta k] = \mathrm{res}(f'g)\ k, \tag{14}$$

where f and g are formal Laurent series, res(f) is the coefficient of t^{-1}, η and ζ are complex numbers, and k denotes the central element. The derivative with respect to t is adjoined as an additional element L_{-1} through

$$[L_{-1}, f \oplus \zeta\ k] = -f'. \tag{15}$$

In terms of the generators $a_n = t^n$, $n \in \mathbb{Z}$, the bracket (14) becomes

$$[a_n, a_m] = n\delta_{n,m} k. \tag{16}$$

Analogous to (2) one associates with (16) a $U(1)$–current $j(z) = \sum_{z\in\mathbb{Z}} a_n\ z^{-n-1}$. We remark that $\widehat{Lg}$ can be decomposed into $n_- \oplus g \oplus n_+$, where $n_\pm = t^\pm\mathbb{C}[[t^\pm]]$. Furthermore, $\widehat{Lg} \oplus \mathbb{C}L_{-1}$ is integer graded by the assignments $\deg(t^n) = n$, $\deg(k) = 0$, $\deg(L_{-1}) = -1$.

The next brick in this construction are highest weight modules $V(\alpha)$, where α is a real number called the charge of the module. $V(\alpha)$ is defined as being generated from a cyclic vector $v(\alpha)$ with defining properties

$$\begin{aligned} a_n v(\alpha) &= 0, \quad n > 0, & (17)\\ a_0 v(\alpha) &= \alpha v(\alpha), & (18)\end{aligned}$$

and $kv(\alpha) = v(\alpha)$. The module is spanned by finite linear combinations of vectors $a_{-n_1}\ a_{-n_2} \cdots a_{-n_k}\ v(\alpha)$ with $n_1 \geq n_2 \geq \cdots \geq n_k$. The space $V(\alpha)$ comes equipped with the structure of a highest weight module over the Virasoro algebra through the following basic construction. One defines

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : -\beta(n+1) a_n, \tag{19}$$

where β is another real parameter. The colons mean normal ordering. They order the annihilators a_n, $n > 0$, to the right. For instance,

$$L_{-1} = \sum_{m \geq 0} a_{-1-m} a_m. \tag{20}$$

On every element of $V(\alpha)$ of finite weight only finitely many terms in the sum contribute. Therefore, (19) is well defined on $V(\alpha)$. It can be shown that (19) satisfy the relations (3) with central charge

$$c = 1 - 12\,\beta^2. \tag{21}$$

This construction is therefore in particular suited for the case of minimal models. $v(\alpha)$ is also Virasoro highest weight vector. It satisfies

$$\begin{aligned} L_n\, v(\alpha) &= 0, \quad n > 0, & (22)\\ L_0\, v(\alpha) &= h(\alpha, \beta)\, v(\alpha), & (23)\end{aligned}$$

with highest weight

$$h(\alpha, \beta) = \frac{1}{2}\alpha^2 - \alpha\beta. \tag{24}$$

We remark that this representations of the Virasoro algebra is not irreducible in the case of minimal models. The irreducible representation can be obtained through a beautiful cohomological construction. We refer to [F89] for this so called BRST–cohomology.

2.5 Conformal Blocks

Conformal blocks were introduces as correlation functions of chiral conformal fields. They depend on the positions of the field operators which we take to be non–coinciding. To describe this situation, we introduce a configuration space

$$\mathbb{C}^{[n]} = \mathbb{C}^n \setminus \bigcup_{i<j} \{z_i = z_j\} \tag{25}$$

of n non–coinciding points in the complex plane. Chiral conformal fields come as conformal multiplets. This is formalized by associating with each point z_j a highest weight module $V(\alpha_j)$. The highest weight vector $v(\alpha_j)$ corresponds to a chiral primary field at z_j. Other vectors correspond to descendant fields. This correspondence is slightly inaccurate because we are using Fock spaces at this moment instead of irreducible Viraso modules. We should better speak of free field or $U(1)$ conformal blocks.

The defining property of these conformal blocks is a $U(1)$–Ward identity which we explain next. Let $\mathcal{M}(z_1, \dots, z_n)$ be the space of meromorphic functions, holomorphic on $\mathbb{C} \setminus \{z_1, \dots, z_n\}$. They can be thought of as g–valued functions and form an abelian Lie algebra. Let f be such a meromorphic function. At each z_j we can perform a Laurant expansion and interpret the result as an element of Lg. But $Lg \subset \widehat{Lg}$, although the inclusion is not a Lie algebra homomorphism. $\widehat{Lg}$ acts on $V(\alpha_j)$. Therefore, we have linear maps

$$\pi_j : \mathcal{M}(z_1, \dots, z_n) \to \mathrm{End}(V(\alpha_j)), \quad \pi_j(f) = f(z_j + t). \tag{26}$$

Doing this for all points $z_1, \dots, z_n$ we obtain $\pi = \pi_1 \oplus \cdots \oplus \pi_n$, a linear map

$$\pi : \mathcal{M}(z_1, \dots, z_n) \to \mathrm{End}(V(\alpha_1) \otimes \cdots \otimes V(\alpha_n)). \tag{27}$$

This map has the remarkable property of being a Lie algebra homomorphism. The reason is that the residues coming from (14),

$$[\pi(f), \pi(g)] = \sum_{j=1}^{n} \mathrm{res}_{z_j}(f'g), \tag{28}$$

sum up to zero thanks to the residue theorem. For this purpose we better formulate the theory on the Riemann sphere $\mathbb{C}P^1$, rather than on the plane $\mathbb{C}$. In other words, we have an action of the space of g-valued meromorphic functions on the tensor product of highest weight modules.

Conformal blocks are distinguished by being invariant under this action. A conformal block at fixed positions $z_1, \dots, z_n$ is a linear form

$$F : V(\alpha_1) \otimes \cdots \otimes V(\alpha_n) \to \mathbb{C}, \tag{29}$$

such that for all $f \in \mathcal{M}(z_1, \dots, z_n)$ and all elements $u \in V(\alpha_1) \otimes \cdots \otimes V(\alpha_n)$

$$\langle F, \pi(f)u \rangle = 0. \tag{30}$$

The bracket $\langle \cdot, \cdot \rangle$ means the evaluation of a linear form on a vector. The space of these invariant linear forms is denoted by

$$E(z_1, \dots, z_n) = \mathrm{Hom}_{\mathcal{M}(z_1,\dots,z_n)}(V(\alpha_1) \otimes \cdots \otimes V(\alpha_n), \mathbb{C}). \tag{31}$$

This is equivalent to a $U(1)$–Ward identity. A case of particular interest is the value of an invariant linear form on the product of all highest weight vectors

$$v = v(\alpha_1) \otimes \cdots \otimes v(\alpha_n). \tag{32}$$

This case corresponds to a product of chiral primary fields. An interesting question is wether the value of a conformal block on general vectors is already fixed by its value on the product of highest weight vectors. This is indeed the case. For $f(z) = (z - z_j)^{-n}$, $n > 0$, it follows that

$$\pi(f)\; v = \left\{ a_{-n}^{(j)} + \sum_{i \neq j} (z_i - z_j)^{-n}\; a_0^{(i)} \right\} v, \tag{33}$$

where $a_{-n}^{(j)}$ is meant to act on the j'th factor of v. But the Ward identity then implies that

$$\left\langle F, a_{-n}^{(j)} v \right\rangle = - \sum_{i \neq j} \alpha_i \; (z_i - z_j)^{-n} \; \langle F, v \rangle \,. \tag{34}$$

Iterating this procedure, we can evaluate the conformal block on any vector in the product module.

The next question to be asked is how a conformal blocks behave under variation of the positions $z_1, \ldots, z_n$. The answer to this question relies on a remarkable differential equation, to which we turn our attention now. The idea is to view conformal blocks as invariant sections of a vector bundle. Let $U \subset \mathbb{C}^{[n]}$ be an open subset. Consider then the infinite rank trivial vector bundle

$$U \otimes \mathrm{Hom}_{\mathbb{C}}(V(\alpha_1) \otimes \cdots \otimes V(\alpha_n), \mathbb{C}). \tag{35}$$

Let $E(U)$ denote the space of sections $F : U \to \mathrm{Hom}_{\mathbb{C}}(V(\alpha_1) \otimes \cdots \otimes V(\alpha_n), \mathbb{C})$ of this vector bundle such that F is holomorphic in $z_1, \ldots, z_n$ and such that $F(z_1, \ldots, z_n)$ is an element of $E(z_1, \ldots, z_n)$ for all $(z_1, \ldots, z_n) \in U$. Elements of $E(U)$ are called holomorphic conformal blocks. The trivial vector bundle (35) comes together with a flat connection

$$\nabla = \sum_{i=1}^{n} \, \mathrm{d}z_i \; \nabla_{z_i}, \quad \nabla_{z_i} F = \partial_{z_i} F - F \; L_{-1}^{(i)}, \tag{36}$$

defined on holomorphic sections. This connection is called Friedan–Shenker connection. A straight forward computation proves that

$$\nabla_{z_i}(F \; \pi(f)) = \nabla_{z_i} F \; \pi(f) + F \partial_{z_i} \pi(f). \tag{37}$$

As a consequence (36) leaves invariant the space of sections $E(U)$. To be precise, an invariant section is mapped to an invariant differential form. Conformal blocks are distinguished by the property of being horizontal sections. That is, a conformal block is an element of $F \in E(U)$ which satisfies

$$\nabla F = 0. \tag{38}$$

In the case of general current algebras, where one starts this construction from a general simple complex Lie algebra, (38) is the celebrated Knizhnik–Zamolodchikov differential equation [KZ84]. A more explicit form of (38) is

$$\partial_{z_i} \langle F(z_1, \dots, z_n), u \rangle = \left\langle F(z_1, \dots, z_n), L_{-1}^{(i)}\, u \right\rangle. \tag{39}$$

We have already seen that it is sufficient to compute the pairing with the product of highest weight vectors (32). In this case we have that

$$L_{-1}^{(i)}\, v = a_{-1}^{(i)} a_0^{(i)}\, v = \alpha_i\, a_{-1}^{(i)}\, v. \tag{40}$$

From (34) we then immediately deduce the $U(1)$–Kniznik–Zamolodchikov equation

$$\partial_{z_i} \langle F(z_1, \dots, z_n), v \rangle = \sum_{j \neq i} \frac{\alpha_i \alpha_j}{z_i - z_j} \langle F(z_1, \dots, z_n), v \rangle. \tag{41}$$

It tells the behavior of conformal blocks upon variation of the configuration $(z_1, \dots, z_n)$. It is not difficult to solve in the special case that

$$\alpha_1 + \dots + \alpha_n = 0. \tag{42}$$

Recall that α_i is the $U(1)$–charge of $V(\alpha_i)$. Eq. (42) therefore requires the total charge to be zero. Notice that charge neutrality is required by invariance of a conformal block under the constant function $\pi(1)$. The result is

$$\langle F(z_1, \dots, z_n), v) \rangle = \prod_{i<j} (z_i - z_j)^{\alpha_i \alpha_j}, \tag{43}$$

choosing an overall normalization constant equal to one. Eq. (43) is a well known expression from many body theory of $U(1)$–charged particles. The point with charge neutrality is that (43) behaves at infinity like

$$z_1^{\alpha_1 \sum_{j \neq 1} \alpha_j} = z_1^{-\alpha_1^2} \tag{44}$$

as a function of z_1 for fixed $z_2, \dots, z_n$, and is therefore regular at infinity. In the non–neutral case (43) is still a solution to (41) but is singular at infinity. Thus it would have to be interpreted as a conformal block with another field located at infinity.

2.6 Integral Representations

With this formalism we are ready to derive integral representations. The case of total charge neutrality has been solved explicitely. The key idea is to reduce the general non–neutral case to the neutral one at the expense of additional so called screening charges. Screening charges are defined by $h(\alpha, \beta) = 1$. That is, they correspond to highest weight modules of conformal weight one. The solution to this quadratic equation defines two charges

$$\alpha_\pm = \beta \pm \sqrt{\beta^2 + 2}. \tag{45}$$

Integral representations can be derived if the excess charge can be compensated for by screening charges. With this we mean that we have charge neutrality in an extended system

$$\sum_{i=1}^{n} \alpha_i + r_+\alpha_+ + r_-\alpha_- = 0. \tag{46}$$

The total number of screening charges will be denoted by $r = r_+ + r_-$. Let $w_1^+, \ldots, w_{r_+}^+$ and $w_1^-, \ldots, w_{r_-}^-$ be the positions of the positive and negative screening charges respectively. To simplify the notation we put

$$z = (z_1, \ldots, z_n), \quad w = (w_1^+, \ldots, w_{r_+}^+, w_1^-, \ldots, w_{r_-}^-), \tag{47}$$

and define furthermore

$$\begin{aligned} v^{\pm} &= \prod_{j=1}^{r_\pm} a_{-1}^{(j)}\, v(\alpha_\pm) \otimes \cdots \otimes v(\alpha_\pm), &(48)\\ v_i^{\pm} &= \prod_{j \neq i} a_{-1}^{(j)}\, v(\alpha_\pm) \otimes \cdots \otimes v(\alpha_\pm). &(49) \end{aligned}$$

Then we define an extended conformal block, which depends on the screening charge positions w, by

$$\left\langle \tilde{F}(z,w), u \right\rangle = \left\langle F(z,w), u \otimes v^+ \otimes v^- \right\rangle. \tag{50}$$

The right hand side is a charge neutral conformal block and is explicitely known. u is an element of $V(\alpha_1) \otimes \cdots \otimes V(\alpha_n)$. As it stands (50) does not satisfy invariance under $\mathcal{M}(z_1, \ldots, z_n)$ due to the presence of sreening charges. The idea is now to integrate out the sreening charges in a way to obtain an invariant expression. Eq. (50) is by construction invariant under the extended symmetry

$$\pi(f) = \pi_z(f) + \pi_w(f) \tag{51}$$

in a self–explanatory notation. The problematic piece is $\pi_w(f)$. Notice that we seek invariance for functions f, which are regular on the positions of the screening charges. The trick is best explained by computing

$$\begin{aligned} \pi(f) a_{-1}\, v(\alpha_\pm) &= \{f(w) a_0 + f'(w) a_1\}\, a_{-1}\, v(\alpha_\pm) \\ &= \{f(w) L_{-1} + f'(w)\}\, v(\alpha_\pm), &(52) \end{aligned}$$

where f is assumed to be regular at w. At this point it is required that the charge be a screening charge, i.e., the conformal weight be one. As a consequence,

$$\begin{aligned} \left\langle F(z,w), \pi_w(f) u \otimes v^+ \otimes v^- \right\rangle &= \sum_{j=1}^{r_+} \partial_{w_j^+} \left\{ f(w_j^+) \left\langle F(z,w), u \otimes v_j^+ \otimes v^- \right\rangle \right\} + \\ &\quad \sum_{j=1}^{r_-} \partial_{w_j^-} \left\{ f(w_j^-) \left\langle F(z,w), u \otimes v^+ \otimes v_j^- \right\rangle \right\}. &(53) \end{aligned}$$

The strategy is then to integrate the screening charges over contours $\Gamma = (\gamma_1, \dots, \gamma_r)$ such that (53) is zero for all functions $f \in \mathcal{M}(z)$. This can indeed be accomplished and is a main theme of this thesis. Naively one would expect that any collection of closed curves encircling the singularities would do the job. This is not the case because the integrand is generally multi–valued and one has to account for phase factors. In the more complicated case of the torus one even has to account for a general case of certain matrices. Once this has been achieved one has an integral representation

$$\langle F_\Gamma(z), u \rangle = \int_\Gamma \left\langle \tilde{F}(z,w), u \right\rangle \, \mathrm{d}w \tag{54}$$

with all desired properties. The integral (54) is then in particular $\mathcal{M}(z)$ invariant and satisfies the flatness condition. The integrand follows directly from the neutral expression. It is a single valued function times

$$\begin{aligned} &\prod_{i<j}(z_i - z_j)^{\alpha_i\alpha_j} \prod_{i,j}(z_i - w_j^+)^{\alpha_i\alpha_+} \prod_{i,j}(z_i - w_j^-)^{\alpha_i\alpha_-} \\ &\prod_{i<j}(w_i^+ - w_j^+)^{\alpha_+^2} \prod_{i<j}(w_i^- - w_j^-)^{\alpha_-^2} \prod_{i<j}(w_i^+ - w_j^-)^{-2}. \end{aligned} \tag{55}$$

Notice that $\alpha_+\alpha_- = -2$. The single valued function comes from evaluating (50) by means of (34). This evaluation is a form of Wicks theorem and is left to the reader. Notice that we do not take the screening charges to be highest weight vectors.

2.7 Local Systems

Our investigation of topological representations of quantum groups begins with this integral representation for minimal models. The function (55) is multi–valued on the configuration space $\mathbb{C}^{[n+r]}$ of charges and screening charges. We define it by analytic continuation from a reference point. Analytical continuation associates with each loop in the configuration space, beginning and ending at the reference point, a phase factor. This is a one dimensional local system. With the local systems from minimal models on $\mathbb{C}P^1$ we associate topologigal representations of the quantum group $U_q(\mathfrak{s}l_2(\mathbb{C}))$. They serve in the first place to explain what kind of integration contours yield physical conformal blocks. They also answer structural questions about the conformal blocks. One such structural question is the behavior of conformal blocks upon analytic continuation of the insertion points. For minimal models on $\mathbb{C}P^1$ one finds an R-representation of the braid group, where the R–matrix belongs to $U_q(\mathfrak{s}l_2(\mathbb{C}))$. The topological representation of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ gives an explanation of this fact. Thus conformal blocks come encoded with quantum group data. The quantum group data can be reconstructed to a large extent from the local system alone. In particular another class of models, the Wess–Zumino–Witten models built on $SU(2)$ possess the same local system and therefore the same quantum group structure. This may be viewed as a kind of quantum group universality. Quantum group symmetry was used on the side of integrable lattice models in [PS90] to identify scaling limits. The connection between conformal field theory and quantum groups is therefore also a contribution to the classification program. The constructions of this thesis are built upon integral representations of the form (54) and generalizations thereof. A surprisingly rich structure emerges, in particular when one considers conformal field theories on a toroidal space time. Connections with quantum groups, infinite dimensional Lie algebras, and complex geometry appear. They form a whole menu of interesting problems among which the topological representations of quantum groups.

Except for the last two chapters, which rely on the theory of current algebras, the basic input from conformal field theory is the local system. Each chapter is preceded by a short introduction to the particular subject under investigation. Chapter one to four form one logical unit, and chapter five and six another one. They can be read independently. Chapter five is a sub–case of chapter six. We have included it in order to make the essence readable to a broader audience.

3 Topological Representations of the Quantum Group $U_q\left(\mathfrak{s}l_2(\mathbb{C})\right)$

We define a topological action of the quantum group $U_q\left(\mathfrak{s}l_2(\mathbb{C})\right)$ on a space of homology cycles with twisted coefficients on the configuration space of the punctured disc. This action commutes with the monodromy action of the braid groupoid, which is given by the R-matrix of $U_q\left(\mathfrak{s}l_2(\mathbb{C})\right)$.

3.1 Introduction

In the free field representation of conformal field theory based on $SU(2)$ one is led to consider integrals of the form [DF84, ZF86]

$$\begin{aligned} G_C(w_1,\ldots,w_s) = \int_C \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_r\; f(z_1,\ldots,z_r,w_1,\ldots,w_s) \\ \prod_{i<j}(z_i-z_j)^{2\nu}\prod_{i,j}(z_i-w_j)^{(1-n_j)\nu}\prod_{i<j}(w_i-w_j)^{\frac{1}{2}(1-n_i)(1-n_j)\nu} \end{aligned} \tag{56}$$

In this formula $n_1,\ldots,n_s$ are positive integers, f is a single valued meromorphic function, symmetric under permutations of the z-variables, with poles on the hyperplanes $\{z_i = w_j\}$. The parameter ν is equal to $1/(k+2)$ for the Wess-Zumino-Witten (WZW) model based on $SU(2)$ at level k, and is equal to p'/p for minimal models with central charge $c = 1-6(p-p')^2/(pp')$.

For each integration cycle C in the rth homology group with coefficients in the local system given by the monodromy of the differential form in (56), G_C is a many valued analytic function on the configuration space $C_{1,\ldots,1}(\mathbb{C}) = \{(w_1,\ldots,w_s)\in\mathbb{C}^s | w_i \neq w_j\ (i\neq j)\}$. To compute its transformation under analytic continuation along paths exchanging the punctures w_i, one needs to know the monodromy action of the braid groupoid on homology. Examples of this computation by *contour deformation* have been worked out by several authors, among others [GN84, DF84, TK88, FFK89, La90], in different languages. It generalizes the computation of Gauss for the hypergeometric function. It has become clear that the monodromy is described by the R-matrix (more precisely by the $6j$-symbols) of the quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$. The topological point of view we adopt here is closest to [PS90].

In this chapter, we propose an *explanation* of this fact. It consists of two parts. First one considers a space of relative cycles on which $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts. The action is described purely in topological terms and commutes with the monodromy action of the braid group. The absolute cycles are then given by the highest weight vectors in the space of relative cycles. We have then schematically the following dictionary between topological and algebraic entities:

Relative cycles	Elements of the tensor product of Verma modules $\otimes_i V_{n_i}$
Absolute cycles	Highest weight vectors in $\otimes_i V_{n_i}$
Intersection pairing	Covariant bilinear form
Monodromy action of the braid groupoid on relative cycles	R-matrix representation of the of braid groupoid on $\otimes_i V_{n_i}$

Moreover, the quotient of the space of absolute cycles by the cycles in the null space of the intersection pairing is closed under braiding and is given by the fusion rule subquotient. More precise definitions and correspondences are explained in the bulk of this chapter.

Our approach is rather elementary and based on the concept of *families of loops* rather than on the (in some sense more natural) homology groups directly. We expect that our construction extends to locally finite homology, but this would require a somewhat more sophisticated machinery.

We point out that part of our results can be understood as a topological version of results known in the literature on free field representation of conformal field theory [PS90, GP89, BMP90] and, in particular, [GS90]. The results in [BMP90] suggest that our construction extends to groups of higher rank. Here we present the purely topological results in this subject, which can be read without knowledge in conformal field theory. See [FS89a, F90, G90] for applications of these concepts to conformal field theory. While this work was completed, we received some interesting preprints [SV90] where related results were obtained.

This chapter is organized as follows: in section 3.2 we introduce the concept of braid groupoid representations and local systems in a rather general context. In section 3.3 we specialize to $SU(2)$, and explain the action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on relative cycles. Section 3.4 contains the discussion on intersection pairing. In section 3.5 we show that the representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on relative cycles is isomorphic to the tensor product of Verma modules, one for each puncture. In section 3.6 we compute the monodromy action of the braid groupoid on relative cycles. The appendix contains a summary of results on $U_q\left(\mathfrak{s}l_2(\mathbb{C})\right)$.

3.2 Local Systems on Configuration Spaces

3.2.1 Colored Braid Groupoids

Let X be a connected two-dimensional manifold, possibly with boundary, k a positive integer (the number of colors), and $n_1, \ldots, n_k$ non-negative integers (the numbers of strands with given color). Set $n = \sum n_i$. Define the configuration spaces

$$\mathcal{C}_{(n_i)}(X) = \mathcal{C}_{n_1 \ldots n_k}(X) = \left(X^n \setminus \cup_{i<j}\{z_i = z_j\}\right) / \left(S_{n_1} \times \ldots \times S_{n_k}\right), \tag{57}$$

where the symmetric group S_{n_1} acts by permutations on the first n_1 variables, S_{n_2} on the subsequent n_2 variables, and so on. It is understood that the factors S_{n_i} with $n_i = 0$ should be omitted in (57). An element of $\mathcal{C}_{n_1 \ldots n_k}(X)$ can also be thought as a sequence $(Z_1, \ldots, Z_k)$ of pairwise disjoint subsets of X with cardinalities $|Z_i| = n_i$.

Fix a base point x of $\mathcal{C}_{n_1 \ldots n_k}(X)$, and let O_x be the orbit of x under the symmetric group[1] S_n. Thus O_x can be identified with the right coset space

$$O_x = S_n / (S_{n_1} \times \ldots \times S_{n_k}). \tag{58}$$

The colored braid groupoid $B_{n_1 \ldots n_k}(X, x)$ is the space of paths in X, starting and ending in O_x, up to homotopies preserving endpoints; viewed as a subgroupoid of the fundamental groupoid of $\mathcal{C}_{n_1 \ldots n_k}(X)$. The groupoid $G = B_{n_1 \ldots n_k}(X, x)$ is indexed by O_x and has components labeled by the endpoints:

$$G = \cup_{\alpha, \beta \in O_x} G_{\alpha\beta}, \tag{59}$$

The multiplication law $G_{\alpha\beta} \times G_{\beta\gamma} \to G_{\alpha\gamma}$ is the composition of paths. Since X is connected, braid groupoids corresponding to different choices of base points are isomorphic. Any such isomorphism can be described as the composition with a homotopy class of paths connecting the base points. If $k = 1$, G is a group, the braid group on n strands on X. The groupoid $G = B_{n_1 \ldots n_k}(X, x)$ can be described in terms of the braid group $B_n(X, x)$. Let $h : B_n(X, x) \to S_n$ be the canonical projection homomorphism. Then for $\alpha, \beta \in O_x$ there is a one-to-one map

$$\phi_{\alpha,\beta} : \{g \in B_n(X, x) : \alpha = h(g)\beta\} \to G_{\alpha\beta}, \tag{60}$$

such that $\phi_{\alpha\beta}(g)\phi_{\beta\gamma}(g') = \phi_{\alpha\gamma}(gg')$.

For $X \subset \mathbb{C}$, call $x \in \mathcal{C}_{n_1 \ldots n_k}(X)$ an *admissible base point* if x is the image of a point in $\mathbb{C}^n$ with

$$\mathfrak{Re}(z_1) < \ldots < \mathfrak{Re}(z_n). \tag{61}$$

[1] Acting as $\pi(z_1, \ldots, z_n) = (z_{\pi^{-1}(1)}, \ldots, z_{\pi^{-1}(n)})$.

Suppose now that $X = \mathbb{C}$. For any two admissible base points there is a unique homotopy class of paths in the space of admissible base points connecting them. Therefore the corresponding colored braid groupoids can be uniquely identified, and we can omit the dependence on x in the notation; with the agreement that $G = B_{n_1 \dots n_k}(\mathbb{C})$ is defined using any admissible base point.

An element α in O_x can be described by a color map

$$\bar{\alpha} : \{1, \dots, n\} \to \{1, \dots, k\} \tag{62}$$

such that $|\bar{\alpha}^{-1}(i)| = n_i$. The correspondence between α and $\bar{\alpha}$ is the following: Let $\alpha = \pi(x)$, $\pi \in S_n$. Then

$$\bar{\alpha}(i) = \lambda \quad \text{iff} \quad \pi^{-1}(i) \in \left\{ \sum_1^{\lambda-1} n_j + 1, \dots, \sum_1^{\lambda} n_j \right\}. \tag{63}$$

Let σ_i, $i = 1, \dots, n-1$ be the standard generator of $B_n(\mathbb{C})$, that exchanges the ith strand with the $(i+1)$st one, and let $\tau_i = h(\sigma_i)$ denote the corresponding transposition. Then the system

$$\sigma_i^\alpha = \phi_{\tau_i \alpha, \alpha}(\sigma_i) \in G_{\tau_i \alpha, \alpha}, \quad i = 1, \dots, n-1, \quad \alpha \in O_x, \tag{64}$$

is a system of generators of G.

Let $\alpha \in \mathbb{R}$. The inclusion $\mathcal{C}_{(n_i)}(\{\mathfrak{Re}(z) < a\}) \subset \mathcal{C}_{(n_i)}(\mathbb{C})$ induces an isomorphism $B_{(n_i)}(\{\mathfrak{Re}(z) < a\}) \to B_{(n_i)}(\mathbb{C})$. The same holds for the subset $\{\mathfrak{Re}(z) > a\}$. Let $n_i = n_i' + n_i''$, $i = 1, \dots, k$. The inclusion

$$\begin{aligned} \phi : \mathcal{C}_{(n_i')}(\{\mathfrak{Re}(z) < a\}) \times \mathcal{C}_{(n_i'')}(\{\mathfrak{Re}(z) > a\}) &\to \mathcal{C}_{(n_i)}(\mathbb{C}) \\ ((Z_1', \dots, Z_k'), (Z_1'', \dots, Z_k'')) &\mapsto (Z_1' \cup Z_1'', \dots, Z_k' \cup Z_k'') \end{aligned} \tag{65}$$

induces an injective homomorphism of groupoids

$$\phi : B_{(n_i')}(\mathbb{C}) \times B_{(n_i'')}(\mathbb{C}) \to B_{(n_i)}(\mathbb{C}) \tag{66}$$

More precisely, we have a map $\phi : O' \times O'' \to O$ defined by restriction to the orbits O', O'' of admissible base points, and maps (in an obvious notation)

$$\phi : G'_{\alpha'\beta'} \times G''_{\alpha''\beta''} \to G_{\alpha\beta}, \tag{67}$$

with $\alpha = \phi(\alpha', \alpha'')$ and $\beta = \phi(\beta', \beta'')$, compatible with the composition law. Intuitively, this homomorphism is simply the juxtaposition of colored braids.

3.2.2 R-Matrix Representations.

A representation of a groupoid $G = \cup_{\alpha\beta \in I} G_{\alpha\beta}$ with index set I, on a family of complex vector spaces $(V_\alpha)_{\alpha \in I}$ is an index preserving homomorphism from G to the groupoid

$\cup_{\alpha\beta\in I}\mathrm{Hom}^*(V_\beta, V_\alpha)$ of invertible linear maps of the vector spaces V_α. In other words, a representation ρ of G is a family of maps

$$\rho_{\alpha\beta} : G_{\alpha\beta} \to \mathrm{Hom}^*(V_\beta, V_\alpha), \tag{68}$$

such that $\rho_{\alpha\beta}(g)\rho_{\beta\gamma}(g') = \rho_{\alpha\gamma}(gg')$. To simplify the notation, we will often omit the label $\alpha\beta$, thinking of $\rho_{\alpha\beta}$ as the restriction of a map ρ defined on G.

Definition: Let $U_\lambda, \lambda = 1, \dots, k$ be vector spaces and for each pair λ, μ let $R_{\lambda\mu}$ be an invertible element of $\mathrm{End}(U_\lambda \otimes U_\mu)$. An R-matrix representation of the groupoid $B_{(n_i)}(\mathbb{C})$ is a representation on the family of vector spaces, labeled by O_x,

$$V_\alpha = U_{\bar\alpha(1)} \otimes \dots \otimes U_{\bar\alpha(r)}, \tag{69}$$

such that on generators

$$\rho(\sigma_i^\alpha) = PR^{i,i+1}_{\bar\alpha(i)\bar\alpha(i+1)}, \qquad Pu \otimes v = v \otimes u, \tag{70}$$

where $PR^{i,j}_{\lambda\mu}$ denotes $PR_{\lambda\mu}$ acting on the ith and jth factor in the tensor product.

Proposition 3.1 *(I) Let k be a fixed positive integer. A family of vector spaces U_λ, $\lambda = 1, \dots, k$ and a family $R_{\lambda\mu}$ of invertible elements of* $\mathrm{End}(U_\lambda \otimes U_\mu)$ *defines a representation $\rho_{(n_i)}$ of $B_{n_1,\dots,n_k}(\mathbb{C})$ for all $n_1, \dots, n_k$ if and only if the Yang-Baxter equation*

$$R^{23}_{\mu\nu}R^{13}_{\lambda\nu}R^{12}_{\lambda\mu} = R^{12}_{\lambda\mu}R^{13}_{\lambda\nu}R^{23}_{\mu\nu} \tag{71}$$

holds on $U_\lambda \otimes U_\mu \otimes U_\nu$. (II) Let ϕ be the homomorphism (66),

$$B_{n'_1\dots n'_k}(\mathbb{C}) \times B_{n''_1\dots n''_k}(\mathbb{C}) \to B_{n_1\dots n_k}(\mathbb{C}), \qquad n_i = n'_i + n''_i, \tag{72}$$

and set $\rho' = \rho_{(n'_i)}$, $\rho'' = \rho_{(n''_i)}$, $\rho = \rho_{(n_i)}$. Then, for all $\alpha', \beta' \in O'$, $\alpha'', \beta'' \in O''$

$$\rho_{\alpha\beta}(\phi(g', g'')) = \rho_{\alpha'\beta'}(g') \otimes \rho_{\alpha''\beta''}(g''), \tag{73}$$

where $\alpha = \phi(\alpha', \alpha'')$ and $\beta = \phi(\beta', \beta'')$.

Example 1: Let $U_\lambda = \mathbb{C}$, $\lambda = 1, \dots, k$, and identify $V_\alpha = \mathbb{C} \otimes \dots \otimes \mathbb{C}$ with $\mathbb{C}$. Let $q_{\lambda\mu}$ be any non zero complex numbers. Then $\rho(\sigma_i^\alpha) = q_{\bar\alpha(i)\bar\alpha(i+1)}$ defines an R-matrix representation of $B_{n_1\dots n_k}(\mathbb{C})$.

Example 2: Let A be a quantum universal enveloping algebra [D86] with universal R-matrix $R \in A \otimes A$, and let ρ_λ be finite dimensional representations of A on spaces U_λ. Then $R_{\lambda\mu} = \rho_\lambda \otimes \rho_\mu(R)$ defines an R-matrix representation of $B_{n_1\dots n_k}(\mathbb{C})$.

3.2.3 Local Systems

Let (M,x) be a topological space with base point, and $\hat{M}$ its universal covering space, with right action of $\pi_1(M,x)$. $\hat{M}$ is the space of homotopy classes of paths in M originating at x. For any representation $\rho:\pi_1(M,x)\to GL(V)$ on a vector space V one defines a local system L as the vector bundle $(\hat{M}\times V)/\sim$ over M with the identification $(\hat{m},\rho(\eta)v)\sim(\hat{m}\eta,v)$, $\eta\in\pi_1(M,x)$, and projection $(\hat{m},v)\mapsto m$, the covering projection on the first argument. Thus a local system is the same as a flat vector bundle with holonomy ρ, and specified trivialization of the fiber over the base point.

This construction has the following slight generalization. Let O be a finite subset of M and G the subgroupoid of the fundamental groupoid of M, consisting of homotopy classes of paths whose endpoints are in O. For $\alpha\in O$, let $\hat{M}_\alpha$ be the universal covering space of the space with base point (M,α). The groupoid G acts on the disjoint union $\sqcup_\alpha\hat{M}_\alpha$ on the right by composition of paths, given by maps $\hat{M}_\alpha\times G_{\alpha\beta}\to\hat{M}_\beta$. Let ρ be a representation of $G=\cup_{\alpha\beta\in O}G_{\alpha\beta}$ on a family of vector spaces $(V_\alpha)_{\alpha\in O}$. These data define a local system as the vector bundle

$$\Big(L=\sqcup_\alpha\hat{M}_\alpha\times V_\alpha\Big)\,/\sim \tag{74}$$

with identification $(\hat{m}_\alpha,\rho_{\alpha\beta}(\eta_{\alpha\beta})v)\sim(\hat{m}_\alpha\eta_{\alpha\beta},v_\beta)$, $\eta_{\alpha\beta}\in G_{\alpha\beta}$. Such a local system is the same as a flat vector bundle over M together with a family of vector spaces (V_α) and isomorphisms of the fibers over $\alpha\in O$ with V_α, such that parallel transport operators are given by ρ. Local horizontal sections are continuous sections which locally can be written as $m\mapsto(\hat{m},v)$, with constant v, and $\hat{m}$ covering m.

Let M_1,M_2 be topological spaces and $O_1\subset M_1$, $O_2\subset M_2$ be finite subsets. A homomorphism of local systems L_1 over M_1 to L_2 over M_2 is a map $L_1\to L_2$ mapping fibers to fibers linearly and sending local horizontal sections to local horizontal sections.

Lemma 3.2 *Let f be a map from M_1 to M_2 such that $f(O_1)\subset O_2$ and let $f_\alpha\in\mathrm{Hom}(V_\alpha\to V_{f(\alpha)})$, be linear maps indexed by O_1 such that the diagram*

$$\begin{tikzcd} V_\alpha \arrow[r, "\rho_1(\eta)"] \arrow[d, "f_\alpha"'] & V_\beta \arrow[d, "f_\beta"] \\ V_{f(\alpha)} \arrow[r, "\rho_2(f\circ\eta)"'] & V_{f(\beta)} \end{tikzcd} \tag{75}$$

is commutative for all $\alpha,\beta\in O_1,\eta\in G_{\alpha\beta}$. Then f lifts uniquely to a homomorphism $L_1\to L_2$ of the local systems associated to ρ_1,ρ_2, also denoted by f, which reduces to f_α on the fiber V_α over $\alpha\in O_1$.

Let ρ be a representation of $B_{n_1\dots n_k}(\mathbb{C})$ and let L be the corresponding local system. Here is an explicit description of L in terms of transition functions. Fix an admissible base point x, and define the cells $C^\alpha_{(n_i)} \subset C_{(n_i)}(\mathbb{C})$ as follows: let $\alpha = \sigma x$, $\sigma \in S_n$ and define

$$C^\alpha_{(n_i)} = \left\{ (z_1, \dots, z_n) \in \mathcal{C}_{(n_i)}(\mathbb{C}) | \Re e(z_{\sigma^{-1}(1)}) < \dots < \Re e(z_{\sigma^{-1}(n)}) \right\}. \qquad (76)$$

The cells $C^\alpha_{(n_i)}$ are pairwise disjoint, their union is dense in $\mathcal{C}_{(n_i)}(\mathbb{C})$, and each cell contains precisely one point in O_x, for any choice of admissible base point x.

Let $\bar{C}^\alpha_{(n_i)}$ be the closure of the cell $C^\alpha_{(n_i)}$. For $y \in \bar{C}^\alpha_{(n_i)} \cap \bar{C}^\beta_{(n_i)}$, let η be any path going from α to y in $C^\alpha_{(n_i)}$ and continuing from y to β in $C^\beta_{(n_i)}$. Define the locally constant transition function $g_{\alpha\beta}(y) = \rho(\eta)$. Then L is the flat vector bundle over $\mathcal{C}_{(n_i)}(\mathbb{C})$,

$$L = \sqcup_{\alpha \in O} \left(\bar{C}^\alpha_{(n_i)} \times V_\alpha \right) / \sim \qquad (77)$$

with identification

$$(y, v_\alpha) \sim (y, v_\beta), \quad y \in \bar{C}^\alpha_{(n_i)} \cap \bar{C}^\beta_{(n_i)}, \quad v_\alpha \in V_\alpha, \quad v_\beta \in V_\beta, \qquad (78)$$

if and only if $v_\alpha = g_{\alpha\beta}(y) v_\beta$.

Let η be any path whose endpoints lie in $\cup_{\alpha \in O} C^\alpha_{(n_i)}$. Then the parallel transport operator along η is an operator in $\mathrm{Hom}(V_\alpha, V_\beta)$, in the trivialization. Therefore, we have an extension of the definition of ρ to all homotopy classes of paths with endpoints in $\cup_{\alpha \in O} C^\alpha_{(n_i)}$.

Let now $\rho_{(n_i)}$ be the representations associated with a family of R-matrices, as in Proposition 3.1, and $L_{(n_i)}$ the corresponding local systems on $\mathcal{C}_{(n_i)}(\mathbb{C})$. Let $a \in \mathbb{R}, \mathbb{C}^+ = \{\Re e(z) > a\}$, $\mathbb{C}^- = \{\Re e(z) < a\}$. Denote by $L^<_{(n_i)}(L^>_{(n_i)})$ the restriction of $L_{(n_i)}$ to $\mathcal{C}_{(n_i)}(\mathbb{C}^-)$ ($\mathcal{C}_{(n_i)}(\mathbb{C}^+)$, respectively). Let $L^<_{(n'_i)} \otimes L^>_{(n''_i)}$ be the flat bundle over $\mathcal{C}_{(n'_i)}(\mathbb{C}^-) \times \mathcal{C}_{(n''_i)}(\mathbb{C}^+)$ defined by taking the tensor products of the fibers.

Proposition 3.3 *The maps $\phi : \mathcal{C}_{(n'_i)}(\mathbb{C}^-) \times \mathcal{C}_{(n''_i)}(\mathbb{C}^+) \to \mathcal{C}_{(n_i)}(\mathbb{C})$ lifts to a homomorphism*

$$\phi : L^<_{(n'_i)} \otimes L^>_{(n''_i)} \to L_{(n_i)} \qquad (79)$$

sending local horizontal sections to local horizontal sections. The lift is fixed by setting the homomorphisms of Lemma 3.2 equal to the canonical homomorphisms $V_\alpha \otimes V_\beta \xrightarrow{\sim} V_{\phi(\alpha,\beta)}$.

Proof: Let $C^\alpha_{(n'_i)}{}^< = C^\alpha_{(n'_i)} \cap \mathcal{C}_{(n'_i)}(\mathbb{C}^-)$ and $C^\alpha_{(n''_i)}{}^> = C^\alpha_{(n''_i)} \cap \mathcal{C}_{(n''_i)}(\mathbb{C}^+)$. Then $L^<_{(n'_i)} \otimes L^>_{(n''_i)}$ is the vector bundle

$$\sqcup_{\alpha \in O', \beta \in O''} (\bar{C}^\alpha_{(n'_i)}{}^< \times \bar{C}^\beta_{(n''_i)}{}^>) \times (V_\alpha \otimes V_\beta) / \sim . \qquad (80)$$

The map ϕ maps $\bar{C}^\alpha_{(n'_i)}{}^< \times \bar{C}^\beta_{(n''_i)}{}^>$ to $\bar{C}^{\phi(\alpha,\beta)}_{(n_i)}$, and the transition functions are given by tensor products of transition functions. The claim follows then from Proposition 3.1 and Lemma 3.2. $\square$

If $n_\lambda = 1$ and $n_\mu = 0$, $\mu \neq \lambda$, then $\mathcal{C}_{0,\dots,1,\dots,0}(\mathbb{C}) = \mathbb{C}$ and the fiber of $L_{0,\dots,1,\dots,0}$ over any point is canonically identified with U_λ. We have the following special case of the preceding Proposition.

Proposition 3.4 *Let $a \in \mathbb{R}$, $\lambda \in \{1, \ldots, k\}$, and z_+, z_- be complex numbers with $\mathfrak{Re}(z_-) < a < \mathfrak{Re}(z_+)$. Then the maps*

$$\phi_-^\lambda : \mathcal{C}_{n_1,\ldots,n_k}(\mathbb{C}^+) \to \mathcal{C}_{n_1,\ldots,n_\lambda+1,\ldots,n_k}(\mathbb{C}), \tag{81}$$

$$\phi_+^\lambda : \mathcal{C}_{n_1,\ldots,n_k}(\mathbb{C}^-) \to \mathcal{C}_{n_1,\ldots,n_\lambda+1,\ldots,n_k}(\mathbb{C}), \tag{82}$$

given by $(Z_1, \ldots, Z_k) \mapsto (Z_1, \ldots, Z_\lambda \cup \{z_\pm\}, \ldots, Z_k)$, lift to homomorphisms

$$\phi_-^\lambda : U_\lambda \otimes L^>_{n_1,\ldots,n_k} \to L_{n_1,\ldots,n_\lambda+1,\ldots,n_k} \tag{83}$$

$$\phi_+^\lambda : L^<_{n_1,\ldots,n_k} \otimes U_\lambda \to L_{n_1,\ldots,n_\lambda+1,\ldots,n_k}. \tag{84}$$

These homomorphisms preserve horizontal sections and are isomorphisms on each fiber.

3.3 The Topological Action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$

3.3.1 The $SU(2)$ case

Let us specialize the general discussion to the case of interest to us. Let D be the unit disc $\{|z| \leq 1\}$, and $w_1, \ldots, w_s$ be s distinct points in its interior. Define $X_r(w_1, \ldots, w_s)$ to be the fiber over $(w_1, \ldots, w_s)$ of the fibration $\mathcal{C}_{r,1,\ldots,1}(D) \to \mathcal{C}_{1,\ldots,1}(D)$. In other words, $X_r(w_1, \ldots, w_r)$ is the space of subsets of $D \setminus \{w_1, \ldots, w_s\}$ with r elements. Let $n_1, \ldots, n_s$ be positive integers, and $q \in \mathbb{C} \setminus \{0\}$. The family of one-dimensional R-matrices

$$R_{11} = -q^2, \qquad R_{1j} = R_{j1} = q^{1-n_j}, \quad j = 2, \ldots, s+1, \tag{85}$$

defines a representation of $B_{r,1,\ldots,1}(\mathbb{C})$, and a local system over $\mathcal{C}_{r,1,\ldots,1}(\mathbb{C})$ (and also on $\mathcal{C}_{r,1,\ldots,1}(D)$, by restriction). Let $L_r(w_1, \ldots, w_s)$ be the restriction of this local system to $X_r(w_1, \ldots w_s)$. We will often omit the w dependence in the notation, and write X_r, L_r when no confusion arises.

In the following construction it is useful to choose also two points on the boundary of D. For definiteness, choose $P_+ = 1$, $P_- = -1$. Denote $X_r^\pm = \{Z \in X_r | Z \ni P_\pm\}$. By Proposition 3.4, the inclusions

$$X_r \setminus X_r^\pm \to X_{r+1}^\pm, \qquad Z \mapsto Z \cup \{P_\pm\}, \tag{86}$$

lift to homomorphisms $\phi_\pm : L_r|_{X_r \setminus X_r^\pm} \to L_{r+1}|_{X_{r+1}^\pm}$.

3.3.2 Families of Loops.

In the following we fix s distinct points $w_1, \ldots, w_s$ in the interior of the unit disc, and denote by X the set $D \setminus \{w_1, \ldots, w_s\}$.

Definition: A non-intersecting family of loops in X, based at the point P_-, is a finite sequence $\gamma_0, \ldots, \gamma_{r-1} : [0, 1] \to X$ of curves in X such that

1. $\gamma_j(0) = \gamma_j(1) = P_-$; $\gamma_j(t) \neq P_-$ for $t \in]0,1[$,

2. If $t, s \in]0,1[$ and $\gamma_j(t) = \gamma_k(s)$ then $t = s$ and $j = k$,

3. For all j, the homotopy class of γ_j is non-trivial.

A non-intersecting family of loops can also be represented as a map Γ from the r-cube $]0,1[^r$ to X_r. It is the restriction of a continuous map $\bar{\Gamma}$ defined on the open r-cube with open $r-1$-faces

$$Q_r =]0,1[^r \cup \bigcup_{i=1}^{r} (]0,1[\times \ldots \times \{0,1\} \times \ldots \times]0,1[) \tag{87}$$

defining an inclusion $\bar{\Gamma} : Q_r \to X_r$ of a closed subset of X_r.

Definition: A *homotopy* of non-intersecting families of loops is defined to be a homotopy $h :]0,1[^r \times [0,1] \to X_r$ such that for all $s \in [0,1]$, $h(\cdot, s)$ is a non-intersecting family of loops. Two families Γ, Γ' are said to be *homotopic* if there is a homotopy h such that $h(\cdot, 0) = \Gamma$ and $h(\cdot, 1) = \Gamma'$.

Consider the space $A_r = A_r(w_1, \ldots, w_r)$ of finite linear combinations

$$\sum_{\Gamma} \lambda_\Gamma [\Gamma], \tag{88}$$

where $[\Gamma] = [\gamma_0, \ldots, \gamma_{r-1}]$ are homotopy classes of families of loops and λ_Γ are horizontal sections of the pull-back bundle $\Gamma^* L_r$ over the contractible space Q_r, modulo the equivalence relations:

1. $\lambda[\Gamma] \sim \pm f^* \lambda[\Gamma \circ f]$, for any orientation preserving $(+)$ or reversing $(-)$ isometry f of the cube.

2. If, for some i, γ_i is homotopic to the composition $\gamma_i' * \gamma_i''$ with homotopy $\tilde{\gamma}_i : [0,1] \times [0,1] \to X$ and $\gamma_0, \ldots, \tilde{\gamma}_i(\cdot, s), \ldots, \gamma_{r-1}$, $(0 \leq s < 1)$; $\gamma_0, \ldots, \gamma_i', \ldots, \gamma_{r-1}$; $\gamma_0, \ldots, \gamma_i'', \ldots, \gamma_{r-1}$ are all non-intersecting families of loops, then

$$\lambda[\gamma_0, \ldots, \gamma_{r-1}] \sim \lambda'[\gamma_0, \ldots, \gamma_i', \ldots, \gamma_{r-1}] + \lambda''[\gamma_0, \ldots, \gamma_i'', \ldots, \gamma_{r-1}], \tag{89}$$

   where λ', λ'' are defined by restriction of λ.

It is understood that horizontal sections over homotopic families of loops are canonically identified by parallel transport, so that the expressions (88) make sense.

Let ϵ be sufficiently small that the closed discs of radius ϵ centered at w_j are disjoint and contained in the interior of the unit disc. Let X_r^ϵ, $X_r^{\epsilon-}$ be the spaces obtained from X_r, X_r^- by removing points $\{z_1, \ldots, z_r\}$ such that $|z_i - w_j| < \epsilon$. Elements of A_r

represent relative locally finite cycles in $H_r^{lf}(X_r^\epsilon, X_r^{\epsilon-}; L_r)$ with coefficients in the local system L_r. Thus we have a linear map

$$\varphi_r : A_r(w_1, \dots, w_s) \to H_r^{lf}(X_r^\epsilon, X_r^{\epsilon-}; L_r). \tag{90}$$

On the other hand, we can also view a family Γ as a map $\tilde{\Gamma}$ from $]0,1[^r$ to $X_r^{\epsilon-}$ by the formula

$$\tilde{\Gamma} : (t_0, \dots, t_{r-1}) \mapsto \{-1, \gamma_0(t_0), \dots, \gamma_{r-1}(t_{r-1})\}, \tag{91}$$

and a section λ of $\Gamma^* L_r$ is mapped under ϕ_- to a section of $\tilde{\Gamma}^* L_{r+1}$, and we also have a linear map

$$\psi_{r+1} : A_r(w_1, \dots, w_s) \to H_{r+1}^{lf}(X_{r+1}^{\epsilon-}; L_r). \tag{92}$$

3.3.3 Operators

We define a set of operators acting on $\oplus_0^\infty A_r$ and then compute their commutation relations.

Let γ be the path

$$\gamma : [0,1] \to X, \qquad t \mapsto -e^{2\pi i t} \tag{93}$$

Let $i :]0,1[^r \to]0,1[^{r+1}$ be the inclusion $(t_0, \dots, t_{r-1}) \mapsto (t_0, \dots, t_{r-1}, 1/2)$. Define a linear operator $F : A_r \to A_{r+1}$ that *adds a loop*:

$$F : \lambda[\gamma_0, \dots, \gamma_{r-1}] \mapsto \lambda'[\gamma_0, \dots, \gamma_{r-1}, \gamma], \tag{94}$$

where λ' is the section over Q_{r+1} such that $\phi_+\lambda = \lambda' \circ i$ on $]0,1[^r$. This definition makes sense since we can assume that the representative $\gamma_0, \dots, \gamma_{r-1}$ does not intersect γ except at the endpoints.

Introduce the face maps $[0,1]^r \to [0,1]^{r+1}$,

$$e_{i,r}^+(t_0, \dots, t_{r-1}) = (t_0, \dots, t_{i-1}, 1, t_i, \dots, t_{r-1}), \tag{95}$$
$$e_{i,r}^-(t_0, \dots, t_{r-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{r-1}), \tag{96}$$

and the linear operator that *kills a loop*

$$E : \lambda[\gamma_0, \dots, \gamma_{r-1}] \mapsto \sum_{i=0}^{r-1} (-1)^i \phi_-^{-1}(\lambda \circ e_{i,r}^+ - \lambda \circ e_{i,r}^-)[\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_{r-1}]. \tag{97}$$

($\hat{}$ denotes omission). The third operator is the diagonal operator K^2, defined on A_r as

$$K^2 = q^{\Sigma(n_i - 1) - 2r} 1_{A_r}. \tag{98}$$

The relation between E and the boundary operator is explained by the

Proposition 3.5 *The diagram*

$$\begin{array}{ccc} A_r & \xrightarrow{\varphi_r} & H_r^{lf}(X_r^\epsilon, X_r^{\epsilon-}; L_r) \\ {\scriptstyle E}\big\downarrow & & \big\downarrow{\scriptstyle \partial_*} \\ A_{r-1} & \xrightarrow[\psi_r]{} & H_{r-1}^{lf}(X_r^{\epsilon-}; L_r) \end{array} \tag{99}$$

is commutative.

3.3.4 Relations

Theorem 3.6 *The operators $E, F, K^{\pm 2}$ obey the relations*

$$K^2E = q^2EK^2, \quad K^2F = q^{-2}FK^2, \quad EF - FE = K^2 - K^{-2}. \tag{100}$$

We have a representation of the quantum group $U_q(\mathfrak{s}l_2(\mathbb{C}))$ on $\oplus_r A_r(w_1, \ldots, w_s)$.

Proof: The first two relations follow from the definition. The third relation is best checked in an explicit trivialization. We can assume that $\{\gamma_0\left(\frac{1}{2}\right), \ldots, \gamma_{r-1}\left(\frac{1}{2}\right)\}$ is in some cell C_r^α. Denote by 1 the horizontal section of $\Gamma^* L_r$, which takes the value 1 over the point $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ in the trivialization over C_r^α. Let $\eta_i^\pm$ be the paths

$$t \mapsto \left\{\gamma_0\left(\frac{1}{2}\right), \ldots, \gamma_i\left(\frac{1}{2}(1 \pm t)\right), \ldots, \gamma_{r-1}\left(\frac{1}{2}\right)\right\}, \tag{101}$$

going from the cell C_r^α to the cell containing the point $\left\{P_-, \gamma_0\left(\frac{1}{2}\right), \ldots, \gamma_{r-1}\left(\frac{1}{2}\right)\right\}$. We have the explicit expressions

$$F\ 1[\gamma_0, \ldots, \gamma_{r-1}] = 1[\gamma_0, \ldots, \gamma_{r-1}, \gamma], \tag{102}$$

$$E\ 1[\gamma_0, \ldots, \gamma_{r-1}] = \sum_{i=0}^{r-1} (-1)^i (\rho_r(\eta_i^+) - \rho_r(\eta_i^-))\ 1[\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r-1}]. \tag{103}$$

Denoting by $\eta^\pm$ the paths $t \mapsto \left\{\gamma_0\left(\frac{1}{2}\right), \ldots, \gamma_{r-1}\left(\frac{1}{2}\right), \gamma\left(\frac{1}{2}(1 \pm t)\right)\right\}$, we find representation factors $\rho_{r+1}(\eta^+) = q^{\Sigma(1-n_i)}(-q^2)^r$ and $\rho_{r+1}(\eta^-) = \rho_{r+1}(\eta^+)^{-1}$. We then compute,

$$\begin{aligned} EF1[\gamma_0, \ldots, \gamma_{r-1}] &= E1[\gamma_0, \ldots, \gamma_{r-1}, \gamma] \\ &= \sum_{i=0}^{r-1} (-1)^i \left(\rho_r(\eta_i^+) - \rho_r(\eta_i^-)\right) 1[\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{r-1}, \gamma] + \\ &\quad (-1)^r \left(\rho_r(\eta^+) - \rho_r(\eta^-)\right) 1[\gamma_0, \ldots, \gamma_{r-1}] \\ &= FE\ 1[\gamma_0, \ldots, \gamma_{r-1}] + \\ &\quad \left(q^{\Sigma(n_i-1)-2r} - q^{-\Sigma(n_i-1)+2r}\right) 1[\gamma_0, \ldots, \gamma_{r-1}]. \end{aligned} \tag{104}$$

The proof is complete. $\square$

From Proposition 3.5 and Theorem 3.6 follows:

Corollary 3.7 *Singular vectors in $A_r(w_1, \ldots, w_s)$ (i.e., vectors in Ker E) represent absolute cycles in $H_r^{lf}(X_r^\epsilon; L_r)$.*

3.4 Intersection Pairing

3.4.1 Reflection and Duality

Let L'_r be the local system dual to L_r, i.e., the flat line bundle with holonomies $\rho'(\eta) = \rho(\eta)^{-1}$, (which is the representation obtained from ρ by replacing q by its inverse), and let θ be the reflection sending $x+iy$ to $-x+iy$. The reflection θ maps orbits of admissible base points to orbits of admissible base points and preserves holonomies,

$$\rho'(\theta \circ \eta) = \rho(\eta), \tag{105}$$

and lifts therefore to an involutive homomorphism of local systems

$$\theta : L_r(w_1, \dots, w_s) \to L'_r(\theta w_1, \dots, \theta w_s). \tag{106}$$

The lift is specified by setting the maps θ_α of Lemma 3.2 equal to the identity.

Denote by $A'_r(w_1, \dots, w_s)$ the space of linear combinations $\sum \lambda_\Gamma[\Gamma]$ with $[\Gamma]$ homotopy classes of non-intersecting families of loops based at P_+, and λ_Γ horizontal sections of $\Gamma^* L'_r(w_1, \dots, w_s)$, modulo the above equivalence relations. The reflection θ induces an isomorphism

$$\Theta : A_r(w_1, \dots, w_s) \to A'_r(\theta w_1, \dots, \theta w_s), \qquad \lambda[\Gamma] \mapsto \theta\lambda[\theta \circ \Gamma], \tag{107}$$

which defines an action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on $\oplus A'_r$.

3.4.2 Intersection Pairing

In this subsection we assume that all families of curves are smooth maps on $]0,1[^r$.

Let Γ be a family of curves based at $P_- = -1$ and Γ' be a family of curves based at P_+. Suppose that Γ and Γ' intersect transversally in a finite number of points lying in the interior of X. Thus the set of (t,t') such that $\Gamma(t) = \Gamma'(t')$ is finite, contained in $]0,1[^r \times]0,1[^r$, and the tangent map $D\Gamma \times D\Gamma'$ is non-singular at any such (t,t'). The intersection index $\sharp(t,t')$ at (t,t') is then defined to be 1 if the tangent map preserves the orientation, and -1 otherwise. The orientation of $T_{\Gamma(t)}X_r = \mathbb{C}^r$ is conventionally defined via the identification

$$(x_1 + iy_1, \dots, x_r + iy_r) \equiv (x_1, \dots, x_r, y_1, \dots, y_r) \tag{108}$$

of $\mathbb{C}^r$ with $\mathbb{R}^{2r}$.

Definition: The intersection pairing is the complex bilinear form

$$(\ ,\ ) : \oplus_r A_r(w_1, \dots, w_r) \times \oplus_r A'_r(w_1, \dots, w_r) \to \mathbb{C}, \tag{109}$$

which is zero on $A_r \times A'_{r'}, r \neq= r'$, and such that

$$([\,],[\,]) = 1, \qquad (\lambda[\Gamma], \lambda'[\Gamma']) = (-1)^r \sum_{(t,t'):\Gamma(t)=\Gamma'(t')} \sharp(t,t')\langle\lambda(t), \lambda'(t')\rangle \tag{110}$$

on $A_r \times A'_r$; $\langle\ ,\ \rangle$ denotes duality of fibers.

It is possible to give a more explicit formula for $(\ ,\ )$. Let $\dot\gamma(t)$ be the tangent vector at t to a smooth curve γ.

Proposition 3.8 *Suppose that $\Gamma = \gamma_0, \dots, \gamma_{r-1}$ and $\Gamma' = \gamma'_0, \dots, \gamma'_{r-1}$ intersect transversally. Let $T_{ij} \subset]0,1[\times]0,1[$ be the set of (t,t') such that $\gamma(t) = \gamma'(t')$ and $\sigma_{ij} =$ sign $\Im m(\bar{\dot\gamma}_i(t)\dot\gamma'_j(t'))$ be the intersection index of γ_i and γ'_j at (t,t'). Let for $\pi \in S_r$,*

$$T_\pi = \left\{(t,t') \in]0,1[^r\times]0,1[^r \mid (t_j, t'_{\pi j}) \in T_{j\pi j},\ j = 0,\dots,r-1\right\}. \tag{111}$$

Then

$$(\lambda[\Gamma], \lambda'[\Gamma']) = (-1)^r \sum_{\pi\in S_r} \text{sign}\ \pi \sum_{(t,t')\in T_\pi} \prod_{j=0}^{r-1} \sigma_{j\pi j} \langle\lambda(t), \lambda'(t')\rangle. \tag{112}$$

Proof: The condition $\Gamma(t) = \Gamma'(t')$ is equivalent to $\gamma_j(t_j) = \gamma'_{\pi j}(t'_{\pi j})$ for all j and some permutation π. Thus

$$\{(t,t')|\Gamma(t) = \Gamma'(t')\} = \cup_{\pi\in S_r} T_\pi, \tag{113}$$

and $T_\pi \cap T_{\pi'} =$ for $\pi \neq \pi'$, by the second property of non-intersecting families of curves. For $(t,t') \in T_\pi$, $\sharp(t,t')$ is the sign of the determinant of the matrix

$$\begin{pmatrix} \delta_{ij}\Re e(\dot\gamma_j(t_j)) & \delta_{\pi i,j}\Re e(\dot\gamma'_j(t_j)) \\ \delta_{ij}\Im m(\dot\gamma_j(t_j)) & \delta_{\pi i,j}\Im m(\dot\gamma'_j(t_j)) \end{pmatrix} \tag{114}$$

which is easily put in block form by permuting rows and column, and the result follows. □

Theorem 3.9 *Fix $w_1, \dots, w_s$, let $(\ ,\ )$ be the intersection pairing corresponding to $w_1, \dots, w_s$ and let $(\ ,\ )_\theta$ be the intersection pairing corresponding to $\theta w_1, \dots, \theta w_s$. (I) For all $a \in A_r(w_1, \dots, w_s)$, $b \in A_r(\theta w_1, \dots, \theta w_s)$,*

$$(a, \Theta b) = (b, \Theta a)_\theta. \tag{115}$$

(II) Let T denote transposition with respect to $(\ ,\ )$. Then

$$E^T = F, \quad F^T = E, \quad K^{2T} = K^2, \tag{116}$$

i.e., $(\ ,\ )$ is a covariant bilinear form.

Proof: (1) Set $a = \lambda_1[\Gamma_1]$ and $b = \lambda_2[\Gamma_2]$. Looking at the definition of intersection pairing, we see that since θ preserves the pairing between fibers, it is sufficient to prove that the intersection index $\sharp(t_1, t_2)$ is the same on both sides of the equation. Let (t_1, t_2) be an intersection point of Γ_1 with $\theta\Gamma_2$. Identify the tangent space at a point of the unit r-cube in a canonical way with $\mathbb{R}^r$, and the tangent space at a point in X_r

with $\mathbb{R}^{2r}$ as above. Then the intersection index occurring on the left hand side is the sign of the determinant of the $r \times r$ matrix $D\Gamma_1 \times \theta_* D\Gamma_2 : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^{2r}$. The matrix θ_* is the diagonal matrix with entries $1, \ldots, 1, -1, \ldots, -1$. We have

$$\begin{aligned} \det(D\Gamma_1 \times \theta_* D\Gamma_2) &= (-1)^r \det(\theta_* D\Gamma_1 \times D\Gamma_2) \\ &= (-1)^{r^2+r} \det(D\Gamma_2 \times \theta_* D\Gamma_1) \\ &= \det(D\Gamma_2 \times \theta_* D\Gamma_1). \end{aligned} \tag{117}$$

The sign of the last determinant is the intersection index occurring on the right hand side. (2) $K^{2T} = K^2$ follows immediately from the definition. Next, we show that $F^T = E$. The third relation follows then from (1). Let $\Gamma = \gamma_0, \ldots, \gamma_{r-1}$ be a family of loops based at P_- and $\Gamma' = \gamma'_0, \ldots, \gamma'_{r-1}$ one based at P_+. Let us, as in the proof of Theorem 3.6, denote by 1 the section of $\Gamma^* L_r(w_1, \ldots, w_r)$, which takes the value 1 over the point $\left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ in the trivialization over $\gamma_0\left(\frac{1}{2}\right), \ldots, \gamma_{r-1}\left(\frac{1}{2}\right)$, which is assumed to be in a cell. Suppose that γ_i intersects γ'_j in a point which is in some cell C^α, and let t, t' denote the value of the parameters at the intersection point. Denote by τ_{ij} the path $s \mapsto \gamma_i\left(\frac{1}{2}(1-s) + ts\right)$ and by τ'_{ij} the path $s \mapsto \gamma'_i\left(\frac{1}{2}(1-s) + t's\right)$. Then we have the explicit expression

$$(1[\Gamma], 1[\Gamma']) = (-1)^r \sum_{\pi \in S_r} \text{sign } \pi \sum_{(t,t') \in T_\pi} \prod_{j=0}^{r-1} \sigma_{j\pi j} \rho(\tau_{j,\pi j}) \rho'(\tau'_{j,\pi j}). \tag{118}$$

Let γ be the path $t \mapsto -\exp(2\pi i t)$. We have to compute

$$(F1[\gamma_0, \ldots, \gamma_{r-1}], 1[\gamma'_0, \ldots, \gamma'_r]) = (1[\gamma_0, \ldots, \gamma_{r-1}, \gamma], 1[\gamma'_0, \ldots, \gamma'_r]) \,. \tag{119}$$

It can be assumed, by possibly applying a homotopy, that γ intersects each γ'_j at exactly two points, namely when the parameter t'_j of γ'_j is close to zero, with positive intersection index, and when t'_j is close to one, with negative index (see Fig. 1). In both cases the parameter t of γ is close to $\frac{1}{2}$. Therefore the corresponding paths τ_{rj}, τ'_{rj}, associated to these intersections, can be replaced by the trivial path and by the paths $\eta'^{\pm}_j$ defined by

$$t \to \left\{ \gamma'_0\left(\frac{1}{2}\right), \ldots, \gamma'_j\left(\frac{1}{2}(1 \pm t)\right), \ldots, \gamma'_r\left(\frac{1}{2}\right) \right\}. \tag{120}$$

We are in position to complete the calculation:

$$\begin{gathered} \Big(F1[\gamma_0, \ldots, \gamma_{r-1}], 1[\gamma'_0, \ldots, \gamma'_{r-1}]\Big) = \\ (-1)^{r+1} \sum_{i=0}^{r} \sum_{\pi \in S_r} (-1)^{r-i} \text{sign } (\pi) \\ \sum_{(t,t') \in T_\pi} \Big(\rho'_r(\eta'^{-}_i) - \rho'_r(\eta'^{+}_i)\Big) \prod_{j=0}^{r-1} \sigma_{j,\pi j} \rho(\tau_{j,\pi j}) \rho'(\tau'_{j,\pi j}) = \\ \Big(1[\gamma_0, \ldots, \gamma_{r-1}], E1[\gamma'_0, \ldots, \gamma'_{r-1}]\Big) \end{gathered} \tag{121}$$

□

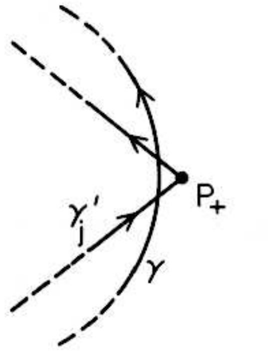

Figure 1: The points of intersection of γ with γ_j'.

3.5 Tensor Products and Coproduct

In this section we give explicitly the structure of $\oplus A_r(w_1, \dots, w_s)$ as a $U_q(\mathfrak{s}l_2(\mathbb{C}))$ module.

3.5.1 The module $\oplus A_r(w_1, \dots, w_s)$

The spaces $A_r(w_1, \dots, w_s)$ constitute a complex vector bundle over $\mathcal{C}_{1,\dots,1}(D)$. This bundle comes with a flat Gauss-Manin connection induced by the connection of $L_{r,1,\dots,1}$. The holonomy of this connection will be computed in the next section. Let us notice that the spaces $A_r(w_1, \dots, w_s)$ are isomorphic (although not canonically isomorphic), and we can fix $w_1, \dots, w_s$ as we like. For definiteness, we choose $w_1, \dots, w_s$ to be admissible, so that $\Re e(w_1) < \dots < \Re e(w_s)$. To describe $A_r(w_1, \dots, w_s)$ as a linear space, we choose a basis as follows. Fix a non-intersecting family of loops $\gamma_1, \dots, \gamma_s$, so that γ_i winds around w_i as shown in Fig. 2a. Introduce the shorthand notation

$$[\gamma_1^{r_1}, \dots, \gamma_s^{r_s}] \tag{122}$$

to denote a homotopy class of non-intersecting families of loops, constructed as follows: Let $\gamma_i^{(j)}$ ($1 \le i \le s$, $1 \le j \le r_i$) be slight homotopic deformations of γ_i such that $\gamma_i^{(j)}$ lies inside $\gamma_i^{(j+1)}$, and such that $\gamma_1^{(1)}, \dots, \gamma_1^{(r_1)}, \dots, \gamma_s^{(1)}, \dots, \gamma_s^{(r_s)}$ is a non-intersecting family of loops. Let $[\gamma_1^{r_1}, \dots, \gamma_s^{r_s}]$ be the associated homotopy class. Define a horizontal section denoted by 1 over this family to be the section which takes the value 1 with respect to the trivialization over a point with coordinates obeying

$$\begin{aligned} &\Re e(w_1) < \Re e(z_1) < \dots < \Re e(z_{r_1}) < \Re e(w_2) < \\ &\Re e(z_{r_1+1}) < \dots < \Re e(z_{r_2}) < \Re e(w_3) < \dots \end{aligned} \tag{123}$$

If $r_1, \dots, r_s$ run over all non-negative integers with total sum r, the families of loops $1[\gamma_1^{r_1}, \dots, \gamma_s^{r_s}]$ form a basis of $A_r(w_1, \dots, w_r)$.

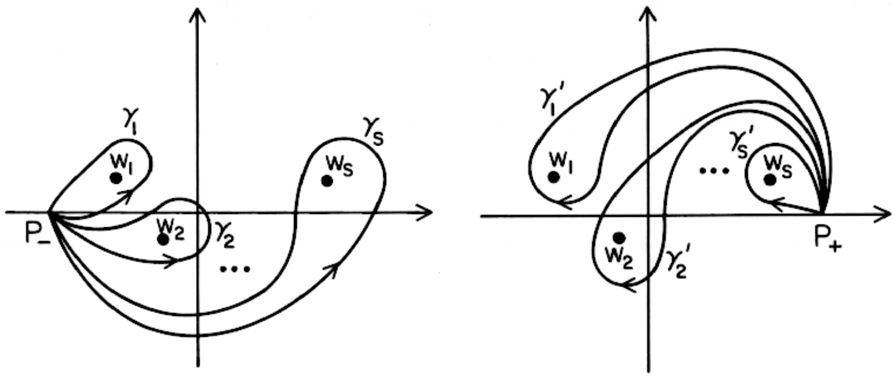

Figure 2: The loops used to define a basis of $A_r(w_1, \ldots, w_s)$ and $A'_r(w_1, \ldots, w_s)$.

Theorem 3.10 *The $U_q(\mathfrak{sl}_2(\mathbb{C}))$-module $\oplus A_r(w_1, \ldots, w_s)$ is isomorphic to the tensor product of Verma modules*

$$V_{n_1} \otimes \ldots \otimes V_{n_s} \tag{124}$$

with action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ given by the s-fold coproduct $\triangle^{(s)}$. If $\mathfrak{Re}(w_1) < \ldots < \mathfrak{Re}(w_s)$, an isomorphism is explicitly given by

$$1[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}] \mapsto F^{r_1} v_{n_1} \otimes \ldots \otimes F^{r_s} v_{n_s}. \tag{125}$$

Proof: For $s = 1$, we have $1[\gamma_1^r] = F1[\gamma_1^{r-1}]$, by definition of F. For higher r, we have to show that the action of the generators on the basis is indeed given by the coproduct. For the (diagonal) generators K^2, K^{-2} this follows from the definition. For the other (raising and lowering) operators, the proof follows from contour deformation. To compute the action of F we must deform the added loop γ to the composition of loops homotopic to $\gamma_s, \ldots, \gamma_1$:

$$F1[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}] = \sum_{i=1}^{s} \alpha_i 1[\gamma_1^{r_1}, \ldots, \gamma_i^{r_i+1}, \ldots, \gamma_s^{r_s}]. \tag{126}$$

The coefficient α_i is, up to a sign, the transition function we pick up by going from the point where the section 1 over $\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}, \gamma$ is trivialized to the point where the section 1 over $\gamma_1^{r_1}, \ldots, \gamma_i^{r_i+1}, \ldots, \gamma_s^{r_s}$ is trivialized. The sign is $(-1)^{\Sigma_{j>i} r_j}$, and comes from reordering the loops. Thus $\alpha_i = q^{\sum_{j>i}(1-n_j-2r_j)}$ and we get the result

$$F = \sum_i 1 \otimes \ldots \otimes 1 \otimes F \otimes K^{-2} \otimes \ldots \otimes K^{-2}. \tag{127}$$

Similarly, by computing the contribution proportional to $1[\gamma_1^{r_1}, \ldots, \gamma_i^{r_i-1}, \ldots, \gamma_s^{r_s}]$ of $E1[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}]$, we see that we get the same terms as in the computation of $E[\gamma_i^{r_i}]$

except for the factor $q^{-\sum_{j<i}(1-n_j-2r_j)}$ that we pick up by going from the vicinity of w_i to P_-, and we obtain the result:

$$E = \sum_i K^2 \otimes \ldots \otimes K^2 \otimes E \otimes 1 \otimes \ldots \otimes 1. \tag{128}$$

This concludes the proof. $\square$

Remark: We see that the tensor product also has a topological interpretation: let S_+, S_- be the upper and lower halves of the unit circle. We can think of $D\backslash\{w_1, \ldots, w_s\}$ (with $w_i \neq w_j$, $i \neq j$ and $w_i \in \text{int } D$) as the result of glueing s punctured discs $D \setminus \{0\}$ in such a way that S_+ of the ith disc is identified with S_- of the $i+1$st disc. This construction gives an identification of $A_r(w_1, \ldots, w_s)$ with $A_1(0) \otimes \ldots \otimes A_1(0)$ so that $1[\gamma_1^{r_1}, \ldots, \gamma_s^{r_s}]$ is identified with $1[\gamma^{r_1}] \otimes 1[\gamma^{r_2}] \otimes \ldots \otimes 1[\gamma^{r_s}]$.

The module $A'_r(w_1, \ldots, w_s)$, being isomorphic to $A_r(\theta w_1, \ldots, \theta w_s)$, also has the structure of a tensor product of Verma modules. In order to achieve compatibility between tensor product structure and bilinear form, one has to choose the isomorphism in a special way. Let γ'_i be the non-intersecting family depicted in Fig. 2b, and, as above, define $[(\gamma'_1)^{r_1}, \ldots, (\gamma'_s)^{r_s}]$ and a horizontal section 1 taking the value 1 with respect to the trivialization over a point with

$$\mathfrak{Re}(z_1) < \ldots < \mathfrak{Re}(z_{r_1}) < \mathfrak{Re}(w_1) < \mathfrak{Re}(z_{r_1+1}) < \ldots < \mathfrak{Re}(z_{r_2}) < \mathfrak{Re}(w_2) < \ldots \tag{129}$$

Let furthermore λ be the automorphism of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ defined on generators by

$$\lambda(H) = H, \quad \lambda(E) = K^{-2}E, \quad \lambda(F) = FK^2. \tag{130}$$

The following dual version of Theorem 3.10 is proven exactly as Theorem 3.10.

Theorem 3.11 *The $U_q(\mathfrak{s}l_2(\mathbb{C}))$-module $A'_r(w_1, \ldots, w_s)$ is isomorphic to the tensor product of Verma modules*

$$V_{n_1} \otimes \ldots \otimes V_{n_s} \tag{131}$$

with action of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ given by the twisted coproduct $\lambda^{-1} \circ \triangle^{(s)} \circ \lambda$. If $\mathfrak{Re}(w_1) < \ldots < \mathfrak{Re}(w_s)$, an isomorphism is explicitly given by

$$1[(\gamma'_1)^{r_1}, \ldots, (\gamma'_s)^{r_s}] \mapsto F^{r_1}v_{n_1} \otimes \ldots \otimes F^{r_s}v_{n_s}. \tag{132}$$

3.5.2 Tensor Products and Intersection Pairing

The isomorphisms described in the preceding Theorems define intersection pairing as a bilinear form $(\ ,\ )$ on $V_{n_1} \otimes \ldots \otimes V_{n_s}$.

Theorem 3.12 *The intersection pairing coincides with the product of the Shapovalov bilinear forms on V_{n_i}. In particular, it is symmetric and degenerate. It reduces to a non-degenerate symmetric bilinear form on the fusion rule subquotient.*

Proof: For $s = 1$, the highest weight vector v_n of V_n has $(v_n, v_n) = 1$ and one has $E^T = F$, $F^T = E$, $K^{2T} = K^2$, which are the characterizing properties of the Shapovalov bilinear form on the Verma module V_n. The choice of identification of A_r, A'_r with the product of Verma modules is chosen in such a way that the weight $(-1)^r \sharp(t,t')\langle\lambda(t), \lambda'(t')\rangle$ of each intersection point factorizes into s factors equal to the weights of the corresponding intersections in $([\gamma_i^{r_i}], [\gamma'^{r'_i}_i])$. $\square$

3.5.3 The Local System

The result of Theorem 3.10 can be cast into the formalism of local systems. To be more precise, introduce the dependence of the labels $n_1, \ldots, n_s$ explicitly in the notation:

$$A_r = A_r(w_1, \ldots, w_s | n_1, \ldots, n_s). \tag{133}$$

Fix a base point $(w_1^0, \ldots, w_s^0)$ such that $\mathfrak{Re}(w_1) < \ldots < \mathfrak{Re}(w_s)$. The spaces (133) define a flat vector bundle over $\mathcal{C}_{1,\ldots,1}(D)$ with (Gauss-Manin) connection induced by the connection on $L_{r,1,\ldots,1}$. The fiber over $(w_1^0, \ldots, w_s^0)$ is identified with $V_{n_1} \otimes \ldots \otimes V_{n_s}$ by the explicit isomorphism of Theorem 3.10. Similarly, for any permutation $\alpha \in S_s$ we can identify the fiber over $\alpha(w_1^0, \ldots, w_s^0)$ with $V_{n_{\alpha(1)}} \otimes \ldots \otimes V_{n_{\alpha(s)}}$ using the trivial identification

$$A_r(w^0_{\alpha^{-1}(1)}, \ldots, w^{(0)}_{\alpha^{-1}(s)} | n_1, \ldots, n_s) = A_r(w_1^0, \ldots, w_s^0 | n_{\alpha(1)}, \ldots, n_{\alpha(s)}). \tag{134}$$

3.6 Monodromy Action of the Braid Groupoid and Universal R-Matrix

In the following we will consider the configuration space $\mathcal{C}_{n_1,\ldots,n_{s+1}}(D)$ with $D = \{z \in \mathbb{C} | \, |z| \leq 1\}, n_1 = r$ and $n_2 = \ldots = n_s = 1$.

Let $p : \mathcal{C}_{r,1,\ldots,1}(D) \to \mathcal{C}_{1,\ldots,1}(D)$ be the projection given by omitting the first r entries of $(z_1, \ldots, z_r, w_1, \ldots, w_s)$. p defines a fiber bundle over $\mathcal{C}_{1,\ldots,1}(D)$ with fibers $p^{-1}(w_1, \ldots, w_s) = X_r(w_1, \ldots, w_s)$. In particular, $X_1(w_1, \ldots, w_s) = D \backslash \{w_1, \ldots, w_s\}$ is the punctured unit disc. In the following we will restrict our attention to $\{w_1, \ldots, w_s\} \subset$ int D.

Fix a base point $x = (w_1, \ldots, w_s) \in \mathcal{C}_{1,\ldots,1}(D)$, with $\mathfrak{Re}(w_1) < \ldots < \mathfrak{Re}(w_s)$. We construct a non-abelian representation ρ of the colored braid groupoid $B_{1,\ldots,1}(D, x) = G$. Note that $O_x = S_s x$ and $G = \cup_{\alpha,\beta \in S_s} G_{\alpha,\beta}$. G is generated by $[\sigma_i^\alpha]$, $i \in \{1, \ldots, s-1\}$ and $\alpha \in S_s$. Here $\sigma_i^\alpha : [0,1] \to \mathcal{C}_{1,\ldots,1}(D)$ is a smooth parametrized curve with $\sigma_i^\alpha(0) = \alpha x$ and $\sigma_i^\alpha(1) = \tau_i \alpha x$, which implements a counter clockwise exchange of $w_{\alpha^{-1}(i)}$ and $w_{\alpha^{-1}(i+1)}$.

Let the representation space V_α associated with $\alpha \in S_s$ be $A_r(w_{\alpha^{-1}(1)}, \ldots, w_{\alpha^{-1}(s)})$. It admits the explicit decription as linear span

$$A_r(w_{\alpha^{-1}(1)}, \ldots, w_{\alpha^{-1}(s)}) = \oplus \mathbb{C}1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha. \tag{135}$$

The sum is over $(j_1,\dots,j_s)\in\{0,\dots,p-1\}^s$ such that $j_1+\dots+j_s=r$. Thus we have an identification $A_r(w_{\alpha^{-1}(1)},\dots,w_{\alpha^{-1}(s)})\cong\mathbb{C}^{N(r,s)}$ with $N(r,s)=\binom{r+s}{s-1}$. The simplest nontrivial case is $s=2$ with $N(r,s)=r+1$. Let $[\sigma_i^\alpha]$ be represented by the deformation homomorphism $\rho([\sigma_i^\alpha]):V_\alpha\to V_{\tau_i\alpha}$ associated with σ_i^α. Introduce the q-number notation $[n]_q=q^n-q^{-n}$, and, for $k=0,\dots,n$,

$$\begin{bmatrix} n\\ k\end{bmatrix}_q=\frac{[n]_q[n-1]_q\dots[n-k+1]_q}{[k]_q[k-1]_q\dots[1]_q}. \tag{136}$$

Proposition 3.13 *(I)*

$$\begin{aligned}&\rho([\sigma_i^\alpha])1[(\gamma_1)^{j_1}\dots(\gamma_s)^{j_s}]_\alpha=\\ &\sum_{k=0}^{j_{i+1}}1[(\gamma_1)^{j_1}\dots(\gamma_i)^{j_{i+1}-k}(\gamma_{i+1})^{j_i+k}\dots(\gamma_s)^{j_s}]_{\tau_i\alpha}\\ &q^{\frac{1}{2}k(k-1)}q^{\frac{1}{2}(n_{\alpha(i+1)}-1-2(j_{i+1}-k))(n_{\alpha(i)}-1-2(j_i+k))}\\ &\begin{bmatrix} j_{i+1}\\ k\end{bmatrix}_q\prod_{l=0}^{k-1}[n_{\alpha(i+1)}-(j_{i+1}-k)]_q\end{aligned} \tag{137}$$

(II)

$$\begin{aligned}&\rho([\sigma_i^\alpha]^{-1})1[(\gamma_1)^{j_1}\dots(\gamma_s)^{j_s}]_\alpha=\\ &\sum_{k=0}^{j_i}1[(\gamma_1)^{j_1}\dots(\gamma_i)^{j_{i+1}+k}(\gamma_{i+1})^{j_i-k}\dots(\gamma_s)^{j_s}]_{\tau_i\alpha}\\ &(-1)^kq^{-\frac{1}{2}k(k-1)}q^{-\frac{1}{2}(n_{\alpha(i+1)}-1-2j_{i+1})(n_{\alpha(i)}-1-2j_i)}\\ &\begin{bmatrix} j_i\\ k\end{bmatrix}_q\prod_{l=0}^{k-1}[n_{\alpha(i)}-(j_i-k)]_q.\end{aligned} \tag{138}$$

Proof: Without loss of generality we can restrict the proof to the case $s=2$, $i=1$, and $\alpha=\mathrm{id}$. The loops used in this proof are represented in Fig. 3. The matrix representation of $[\sigma]$ is computed by consecutive deformations and subdivisions of the individual loops in

$$\rho([\sigma])1[(\gamma_1)^{j_1}(\gamma_2)^{j_2}]=q^{\frac{1}{2}(n_1-1-2j_1)(n_2-1-2j_2)}q^{j_2(n_1-1)-2j_1j_2}1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau. \tag{139}$$

Subdivide the last β_1-loop in a γ_2- and a β_2-loop to obtain

$$\begin{aligned}&1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau=\\ &-q^{-2(n_2-1)+2(j_2-1)}1[(\gamma_2)^{j_1+1}(\beta_1)^{j_2-1}]_\tau+1[(\gamma_2)^{j_1}(\beta_1)^{j_2-1}\beta_2]_\tau.\end{aligned} \tag{140}$$

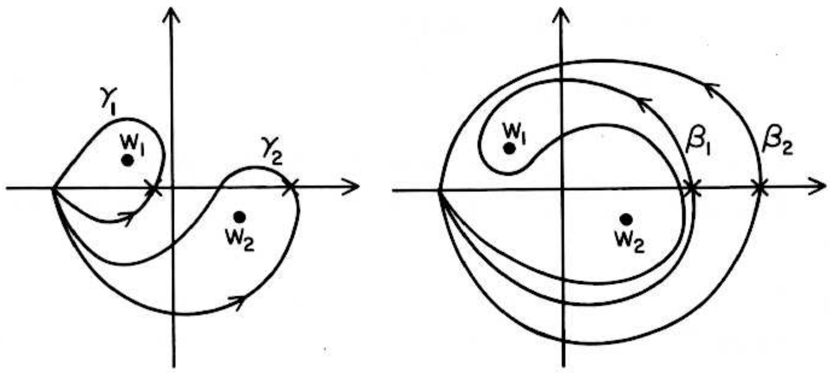

Figure 3: The loops appearing in the proof of Proposition 3.13. The points marked with a cross are the points used to define the section 1.

Iterate this subdivision until there is no β_1-loop left over. The result is

$$1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau = \sum_{k=0}^{j_2}(-1)^k q^{-2k(n_2-1)} \sum_{0\leq i_1<\ldots<i_k\leq j_2-1} q^{2\sum_{l=1}^k i_l} 1[(\gamma_2)^{j_1+k}(\beta_2)^{j_2-k}]_\tau. \tag{141}$$

The sum over ordered k-tuples is now performed with Gauss' formula

$$\sum_{0\leq i_1<\ldots<i_k\leq j_2-1} q^{2\sum_{l=1}^k i_l} = q^{k(j_2-1)} \begin{bmatrix} j_2 \\ k \end{bmatrix}_q . \tag{142}$$

Then subdivide the β_2-loops in γ_1- and γ_2-loops. This decomposition yields

$$1[(\gamma_2)^{j_1+k}(\beta_2)^{j_2-k}]_\tau = \sum_{l=0}^{j_2-k} q^{-(j_2-k-l)(n_1-1-2(j_1+k)-l)} \begin{bmatrix} j_2-k \\ l \end{bmatrix}_q 1\left[(\gamma_1)^{j_2-k-l}(\gamma_2)^{j_1+k+l}\right]_\tau . \tag{143}$$

Insert (142) and (143) into (141), and reorder the double sum to obtain

$$1[(\gamma_2)^{j_1}(\beta_1)^{j_2}]_\tau = \sum_{k=0}^{j_2}(-1)^k q^{k(j_2-2n_2+1)-(j_2-k)(n_1-1-2(j_1+k))} \begin{bmatrix} j_2 \\ k \end{bmatrix}_q \sum_{l=0}^{k}(-1)^l q^{-l(2j_2-2n_2-k+1)} \begin{bmatrix} k \\ l \end{bmatrix}_q 1\left[(\gamma_1)^{j_2-k}(\gamma_2)^{j_1+k}\right]_\tau . \tag{144}$$

Then perform the second sum with the q-binomial formula

$$\sum_{l=0}^{k}(-1)^l q^{-l(2j_2-2n_2-k+1)} \begin{bmatrix} k \\ l \end{bmatrix}_q = (-1)^k q^{k(n_2-j_2)+\frac{1}{2}k(k-1)} \prod_{l=0}^{k-1}[n_2-(j_2-l)]_q. \tag{145}$$

Insert (144) and (145) into (139) to find (137). The matrix representation of $[\sigma]^{-1}$ is computed following the same lines. $\square$

The important consequence of Proposition 3.13 is that the deformation homomorphism $\rho([\sigma_i^\alpha]) : V_\alpha \to V_{\tau_i \alpha}$, written as an operator, has a universal form which resembles the universal R-matrix of the quantum group algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$. Let $p \in \mathbb{N} \cup \{\infty\}$ be the smallest positive integer such that $q^{2p} = 1$.

Theorem 3.14 *Denote by X_i the operator $1 \otimes \ldots \otimes 1 \otimes X \otimes 1 \otimes \ldots \otimes 1$, acting on the i'th factor of $V_{n_1} \otimes \ldots \otimes V_{n_s}$. Take $1 \leq n_1, \ldots, n_s \leq p-1$. Then*

$$\rho([\sigma_i^\alpha]) = \sum_{k=0}^{p-1} q^{\frac{1}{2}k(k-1)} \frac{(q-q^{-1})^k}{[k]_q!} q^{\frac{1}{2}H_i H_{i+1}} E_i^k F_{i+1}^k \tau_i, \tag{146}$$

and

$$\rho([\sigma_i^\alpha]^{-1}) = \sum_{k=0}^{p-1} (-1)^k q^{-\frac{1}{2}k(k-1)} \frac{(q-q^{-1})^k}{[k]_q!} F_i^k E_{i+1}^k q^{-\frac{1}{2}H_i H_{i+1}} \tau_i. \tag{147}$$

Proof: Recall the definition of the operators E_i, F_i, H_i, and of τ_i. They act as

$$E_i 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha = \frac{[j_i]_q [n_{\alpha(i)} - j_i]_q}{q-q^{-1}} 1[(\gamma_1)^{j_1} \ldots (\gamma_i)^{j_i - 1} \ldots (\gamma_s)^{j_s}]_\alpha, \tag{148}$$

$$F_i 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha = 1[(\gamma_1)^{j_1} \ldots (\gamma_i)^{j_i+1} \ldots (\gamma_s)^{j_s}]_\alpha, \tag{149}$$

$$H_i 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha = (n_{\alpha(i)} - 1 - 2j_i) 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha. \tag{150}$$

and

$$\tau_i 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_\alpha = 1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}]_{\tau_i \alpha}. \tag{151}$$

Compare (137) and (138) with (150) and (151) to conclude (146) and (147). $\square$

The content of Theorem 3.14 exceeds pure nomenclature since the operators E_i, F_i, and H_i have a topological interpretation. They satisfy the commutation relations

$$[H_i, E_j] = 2E_j \delta_{j,i}, \quad [H_i, F_j] = -2F_j \delta_{j,i}, \quad [E_i, F_j] = [H_j]_q \delta_{j,i}. \tag{152}$$

We have identified $A_r(w_{\alpha(1)}, \ldots, w_{\alpha(s)})$ with the tensor product $V_{n_{\alpha(1)}} \otimes \ldots \otimes V_{n_{\alpha(s)}}$ of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ Verma modules, the identification being

$$1[(\gamma_1)^{j_1} \ldots (\gamma_s)^{j_s}] \mapsto F^{j_1} v_{n_{\alpha(1)}} \otimes \ldots \otimes F^{j_s} v_{n_{\alpha(s)}}. \tag{153}$$

Moreover, we have identified $E_i \in \mathrm{Hom}(A_r(w_{\alpha(1)}, \ldots, w_{\alpha(s)}), A_{r-1}(w_{\alpha(1)}, \ldots, w_{\alpha(s)}))$ with the element

$$E_i \mapsto 1 \otimes \ldots \otimes E \otimes \ldots \otimes 1 \tag{154}$$

of $U_q(\mathfrak{s}l_2(\mathbb{C}))^{\otimes s}$. Here E stands in the ith entry. Similarly we have proceeded with F_i and E_i. τ_i is identified with the ith transposition. We have proved that this identification is a quantum group algebra homomorphism and a module isomorphism.

The observation of this section is now that

$$\rho([\sigma_i^\alpha]^{\pm 1}) \mapsto 1 \otimes \ldots \otimes (RP)^{\pm 1} \otimes \ldots \otimes 1 \tag{155}$$

with $R \in U_q(\mathfrak{s}l_2(\mathbb{C}))^{\otimes 2}$ the universal R-matrix in an obvious normalization and R^{-1} its inverse, acting on the ith and $(i+1)$st entry.

It follows that ρ defines an R-matrix representation of $B_{1,\ldots,1}(X)$. The Yang-Baxter equations for the topological R-matrix follows from properties of the universal R-matrix.

Having constructed an $N(r,s)$-dimensional R-matrix representation of $B_{1,\ldots,1}(D)$, we also have a rank $N(r,s)$ local system $L^r_{1,\ldots,1}(D)$ over $\mathcal{C}_{1,\ldots,1}(D)$. Let $C^\alpha_{1,\ldots,1}(D)$ be the intersection of the cell $C^\alpha_{1,\ldots,1}$ with $\mathcal{C}_{1,\ldots,1}(D)$.

$$L^r_{1,\ldots,1}(D) = \left(\sqcup_{\alpha \in S_s} \overline{C^\alpha_{1,\ldots,1}(D)} \times \mathbb{C}^{N(r,s)}\right) \Big/ \sim \tag{156}$$

with the equivalence relation over $\overline{C^\alpha_{1,\ldots,1}(D)} \cap \overline{C^{\tau_i\alpha}_{1,\ldots,1}(D)}$ given by multiplication with the matrices (137) and (138) respectively. The fiber space $p^{-1}(w_{\alpha^{-1}(1)},\ldots,w_{\alpha^{-1}(s)})$ is $A_r(w_{\alpha^{-1}(1)},\ldots,w_{\alpha^{-1}(s)})$. The parallel transport matrix associated with σ_i^α in the basis (136) is the universal R-matrix in the representation $n_{\alpha(i)} \otimes n_{\alpha(i+1)}$.

This representation of the braid groupoid is not irreducible in general. In particular, it has as invariant subspaces the null space of the bilinear form (,), which defines a subbundle of our flat bundle, invariant under parallel transport.These subspaces are described explicitly in the Appendix.

3.7 Locally finite homology

We have worked on the spaces A_r rather than on homology groups directly. We now formulate some conjectures on the relation to homology, and the structure of the corresponding locally finite homology groups. These conjectures follow from the assumption that our quantum group action extends to an action on homology, and by applying the computations of [FS89a, F90], which are not completely rigorous, to the situation studied here. As usual, we assume that s distinct points $w_1,\ldots,w_s$ in the interior of the unit circle, s positive integers $n_1,\ldots,n_s$, and a complex number $q \neq -1,0,1$ are given. If q is a root of unity, we furthermore assume that $1 \leq n_i \leq p-1$, where p is the smallest positive integer such that $q^{2p}=1$. For ϵ small enough, the locally compact spaces $X_r^\epsilon \supset X_r^{\epsilon -}$ are defined as in (86), and we have a local system L_r over X_r^ϵ.

Conjecture 3.15 *If q is not a root of unity, the map*

$$\varphi_r : A_r(w_1,\ldots,w_s) \to H_r^{lf}(X_r^\epsilon, X_r^{\epsilon -}; L_r), \tag{157}$$

is an isomorphism of vector spaces.

If q is a root of unity, let $U_q^L(\mathfrak{s}l_2(\mathbb{C}))$ be Lusztig's version of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ [Lu90], with generators $H, E, F, E^p/[p]!, F^p/[p]!$. Let V_n^L be the Verma module over the quantum group $U_q^L(\mathfrak{s}l_2(\mathbb{C}))$ with vacuum vector v_n, so that $Hv_n = (n-1)v_n$, and $Ev_n = E^p/[p]!v_n = 0$. There is a canonical Hopf algebra homomorphism $U_q(\mathfrak{s}l_2(\mathbb{C})) \to U_q^L(\mathfrak{s}l_2(\mathbb{C}))$, so that V_n^L is also an $U_q(\mathfrak{s}l_2(\mathbb{C}))$ module. For any H-diagonalizable $U_q(\mathfrak{s}l_2(\mathbb{C}))$ module M, denote by $(M)_n$ the eigenspace of H to the eigenvalue n.

Conjecture 3.16 *If q is not a root of unity, there are isomorphisms*

$$H_r^{lf}(X_r^\epsilon, X_r^{\epsilon-}; L_r) \xrightarrow{\sim} (V_{n_1} \otimes \ldots \otimes V_{n_s})_{\Sigma n_i - s - 2r}, \tag{158}$$
$$H_r^{lf}(X_r^\epsilon; L_r) \xrightarrow{\sim} \mathrm{Ker}\ E|(V_{n_1} \otimes \ldots \otimes V_{n_s})_{\Sigma n_i - s - 2r}. \tag{159}$$

If q is a root of unity, there are isomorphisms

$$H_r^{lf}(X_r^\epsilon, X_r^{\epsilon-}; L_r) \xrightarrow{\sim} (V_{n_1}^L \otimes \ldots \otimes V_{n_s}^L)_{\Sigma n_i - s - 2r}, \tag{160}$$
$$H_r^{lf}(X_r^\epsilon; L_r) \xrightarrow{\sim} \mathrm{Ker}\ E|(V_{n_1}^L \otimes \ldots \otimes V_{n_s}^L)_{\Sigma n_i - s - 2r}. \tag{161}$$

3.8 Appendix

We summarize some known facts about $U_q(\mathfrak{s}l_2(\mathbb{C}))$, following essentially [D86, RT91, Lu90].

Fix a non-zero complex number q. Let $U_q(\mathfrak{s}l_2(\mathbb{C}))$ be the algebra with unit over $\mathbb{C}$ with generators E, F, H, and relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = q^H - q^{-H}. \tag{162}$$

We often denote $K^2 = q^H$. Of course q^H is not well-defined in the algebra, but its action on modules where H takes integer values is. A more precise definition is the following: let $U_q(\mathfrak{s}l_2(\mathbb{C}))$ be the complex algebra with unit with generators E, F, H, K^2, K^{-2} and relations

$$\begin{gathered}[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = K^2 - K^{-2}, \\ K^2K^{-2} = K^{-2}K^2 = 1, \quad K^2H = HK^2.\end{gathered} \tag{163}$$

$U_q(\mathfrak{s}l_2(\mathbb{C}))$ is a $\mathbb{Z}$-graded algebra, with the assignment $\deg(E) = -\deg(F) = 1$, $\deg(H) = \deg(K^{\pm 2}) = 0$. Let G_q be the category of $\mathbb{Z}$-graded left $U_q(\mathfrak{s}l_2(\mathbb{C}))$-modules $M = \oplus_{n\in\mathbb{Z}} M_n$ such that:

1. For all $\xi \in M$ there exists an N such that $E^N\xi = 0$.

2. $HM_n = nM_n$ and $K^2M_n = q^nM_n$.

The degree of homogeneous element of a module in G_q is called *weight*. Following common usage, we refer to objects in G_q as ($\mathbb{Z}$-graded) $U_q(\mathfrak{s}l_2(\mathbb{C}))$-modules.

Let n be an integer, and $q \in \mathbb{C} \setminus \{0\}$. The *Verma module* V_n is the quotient of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ by the left ideal generated by $E, K^2 - q^{n-1}$ and $H - (n-1)$, with left action of $U_q(\mathfrak{s}l_2(\mathbb{C}))$. The module V_n is in G_q and is generated by a highest weight vector v_n (=image of 1) of weight $n-1$. A basis of V_n is given by the vectors $F^j v_n$, $j=0,1,\ldots$, and one has the explicit formulae

$$EF^j v_n = \frac{[j][n-j]}{q-q^{-1}} F^{j-1} v_n, \quad HF^j v_n = (n-1-2j)F^j v_n. \tag{164}$$

The notation we use here for q-numbers are

$$[j] \equiv [j]_q = q^j - q^{-j}, \quad \begin{bmatrix} j \\ l \end{bmatrix}_q = \frac{[j][j-1]\ldots[j-l+1]}{[l][l-1]\ldots[1]},$$
$$[j]! = [j][j-1]\ldots[2][1], \quad [0]! = 1. \tag{165}$$

If q is a root of unity we define a number p as the smallest positive integer such that

$$q = e^{\pi i p'/p} \tag{166}$$

for some integer $p' > 0$. If q is not a root of unity we set $p = \infty$.

Proposition 3.17 *(I) If q is not a root of unity, V_n is irreducible for $n \leq 0$. It contains a proper submodule SV_n generated by the singular vector*[2] *$F^n v_n$, if $n \geq 1$. The quotient V_n/SV_n is an irreducible n-dimensional representation.*
(II) If $q = e^{\pi i p'/p}$ is a root of unity, then V_n contains a proper submodule SV_n generated by the singular vector $F^{\bar{n}} v_n$, where $1 \leq \bar{n} \leq p$ and $\bar{n} \equiv n (\mathrm{mod}\ p)$. The quotient V_n/SV_n is irreducible, of dimension $\bar{n}$.

The *Shapovalov form* on V_n is the symmetric bilinear form $(\ ,\ )$: $V_n \times V_n \to \mathbb{C}$, uniquely characterized by

1. $(v_n, v_n) = 1$ and
2. $(E\xi, \eta) = (\xi, F\eta)$, $(H\xi, \eta) = (\xi, H\eta), \xi, \eta \in V_n$.

The null space of $(\ ,\ )$ is SV_n.

The action of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ on tensor products of modules in G_q is defined by the coassociative coproduct $\triangle : U_q(\mathfrak{s}l_2(\mathbb{C})) \to U_q(\mathfrak{s}l_2(\mathbb{C})) \otimes U_q(\mathfrak{s}l_2(\mathbb{C}))$ defined on generators as

$$\triangle(H) = H \otimes 1 + 1 \otimes H, \qquad \triangle(K^{\pm 2}) = K^{\pm 2} \otimes K^{\pm 2},$$
$$\triangle(E) = E \otimes 1 + K^2 \otimes E, \qquad \triangle(F) = F \otimes K^{-2} + 1 \otimes F. \tag{167}$$

The action on tensor products with s factors is given by $\triangle^{(s)} : U_q(\mathfrak{s}l_2(\mathbb{C})) \to U_q(\mathfrak{s}l_2(\mathbb{C})) \otimes \ldots \otimes U_q(\mathfrak{s}l_2(\mathbb{C}))$ with $\triangle^{(s+1)} = (\triangle^{(s)} \otimes 1)\triangle$, $\triangle^{(2)} = \triangle$. The universal R-matrix of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ is the formal series

$$R = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-1)} \frac{(q-q^{-1})^k}{[k]!} q^{\frac{1}{2}H \otimes H} E^k \otimes F^k. \tag{168}$$

[2]A singular vector is a vector annihilated by E.

This series is well defined on any tensor product module in G_q since only finitely many terms are non-vanishing when R acts on a vector. Also, singular denominators cancel.

Let $(\ ,\ )$ denote the product of Shapovalov forms: $(\ ,\ ) : V_{n_1} \otimes \ldots \otimes V_{n_s} \times V_{n_1} \otimes \ldots \otimes V_{n_s} \to \mathbb{C}$.

Proposition 3.18 *Let $R_{i,i+1} = 1 \otimes \ldots \otimes 1 \otimes R \otimes 1 \otimes \ldots \otimes 1$ be the R matrix acting on the ith and $(i+1)$'st factor in $V_{n_1} \otimes \ldots \otimes V_{n_s}$ and $P_{i,i+1}$ the transposition $\xi_1 \otimes \ldots \otimes \xi_s \mapsto \xi_1 \otimes \ldots \otimes \xi_{i+1} \otimes \xi_i \otimes \ldots \otimes \xi_s$. Then*

$$(R_{i,i+1}\xi, \eta) = (\xi, P_{i,i+1}R_{i,i+1}P_{i,i+1}\eta) \tag{169}$$

for all $\xi, \eta \in V_{n_1} \otimes \ldots \otimes V_{n_s}$.

Let $W_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$ be the space of singular vectors of weight $n-1$ in $V_{n_1} \otimes \ldots \otimes V_{n_s}$. The family of vector spaces $W_n(V_{\alpha(n_1)} \otimes \ldots \otimes V_{\alpha(n_s)})$, $\alpha \in S_s$ carries an R-matrix representation of the colored braid groupoid $B_{1,\ldots,1}$. As a consequence of Proposition 3.18 we have

Proposition 3.19 *Let $F_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$ be the quotient of $W_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$ by the null space $\mathcal{N}$ of $(\ ,\ )$ restricted to $W_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$. The representation of $B_{1,\ldots,1}$ on $\{W_n(V_{\alpha(n_1)} \otimes \ldots \otimes V_{\alpha(n_s)})\}_{\alpha\in S_s}$ reduces to a well-defined representation on $\{F_n(V_{\alpha(n_1)} \otimes \ldots \otimes V_{\alpha(n_s)})\}_{\alpha\in S_s}$.*

The subquotient $F_n(V_{n_1} \otimes \ldots \otimes V_{n_s})$ is called the *fusion rule subquotient* of $V_{n_1} \otimes \ldots \otimes V_{n_s}$ with weight $n-1$. It can be characterized more explicitly.

Proposition 3.20 *Let $p-1 \geq n_1, n_2, n \geq 1$. Then*

$$\begin{aligned} N^n_{n_1 n_2} &= \dim F_n(V_{n_1} \otimes V_{n_2}) \\ &= \begin{cases} 1 & \text{if} |n_1 - n_2| + 1 \leq n \leq \min(n_1 + n_2 - 1, 2p - n_1 - n_2 - 1), \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{170}$$

Thus, if $N^n_{n_1 n_2} = 1$, there is a singular vector in $V_{n_1} \otimes V_{n_2}$ of weight $n-1$, which is not in the null space of $(\ ,\ )$. Correspondingly we have a homomorphism

$$C^n_{n_1,n_2} : V_n \to V_{n_1} \otimes V_{n_2}. \tag{171}$$

Suppose in the following that $p-1 \geq n_1, \ldots, n_s, n \geq 1$. Introduce the *path space* $P^n_{n_1}, \ldots, n_s$ as the space of complex linear combinations of sequences $(m_1, \ldots, m_{s-2})$ of integers in $[1, p-1]$ such that $N^n_{n_1,m_1} = N^{m_{i-1}}_{n_i m_i} = N^{m_{s-2}}_{n_{s-1} n_s} = 1$, $(2 \leq i \leq s-2)$.

Proposition 3.21 *The homomorphism*

$$\begin{aligned} P^n_{n_1,\ldots,n_s} &\to W_n(V_{n_1} \otimes \ldots \otimes V_{n_s}), \\ (m_1, \ldots, m_{s-2}) &\mapsto (1 \otimes \ldots \otimes 1 \otimes C^{m_{s-1}}_{n_{s-1} n_s}) \ldots (1 \otimes C^{m_1}_{n_2 m_2}) C^n_{n_1 m_1} v_n, \end{aligned} \tag{172}$$

composed with the canonical projection $W_n \to F_n$, gives an isomorphism

$$P^n_{n_1,\ldots,n_s} \xrightarrow{\sim} F_n(V_{n_1} \otimes \ldots \otimes V_{n_s}). \tag{173}$$

The proofs of the last two propositions can be extracted from [FK93], noticing that since the vectors of the form $\xi_1 \otimes \ldots \otimes \xi_s$ with some $\xi_j \in SV_{n_j}$ are in the null space of $(\, , \,)$, we can replace everywhere V_n by the irreducible quotient V_n/SV_n.

4 Fock Space Representations of $A_1^{(1)}$ and Topological Representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$

We apply the topological representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ to the Fock space representations of the untwisted affine Kac-Moody algebra $A_1^{(1)}$. We show how singular vectors in quantum group Verma modules determine Fock space representations of BRST operators, primary fields, and conformal blocks.

4.1 Fock Space Representations of $A_1^{(1)}$

Let us recall the basic facts about Fock space representations of $A_1^{(1)}$. The algebra $A_1^{(1)}$ is generated by J_n^a, $a \in \{+,-,0\}$ and $n \in \mathbb{Z}$, and a central element K with relations

$$\begin{aligned} [J_n^0, J_m^{\pm}] = \pm J_{n+m}^{\pm}, \quad [J_n^0, J_m^0] = \frac{n}{2}\delta_{n,-m}K, \\ [J_n^+, J_m^-] = 2J_{n+m}^0 + n\delta_{n,-m}K, \quad [K, J_n^a] = 0, \end{aligned} \tag{174}$$

supplemented by a derivation d satisfying

$$[d, J_n^a] = -nJ_n^a, \quad [d, K] = 0. \tag{175}$$

Following [Wa86, BF90, FF89], we then require a bosonic $\omega - \omega^+$ system together with a free bosonic field ϕ. Let Γ^1 be the algebra generated by ω_n and ω_n^+, $n \in \mathbb{Z}$, with relations

$$[\omega_n, \omega_m^+] = \delta_{n,-m}, \quad [\omega_n, \omega_m] = [\omega^+, \omega_m^+] = 0. \tag{176}$$

Furthermore, we require a Fock space F^1 with vacuum vector v^1 such that, for $0 \leq n$,

$$\omega_{n+1}v^1 = \omega_n^+ v^1 = 0. \tag{177}$$

Let Γ^2 be the algebra generated by a_n, $n \in \mathbf{Z}$, with relations

$$[a_n, a_m] = n\delta_{n,-m}, \tag{178}$$

and $F_{J,k}^2$ the Fock space of charge J/γ, where $\gamma = \sqrt{\frac{k+2}{2}}$, generated from a vacuum vector $v_{J,k}^2$ satisfying, for $0 \leq n$,

$$a_n v_{J,k}^2 = \frac{J}{\gamma}\delta_{n,0}v_{J,k}^2. \tag{179}$$

Then define $\Gamma = \Gamma^1 \otimes \Gamma^2$ and $F_{J,k} = F^1 \otimes F^2_{J,k}$. This tensor product of Fock spaces comes equipped with the structure of a highest weight module over $A_1^{(1)}$ at level k through the following construction due to [Wa86]. Let us introduce field operators

$$\omega(z) = \sum_{n=-\infty}^{\infty} \omega_n z^{-n}, \quad \omega^+(z) = \sum_{n=-\infty}^{\infty} \omega_n^+ z^{-n-1}, \tag{180}$$

$$j(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n-1}, \tag{181}$$

and $\phi(z)$ such that $j(z) = i\partial\phi(z)$. In terms of these we can compose currents

$$J^+(z) = \omega^+(z) \otimes id, \tag{182}$$

$$J^0(z) =: \omega(z)\omega^+(z) : \otimes id + \gamma \, id \otimes j(z), \tag{183}$$

$$J^-(z) = -\Big(: \omega(z)^2\omega^+(z) : + k\,\partial\omega(z)\Big) \otimes id - 2\gamma\,\omega(z) \otimes j(z). \tag{184}$$

Then it can be shown that the coefficients of the developments

$$J^a(z) = \sum_{-\infty}^{\infty} J^a_n z^{-n-1} \tag{185}$$

satisfy the relations (174). The free field stress energy tensor

$$T(z) = - : \partial\omega(z)\omega^+(z) : \otimes id + \frac{1}{2} id \otimes \left(: j(z)^2 : - \frac{1}{\gamma}\partial j(z)\right) \tag{186}$$

agrees with the Sugawara form. Expanding, as usual,

$$T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2} \tag{187}$$

$F_{J,k}$ also becomes a highest weight module over the Virasoro algebra Vir with central charge $c = 3k/(k+2)$. The generators of $A_1^{(1)}$ have the explicit form

$$J_n^+ = \omega_n^+ \otimes id, \tag{188}$$

$$J_n^0 = \sum_{m=-\infty}^{\infty} : \omega_m \omega^+_{n-m} : \otimes id + \gamma \otimes a_n, \tag{189}$$

$$J_n^- = -\left(\sum_{m,l=-\infty}^{\infty} : \omega_m \omega_n \omega^+_{n-m-l} : - kn\,\omega_n \right) \otimes id - 2\gamma \sum_{m=-\infty}^{\infty} \omega_m \otimes a_{n-m}. \tag{190}$$

The generators with $n = 0$ generate an sl_2 subalgebra. $F_{J,k}$ is, in particular, a module over sl_2. The vacuum is also an sl_2 highest weight vector with

$$J_0^+ v_{J,k} = 0, \quad J_0^0 v_{J,k} = J v_{J,k}. \tag{191}$$

As an sl_2-module, $F_{J,k}$ thus contains a highest weight representation with spin J generated from the vacuum. This representation is irreducible since $(J_0^-)^{2J+1} = 0$. The basic objects in the following constructions are vertex operators, which we take to be defined as

$$V_{I,k}(z) =$$
$$\exp\left(\frac{I}{\gamma}b_0\right)\exp\left(\frac{I}{\gamma}\ln(z)a_0\right)\exp\left(\frac{I}{\gamma}\sum_{n=1}^{\infty}a_{-n}\frac{z^n}{n}\right)\exp\left(-\frac{I}{\gamma}\sum_{n=1}^{\infty}a_n\frac{z^{-n}}{n}\right). \tag{192}$$

Here $[a_n, b_0] = \delta_{n,0}$ and we have an operator from $F^2_{J,k}$ to $F^2_{J+I,k}$. To be precise, $V_{J,k}(z) =: \exp(i\frac{I}{\gamma}\phi(z))$: defines a bilinear form on $(F^2_{J+K,k})^* \otimes F^2_{J,k}$ (restricted dual). Finally let

$$V_{I,k}(z) = \tilde{V}_{I,k}(z)z^{I\gamma^{-1}a_0}, \tag{193}$$

splitting off the factor which depends on the zero mode of $j(z)$.

4.2 Intertwiners

As a first application of the topological representations, we construct operators mapping one Fock space to another which intertwine the representations of $A_1^{(1)}$ and, consequently, *Vir*. As a basic ingredience, introduce

$$V^-(z) = \omega^+(z) \otimes V_{-1,k}(z)\,, \tag{194}$$

the screening operator. $V^-(z)$ maps $F_{J,k}$ to $F_{J-1,k}$. Its operator product expansion with the stress-energy tensor (186) is given by

$$T(z)V^-(w) = \frac{1}{(z-w)^2}V^-(w) + \frac{1}{z-w}\frac{\partial}{\partial w}V^-(w) + O(1)\,, \tag{195}$$

where $O(1)$ sums up regular terms. It shows that V^- has conformal weight 1. The operator product expansions of V^- with the currents have the form

$$J^+(z)V^-(w) = O(1), \tag{196}$$
$$J^-(z)V^-(w) = 2\gamma^2\frac{\partial}{\partial w}\Big\{\frac{1}{z-w}id \otimes V_{-1,k}(w)\Big\}, \tag{197}$$
$$J^0(z)V^-(w) = O(1). \tag{198}$$

They follow from basic operator product expansions of the free field, diverging terms cancelling each other neatly. Due to the total derivative, $V^-(z)$ does not intertwine the action of $A_1^{(1)}$ on $F_{J,k}$ with that on $F_{J-1,k}$. To produce intertwiners we have to integrate products of screening operators. This gives

$$\begin{aligned} Q_C^R &= \int_C V^-(z_1)\dots V^-(z_R)\,dz_1\dots dz_R \\ &= \int_C :V^-(z_1)\dots V^-(z_R): \prod_{1\le i<j\le R}(z_i-z_j)^{\gamma^{-2}}\,dz_1\dots dz_R \\ &= \int_C :\tilde V^-(z_1)\dots \tilde V^-(z_R): \prod_{i=1}^R z_i^{-\gamma^{-1}a_0} \prod_{1\le i<j\le R}(z_i-z_j)^{\gamma^{-2}}\,dz_1\dots dz_R\,, \end{aligned} \tag{199}$$

a map from $F_{J,k}$ to $F_{J-R,k}$.

Taking matrix elements, these integrals give rise to topological representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ at $q=\exp(\frac{\pi i}{2\gamma^2})$, possibly a root of 1. Let $X_R(0)=(D\backslash\{0\})^R\backslash\cup_{1\le i\le j\le R}\{z_i=z_j\}/S_R$ be the configuration space of R indistinguishable screening charges in the disc $D=\{|z|\le 1\}$, punctured at 0. Fix a point p_- on the boundary of D, e.g., $p_-=-1$. Let L_R be the local system determined by the multi valued form on X_R, obtained by evaluating the integrand on Fock vacua. Note that different values of J yield different L_R.

Let $A_R(0)$ be the space of linear combinations of families Γ of non-intersecting loops in X_R based at p_- together with sections of $\Gamma^* L_R$ modulo equivalence relations reflecting the possibility to homotopically deform or reparametrize Γ. Let $\hat E: A_R(0)\to A_{R-1}(0)$, $\hat F: A_R(0)\to A_{R+1}(0)$, and $\hat K^2: A_R(0)\to A_R(0)$ be the topological operators introduced above. They satisfy the $U_q(\mathfrak{sl}_2(\mathbb{C}))$ relations. $\hat E$ is the combinatorial boundary operator composed with a map which identifies $(R-1)$-chains in X_R with $(R-1)$-chains in X_{R-1}. We have shown that

$$\hat K^2\hat E = q^2\hat E\hat K^2,\quad \hat K^2\hat F=q^{-2}\hat F\hat K^2,\quad [\hat E,\hat F]=\hat K^2-\hat K^{-2}, \tag{200}$$

endowing $\bigoplus_{R=0}^\infty A_R(0)$ with the structure of a module over the quantum group algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$. In the application we have in mind, $q^p=1$. In this case, $\hat E^p=0$ and $\hat K^{4p}=1$. This is proved by an explicit computation using a basis to describe $A_R(0,w)$ as a space. Families of loops which contain p homotopic deformations of a single loop represent null homologous cycles. To prove this one retracts the p loops to p points on a single loop path ordering the arguments. The prefactor from the path ordering vanishes. This can be taken into account by adding another relation in the definition of the space $A_R(0)$. This relation puts families of loops which contain p homotopic loops equivalent to the null family. As a consequence, then also $\hat F^p=0$. Using a generalized version of Poincare duality this could possibly be understood in terms of the kernel of the topological intersection pairing. For q a root of unity the conclusion is that we find a representation of the reduced quantum group algebra $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$, the algebra obtained from $U_q(\mathfrak{sl}_2(\mathbb{C}))$ dividing by the ideal generated by the central elements E^p, F^p, and $K^{4p}-1$. Let us only consider this case in the following. For a detailed account on $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ and its representation theory we recommend [FK93].

As a quantum group module, $\bigoplus_{R=0}^{p-1} A_R(0)$ is isomorphic to the $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ Verma module $V(n)$, $n = 2J+1$, generated from a singular vector $v_0(n)$ such that

$$Ev_0(n) = 0\,, \quad K^2 v_0(n) = q^{n-1} v_0(n)\,. \tag{201}$$

Let $\varphi : \bigoplus_{R=0}^{p-1} A_R(0) \to V(n)$ be the isomorphism. For $0 \leq R \leq p-1$, we define weight spaces $V(n)_R = \mathbb{C}F^R v_0(n)$. K^2 acts on $V(n)_R$ by multiplication with q^{n-1-2R}.

Restricting our attention to the most interesting case, let $2\gamma^2 = k+2 = p$ be a positive integer and let $n \in \mathbf{Z}$. The BRST construction of [BF90, FF89] then produces unitary integrable irreducible highest weight representations of $A_1^{(1)}$ with $1 \leq n \leq p-1$, $n = 2J+1$, on the homology. Here we work directly with the representations on the Fock spaces.

Theorem 4.1 *(I) Consider $Q^{(R)}$ as a map from $V(n)$ to the space of linear operators* $\mathrm{Hom}_{\mathbf{C}}(F_{J,k}, F_{J-R,k})$. *Then*

$$Q^{(R)} : \ker\left(E : V(n)_R \to V(n)_{R-1}\right) \to \mathrm{Hom}_{A_1^{(1)}}(F_{J,k}, F_{J-R}) \tag{202}$$

maps to the space of $A_1^{(1)}$ intertwiners. (II) Let $n = 2J+1$ and $\bar{n} = n\, mod$, $0 \leq \bar{n} \leq p-1$. For $\bar{n} = 0$,

$$\ker(E) = \mathbf{C}v_0(n). \tag{203}$$

For $1 \leq \bar{n} \leq p-1$,

$$\ker(E) = \mathbf{C}v_0(n) \oplus \mathbf{C}F^{\bar{n}} v_0(n). \tag{204}$$

(III) For $1 \leq \bar{n} \leq p-1$, the only non-vanishing intertwiner besides the identity is $Q_C^{(R)}$ with $R = \bar{n}$ and $C = \varphi^{-1}(F^{\bar{n}} v_0(n))$, unique up to a normalization constant.

Thus the structure of singular vectors in $U_q(\mathfrak{sl}_2(\mathbb{C}))$-Verma modules, in our case a root of 1, provides all data needed to decide wether intertwiners exist and, furthermore, gives explicit formulas for them.

Corollary 4.2 *(I) For $n = 2J+1$, let $F_n = F_{J,k}$. For $1 \leq \bar{n} \leq p-1$, the non-vanishing intertwiners form an infinite sequence*

$$\cdots \to F_{2p-\bar{n}} \stackrel{Q^{(p-\bar{n})}}{\to} F_{\bar{n}} \stackrel{Q^{(\bar{n})}}{\to} F_{-\bar{n}} \to \ldots \tag{205}$$

(II) This sequence is a complex.

The homology of this complex is isomorphic to the irreducible $A_1^{(1)}$ highest weight module of weight J and level k. Further intertwiners can be constructed out of cycles with np path ordered arguments, $n \in \{1, 2, \dots\}$. They correspond to powers of generators $F^p[p]!^{-1}$ in Lusztig's version of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. They can be used to construct complexes whose homology gives non-integrable $A_1^{(1)}$ modules [BMP90]. Here $[n] = q^n - q^{-n}$ and $[n]! = [1]\ldots[n]$.

4.3 Chiral Primary Fields

The second application of topological representations concerns the construction of chiral primary fields $\phi^K_{J,I}(w) : F_{J,k} \otimes \tilde{U}_I \to F_{K,k}$. $\tilde{U}_I$ is a representation of $A_1^{(1)}$ at level 0 to be defined below. Consider operators

$$\psi_{I,\mu}(z) =: \omega(z)^{I-\mu} : \otimes V_{I,k}(z), \tag{206}$$

mapping $F_{J,k}$ to $F_{J+I,k}$. The operator product expansion with $T(z)$

$$T(z)\psi_{I,\mu}(w) = \frac{I(I+1)/(2\gamma^2)}{(z-w)^2}\psi_{I,\mu}(w) + \frac{1}{z-w}\partial\psi_{I,\mu}(w) + O(1) \tag{207}$$

shows that $\psi_{I,\mu}(z)$ has conformal weight $h_I = I(I+1)/(2\gamma^2)$. The operator product expansion with the current $J^a(z)$ has the form

$$J^a(z)\psi_{I,k}(w) = \frac{1}{z-w}\sum_{\nu=-I}^{I}\psi_{I,\nu}(w)D^{(I)}_{\nu,\mu}(J^a) + O(1). \tag{208}$$

$D^{(I)}(J^a)$ is a spin I representation matrix of sl_2. For $u \in U_I$, the representation space of the representation $D^{(I)}$, define

$$\psi_I(u;w) = \sum_{\mu=-I}^{I}\psi_{I,\mu}(w)u_\mu. \tag{209}$$

The operator product expansion implies that

$$[J^a_n, \psi_I(u;w)] = \psi\left(D^{(I)}(J^a)u;w\right)w^n. \tag{210}$$

With these preparations in mind let us consider the operators

$$\phi_{I,\mu}(w)^{(R)}_C = \int_C \psi_{I,\mu}(w)V^-(z_1)\dots V^-(z_R)dz_1\dots dz_R \tag{211}$$

from $F_{J,k}$ to $F_{K,k}$ with $K = J+I-R$. Using Wick's theorem

$$\begin{aligned} :\omega(w)^{I-\mu}:\prod_{i=1}^{R}\omega^+(z_i) &= \sum_{\nu=0}^{min\{I-\mu,R\}} C(I-\mu,\nu) \sum_{\substack{\mathcal{I}\subset\{1,\dots,R\}\\ |\mathcal{I}|=\nu}} \\ &:\omega(w)^{I-R-\mu+\nu}\prod_{i\in\mathcal{I}}\omega^+(z_i): \prod_{i\in\{1,\dots,R\}\setminus\mathcal{I}}(z-w)^{-1} \end{aligned} \tag{212}$$

we normal order the integrand. $C(I-\mu,\nu)$ is a combinatorial factor counting the number of contractions. The result is

$$\phi_{I,\mu}(w)^{(R)}_C = \int_C dz_1\dots dz_R \sum_{\nu=0}^{min\{I-\mu,R\}} C(I-\mu,\nu) \sum_{\substack{\mathcal{I}\subset\{1,\dots,R\}\\ |\mathcal{I}|=\nu}}$$
$$:\omega(w)^{I-R-\mu+\nu}\prod_{i\in\mathcal{I}}\omega^+(z_i):\otimes:\tilde{V}_{I,k}(w)\tilde{V}_{-1,k}(z_1)\dots\tilde{V}_{-1,k}(z_R):w^{I\gamma^{-1}a_0}$$
$$\prod_{i=1}^{R} z^{-\gamma^{-1}a_0}\prod_{i\in\{1,\dots,R\}\setminus\mathcal{I}}(w-z_i)^{-1}\prod_{i=1}^{R}(w-z_i)^{-I\gamma^{-2}}\prod_{1\le i<j\le R}(z_i-z_j)^{\gamma^{-2}}. \tag{213}$$

The chain C is taken to be an element of $A_R(0,w)$. The local system $L_R(0,w)$ involved in the definition of $A_R(0,w)$ is determined by the monodromy of the multi valued R-form obtained by evaluating the integrand on Fock vacua. As a $U_q(\mathfrak{sl}_2(\mathbb{C}))$-module, the space $\bigoplus_{R=0}^{2p-2}A_R(0,w)$ has been shown to be isomorphic to the tensor product of Verma modules $V(n)\otimes V(m)$. Let $(V(n)\otimes V(m))_R$ be the weight space on which K^2 acts by multiplication with $q^{n+m-2-2R}$.

Lemma 4.3 *Consider $\phi_I(w)^{(R)}$ as a map from $V(n)\otimes V(m)$ to the space of bilinear operators* $\mathrm{Hom}_{\mathbb{C}}(F_{J,k}\otimes U_I, F_{K,k})$ *with $K=J+I-R$. Here U_I is the sl_2 representation space. For $\varphi(C)\in\ker\left(E:(V(n)\otimes V(m))_R\to(V(n)\otimes V(m))_{R-1}\right)$, we find the commutation relation*

$$[J^a_n,\phi_I(u;w)^{(R)}_C]=\phi_I\left(D^{(I)}(J^a)u;w\right)^{(R)}_C w^n. \tag{214}$$

For C a chain, which represents an absolute cycle in homology, we obtain a chiral primary field. Its construction goes as follows. Expanding

$$\phi_I(u;w)^{(R)}_C=\sum_{n=-\infty}^{\infty}\phi_{I,n}(u)^{(R)}_C w^{-n+h_K-h_I-h_J} \tag{215}$$

it follows that

$$[J^a_m,\phi_{I,n}(u)^{(R)}_C]=\phi_{I,m+n}\left(D^{(I)}(J^a)u\right)^{(R)}_C. \tag{216}$$

Let

$$\tilde{U}_I=U_I\otimes w^{-h_K+h_I+h_J}\mathbb{C}[w,w^{-1}] \tag{217}$$

be the $A_1^{(1)}$-module with J^a_n acting as $D^{(I)}(J^a)\otimes w^n$, d as $w\partial/(\partial w)$, and K as zero. Then

$$\xi\otimes\left(u\otimes w^{n-h_K+h_I+h_J}\right)\to\phi_{I,n}(u)^{(R)}_C\xi \tag{218}$$

defines a map from $F_{J,k} \otimes \tilde{U}_I$ to $F_{K,k}$. To be precise, for $u \in U_I$,

$$\phi_I(u;w)_C^{(R)} \in \mathrm{Hom}_{\mathbf{C}}(F_{J,k}, F_{K,k}) \otimes w^{-h_K+h_J+h_I}\mathbb{C}(w, w^{-1}), \tag{219}$$

is a formal power series in w whose coefficients are linear operators from the Fock space $F_{J,k}$ to the Fock space $F_{K,k}$. For $u \otimes \epsilon \in \tilde{U}_I$, we obtain a well defined operator through

$$\phi_I(u \otimes \epsilon)_C^{(R)} = \frac{1}{2\pi} \oint \phi_I(u;w)_C^{(R)} \epsilon(w) dw. \tag{220}$$

Here C is moved with w through the Gauss-Manin connection on $\mathbb{C} \setminus \{0\}$. That is, we associate operators to Laurent polynomials ϵ. This map is an $A_1^{(1)}$ homomorphism.

Theorem 4.4 *Consider $\phi_I^{(R)}$ as a map from $(V(n) \otimes V(m))_R$ to the space of bilinear operators $\mathrm{Hom}_{\mathbf{C}}(F_{J,k} \otimes \tilde{U}_I, F_{K,k})$ with $K = J + I - R$. Then*

$$\phi_I^{(R)} : \ker\Big(E : (V(n) \otimes V(m))_R \to (V(n) \otimes V(m))_{R-1}\Big) \to \mathrm{Hom}_{A_1^{(1)}}(F_{J,k} \otimes \tilde{U}_I, F_{K,k}) \tag{221}$$

maps to the space of $A_1^{(1)}$-intertviners.

The general decomposition of the tensor product of Verma modules $V(n) \otimes V(m)$, which contains among other things the structure of singular vectors, has been carried out in [FK93]. For the sake of brevity, it cannot be reproduced here. Let us only mention the following partial results.

Proposition 4.5 *Let $n = 2J + 1$, $m = 2I + 1$, $1 \leq n, m \leq p - 1$, and $0 \leq R < n, m$. Then*

$$\ker\Big(E : (V(n) \otimes V(m))_R \to (V(n) \otimes V(m))_{R-1}\Big) = \mathbb{C}\left(\sum_{l=0}^{R} x_l F^{R-l} v_0(n) \otimes F^l v_0(m)\right) \tag{222}$$

with x_l, $0 \leq l \leq R$, determined by $x_0 = 1$ and the recursion relation

$$[R - l]_q [n - R + l]_q x_l + q^{n-1-2(R-l-1)} [l + 1]_q [n - l - 1]_q x_{l+1} = 0, \tag{223}$$

where $[x]_q = q^x - q^{-x}$.

The intertwiner corresponding to this cycle is denoted by $\phi_{J,I}^K(u \otimes \epsilon)$ with $K = J + I - R$. It is unique up to a normalization constant which can be fixed as follows using the three point function

$$\langle v_{K,k,\lambda}, \phi_{J,I}^K(e_\mu; w) v_{J,k,\nu}\rangle = w^{h_K - h_I - h_J} N_{J,I}^K \begin{bmatrix} K & J & I \\ \lambda & \nu & \mu \end{bmatrix}, \tag{224}$$

involving a (classical) sl_2 Clebsch-Gordan coefficient and the fusion coefficient $N_{J,I}^K$. The three point function is conveniently normalized such that the fusion coefficient takes the values 0 or 1. $\phi_{J,I}^K(u \otimes \epsilon)$ is the chiral primary field in the Fock space representation.

Proposition 4.6 *Let $n = 2J+1$, $m = 2I+1$, and $l = 2K+1$ be such that $1 \leq n, m, l \leq p-1$. Define a triple $\begin{pmatrix} K \\ J\, I \end{pmatrix}$ to be admissible if $|I-J| \leq K \leq I+J$, $I+J+K \in \mathbf{Z}$, and $I+J+K \leq p-2$. Then the chiral primary field has the following properties: (I) For $\begin{pmatrix} K \\ J\, I \end{pmatrix}$ not admissible, $N_{J,I}^K = 0$. (II) For $\begin{pmatrix} K \\ J\, I \end{pmatrix}$ admissible, $N_{J,I}^K = 1$.*

The quantum group data encoded in the fusion rules of $\phi_{J,I}^K(w)$ is here inherited from the representation $V(n) \otimes V(m)$. The most important property of $\phi_{J,I}^K(w)$ is BRST invariance. It is a necessary condition to proceed through the BRST construction. Let us mention that BRST invariance also has a cohomological meaning on the quantum group level.

Conformal blocks are vacuum expectation values of products of chiral primary fields. A product

$$\phi_{J_s,I_s}^{J_s+1}(u_s \otimes \epsilon_s) \dots \phi_{J_2,I_2}^{J_3}(u_2 \otimes \epsilon_2) \phi_{J_1,I_1}^{J_2}(u_1 \otimes \epsilon_1) \tag{225}$$

defines an element of

$$\mathrm{Hom}_{A_1^{(1)}}(F_{J_1,k} \otimes \tilde{U}_{I_1} \otimes \cdots \otimes \tilde{U}_{I_s}, F_{J_{s+1},k}). \tag{226}$$

Using (213), such operators can be expressed in terms of free fields. Products of expressions of the form (213) are well defined (as absolutely convergent matrix product) and give rise to elements of (226) when integrated as in (220), provided the time ordering condition is satisfied. This condition states that $: U_1(z_1) \dots U_p(z_p) :\times: U_{p+1}(z_1') \dots U_{p+q}(z_q') :$ is well defined whenever $|z_i| \geq |z_j'|$ for all i, j. Here U_i stands for ω, ω^+ or a vertex operator. This condition gives the cycle C of integration for the expression (45) as a product of nested cycles $C_{J_i,I_i}^{J_{i+1}}$ used to integrate the factors. The resulting cycle is identified with a singular vector of $V(m_0) \otimes \cdots \otimes V(m_s)$, with $m_i = 2J_i + 1$, $I_0 = J_1$, in the topological representation space. Algebraically, the expression of C in terms of $C_{J_i,I_i}^{J_{i+1}}$ is understood as follows: Identify $C_{J_i,I_i}^{J_{i+1}}$ with an element of

$$\ker|_{(V(n_i) \otimes \dots V(m_i))_{R_i}} \cong \mathrm{Hom}_{U_q(sl_2)}(V(n_{i+1}), V(n_i) \otimes V(m_i)) \tag{227}$$

where $m_i = 2J_i + 1$ and $R_i = I_i + J_i - J_{i+1}$. Then

$$\begin{aligned} C &= \left(C_{J_1,I_1}^{J_2} \otimes 1 \otimes \cdots \otimes 1\right) \dots \left(C_{J_{s-1},I_{s-1}}^{J_s} \otimes 1\right) C_{J_s,I_s}^{J_{s+1}} \\ &\in \mathrm{Hom}_{U_q(sl_2)}(V(n_{s+1}), V(m_0) \otimes V(m_1) \otimes \cdots \otimes V(m_s)) \\ &= \ker E|_{(\bigotimes V(m_i))_{\sum R_i}}. \end{aligned} \tag{228}$$

These quantum group intertwiners and path spaces have been investigated in [FK93].

5 Generalized Hypergeometric Functions on the Torus and the Adjoint Representation of $U_q\left(\mathfrak{sl}_2(\mathbb{C})\right)$

We study the homology groups with coefficient in local systems arising in the free field representation of minimal models of conformal field theory on an elliptic curve with punctures. We define an action of the quantum enveloping algebra $U_q\left(\mathfrak{sl}_2(\mathbb{C})\right)$ on a space of relative cycles, extending the results obtained previously for the sphere. Absolute cycles are identified with singular vectors. In the case of one puncture, we find that the resulting topological representation is essentially the adjoint representation.

5.1 Introduction

The studies [La90, FW91, SV90] indicate that there exists a dictionary between homology of certain configuration spaces with coefficients in local systems and representation theory of quantum enveloping algebras [D86]. The examples of local systems providing such connections come from integral representation of conformal blocks of conformal field theory [BPZ84, DF84, GN84, ZF86, M90, CF87, FGK91]. The idea is that (in some sense) the charges generating (half of) the quantum group symmetry in the free field representation in conformal field theory are given by integrals over screening operators [PS90, BMP90, GP89, GS90]. In the previous chapter [FW91], we have shown the existence of an action of $U_q\left(\mathfrak{sl}_2(\mathbb{C})\right)$ on certain relative locally finite homology groups on configuration spaces on the sphere. In this case, the local system is given by the integrand of the free field representation of conformal blocks of the SU(2) WZW models or minimal models.

In this chapter, we consider the situation of the torus, for which one knows explicit integral representations [F89, BF90, FG92]. We restrict our attention to the case of minimal models, which is the simplest. The main difference is that the local system is not given by a line bundle as in the case of the sphere, but rather a vector bundle. From the point of view of free fields, this follows from the fact that the space of free field conformal blocks on the torus is higher dimensional.

We find again an action of the quantum enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$ on relative cycles, in such a way that absolute cycles are identified with singular vectors, as in the case of the sphere. The resulting representation is a tensor product of Verma modules with $U_q\left(\mathfrak{sl}_2(\mathbb{C})\right)$ with the adjoint action. We hope that this work will lead to a clearer understanding of the role of quantum groups for higher genus Riemann surfaces.

5.2 Generalized Hypergeometric Functions on the Torus

In the free field representation of minimal models with central charge $c = 1 - 6(p' - p)^2/(pp')$ on the torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ one is led to consider integrals of the form

$$G_C(z_1, \ldots, z_s|\tau) =$$
$$\sum_{\mu=0}^{2p'p-1} \lambda_\mu \int_{C_\mu} \mathrm{d}z_{s+1} \wedge \cdots \wedge \mathrm{d}z_{s+r+r'}\ \Delta_\mu(W|\tau) \prod_{1\le i<j\le s+r+r'} E(z_i, z_j|\tau)^{\alpha_i\alpha_j}, \tag{229}$$

where

$$\alpha_i = \begin{cases} \alpha_{n_i' n_i}, & 1 \le i \le s, \\ \alpha_{1,-1}, & s+1 \le i \le s+r, \\ \alpha_{-1,1}, & s+r+1 \le i \le s+r+r', \end{cases} \tag{230}$$

$$\alpha_{m',m} = \frac{(1-m)p' - (1-m')p}{\sqrt{2p'p}}, \tag{231}$$

parametrize the exponents, $\alpha_\pm = \alpha_{\pm1,\mp1}$ belonging to integrated screening variables, and

$$\Delta_\mu(W|\tau) = \frac{1}{\eta(\tau)} \exp\left\{\frac{\pi i}{2p'p}\mu(2W + \tau\mu)\right\}\ \theta_3(W + \tau\mu|2p'p\tau), \tag{232}$$

$$W = \sqrt{2p'p} \sum_{i=1}^{s+r+r'} \alpha_i z_i, \tag{233}$$

$$E(z_i, z_j|\tau) = 2\pi i \frac{\theta_1(z_i - z_j|\tau)}{\theta_1'(0|\tau)}, \tag{234}$$

with

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n}), \tag{235}$$

$$\theta_3(z|\tau) = \sum_{n=-\infty}^{\infty} e^{2\pi i z n - \pi i \tau n^2}, \tag{236}$$

$$\theta_1(z|\tau) = -\sum_{n=-\infty}^{\infty} e^{2\pi i(z+\frac{1}{2})(n+\frac{1}{2}) - \pi i\tau(n+\frac{1}{2})^2}, \tag{237}$$

the Dedekind eta function and the Jacobi theta functions. The values of the exponents are constrained to satisfy

$$\sum_{i=1}^{s}(1 - n_i) + 2r = xp, \qquad \sum_{i=1}^{s}(1 - n_i') + 2r' = xp', \tag{238}$$

for some integer x, reflecting charge conservation. In the following we restrict our attention to the case when $r' = 0$ and $n_1' = \cdots = n_s' = 1$, admitting α_+ screening charges only. That is, we assume that

$$\alpha_i = \begin{cases} \frac{(1-n_i)p'}{\sqrt{2p'p}}, & 1 \le i \le s, \\ \frac{2p'}{\sqrt{2p'p}} & s+1 \le i \le s+r, \end{cases} \tag{239}$$

with the neutrality condition

$$\sum_{i=1}^{s+r} \alpha_i = 0. \tag{240}$$

When studying integrals of this form, one is entering the following kind of problems. To begin with, assume that $\alpha_1 = \cdots = \alpha_{n_1}$, $\alpha_{n_1+1} = \cdots = \alpha_{n_1+n_2}, \ldots, \alpha_{n_1+\cdots+n_{k-1}+1} = \cdots = \alpha_{n_1+\cdots+n_k}$, for some k and $(n_1, \ldots, n_k) \in \{1,\ 2, \ldots\}^k$, such that $n_1+\cdots+n_{k-1} = s$ and $n_k = r$. Let $\Sigma = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ be the torus with modular parameter τ. Then we have a vector of multivalued forms with components

$$\omega_\mu(z_1, \ldots, z_s|z_{s+1}, \ldots, z_{s+r}|\tau) = \\ \Delta_\mu(W|\tau) \times \prod_{1\le i<j\le s+r} E(z_i, z_j|\tau)^{\alpha_i\alpha_j}\, \mathrm{d}z_{s+1} \wedge \cdots \wedge \mathrm{d}z_{s+r}, \tag{241}$$

on the configuration space

$$X_n = (\Sigma^{s+r} \setminus \cup_{i<j}\{z_i = z_j\})/(S_{n_1} \times \cdots \times S_{n_k}). \tag{242}$$

Since $\Delta_{\mu+2p'p}(W|\tau) = \Delta_\mu(W|\tau)$, we can restrict μ to the range $0 \le \mu \le 2p'p-1$. Other properties of $\Delta_\mu(W|\tau)$ are summarized in Appendix A.

We will often use the identification $\mu \mapsto (m, m')$, $\mu = m'p - mp'$, of $\mathbb{Z}/(2pp'\mathbb{Z})$ with $\mathbb{Z}^2/\Lambda$, where Λ is the lattice generated by (p, p') and $(2p, 0)$.

Let $n' = (n_1, \ldots, n_{k-1})$ and

$$X_{n'} = (\Sigma^s \setminus \cup_{i<j}\{z_i = z_j\})/(S_{n_1} \times \cdots \times S_{n_{k-1}}). \tag{243}$$

The projection $p : X_n \to X_{n'}$ on the first n variables is a fibration with fibers

$$X_r(z_1, \cdots, z_s) = p^{-1}(z_1, \ldots, z_s). \tag{244}$$

These are configuration spaces of r indistinguishable particles on the punctured torus $\Sigma \setminus \{z_1, \ldots, z_s\}$. Fix τ and $(z_1, \ldots, z_s) \in X_{n'}$ to obtain a vector (241) of multi-valued r-forms on (244). The positions $z_1, \ldots, z_s$, which are presently kept fixed, should not be confused with the positions of the screening charges $z_{s+1}, \ldots, z_{s+r}$. Let us suppress the dependence on the former and the modular parameter in our notation. The r-forms (241) are multivalued on X_r, single valued on the universal covering space $\tilde{X}_r(*)$ with base point $*$, and define a $(2p'p)$-dimensional representation ρ of $\pi_1(X_r, *)$ through

$$\phi_\sigma^*(\omega_\mu) = \sum_{\nu=0}^{2p'p-1} \omega_\nu\ \rho_{\nu\mu}(\sigma), \qquad \sigma \in \pi_1(X_r, *), \tag{245}$$

with $\phi_\sigma(x) = x\sigma$ the right action of the fundamental group on the universal covering space. The representation matrices can be computed by analytic continuation. Let c_μ, $\mu = 0, \ldots, 2p'p-1$, be singular r-chains in the universal covering space. The equivalence relation $\phi_\sigma(c_\mu) \sim \sum_\nu \rho_{\mu\nu} c_\nu$ is compatible with the pairing

$$< \omega, c >= \sum_{\mu=0}^{2p'p-1} \int_{c_\mu} \omega_\mu. \tag{246}$$

In other words we can view c as a singular r-chain with coefficients in the space of local horizontal sections of the vector bundle of rank $2p'p$

$$L_r = \tilde{X}_r(*) \times \mathbb{C}^{2p'p} / \sim, \qquad (x\sigma, v) \sim (x, \rho(\sigma)v). \tag{247}$$

We thus need to examine the singular homology group $H_r^{lf}(X_r, L_r)$ with coefficients in the local system associated to the representation ρ of $\pi_1(X_r, *)$. As a support condition we will require that the chains are locally finite (lf), [DM86, ES52, BM60], possibly infinite linear combinations of simplices, on $X_r^\epsilon = \{(w_1, \ldots, w_r) \in X_r | \quad |w_i - z_j| \geq \epsilon\}$. Elements of $H_r^{lf}(X_r, L_r)$ produce, when paired with (241) a generalized hypergeometric function on the torus (provided the integral is convergent).

5.3 Local Systems over Configuration Spaces on the Torus

5.3.1 Braid Group on the Torus

Let $T = S^1 \times S^1$, $n \in \{1, 2, \ldots\}$, and define

$$\mathcal{C}_n(T) = (T^n \setminus \cup_{i<j}\{x_i = x_j\})/S_n, \tag{248}$$

the configuration space of n indistinguishable particles on the torus. Here S_n denotes the symmetric group acting as $\pi(x_1, \ldots, x_n) = (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$. Let

$$B_n(T, *) = \pi_1(\mathcal{C}_n(T), *), \tag{249}$$

be the braid group on the torus. A convenient choice of base point is

$$* = [(\frac{1}{N}, \frac{1}{N}), \ldots (\frac{n}{N}, \frac{n}{N})] \tag{250}$$

for some $N > n$. For $1 \leq i \leq n$, define elements $\alpha_i, \beta_i \in B_n(T, *)$ as represented by the paths $[0, 1] \to \mathcal{C}_n(T)$,

$$\alpha_i(t) = [\ldots, (\frac{i}{N} + t, \frac{i}{N}), \ldots], \quad \beta_i(t) = [\ldots, (\frac{i}{N}, \frac{i}{N} + t), \ldots], \tag{251}$$

moving the particle in position i along an $A-$ and $B-$cycle, respectively. For $1 \leq i \leq n-1$, define elements $\sigma_i \in B_n(T,*)$ as represented by

$$\begin{aligned}
\sigma_i &= [\ldots, x_i(t), x_{i+1}(t), \ldots], \\
x_i(t) &= \frac{1}{2N}(2i+1, 2i+1) - \frac{1}{\sqrt{2}N}\Delta(t), \\
x_{i+1}(t) &= \frac{1}{2N}(2i+1, 2i+1) + \frac{1}{\sqrt{2}N}\Delta(t), \\
\Delta(t) &= (\cos(2\pi t - \frac{\pi}{4}), \sin(2\pi t - \frac{\pi}{4})),
\end{aligned} \tag{252}$$

implementing a counter clockwise exchange of the particle in position i with that in position $i+1$. It is convenient also to introduce the abbreviations

$$\alpha = \sigma_1 \cdots \sigma_{n-1}\alpha_n, \qquad \beta = \beta_n, \tag{253}$$

in terms of which

$$\alpha_i = \sigma_{i-1}^{-1} \cdots \sigma_1^{-1}\alpha\sigma_{n-1} \cdots \sigma_i, \qquad \beta_i = \sigma_i^{-1} \cdots \sigma_{n-1}^{-1}\beta\sigma_{n-1}^{-1} \cdots \sigma_i^{-1}. \tag{254}$$

The group $B_n(T,*)$ is generated by α, β, and σ_i, $1 \leq i \leq n-1$. A detailed investigation of $B_n(T,*)$ can be found in [Bi69, S70, Cr93].

5.3.2 Coloured Braid Groupoid and Local System

Let $n = (n_1, \ldots, n_k) \in \{1, 2, \ldots\}^k$, $|n| = n_1 + \cdots + n_k$, and define

$$\mathcal{C}_n(T) = (T^{|n|} \setminus \cup_{i<j}\{x_i = x_j\})/(S_{n_1} \times \cdots \times S_{n_k}), \tag{255}$$

the configuration space of particles with colours $\{1, \ldots, k\}$, identically coloured particles being indistinguishable. Let $* \in \mathcal{C}_n(T)$ be the base point (250). The orbit of $*$ under the action of $S_{|n|}$ can be identified with the right coset space $I_n = S_n/S_{n_1} \times \cdots \times S_{n_k}$. An element $[\pi] \in I_n$ can in turn be described by a colour map $\bar{\pi} : \{1, \ldots, |n|\} \to \{1, \ldots, k\}$. For $[\pi]$, $[\sigma] \in I_n$, let $B_{|n|}^{\bar{\sigma},\bar{\pi}}(T,*)$ be the space of paths starting in $\pi*$ and ending in $\sigma*$, up to homotopies preserving the end points. Define

$$B_n(T,*) = \cup_{[\pi],[\sigma] \in I_n} B_{|n|}^{\bar{\sigma},\bar{\pi}}(T,*), \tag{256}$$

the coloured braid groupoid on the torus. Multiplication is composition of paths. In particular, $B_{|n|}^{\bar{\mathrm{id}},\bar{\mathrm{id}}}(T,*) = \pi_1(\mathcal{C}_n(T),*)$. The coloured braid groupoid $B_n(T,*)$ can be described in terms of the braid group $B_n(T,*)$. Let $\psi : B_n(T,*) \to S_{|n|}$ be the canonical homomorphism. There exist one-to-one maps

$$\phi_{\overline{\sigma},\overline{\pi}} : \{g \in B_{|n|}(T,*) | [\sigma] = [\psi(g)\pi]\} \to B_n^{\bar{\sigma},\bar{\pi}}(T,*), \tag{257}$$

having the property

$$\phi_{\overline{\nu},\overline{\sigma}}(g')\phi_{\overline{\sigma},\overline{\pi}}(g) = \phi_{\overline{\nu},\overline{\sigma}}(g'g). \tag{258}$$

Using this, we can write down generators for $B_n(T,*)$. The generators are

$$\alpha^{[\pi]} = \phi_{\bar{\epsilon\pi},\bar{\pi}}(\alpha), \qquad \beta^{[\pi]} = \phi_{\bar{\pi},\bar{\pi}}(\beta),$$
$$\sigma_i^{[\pi]} = \phi_{\bar{\tau_i\pi},\bar{\pi}}(\sigma_i), \qquad 1 \le i \le |n|-1. \tag{259}$$

ϵ is the cyclic permutation $\epsilon(i) = i+1 \mod |n|$ and τ_i is the ith transposition. A representation of $B_n(T,*)$ on a family of finite dimensional vector spaces $V_{\bar{\pi}}$ indexed by $[\pi] \in I_n$, is a family of maps

$$\rho_{\bar{\sigma},\bar{\pi}} : B_n^{\bar{\sigma},\bar{\pi}}(T,*) \to \mathrm{Hom}^*(V_{\bar{\pi}}, V_{\bar{\sigma}}), \tag{260}$$

from $B_n(T,*)$ to $\cup_{[\pi],[\sigma]\in I_n}\mathrm{Hom}^*(V_{\bar{\pi}}, V_{\bar{\sigma}})$, the groupoid of invertible linear mappings between the vector spaces $V_{\bar{\pi}}$, satisfying the representation property[3]

$$\rho_{\bar{\nu},\bar{\sigma}}(g')\rho_{\bar{\sigma},\bar{\pi}}(g) = \rho_{\bar{\nu},\bar{\sigma}}(g'g). \tag{261}$$

The dimension of a representation ρ is $d = \dim(V_{\bar{\pi}})$. A d-dimensional representation ρ of $B_n(T,*)$ defines a flat rank d vector bundle over $\mathcal{C}_n(T)$ with distinguished trivializations over the points $\pi*$, $[\pi] \in I_n$.

The representation ρ, restricted to $\pi_1(\mathcal{C}_n(T),*)$, gives a flat vector bundle $\tilde{\mathcal{C}}_n(T) \times_{\pi_1} V_{\bar{\mathrm{id}}}$. It comes with an identification of the fiber over $*$ with $V_{\bar{\mathrm{id}}}$. The identification of the fiber over $\pi*$ with $V_{\bar{\pi}}$ is uniquely given by the condition that the parallel transport along any path η from $*$ to $\pi*$ is $\rho_{\bar{\mathrm{id}},\bar{\pi}}(\eta)$

To do explicit calculations, it is convenient to introduce local trivializations of L_r. Define cells labeled by elements of I_n:

$$\mathcal{C}_{n,\pi}(T) = \{[x_1,\ldots,x_{|n|}] \in \mathcal{C}_n(T) | 0 < x^1_{\pi(1)} < \cdots < x^1_{\pi(|n|)} < 1, 0 < x_i^2 < 1\}. \tag{262}$$

The union of the closures of these cells is $\mathcal{C}_n(T)$, and every cell contains precisely one of the points in the $S_{|n|}$ orbit of $*$. Since cells are contractible, we have an identification of the restriction of L_r to $\mathcal{C}_{n,\bar{\pi}}(T)$ with the trivial flat bundle $\mathcal{C}_{n,\bar{\pi}}(T) \times V_{\bar{\pi}}$. This trivialization will be used often below.

5.3.3 Torus with punctures

Let $n' = (n_1,\ldots,n_{k-1})$, $s = |n'|$, and $r = n_k$. The projection $p : \mathcal{C}_n(T) \to \mathcal{C}_{n'}(T)$ on the first s variables is a fibration with fibres

$$\mathcal{C}_r(T \setminus \{x_1,\ldots,x_s\}) = p^{-1}(x_1,\ldots,x_s). \tag{263}$$

Note that $\mathcal{C}_r(T \setminus \{x_1,\ldots,x_s\})$ is the configuration space of r indistinguishable particles on $T\setminus\{x_1,\ldots,x_s\}$, the punctured torus. Choose a base point $* = [x_1,\ldots,x_s] \in \mathcal{C}_{n,\mathrm{id}}(T)$

[3] In the language of categories, $B_n(T,*)$ is the set of morphisms of a category whose objects are elements of I_n. A representation is a functor to the category of finite dimensional vector spaces.

and let $* = [x_{s+1}, \ldots, x_{s+r}]$ be the base point of $\mathcal{C}_r(T \setminus \{x_1, \ldots, x_s\})$. Then $B_{n',r}(T, *) = \pi_1(\mathcal{C}_r(T \setminus \{x_1, \ldots, x_s\}), *)$ is a subgroupoid of $B_n(T, *)$, and we have a homomorphism $B_{n',r}(T, *) \to B_n(T, *)$. The flat vector bundle corresponding to the pull back of a representation ρ is just the restriction to $\mathcal{C}_r(T \setminus \{x_1, \ldots, x_s\})$ of the flat vector bundle over $\mathcal{C}_n(T)$ associated to ρ.

5.4 Topological Representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$

5.5 Local system from multivalued forms

The multipliers of the multivalued r-forms upon analytic continuation define a particular representation of the coloured braid groupoid. This representation defines in turn a local system over the configuration space. The singular homology groups we investigate have coefficients in this local system.

Recall the basic data which we start from: $n = (n_1, \ldots, n_k) \in \{1, 2, \ldots\}^k$, $|n| = s + r$, $n_k = r$, and $(\alpha(1), \ldots, \alpha(k)) \in \mathbb{Q}^k$, $\alpha(k) = \alpha_+$ such that $\sum_{j=1}^k n_k \alpha(k) = 0$. Then $\alpha_i = \alpha(\bar{\mathrm{id}}(i))$. Remember that $\bar{\pi} : \{1, \ldots, s + r\} \to \{1, \ldots, k\}$ denotes the colour map associated with $[\pi] \in I_n$. Given these data, we consider

$$f_\mu^\Sigma(z_1, \ldots, z_{s+r}|\tau) = \Delta_\mu(\sqrt{2p'p}\sum_{i=1}^{s+r} \alpha_i z_i|\tau) \prod_{1 \leq i < j \leq s+r} E(z_i, z_j)^{\alpha_i \alpha_j}, \tag{264}$$

$0 \leq \mu \leq 2p'p - 1$. Fix the modular parameter τ. Then $f^\Sigma = (f_\mu^\Sigma)_{0 \leq \mu \leq 2p'p-1}$ is a multivalued analytic function on the configuration space $\mathcal{C}_n(\Sigma)$ with values in $\mathbb{C}^{2p'p}$.

Let $\phi : T \to \Sigma$, $\phi(x^1, x^2) = x^1 + \tau x^2$, to obtain a diffeomorphism $\phi : \mathcal{C}_n(T) \to \mathcal{C}_n(\Sigma)$. Define $f^T = \phi^* f^\Sigma$. Fix a base point $* = [x_1, \ldots, x_{s+r}]$ in $Y_n := \mathcal{C}_n(T)$ such that $0 < x_1^1 < \cdots < x_{s+r}^1 < 1$. For $[\pi] \in I_n$, let $f^{T,\bar{\pi}}$ be the single-valued function on the universal covering space $\tilde{Y}_n(\pi *)$ with values in $\mathbb{C}^{2p'p}$, defined as the analytic continuation of f^T from the base point $\pi *$ where it takes the value $f^{T,\bar{\pi}}(\pi *) = f^{T,\bar{\mathrm{id}}}(*)$. An element $g \in B_n^{\bar{\sigma},\bar{\pi}}(T, *)$ induces a map $\lambda_g : \tilde{Y}_n(\pi *) \to \tilde{Y}_n(\sigma *)$ through $\lambda_g(x) = xg$. The point xg is represented by a path from $\pi *$ to $\sigma *$, composed with a path from $\sigma *$ to $p(x)$, $p : \tilde{Y}_n(\pi *) \to Y_n$ being the covering projection. Then

$$\lambda_g^*(f_\mu^{T,\bar{\pi}}) = \sum_{\nu=0}^{2p'p-1} f_\nu^{T,\bar{\sigma}} \; M_{\nu,\mu}^{\bar{\sigma},\bar{\mu}}(g) \tag{265}$$

defines a $(2p'p)$-dimensional representation of $B_n(T, *)$ on $V_{\bar{\pi}} = \mathbb{C}^{2p'p}$. An explicit calculation by analytic continuation yields

$$M_{\nu,\mu}^{\overline{\epsilon\pi},\bar{\pi}}(\alpha^{[\pi]}) = \delta_{\nu,\mu} \exp\left\{2\pi i \frac{\alpha(\bar{\pi}(s+r))}{\sqrt{2p'p}} \mu\right\}, \tag{266}$$

$$M_{\nu,\mu}^{\bar{\pi},\bar{\pi}}(\beta^{[\pi]}) = \delta_{\nu,\mu + \sqrt{2p'p}\,\alpha(\bar{\pi}(s+r))} \exp\left\{-\pi i \alpha(\bar{\pi}(s+r))^2\right\}, \tag{267}$$

$$M_{\nu,\mu}^{\overline{\tau_i\pi},\bar{\pi}}(\sigma_i^{[\pi]}) = \delta_{\nu,\mu} \exp\left\{\pi i \alpha(\bar{\pi}(i)) \alpha(\bar{\pi}(i+1))\right\}. \tag{268}$$

If $\alpha(\bar{\pi}(s+r)) = \alpha_+$ is a screening charge, it follows that

$$M^{\overline{\epsilon\pi},\bar{\pi}}_{\nu,\mu}(\alpha^{[\pi]}) = \delta_{\nu,\mu} \exp\left\{2\pi i \frac{\mu}{p}\right\}, \tag{269}$$

$$M^{\bar{\pi},\bar{\pi}}_{\nu,\mu}(\beta^{[\pi]}) = \delta_{\nu,\mu+2p'} \exp\left\{-2\pi i \frac{p'}{p}\right\}. \tag{270}$$

These matrices deserve an abbreviation since they will occur frequently below. Let $q = \exp(\pi i p'/p)$ and

$$A_{\nu,\mu} = \delta_{\nu,\mu}\, q^{2\frac{\mu}{p'}}, \qquad B_{\nu,\mu} = \delta_{\nu,\mu+2p'}\, q^{-2}. \tag{271}$$

with the convention $q^{1/p'} = \exp(\pi i/p)$. If $\alpha(\bar{\pi}(i)) = (1 - n(\bar{\pi}(i)))p'/\sqrt{2p'p}$ and $\alpha(\bar{\pi}(i+1)) = \alpha_+$, it follows that

$$M^{\overline{\tau_i\pi},\bar{\pi}}_{\nu,\mu}(\sigma_i^{[\pi]}) = \delta_{\nu,\mu}\, q^{1-n(\bar{\pi}(i))}. \tag{272}$$

If $\alpha(\bar{\pi}(i)) = \alpha(\bar{\pi}(i+1)) = \alpha_+$, this representation matrix takes the form

$$M^{\overline{\tau_i\pi},\bar{\pi}}_{\nu,\mu}(\sigma_i^{[\pi]}) = \delta_{\nu,\mu}\, q^2. \tag{273}$$

completing the description of the representation of $B_n(T,*)$ associated with the multipliers of f^T.

Let $n' = (n_1, \ldots, n_{k-1})$, $|n'| = s$, $p : \mathcal{C}_n(T) \to \mathcal{C}_{n'}(T)$, and $\mathcal{C}_r(T \setminus \{x_1, \ldots, x_s\}) = p^{-1}(x_1, \ldots, x_s)$. In the following, $[x_1, \ldots, x_s]$ will be fixed as above. Pulling back the above representation of $B_n(T,*)$ with the homomorphism $B_{n',r}(T,*) \to B_n(T,*)$, we obtain a representation of $B_{n',r}(T,*)$. We take the tensor product of this representation with the pull-back of the totally antisymmetric representation of S_r by the canonical homomorphism $B_{n',r}(T,*) \to S_r$. The result is the representation, denoted by ρ, associated with the multi-valued r-forms (241). This representation induces a local system (flat vector bundle) $\pi : L_r(x_1, \ldots, x_s) \to \mathcal{C}_r(T \setminus \{x_1, \ldots, x_s\})$.

Let $\epsilon > 0$ be a small number, D_i^ϵ the open disc of radius ϵ centered at x_i, $i = 1, \ldots, s$, $Y^\epsilon = T \setminus (\cup_{i=1}^s D_i)$. Denote by Y_r^ϵ the configuration space $\mathcal{C}_r(Y^\epsilon)$ of r indistinguishable points on Y^ϵ. Thus elements of Y_r^ϵ are subsets $Z \subset Y^\epsilon$ of cardinality r. Fix points y_-, $y_+ \in \partial D_1^\epsilon$ such that $y_+^1 < x_1 < y_-^1$, and define $Y_r^{\epsilon\pm} = \{Z \in Y_r | y_\pm \in Z\}$. The bijections

$$\phi_\pm : Y_r^\epsilon \setminus Y_r^{\epsilon\pm} \to Y_{r+1}^{\epsilon\pm}, \qquad Z \mapsto Z \cup \{y_\pm\} \tag{274}$$

lift to isomorphisms $\phi_\pm : L_r|Y_r^\epsilon \setminus Y_r^{\epsilon\pm} \to L_{r+1}|Y_{r+1}^{\epsilon\pm}$. The lift is of course not unique. To fix it it is sufficient to define the isomorphism from the fiber of the base point to the fiber of its image. We define it to be the identity map in the distinguished trivialization introduced above.

5.6 Families of Loops and Operators

We generalize the previous construction on the sphere to the torus. To make this chapter self-contained we briefly recall the notion of families of loops and topological operators with quantum group relations, adapted to the torus. Let $[x_1, \ldots, x_s] \in \mathcal{C}_{n',\mathrm{id}}(T)$, $\epsilon > 0$, $Y_r^\epsilon = \mathcal{C}_r(Y^\epsilon)$, and $y_\pm \in \partial Y^\epsilon$ be as above. The position of the punctures will be kept fixed in the following.

A non-intersecting family of loops based at y_- is a family $\gamma_0, \ldots, \gamma_{r-1}$ of smooth homotopically non-trivial embedded loops starting and ending at y_-, with no mutual intersections except at the endpoints. Homotopies of families of loops are defined. Non-intersecting families of loops can be represented by embeddings Γ of the open r-cube with open $(r-1)$-faces Q_r into X_r.

Let $\tilde{A}_r^\epsilon$ be the space of linear combinations $\sum \lambda_\Gamma[\Gamma]$ of homotopy classes $[\Gamma] = [\gamma_0, \ldots, \gamma_{r-1}]$ of non-intersecting families of loops, with coefficients λ_Γ in the space of horizontal sections of $\Gamma^* L_r$ over Q_r. Horizontal sections corresponding to homotopic families of loops are canonically identified by parallel transport, so the definition makes sense. The elements of $\tilde{A}_r^\epsilon$ represent locally finite relative homology classes in $H_r^{lf}(Y_r^\epsilon, Y_r^{\epsilon -}; L_r)$. We consider a quotient of $\tilde{A}_r^\epsilon$ by a subspace which maps to zero in homology. Let $A_r^\epsilon = \tilde{A}_r^\epsilon / \sim$, where the equivalence relation $\sim$ is given by

1. $\lambda[\Gamma] \sim \pm f^* \lambda[\Gamma \circ f]$ for any orientation preserving (+) or reversing (-) isometry of the cube Q_r.

2. $\lambda[\gamma_0, \ldots, \gamma_i, \ldots, \gamma_{r-1}] = \lambda'[\gamma_0, \ldots, \gamma_i', \ldots, \gamma_{r-1}] + \lambda''[\gamma_0, \ldots, \gamma_i'', \ldots, \gamma_{r-1}]$, whenever γ_i is homotopic to the composition $\gamma_i' \circ \gamma_i''$ in such a way that if the homotopy is denoted by $h(\cdot, s)$, $s \in [0,1]$, $\gamma_0, \ldots, h(\cdot, s), \ldots, \gamma_{r-1}$ is a non-intersecting family of loops for all $s \in [0, 1[$. The sections λ', λ'' are the restrictions of λ.

3. $\lambda[\gamma_0, \ldots, \gamma_{r-1}] \sim 0$, whenever at least p loops in the family are in the same class in $\pi_1(Y^\epsilon, y_-)$.

The third identification is peculiar to the case when q is a root of unity: if n loops, say $\gamma_0, \ldots, \gamma_{n-1}$ in a non-intersecting family $\gamma_0, \ldots, \gamma_{r-1}$ are all homotopic to a loop γ, then the corresponding locally finite homology class is proportional to the class of a relative cycle parametrized by $t_0 < t_1 < \cdots < t_{n-1}, t_n, \ldots, t_{r-1} \in Q_r$ as

$$(t_0, \ldots, t_{r-1}) \mapsto (\gamma(t_0), \ldots, \gamma(t_{n-1}), \gamma(t_n), \ldots, \gamma(t_{r-1})). \tag{275}$$

The proportionality factor is

$$\prod_{j=1}^{n} \frac{q^{2j} - 1}{q^2 - 1}, \tag{276}$$

and vanishes if $n \geq p$. We define now operators $\hat{E}$, $\hat{F}$, and $\hat{K}$, acting on the space $\oplus_{r=0}^{\infty} A_r^\epsilon$ and compute their commutation relations. The operator $\hat{E}$ is a close relative of the boundary operator. Define

$$\hat{E}:\lambda_\Gamma[\gamma_0,\ldots,\gamma_{r-1}]\mapsto\sum_{i=0}^{r-1}(-1)^i\phi_-^{-1}\left(\lambda_\Gamma\circ e_{i,r}^+-\lambda_\Gamma\circ e_{i,r}^-\right)[\gamma_0,\ldots,\hat{\gamma}_i,\ldots,\gamma_{r-1}], \tag{277}$$

with $e_{i,r}^\pm:[0,1]^{r-1}\to[0,1]^r$ the standard face maps. $\hat{\gamma}_i$ denotes the omission of γ_i. Intuitively the ith particle is moved to y_- and then taken out. The operator $\hat{F}$ adds a loop along the boundary of the hole around the first puncture.

$$\hat{F}:\lambda[\gamma_0,\ldots,\gamma_{r-1}]\mapsto\lambda'[\gamma_0,\ldots,\gamma_{r-1},\gamma_C] \tag{278}$$

with $\gamma_C:[0,1]\to Y^\epsilon$, $\gamma_C(t)=x_1+\frac{\epsilon}{\sqrt{2}}\Delta(t)$. Here $\Gamma'=\gamma_0,\ldots,\gamma_{r-1},\gamma_C$ and λ' is the horizontal section of $\Gamma'^*L_{r+1}^\epsilon$ with $\phi_+\lambda=\lambda'\circ i$, where i is the inclusion $(t_0,\ldots,t_{r-1},\frac{1}{2})$. This definition makes sense since we can assume that $\gamma_0,\ldots,\gamma_{r-1}$ do not intersect γ_C except at the endpoints. The operator $\hat{K}$ is simply defined as

$$\hat{K}^2|_{A_r^\epsilon}=q^{-1-n_1(r)}. \tag{279}$$

Charge neutrality requires that $1-n_1(r)=\sum_{i=2}^s(n_i-1)-2r$. Note that both the construction of $\hat{E}$ and $\hat{F}$ make use of the isomorphisms (274), relating local systems over different configuration spaces.

Theorem 5.1 *The operators $\hat{E}$, $\hat{F}$ and $\hat{K}$ satisfy the relations*

$$\hat{K}^2\hat{E}=q^2\hat{E}\hat{K}^2,\quad \hat{K}^2\hat{F}=q^{-2}\hat{F}\hat{K}^2,\quad [\hat{E},\hat{F}]=\hat{K}^2-\hat{K}^{-2}. \tag{280}$$

In other words, the operators $\hat{E}$, $\hat{F}$, $\hat{K}^2$ and $\hat{K}^{-2}$ define a representation of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ on $\oplus_{r=0}^\infty A_r^\epsilon$.

Proof: The proof is a repetition of that of Theorem 3.6. The first and the second relation are immediate consequences of the definition of $\hat{E}$, $\hat{F}$, and $\hat{K}$. The third relation is best proved using an explicit trivialization. Without loss of generality, we can assume that $[x_1,\ldots,x_s,\gamma_0(\frac{1}{2}),\ldots,\gamma_{r-1}(\frac{1}{2})]\in\mathcal{C}_{n,\pi}(T)$ for some $[\pi]\in I_n$. Denote by $\lambda(v)$ the section with the value v in the trivialization over $\mathcal{C}_{n,\pi}(T)$. We can further assume that $[x_1,\ldots,x_s,\gamma_0(\frac{1}{2}),\ldots,y_-,\ldots,\gamma_{r-1}(\frac{1}{2})]\in\mathcal{C}_{n,\sigma_i}(T)$ for some $[\sigma_i]\in I_n$, (y_- in position i). Let $\eta_i^\pm$ be the paths $t\mapsto[x_1,\ldots,x_s,\gamma_0(\frac{1}{2}),\ldots,\gamma_i(\frac{1}{2}(1\pm t)),\ldots\gamma_{r-1}(\frac{1}{2})]$. Using

$$\hat{E}\lambda(v)[\gamma_0,\ldots,\gamma_{r-1}]=$$
$$\sum_{i=0}^{r-1}(-1)^i(\rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^+)-\rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^-))\lambda(v)[\gamma_0,\ldots,\hat{\gamma}_i,\ldots,\gamma_{r-1}], \tag{281}$$

it follows that

$$
\begin{aligned}
\hat{E}\hat{F}\lambda(v)[\gamma_0,\dots,\gamma_{r-1}] &= \hat{E}\lambda(v)[\gamma_0,\dots,\gamma_{r-1},\gamma_C] \\
&= \sum_{i=0}^{r}(-1)^i \left(\rho_{\bar{\sigma}_i',\bar{\pi}'}(\eta_i^{+\prime}) - \rho_{\bar{\sigma}_i',\bar{\pi}'}(\eta_i^{-\prime})\right)\lambda(v)[\gamma_0,\dots,\hat{\gamma}_i,\dots,\gamma_r] \\
&= \sum_{i=0}^{r-1}(-1)^i \left(\rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^{+}) - \rho_{\bar{\sigma}_i,\bar{\pi}}(\eta_i^{-})\right)\lambda(v)[\gamma_0,\dots,\hat{\gamma}_i,\dots,\gamma_r-1,\gamma_C] + \\
&\quad (-1)^r \left(\rho_{\bar{\sigma}_r',\bar{\pi}'}(\eta_r^{+}) - \rho_{\bar{\sigma}_r',\bar{\pi}'}(\eta_r^{-})\right)\lambda(v)[\gamma_0,\dots,\gamma_{r-1}] \\
&= \hat{F}\hat{E}\lambda(v)[\gamma_0,\dots,\gamma_{r-1}] + \\
&\quad \left(q^{1-n_1(r+1)} - q^{n_1(r+1)-1}\right)\lambda(v)[\gamma_0,\dots,\gamma_{r-1}],
\end{aligned} \tag{282}
$$

proving the third relation. □

5.7 The Torus with One Puncture

We have proved above that $\oplus_{r=0}^{\infty}A_r^{\epsilon}$ comes equipped with the structure of a $U_q(\mathfrak{s}l_2(\mathbb{C}))$ module. A legitimate question to address is what kind of module this is. In this section we will consider the torus with a single puncture and find the adjoint representation of $U_q(\mathfrak{s}l_2(\mathbb{C}))$.

To describe A_r^{ϵ} as a space., we choose a basis as follows. Let γ_A and γ_B be loops in Y^{ϵ} based at y_- such that γ_A winds around an A-cycle and γ_B winds around a B-cycle. For $j_A,\, j_B \in \{0,1,\dots\}$ such that $j_A+j_B=r$, let $\gamma_B^{(1)},\dots,\gamma_B^{(j_B)}$ be homotopic deformations of γ_B and $\gamma_A^{(1)},\dots,\gamma_A^{(j_A)}$ homotopic deformations of γ_A such that $\gamma_B^{(i)}$ lies on the left of $\gamma_B^{(i+1)}$ and $\gamma_A^{(i)}$ above $\gamma_A^{(i+1)}$, and $\gamma_B^{(1)},\dots,\gamma_A^{(j_A)}$ is a non-intersecting family of loops, also denoted by $\gamma_B^{j_B},\gamma_A^{j_A}$. This family of loops is drawn in Fig. 4. We choose representatives in such a way that $[x_1,\gamma_1(\frac{1}{2}),\dots,\gamma_r(\frac{1}{2})]\in\mathcal{C}_{n,\mathrm{id}}(T)$. We define $\lambda(v)$ to be the horizontal section over $[\gamma_B^{j_B},\gamma_A^{j_A}]$ which takes the value v in the trivialization over $\mathcal{C}_{n,\mathrm{id}}(T)$. Let e_μ, $1\le\mu\le 2p'p$ be the standard basis of $\mathbb{C}^{2p'p}$. Then

$$
A_r^{\epsilon} = \oplus_{\mu=1}^{2p'p}\oplus_{j_A+j_B=r}\mathbb{C}\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A}]. \tag{283}
$$

In this basis, the operators $\hat{E}$, $\hat{F}$ and $\hat{K}$ are represented by the following matrices.

Lemma 5.2 *Applying $\hat{E}$, $\hat{F}$ and $\hat{K}$ to the elements of the basis (283) of A_r^{ϵ} we obtain*

$$
\begin{aligned}
\hat{E}\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A}] &= \frac{[j_B]_q}{q-q^{-1}}\left(q^{-4j_A-3j_B+3}B - q^{-j_B+1}\right)\lambda(e_\mu)[\gamma_B^{j_B-1},\gamma_A^{j_A}] + \\
&\quad \frac{[j_A]_q}{q-q^{-1}}\left(q^{j_A+2j_B+1}A - q^{-j_A-2j_B+1}\right)\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A-1}],
\end{aligned} \tag{284}
$$

$$
\begin{aligned}
\hat{F}\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A}] &= \left(q^{2j_A+2j_B-2}B^{-1} - q^{2j_A}B^{-1}A^{-1}\right)\lambda(e_\mu)[\gamma_B^{j_B+1},\gamma_A^{j_A}] + \\
&\quad \left(q^{2j_A}B^{-1}A^{-1} - A^{-1}\right)\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A+1}],
\end{aligned} \tag{285}
$$

$$
\hat{K}^2\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A}] = q^{-2j_A-2j_B-2}\lambda(e_\mu)[\gamma_B^{j_B},\gamma_A^{j_A}], \tag{286}
$$

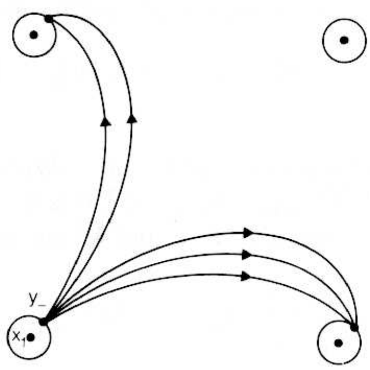

Figure 4: The family of loops $\gamma_B^{j_B}, \gamma_A^{j_A}$.

with the convention that $[\gamma_B^{j_B}, \gamma_A^{j_A}] = 0$ *if* $j_B < 0$ *or* $j_A < 0$.

Proof: The action of $\hat{E}$ is explicitly computed from

$$\hat{E}\lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] = \sum_{i=1}^{j_B}(-1)^i \left(\rho_{\bar{\sigma}_i,\bar{\mathrm{id}}}(\eta_i^+) - \rho_{\bar{\sigma}_i,\bar{\mathrm{id}}}(\eta_i^-)\right) \lambda(e_\mu)[\gamma_B^{j_B-1}, \gamma_A^{j_A}] + \sum_{i=j_B+1}^{j_A+j_B}(-1)^i \left(\rho_{\bar{\sigma}_i,\bar{\mathrm{id}}}(\eta_i^+) - \rho_{\bar{\sigma}_i,\bar{\mathrm{id}}}(\eta_i^-)\right) \lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A-1}]. \tag{287}$$

The paths $\eta_i^\pm$ have representation matrices

$$\rho_{\bar{\sigma}_i,\bar{\mathrm{id}}}(\eta_i^+) = \begin{cases} (-q^{-2})^{r-1}B(-q^{-2})^{r-i}, & 1 \le i \le j_B, \\ (-1)^{r-1}q^{n_1(r)-1}A(-q^{-2})^{r-i}, & j_B+1 \le i \le j_A + j_B, \end{cases} \tag{288}$$

$$\rho_{\bar{\sigma}_i,\bar{\mathrm{id}}}(\eta_i^-) = (-q^{-2})^{i-1}. \tag{289}$$

To compute the action of $\hat{F}$, we have to deform the added loop to the composition of two γ_A and two γ_B loops. The result is

$$\hat{F}\lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A}] = \rho_{\bar{\sigma}_1,\bar{\pi}}(\eta_1)\lambda(e_\mu)[\gamma_B^{j_B}, \gamma_A^{j_A+1}] + \rho_{\bar{\sigma}_2,\bar{\pi}}(\eta_2)\lambda(e_\mu)\Big\{[\gamma_B^{(1)}, \ldots, \gamma_B^{(j_B)}, \gamma_A^{(1)}, \ldots, \gamma_A^{(j_A)}, \gamma_B^{(j_B+1)}] - [\gamma_B^{(1)}, \ldots, \gamma_B^{(j_B)}, \gamma_A^{(2)}, \ldots, \gamma_A^{(j_A+1)}, \gamma_A^{(1)}]\Big\} - \rho_{\bar{\sigma}_3,\bar{\pi}}(\eta_3)\lambda(e_\mu)[\gamma_B^{(2)}, \ldots, \gamma_B^{(j_B+1)}, \gamma_A^{(1)}, \ldots, \gamma_A^{(j_A)}, \gamma_B^{(1)}]. \tag{290}$$

A transition function is picked up when the cell on which the bundle is trivialized changes. The paths η_i have braid groupoid representation matrices

$$\rho_{\bar{\sigma}_1,\bar{\pi}(\eta_1)} = A^{-1}, \tag{291}$$

$$\rho_{\bar{\sigma}_2,\bar{\pi}(\eta_2)} = (-q^2)^{j_A}B^{-1}A^{-1}, \tag{292}$$

$$\rho_{\bar{\sigma}_3,\bar{\pi}(\eta_3)} = (-1)^r q^{n_1(r+1)-1}AB^{-1}A^{-1}. \tag{293}$$

The last representation matrix can be simplyfied using $AB = q^4 BA$. Finally, the loops are reordered with the help of an isometry of the unit cube. The action of $\hat{K}$ follows immediately from its definition. $\Box$

The matrices acting on the sections are

$$Ae_\mu = q^{\frac{2\mu}{p'}} e_\mu, \quad Be_\mu = q^{-2} e_{\mu+2p'}. \tag{294}$$

Using these formulae, we conclude that the representation of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on $\oplus A^\epsilon_r$ can be split into a direct sum of isomorphic representations.

Define for $n = 0, 1, 2, \ldots$, $[n]_q = q^n - q^{-n}$, $[n]_q! = [n]_q [n-1]_q \cdots [2]_q [1]_q$, $[0]_q! = 1$.

Lemma 5.3 *(I) A^ϵ decomposes into a direct sum of subspaces*

$$A^{\epsilon,n'} = \oplus_{n=0}^{2p-1} \oplus_{j_A,j_B=0}^{p-1} \mathbb{C}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}], \quad 0 \le n' \le p'-1, \tag{295}$$

invariant under the action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. (II) The action of $\hat{E}$, $\hat{F}$ and $\hat{K}$ on $A_r^{\epsilon,n'}$ takes the explicit form

$$\begin{gathered}\hat{E}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] = \\ \frac{[j_B]_q}{q-q^{-1}} \left\{ q^{-4j_A-3j_B+1} \lambda(e_{-n'p+(n+2)p'}) - q^{-j_B+1} \lambda(e_{-n'p+np'}) \right\} [\gamma_B^{j_B-1}, \gamma_A^{j_A}] + \\ q^{-n'p/p'+n+1} \frac{[j_A]_q [j_A + 2j_B - n'p/p' + n]_q}{q-q^{-1}} \lambda(e_{-n'p+np'}) [\gamma_B^{j_B}, \gamma_A^{j_A-1}],\end{gathered} \tag{296}$$

$$\begin{gathered}\hat{F}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] = \\ q^{2j_A+j_B+n'p/p'-n+1} [j_B - n'p/p' + n - 1]_q \lambda(e_{-n'p+(n-2)p'}) [\gamma_B^{j_B+1}, \gamma_A^{j_A}] + \\ \left\{ q^{2j_A+2n'p/p'-2n+2} \lambda(e_{-n'p+(n-2)p'}) - q^{2n'p/p'-2n} \lambda(e_{-n'p+np'}) \right\} [\gamma_B^{j_B}, \gamma_A^{j_A+1}],\end{gathered} \tag{297}$$

$$\hat{K}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] = q^{-2j_A-2j_B-2} \lambda(e_{-n'p+np'}) [\gamma_B^{j_B}, \gamma_A^{j_A}]. \tag{298}$$

Denote $A_r^{\epsilon,n'} = A_r \cap A^{\epsilon,n'}$. Let $U_q(\mathfrak{sl}_2(\mathbb{C}))$ be the Hopf algebra with generators E, F, $K^{\pm 2}$, with the relations of Theorem 4.1, and coproduct $\Delta(E) = E \otimes K^2 + 1 \otimes E$, $\Delta(F) = F \otimes K^{-2} + 1 \otimes F$, $\Delta(K^{\pm 2}) = K^{\pm 2} \otimes K^{\pm 2}$ (see Section 5). Let I_p be the ideal generated by the central elements E^p, F^p and $(K^2)^{2p} - 1$, and $U_q^0(\mathfrak{sl}_2(\mathbb{C})) = U_q(\mathfrak{sl}_2(\mathbb{C}))/I_p$. $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts on $U_q^0(\mathfrak{sl}_2(\mathbb{C}))$ by the adjoint action. We define the idempotents

$$T_{n,\omega} = \frac{1}{2p} \sum_{m=0}^{2p-1} (\omega q^n)^{-m} K^{2m}, \quad n \in \frac{1}{p'}\mathbb{Z}, \quad \omega = \pm 1. \tag{299}$$

As before we set $q^{1/p'} = \exp(\pi i/p)$. These idempotents have the following properties

$$\begin{gathered}K^2 T_{n,\omega} = \omega q^n T_{n,\omega}, \quad T_{n+p,\omega} = T_{n,q^p\omega}, \quad T_{n+n'p/p',\omega} = T_{n,(-1)^{n'}\omega}, \quad n, n' \in \mathbb{Z}, \\ T_{n+n'p/p',\omega} T_{m+n'p/p',\omega'} = \delta_{n,m}\delta_{\omega,\omega'} T_{n+n'p/p',\omega}, \quad 0 \le n, m \le p-1, \quad n' \in \mathbb{Z}.\end{gathered} \tag{300}$$

For each $n' \in \mathbb{Z}$, the elements $T_{n+n'p/p',\omega}$, $n = 0, \dots, p-1$, $\omega = \pm 1$ build a basis of the subalgebra generated by K^2.

Definition: For $0 \le n' \le p'-1$, define linear maps $\phi_{n'} : A^{\epsilon,n'} \to U_q^0(\mathfrak{sl}_2(\mathbb{C}))$ as follows:

$$\phi_{n'}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A}] =$$
$$q^{-\frac{1}{2}j_B(j_B-1)+(j_A-1)(-n'p/p'+n+1)} \frac{[j_B]_q!}{(q-q^{-1})^{j_B}} F^{j_A} E^{p-1-j_B} T_{1+n'p/p'-n,1}. \tag{301}$$

Having introduced the maps $\phi_{n'}$, we are ready to state the main result of this section.

Theorem 5.4 *(I) For $X \in U_q(\mathfrak{sl}_2(\mathbb{C}))$, and $n' = 0, \dots, p'-1$, the diagram*

$$\begin{tikzcd} A^{\epsilon,n'} \arrow[r, "\hat{X}"] \arrow[d, "\phi_{n'}"'] & A^{\epsilon,n'} \arrow[d, "\phi_{n'}"] \\ U_q^0(\mathfrak{sl}_2(\mathbb{C})) \arrow[r, "ad_X"'] & U_q^0(\mathfrak{sl}_2(\mathbb{C})) \end{tikzcd} \tag{302}$$

commutes. That is, $\phi_{n'}$ is a homomorphism from $A^{\epsilon,n'}$ to the module $U_q^0(\mathfrak{sl}_2(\mathbb{C}))$ with the adjoint action. (II) If p' is odd, $\phi_{n'}$ is an isomorphism. If p' is even, $\phi_{n'}$ is two-to-one, with image the submodule $\{X \in U_q^0(\mathfrak{sl}_2(\mathbb{C})) \,|K^{2p}X = (-1)^{n'}X\}$.

Proof: (I) is checked by an explicit calculation using Lemma 5.3 and Lemma 5.2. To prove (II), we notice that $F^jE^lT_{n,\omega}$, with $j, l, n = 0, \dots, p-1$ and $\omega = \pm 1$, build a basis of $U_q(\mathfrak{sl}_2(\mathbb{C}))$. For p' odd, we see from Definition 4.4 using the third eq. of (300) that $\phi_{n'}$ is bijective. If p' is even, the image is the subspace spanned by the basis vectors with $\omega = (-1)^{n'}$. $\square$

Let us work out the interplay between topological and algebraic objects a little further. We have introduced $\hat{F} : A_r^\epsilon \to A_{r+1}^\epsilon$ as the operator which adds a γ_C-loop and identifies the the section as described above, using the point y_+. With any loop $\gamma : [0,1] \to Y^\epsilon$ based at y_- such that $\gamma(\frac{1}{2}) = y_+$, we can associate an operator $\hat{L}(\gamma)$: $A_r^\epsilon \to A_{r+1}^\epsilon$, $\lambda[\gamma_0, \dots, \gamma_{r-1}] \mapsto \lambda'[\gamma_0, \dots, \gamma_{r-1}, \gamma]$, such that $\phi_+\lambda = \lambda' \circ i$, generalizing $\hat{F} = \hat{L}(\gamma_C)$. Two special cases are $\hat{F}_L = \hat{L}(\gamma_A)$ and $\hat{F}_R = \hat{L}(\gamma_D)$. See Fig. 5 for a graphical representation.

Theorem 5.5 *(I) For $0 \le n' \le p'-1$, $\hat{F}_{L,R}$ maps $A_r^{\epsilon,n'}$ to $A_{r+1}^{\epsilon,n'}$ and $\hat{F} = \hat{F}_L - \hat{F}_R$. (II) The diagrams*

$$\begin{tikzcd} A_r^{\epsilon,n'} \arrow[r, "\hat{F}_L"] \arrow[d, "\phi_{n'}"'] & A_r^{\epsilon,n'} \arrow[d, "\phi_{n'}"] \\ U_q^0(\mathfrak{sl}_2(\mathbb{C})) \arrow[r, "X \mapsto FXK^2"'] & U_q^0(\mathfrak{sl}_2(\mathbb{C})) \end{tikzcd} \tag{303}$$

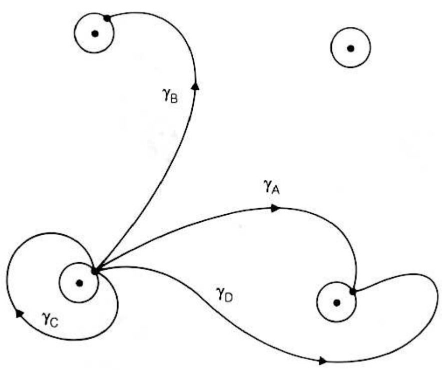

Figure 5: The loops $\gamma_A, \ldots, \gamma_D$.

and

$$\begin{tikzcd} A_r^{\epsilon,n'} \arrow[r, "\hat{F}_R"] \arrow[d, "\phi_{n'}"'] & A_r^{\epsilon,n'} \arrow[d, "\phi_{n'}"] \\ U_q^0\left(\mathfrak{s}l_2(\mathbb{C})\right) \arrow[r, "X\mapsto XFK^2"'] & U_q^0\left(\mathfrak{s}l_2(\mathbb{C})\right) \end{tikzcd} \tag{304}$$

commute.

Proof: (I) γ_C is homotopic to the composition $(\gamma_D^{-1}) \circ \gamma_A$. Using the equivalence relations imposed on $A_{r+1}^{\epsilon,n'}$ it follows that

$$\lambda[\gamma_0, \ldots, \gamma_{r-1}, \gamma_C] = \lambda'[\gamma_0, \ldots, \gamma_{r-1}, \gamma_A] + \lambda''[\gamma_0, \ldots, \gamma_{r-1}, \gamma_D] \tag{305}$$

the sections being identified as above. (II) The counterpart of $\hat{F}_L$ on $U_q^0\left(\mathfrak{s}l_2(\mathbb{C})\right)$ follows from

$$\lambda(e_{-n'p+np'})'[\gamma_B^{j_B}, \gamma_A^{j_A+1}] = A^{-1}\lambda(e_{-n'p+np'})[\gamma_B^{j_B}, \gamma_A^{j_A+1}] \tag{306}$$

using the explicit form (301) of $\phi_{n'}$. The action of $\hat{F}_R$ on $U_q^0\left(\mathfrak{s}l_2(\mathbb{C})\right)$ is computed with $\hat{F} = \hat{F}_L - \hat{F}_R$, and $\mathrm{ad}_F(X) = FXK^2 - XFK^2$. $\square$

5.7.1 The Torus with Many Punctures

Combining the above results with previous work on topological representations of $U_q\left(\mathfrak{s}l_2(\mathbb{C})\right)$ on the disc, the representation on $\bigoplus_{r=o}^{\infty} A_r^{\epsilon}$ can be identified. The result is a tensor product of $U_q^0\left(\mathfrak{s}l_2(\mathbb{C})\right)$ Verma modules, one for every additional puncture, with the algebra itself. The latter is understood as the representation space for the adjoint action.

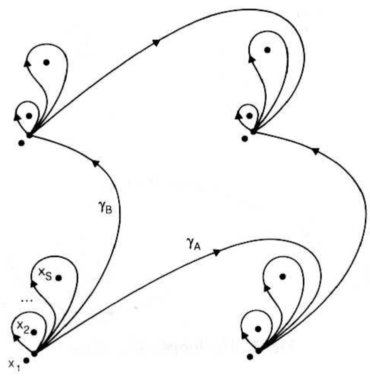

Figure 6: The family of loops of eq. (307).

The starting point is again an explicit description of A_r^ϵ as a space in terms of a basis. Fix a non-intersecting family of loops

$$\begin{aligned}&[\gamma_2^{j_2},\dots,\gamma_s^{j_s},\gamma_B^{j_B},\gamma_A^{j_A}]:=\\ &[\gamma_2^{(1)},\dots,\gamma_2^{(j_2)},\dots,\gamma_s^{(1)},\dots,\gamma_s^{(j_s)},\gamma_B^{(1)},\dots,\gamma_B^{(j_B)},\gamma_A^{(1)},\dots,\gamma_A^{(j_A)}].\end{aligned} \tag{307}$$

It is understood that $\gamma_i^{(k)}, 2\le i\le s$ and $1\le k\le j_i$, are homotopic deformations of γ_i such that $\gamma_i^{(k+1)}$ lies inside $\gamma_i^{(k)}$. See Fig. 6. Let this family be parametrized such that

$$[x_1,\dots,x_s,\gamma_2^{(1)}(\frac{1}{2}),\dots,\gamma_A^{(j_A)}(\frac{1}{2})]\in\mathcal{C}_{n,\overline{id}}(T). \tag{308}$$

Define a horizontal section over this family, denoted by $\lambda(v)$ giving it the value v in the distinguished trivialization over $\mathcal{C}_{n,\overline{id}}(T)$. Then

$$A_r^\epsilon=\bigoplus_{\mu=1}^{2p'p}\bigoplus_{j_2+\cdots+j_A=r}\mathbb{C}\lambda(e_\mu)[\gamma_2^{j_2},\dots,\gamma_s^{j_s},\gamma_B^{j_B},\gamma_A^{j_A}] \tag{309}$$

Generalizing the case with one puncture, we define the following map.

Definition: For $0\le n'\le p'-1$, define maps $\phi_{n'}^{(s)}:A^{\epsilon,n'}\to V(n_2)\otimes\cdots\otimes V(n_s)\otimes U_q^0(\mathfrak{sl}_2(\mathbb{C}))$ as follows:

$$\begin{aligned}&\phi_{n'}^{(s)}\left(\lambda(e_{-n'p+np'})[\gamma_2^{j_2},\dots,\gamma_s^{j_s},\gamma_B^{j_B},\gamma_A^{j_A}]\right):=\\ &F^{j_2}v_0(n_2)\otimes\cdots\otimes F^{j_s}v_0(n_s)\otimes\phi_{n'}^{(1)}\left(\lambda(e_{-n'p+np'})[\gamma_B^{j_B},\gamma_A^{j_A}]\right),\end{aligned} \tag{310}$$

where $\phi_{n'}^{(1)}$ denotes the map of Definition 5.7 for the torus with one puncture. $V(h)$ is the $U_q^0(\mathfrak{sl}_2(\mathbb{C}))$ Verma module generated from a singular vector $v_0(h)$ with $Ev_0(h)=0$ and $K^2v_0(h)=q^{h-1}v_0(h)$.

Theorem 5.6 *(I) For $0 \leq n' \leq p'-1$, the maps $\phi^{(s)}_{n'} : A^{\epsilon,n'} \to V(n_2)\otimes\cdots\otimes V(n_s)\otimes U^0_q(\mathfrak{sl}_2(\mathbb{C}))$ are one-to-one and onto if p' is odd, and two-to-one if p' is even. (II) For $X \in U^0_q(\mathfrak{sl}_2(\mathbb{C}))$ let*

$$\Delta^{(s)}(X) = \sum_i X_i^{(1)} \otimes \cdots \otimes X_i^{(s)}. \tag{311}$$

Denote $B^{n'} := \otimes_{j=1}^{s} V(n_j) \otimes U^0_q(\mathfrak{sl}_2(\mathbb{C}))$. Then the diagram

$$\begin{tikzcd} A^{\epsilon,n'} \arrow[r, "\hat{X}"] \arrow[d, "\phi^{(s)}_{n'}"'] & A^{\epsilon,n'} \arrow[d, "\phi^{(s)}_{n'}"] \\ B^{n'} \arrow[r, "\sum_i X_i^{(1)}\otimes\cdots\otimes X_i^{(s-1)}\otimes ad(X_i^{(s)})"'] & B^{n'} \end{tikzcd} \tag{312}$$

commutes. That is, the topological action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on $A^{\epsilon,n'}$ is given by the coproduct on the tensor product of Verma modules with the algebra itself.

Proof: We will give an explicit proof for $s = 2$. The generalization to $s > 2$ is obvious and will be omitted. Since

$$\begin{gathered} \hat{E}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}] = \\ \frac{[j_2]_q\,[n_2-j_2]_q}{q-q^{-1}}\lambda(e_{-n'p+np'})[\gamma_2^{j_2-1}, \gamma_B^{j_B}, \gamma_A^{j_A}] + \\ q^{n_2-1-2j_X}\left\{\frac{[j_B]_q}{q-q^{-1}}\left(q^{-4j_A-3j_B+3}B - q^{1-j_B}\right)\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B-1}, \gamma_A^{j_A}] + \right. \\ \left. \frac{[j_A]_q}{q-q^{-1}}\left(q^{j_A+2j_B+1}A - q^{-j_A-2j_B+1}\right)\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A-1}]\right\}, \end{gathered} \tag{313}$$

it follows that

$$\begin{gathered} \phi^{(2)}_{n'}\left(\hat{E}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}]\right) = \\ (E\otimes\mathbf{1} + K^2\otimes E)\phi^{(2)}_{n'}\left(\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}]\right). \end{gathered} \tag{314}$$

Where we identify the coproduct $\Delta(E) = E\otimes\mathbf{1} + K^2\otimes E$. To compute the action of $\hat{F}$, the added loop has to be homotopically deformed and split into a composition of loops γ_2, γ_B, and γ_A. By a deformation procedure it is shown that

$$\begin{gathered} \hat{F}\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A}] = \\ q^{2j_A+2j_B+2}\lambda(e_{-n'p+np'})[\gamma_2^{j_2+1}, \gamma_B^{j_B}, \gamma_A^{j_A}] + \\ \left(q^{2j_A}B^{-1}A^{-1} - q^{2j_A+2j_B-2}B^{-1}\right)\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B+1}, \gamma_A^{j_A}] + \\ (A^{-1} - q^{2j_A}\,B^{-1}A^{-1})\lambda(e_{-n'p+np'})[\gamma_2^{j_2}, \gamma_B^{j_B}, \gamma_A^{j_A+1}]. \end{gathered} \tag{315}$$

As a consequence

$$\phi^{(2)}_{n'}\left(\hat{F}\lambda(e_{-n'p+np'})[\gamma_2^{j_2},\gamma_B^{j_B},\gamma_A^{j_A}]\right) =$$
$$(F\otimes K^{-2}+\mathbf{1}\otimes F)\phi^{(2)}_{n'}\left(\lambda(e_{-n'p+np'})[\gamma_2^{j_2},\gamma_B^{j_B},\gamma_A^{j_A}]\right), \tag{316}$$

where $\Delta(F)=F\otimes K^{-2}+\mathbf{1}\otimes F$. Finally,

$$\hat{K}^2\lambda(e_{-n'p+np'})[\gamma_2^{j_2},\gamma_B^{j_B},\gamma_A^{j_A}] =$$
$$q^{n_2-1-2j_x}q^{-2j_A-2j_B-2}\lambda(e_{-n'p+np'})[\gamma_2^{j_2},\gamma_B^{j_B},\gamma_A^{j_A}], \tag{317}$$

so that

$$\phi^{(2)}_{n'}\left(\hat{K}^2\lambda(e_{-n'p+np'})[\gamma_2^{j_2},\gamma_B^{j_B},\gamma_A^{j_A}]\right) =$$
$$K^2\otimes K^2\phi^{(2)}_{n'}\left(\lambda(e_{-n'p+np'})[\gamma_2^{j_2},\gamma_B^{j_B},\gamma_A^{j_A}]\right) \tag{318}$$

proves the assertion since $\Delta(K^2)=K^2\otimes K^2$. The right action of $\hat{K}^2$ is a consequence of the charge neutrality condition. □

Thus we have proved that the topological action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on the torus with many punctures algebraically reproduces the coproduct.

5.8 On the adjoint representation

Let $q=\exp(i\pi p'/p)$, where p and p' are relative prime integers with $p\geq 2$. $U_q(\mathfrak{sl}_2(\mathbb{C}))$ is defined as the unital algebra over $\mathbb{C}$ generated by E,F and $K^{\pm 2}$ subject to the relations

$$K^{\pm 2}K^{\mp 2}=1,\quad K^2E=q^2EK^2,$$
$$K^2F=q^{-2}FK^2,\quad [E,F]=K^2-K^{-2}. \tag{319}$$

In the following, we will consider the quotient $U^0_q(\mathfrak{sl}_2(\mathbb{C}))=U_q(\mathfrak{sl}_2(\mathbb{C}))/I_p$ obtained by dividing by the Ideal I_p generated by the central elements $(K^2)^{2p}-1$, E^p and F^p. From $U_q(\mathfrak{sl}_2(\mathbb{C}))$ it inherits the coproduct

$$\Delta(K^{\pm 2})=K^{\pm 2}\otimes K^{\pm 2},$$
$$\Delta(E)=E\otimes 1+K^2\otimes E,\quad \Delta(F)=F\otimes K^{-2}+1\otimes F, \tag{320}$$

and the antipode

$$S(E)=-K^{-2}E,\quad S(F)=-FK^2,\quad S(K^{\pm 2})=K^{\mp 2}. \tag{321}$$

Theorem 5.7 *The monomials $F^jE^lK^{2n}$, $0\leq j,l\leq p-1$, $0\leq n\leq 2p-1$, form a PBW-basis of $U^0_q(\mathfrak{sl}_2(\mathbb{C}))$.*

Using the notation $\Delta(X) = \sum_i X_i' \otimes X_i''$, the adjoint representation of $U_q(\mathfrak{s}l_2(\mathbb{C}))$ acting on $U_q^0(\mathfrak{s}l_2(\mathbb{C}))$ is given by

$$\mathrm{ad}_X(Y) = \sum_i X_i' Y S(X_i''). \tag{322}$$

In particular, the action of the generators E, F and $K^{\pm 2}$ is

$$\begin{aligned} \mathrm{ad}_E(X) &= EX - K^2 X K^{-2} E, \\ \mathrm{ad}_F(X) &= FXK^2 - XFK^2, \\ \mathrm{ad}_{K^{\pm 2}}(X) &= K^{\pm 2} X K^{\mp 2}. \end{aligned} \tag{323}$$

In order to identify the $U_q(\mathfrak{s}l_2(\mathbb{C}))$-modules $A^{\epsilon,n'}$, $0 \le n' \le p'-1$, with $U_q^0(\mathfrak{s}l_2(\mathbb{C}))$, we consider for each n' the basis given by $F^j E^l T_{n+\frac{n'p}{p'},\omega}$, $0 \le j, l, n \le p-1$, $\omega = \pm 1$. In addition to the properties (300), the idempotents $T_{n,\omega}$ satisfy

$$ET_{n,\omega} = T_{n+2,\omega} E, \quad FT_{n,\omega} = T_{n-2,\omega} F, \tag{324}$$

which, together with

$$\begin{aligned} [F^n, E] &= F^{n-1} [n]_q \frac{q^{n-1}K^{-2} - q^{-n+1}K^2}{q - q^{-1}}, \\ [E^n, F] &= E^{n-1} [n]_q \frac{q^{n-1}K^{2} - q^{-n+1}K^{-2}}{q - q^{-1}}, \end{aligned} \tag{325}$$

and (319), allow us to compute explicitly the action of the generators. The result is

Lemma 5.8 *Let $0 \le n' \le p'-1$. The action of E, F, and K^2 on the basis $F^j E^l T_{n+\frac{n'p}{p'},\omega}$, $0 \le j, l, n \le p-1$, $\omega = \pm 1$, is given explicitly by*

$$\begin{aligned} \mathrm{ad}_E(F^j E^l T_{n+n'p/p',\omega}) &= F^j E^{l+1} \left(T_{n+n'p/p',\omega} - q^{2(l-j)} T_{n+n'p/p'-2,\omega} \right) - \\ &\quad \omega \frac{[j]_q [j - n - 2l - n'p/p' - 1]_q}{q - q^{-1}} F^{j-1} E^l T_{n+n'p/p',\omega}, \end{aligned} \tag{326}$$

$$\begin{aligned} \mathrm{ad}_F(F^j E^l T_{n+n'p/p',\omega}) &= F^{j+1} E^l \left(\omega q^{n+n'p/p'} T_{n+n'p/p',\omega} - \omega q^{n+n'p/p'+2} T_{n+n'p/p'+2,\omega} \right) - \\ &\quad q^{n+n'p/p'+2} \frac{[l]_q [l + n + n'p/p' + 1]_q}{q - q^{-1}} F^j E^{l-1} T_{n+n'p/p'+2,\omega}, \end{aligned} \tag{327}$$

$$\mathrm{ad}_{K^2}(F^j E^l T_{n+n'p/p',\omega}) = q^{2(l-j)} F^j E^l T_{n+n'p/p',\omega}. \tag{328}$$

5.9 Conjecture on locally finite homology

We conclude by stating a conjecture on the locally finite middle-dimensional homology groups with coefficients in the local systems L_r. Let as before q be a root of unity, and p be the smallest positive integer such that $q^{2p} = 1$. Define quantum binomial coefficients as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \lim_{\epsilon \downarrow 0} \frac{[n]_{q_\epsilon}!}{[n-m]_{q_\epsilon}![m]_{q_\epsilon}!}, \qquad q_\epsilon = q(1+\epsilon). \tag{329}$$

Let A be the associative $\mathbb{Z}$-graded algebra with unit, generated by K^2, K^{-2} of degree zero, E of degree -1, and F_n of degree n, $n = 0, 1, \ldots$, with relations

$$\begin{aligned} K^2 E K^{-2} = q^2 E, \qquad K^2 F_n K^{-2} = q^{-2n} F_n, \\ EF_n - F_n E = F_{n-1}(q^{-n+1}K^2 - q^{n-1}K^{-2}), \quad n \geq 1, \\ F_n F_m = \begin{bmatrix} n+m \\ m \end{bmatrix}_q F_{n+m}, \\ K^{\pm 2} K^{\mp 2} = K^{\mp 2} K^{\pm 2} = F_0 = 1. \end{aligned} \tag{330}$$

This is "half" of Lusztig's construction of the quantum group at root of unity. It is obtained by formally setting $F_n = F^n [1]_q / [n]_q!$, for q generic, and taking the limit when q goes to a root of unity.

There is a homomorphism $\iota : U_q(\mathfrak{sl}_2(\mathbb{C})) \to A$ given by $E \mapsto E$, $K^{\pm 2} \mapsto K^{\pm 2}$, and $F \mapsto F_1$. Thus $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts on A via the adjoint action $U_q(\mathfrak{sl}_2(\mathbb{C})) \times A \to A$, $(x, a) \mapsto \Sigma_j \iota(x'_j) a \iota(S(x''_j))$, where $\Delta(x) = \Sigma x'_j \otimes x''_j$. The $U_q(\mathfrak{sl}_2(\mathbb{C}))$ module A is $\mathbb{Z}$ graded for the grading of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ defined by $\deg(E) = -1$, $\deg(F) = 1$, $\deg(K^{\pm 2}) = 0$.

Let A_N be the quotient of A by the ideal generated by the central element E^{Np}, $N = 1, 2, \ldots$. The algebra $U_q(\mathfrak{sl}_2(\mathbb{C}))$ acts on A_N since E^{Np} commutes with the action on A. Multiplication by E^p defines embeddings of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ modules

$$\cdots \hookrightarrow A_N \hookrightarrow A_{N+1} \hookrightarrow \cdots \tag{331}$$

These maps are of degree zero for the shifted degree on A_N given by $\overline{\deg}(x) = \deg(x) + Np$. Define the graded $U_q(\mathfrak{sl}_2(\mathbb{C}))$ module A_∞ to be the direct limit of the modules A_N with the shifted degree. A basis of A_∞ is given by the classes of

$$F_j E^{Np-l-1} T_{n-1,\omega} \in A_N, \quad j, l \in \mathbb{N}, \quad n \in \{0, 1, \ldots, p-1\} \quad \omega = \pm 1. \tag{332}$$

In this expression N is any number such that $Np - l - 1 \geq 0$. The degree of (332) is $j + l - 1$. Denote by A^d_∞ the subspace of homogeneous elements of degree d.

An alternative description of the $U_q(\mathfrak{sl}_2(\mathbb{C}))$ module A_∞ was essentially suggested to us by D. Kazhdan: Let Z be the subalgebra of the center of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ generated by E^p. Then $A_\infty = A \otimes_Z \mathbb{C}[t]$, with adjoint action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$, where E^p acts on A by multiplication and on $\mathbb{C}[t]$ as $\mathrm{d}/\mathrm{d}t$. The isomorphism relating the two definitions is $\mathrm{cl}(x \in A_N) \mapsto x \otimes t^{N-1}/(N-1)!$.

For simplicity, we state our conjecture in the case of p' odd.

Conjecture 5.9 *Suppose that p' is odd. (I) The action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on families of loops extends to an action on $H_r^{lf}(X_r, X_r^-; L_r)$, and there is a degree zero isomorphism of graded $U_q(\mathfrak{sl}_2(\mathbb{C}))$ modules $H_r^{lf}(X_r, X_r^-; L_r) \simeq \oplus_{n'=0}^{p'-1} A_\infty^{r-1}$. (II) There is a degree zero isomorphism of graded vector spaces $H_r^{lf}(X_r) \simeq \oplus_{n'=0}^{2p'-1} \mathrm{Ker}(E : A_\infty^{r-1} \to A_\infty^{r-2})$.*

This conjecture is parallel to the one formulated in [FW91] for the case of the sphere. To prove it one should understand better locally finite homology. In the remaining of this section we describe these isomorphisms. Let, as in (268), γ_A, $\gamma_B : [0,1] \to X$ be A and B loops on the one-holed torus X based at a point y_- on the boundary of the hole. Consider the locally compact cells in X_r

$$\begin{aligned} C_{l,r} = \{ & (\gamma_B(t_1), \dots, \gamma_B(t_l), \gamma_A(t_{l+1}), \dots, \gamma_A(t_r)) \in X_r | \\ & 0 < t_1 < \dots < t_l < 1, 0 < t_{l+1} < \dots < t_r < 1\}^-, \end{aligned} \tag{333}$$

where $^-$ denotes closure in X_r. We orient $C_{l,r}$ using the standard orientation of the parameter space $\mathbb{R}^r \ni t$, and choose as section over it the section taking the value e_μ over a point in one of the cells defined in (249), where a trivialization is fixed. The class in $H_r^{lf}(X_r, X_r^-; L_r)$ represented by $C_{l,r}$ with this section will be denoted by $C_{\mu,l,r}$. The $U_q(\mathfrak{sl}_2(\mathbb{C}))$ module $\oplus_{r=0}^{\infty} H_r^{lf}(X_r, X_r^-; L_r)$ is a direct sum of submodules labeled by $n' = 0, \dots, p'-1$, spanned by $C_{\mu,l,r}$ with $\mu = -n'p + np'$, $n = 0, \dots, 2p-1$, $l = 0, 1, \dots$. Each of these submodules is isomorphic to A_∞. The isomorphism is

$$C_{\mu,l,r} \mapsto \eta F_{r-l} E^{pN-l-1} T_{1-\mu/p',1}, \tag{334}$$

for some root of unity η depending on the choice of trivialization. The second isomorphism is obtained through the identification of $\mathrm{Ker}(E)$ with $\mathrm{Ker}(\partial_*)$.

The space of cycles obtained here is bigger than the space of cycles relevant for conformal field theory. The cycles for conformal field theory should be computed using the cohomological methods of [FS89a, FS92] but the details remain to be understood. By construction, there is a projective action of the mapping class group $\mathrm{PSL}(2, \mathbb{Z})$ on relative homology, which commutes with the action of the quantum group. It will be described below.

5.10 Properties of $\Delta_\mu(W|\tau)$

Define

$$\begin{aligned} \Delta_\mu(W|\tau) &:= \frac{e^{\pi i \mu(2W+\tau\mu)/(2p'p)}}{\eta(\tau)} \theta_3(W + \tau\mu | 2p'p\tau) \\ &= \frac{1}{\eta(\tau)} \sum_{-\infty}^{\infty} e^{2\pi i W(n+\mu/(2p'p)) + \pi i 2p'p\tau(n+\mu/(2p'p))^2}, \end{aligned} \tag{335}$$

$$W := p' \sum_{i=1}^{s} (1 - n_i) z_i + 2p' \sum_{i=s+1}^{s+r} z_i. \tag{336}$$

A straightforward computation yields

$$\Delta_\mu(W + lp'|\tau) = e^{\pi i\mu l/p}\Delta_\mu(W|\tau), \tag{337}$$

$$\Delta_\mu(W + lp'\tau|\tau) = e^{-\pi i(lW+l^2p'\tau/2)/p}\Delta_{\mu+lp'}(W|\tau). \tag{338}$$

and

$$\Delta_\mu(W|\tau+1) = e^{\pi i(\mu^2/(2p'p)-1/12)}\Delta_\mu(W|\tau), \tag{339}$$

$$\Delta_\mu(W|\frac{-1}{\tau}) = \frac{1}{\sqrt{2p'p}}e^{\pi i\tau W^2/(2p'p)}\sum_{\nu=0}^{2p'p-1} e^{\pi i\mu\nu/(p'p)}\Delta_{2p'p-\nu}(W\tau|\tau). \tag{340}$$

To verify the last identity, note that

$$\sum_{\mu=0}^{2p'p-1} e^{-\pi i\mu\nu/(p'p)}\Delta_\mu(W|\tau) = \frac{1}{\eta(\tau)}\theta_3\left(\frac{W-\nu}{2p'p}|\frac{\tau}{2p'p}\right), \tag{341}$$

$$\Delta_\mu(W|\tau) = \frac{1}{2p'p}\sum_{\nu=0}^{2p'p-1} e^{\pi i\mu\nu/(p'p)}\frac{1}{\eta(\tau)}\theta_3\left(\frac{W-\nu}{2p'p}|\frac{\tau}{2p'p}\right), \tag{342}$$

and

$$\eta(\frac{-1}{\tau}) = \sqrt{-i\tau}\eta(\tau), \tag{343}$$

$$\theta_3(z|\frac{-1}{\tau}) = \sqrt{-i\tau}e^{\pi iz^2\tau}\theta_3(z\tau|\tau). \tag{344}$$

5.11 Properties of $\theta_1(z|\tau)$

Following Jacobi one defines

$$\theta_1(z|\tau) := -\sum_{n=-\infty}^{\infty} e^{2\pi i(z+1/2)(n+1/2)+\pi i\tau(n+1/2)^2}. \tag{345}$$

In terms of an infinite product

$$\theta_1(z|\tau) = 2e^{\pi i\tau/6}\eta(\tau)\sin(\pi z)\prod_{n=1}^{\infty}(1 - 2e^{2\pi i\tau n}\cos(2\pi z) + e^{4\pi i\tau n}). \tag{346}$$

It satisfies the following identities

$$\theta_1(z+1|\tau) = -\theta_1(z|\tau), \quad \theta_1(z+\tau|\tau) = -e^{2\pi iz-\pi i\tau}\theta_1(z|\tau), \quad \theta_1(-z|\tau) = -\theta_1(z|\tau),$$
$$\theta_1(z|\tau+1) = \sqrt{i}\theta_1(z|\tau), \quad \theta_1(z|\frac{-1}{\tau}) = \sqrt{i\tau}e^{\pi iz^2\tau}\theta_1(z\tau|\tau). \tag{347}$$

It has simple zeroes on the lattice $\mathbb{Z}\oplus\mathbb{Z}\tau$ and no others. Consider the fractional power $\theta_1(z|\tau)^\alpha, \alpha\in\mathbb{Q}\backslash\mathbb{Z}$, a multi valued function on $\mathbb{C}\backslash\mathbb{Z}\oplus\mathbb{Z}\tau$. Upon analytic continuation along straight paths from z to $z+1$ and $z+\tau$ respectively, it has the property that

$$\theta_1(z+1|\tau)^\alpha = e^{2\pi i\alpha\varphi_A(z)}\theta_1(z|\tau)^\alpha, \tag{348}$$

$$\theta_1(z+\tau|\tau)^\alpha = e^{2\pi i\alpha\varphi_B(z)-2\pi i\alpha z-\pi i\alpha\tau}\theta_1(z|\tau)^\alpha, \tag{349}$$

with

$$\varphi_A(z) = -n - \frac{1}{2}, \quad \varphi_B(z) = m + \frac{1}{2}, \tag{350}$$

$$z \in \{x + y\tau \in \mathbb{C} | m < x < m+1, n < y < n+1\}, \tag{351}$$

as follows from an explicit calculation. Note that $\varphi_\#(z)$ is constant on every translated fundamental domain.

6 Topological Representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ on the Torus and the Mapping Class Group

We compute the mapping class group action on cycles on the configuration space of the torus with one puncture, with coefficients in a local system arising in conformal field theory. This action commutes with the topological action of the quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$, and is given in vertex form.

6.1 Introduction

We consider topological representations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ appearing in free field representations of conformal field theories on the torus based on $SU(2)$. Topological representations of quantum groups on the complex plane were introduced in [FW91, SV91, La90]. The torus has been investigated in [CFW93].

Fock space traces of products of vertex operators yield multi-valued holomorphic differential forms on configuration spaces over the torus. Quantum groups [D86, Lu90, FK93, RT91] enter through their action on a certain space A of linear forms on the space of holomorphic multivalued differential forms with given monodromy. These forms are given by integration on products of loops. Singular vectors with respect with this action give cycles, and define thus linear forms on cohomology. We consider the torus with one puncture together with the local system given in [CFW93], associated to the monodromy of the differential forms. We restrict our attention to the quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$ at q a $2p$th root of unity. The topological action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ has been identified in [CFW93] with the adjoint representation in the sense that the space A is isomorphic to $U_q(\mathfrak{sl}_2(\mathbb{C}))$ as $U_q(\mathfrak{sl}_2(\mathbb{C}))$-module with the adjoint action. The main input from conformal field theory [BPZ84, DF84, ZF86, F89, BF90, SV89, G89, GK90, FS89a, FS89b] is the form of the local system.

An important feature of conformal field theories on the torus is modular invariance [FS89b]. A natural question to pose is the meaning of modular transformations on the side of topological representations. The first observation is that the local system coming from conformal field theory is compatible with modular transformations in a sense to be defined below. As a consequence the modular group acts on the space A. The second observation is that we can explicitly compute the action of the modular

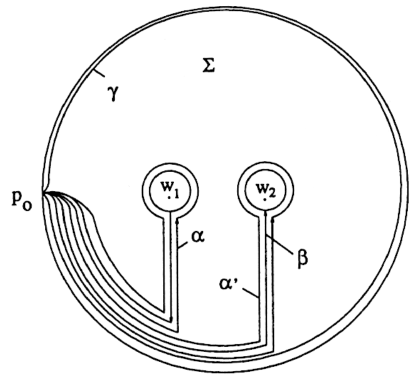

Figure 7: The loops α, α', β, and γ.

group on A by contour deformation methods. Using the identification of A with the quantum group algebra, we obtain the action of the modular group on the latter.

Since the action of the modular group commutes with the topological action of $U_q(\mathfrak{sl}_2(\mathbb{C}))$, it commutes on the algebraic side with the adjoint action. The result is a "vertex" form of the generators T and S of the modular group. These generators are expressed in terms of the universal R-matrix of an enlarged version of $U_q(\mathfrak{sl}_2(\mathbb{C}))$ (the "K-generated algebra") and the Haar measure on $U_q(\mathfrak{sl}_2(\mathbb{C}))$. A representation of the mapping class group in "SOS" form arises in the study of three-manifold invariants of Reshetikhin and Turaev [RT91].

We obtain as a byproduct the quantum group interpretation of modular invariance. Namely, the action of the modular group leaves invariant the subspace of singular vectors in the adjoint representation.

Similar formulas have been discovered independently, in the context of braided groups of monoidal categories, by Lyubashenko and Majid [LM91].

6.2 Configuration spaces and local systems on the torus

Let X be a torus with one puncture.

6.2.1 Representation of X

Let $D_R(w) = \{z \in \mathbb{C} | |z - w| < r\}$, the open disc. We represent $X = (\overline{D_R(0)} \setminus \cup_{i=1}^{2} D_{R'}(w_i))/\sim$, the disc with two holes, the boundaries of which we identify. E.g., we take $w_1 = -R/3$ and $w_2 = R/3$, define $\phi(w_1 + R'e^{i(\varphi-\pi)}) = w_2 + R'e^{-i\varphi}$, and identify $\phi(z) \sim z$. Let $p_0 = -R$ serve as a base point. $\pi_1(X, p_0)$ is generated by elements α and β as represented by the loops in X based at p_0 shown in Fig. 7. For later purpose, we introduce the abbreviations $\alpha' = \beta^{-1} \circ \alpha^{-1} \circ \beta$ and $\gamma = \alpha \circ \beta^{-1} \circ \alpha^{-1} \circ \beta$.

6.2.2 Configuration spaces and braid groups on X

For $r \geq 1$, we define configuration spaces

$$X_r = \left(X^r \setminus \cup_{1\leq i<j\leq r}\{(z_1, \dots, z_r) \in X^r | z_i = z_j\}\right)/S_r. \tag{352}$$

S_r is the symmetric group, acting from the right. Let $*_r = [x_1, \dots, x_r]$ be a base point in X_r. $\pi_1(X_r, *_r)$ is the braid group with r strings on X. We call a base point admissible if $-R \leq x_1 < \cdots < x_r \leq -R/3 - R'$. Braid groups defined with respect to different admissible base points are canonically isomorphic. We will always assume $*_r$ to be an admissible base point such that $\{x_1, \dots, x_r\} \subset \partial X$. $\pi_1(X_r, *_r)$ is generated by elements σ_i, $1 \leq i \leq r-1$, α, and β. Intuitively, σ_i interchanges x_i with x_{i+1} counterclockwise, while α and β move x_r along the respective loops, other components of $*_r$ being kept fixed. An abundance of relations hold among these generators. We will not present them here.

6.2.3 Local systems over X_r

Let p be an odd positive integer. Put $q = e^{\pi i/p}$ and define $2p$ by $2p$ matrices A and B with entries

$$A_{m,n} = q^{1-m}\delta_{m,n}, \quad B_{m,n} = \sum_{l\in\mathbb{Z}} \delta_{m,n+2pl+1}. \tag{353}$$

They satisfy $AB = q^{-1}BA$. Let $V = \mathbb{C}^{2p}$. The assignments

$$\rho_r(\sigma_i) = -q^2,\ 1 \leq i \leq r-1, \quad \rho_r(\alpha) = A^2, \quad \rho_r(\beta) = B^2 \tag{354}$$

define a $2p$-dimensional representation $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$. It is the monodromy representation associated with multivalued differential forms on X_r mentioned above. ρ_r is the direct sum of two equivalent p-dimensional irreducible representations.

Let X_r^0 be the subspace of X_r consisting of configurations which contain p_0 among their components. We then define $\phi_r : X_{r-1} \setminus X_{r-1}^0 \to X_r^0$ to be the bijection which inserts p_0. The family of representations ρ_r, $r \geq 1$, is compatible in the following sense. Let $\pi_1(\phi_r) : \pi_1(X_{r-1} \setminus X_{r-1}^0, *_{r-1}) \to \pi_1(X_r^0, \phi_r(*_{r-1}))$ be the isomorphism induced by ϕ_r, then $\rho_r \circ \pi_1(\phi_r) = \rho_{r-1}$.

With ρ_r we associate the local system $L_r(X) = \hat{X}_r(*_r) \otimes_{\pi_1(X_r,*_r)} V$, a flat vector bundle over X_r with distinguished trivialization over $*_r$, the holonomy associated with elements of $\pi_1(X_r, *_r)$ being ρ_r. Due to the compatibility, ϕ_r can be lifted to $L_r(\phi_r)$: $L_{r-1}(X)\Big|_{X_{r-1}\setminus X_{r-1}^0} \to L_r(X)\Big|_{X_r^0}$. We define $L_r(\phi_r)([x, v]) = [\phi_r(x), v]$, which is checked to be well defined.

6.3 Topological representations of $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$

We summarize briefly the constructions leading to topological representations of the quantum group $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ adjusting the notations to the present setup.

6.3.1 Families of nonintersecting loops with values in the local system

Let $Q_r =]0,1[^r \cup \bigcup_{i=1}^r]0,1[\times \cdots \times \{0,1\} \times \cdots \times]0,1[$ and $\gamma_1, \dots, \gamma_r$ be loops $[0,1] \to X$ starting and ending at p_0, nonintersecting except at p_0. Define $[\gamma_1, \dots, \gamma_r] : Q_r \to X_r$ be the corresponding embedding. Denote by $[\beta^j, \alpha^k] : Q_r \to X_r$, $j+k=r$ a family of nonintersecting loops obtained by homotopic deformation of j β-loops and k α-loops given by

$$[\beta^j, \alpha^k](t_0, \dots, t_{r-1}) = [\beta^{(0)}(t_0), \dots, \beta^{(j-1)}(t_{j-1}), \alpha^{(j)}(t_j), \dots, \alpha^{(r-1)}(t_{r-1})]. \tag{355}$$

It represents a locally finite r-chain in X_r with boundary in X_r^0.

We lift it to take values in $\hat{X}_r(*_r)$. We specify the lift by choosing an admissible point on its image, connecting this point to the base point by an admissible path. Then the equivalence class $[[\beta^j, \alpha^k], v]$, $v \in V$, defines a family of nonintersecting loops in X with values in $L_r(X)$. The space of families of nonintersecting loops in X with values in $L_r(X)$ is denoted by $A_r(X_r, X_r^0; L_r)$ or shorter by A_r. Its precise definition contains equivalence relations reflecting the possibility of homotopic deformation, reparametrization, and splitting of loops (see [FW91]). The elements $[[\beta^j, \alpha^k], e_n]$, $0 \le j, k \le \min\{r, p-1\}$ such that $j+k=r$, and $1 \le n \le 2p$, constitute a basis. Here $(e_n)_m = \delta_{n,m}$. A family which contains p homotopic loops is put equivalent to zero. Therefore, we restrict ourselfs to $r \le 2p-2$.

6.3.2 Topological action of $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$

The basic ingredience of topological representations are operators

$$E : A_r \to A_{r-1}, \quad F : A_r \to A_{r+1}, \quad K^2 : A_r \to A_r \tag{356}$$

defined by

$$E[[\beta^j, \alpha^k], e_n] = -L_r(\phi_r)^{-1} \partial [[\beta^j, \alpha^k], e_n], \tag{357}$$

$$F[[\beta^j, \alpha^k], e_n] = q^{-2j-2k-2}[[\beta^j, \alpha^k, \gamma], e_n], \tag{358}$$

$$K^2[[\beta^j, \alpha^k], e_n] = q^{-2r-2}[[\beta^j, \alpha^k], e_n]. \tag{359}$$

They are shown to satisfy the relations of $U_q(\mathfrak{sl}_2(\mathbb{C}))$:

$$K^2 E = q^2 E K^2, \quad K^2 F = q^{-2} F K^2, \quad [E, F] = K^2 - K^{-2}, \tag{360}$$

and also the additional relations

$$E^p = 0, \quad F^p = 0, \quad (K^2)^{2p} - 1 = 0, \tag{361}$$

defining $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$. Thus $\bigoplus_{r=0}^{2p-2} A_r$ comes equipped with the structure of a module over $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$. We identify this representation as the adjoint representation. Let

$\phi : \bigoplus_{r=0}^{2p-2} A_r \to U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ be the map

$$\phi[[\beta^j, \alpha^k], e_n] = N(j,k,n) F^k T_{n-1} E^{p-1-j}, \tag{362}$$

$$N(j,k,n) = (-1)^j q^{2j+2n} q^{j(j-1)/2} \frac{[j]_q!}{[1]_q^j} q^{(j+k)(j+k-1)+(j+k)(1-n)+j(1+n)}. \tag{363}$$

An explicit computation proves that ϕ is a homomorphism of $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ modules. Moreover, it is one-to-one and onto. Here

$$T_n = \frac{1}{2p} \sum_{m=0}^{2p-1} q^{-nm} K^{2m}. \tag{364}$$

The actions of $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ on itself by left multiplication and by right multiplication, twisted with the antipode, also have topological counterparts. The operator which implements left multiplication by F is called F_L. The operator which corresponds to right multiplication by $\eta(F)$ is denoted by F_R. On the topological side they are given by

$$F_L[[\beta^j, \alpha^k], e_n] = q^{n-1-2j-2k}[[\beta^j, \alpha^{k+1}], e_n], \tag{365}$$

$$F_R[[\beta^j, \alpha^k], e_n] = q^{-2j-2k-2}[[\alpha', \beta^j, \alpha^k], e_n]. \tag{366}$$

Intuitively, F adds a γ-loop, while F_L and F_R add α- and α'-loops respectively. The interpretation of $\bigoplus_{r=0}^{2p-2} A_r$ as a bimodule will not be worked out here.

The important formula of this section to keep in mind is (363).

6.4 Action of the Mapping Class Group

Let Diff(X) be the group of diffeomorphisms which leave $\partial X = \{z \in \mathbb{C} \,|\, |z| = R\}$ invariant. Let Diff$_0(X)$ be the subgroup of Diff(X) consisting of diffeomorphisms homotopic to the identity. The mapping class group of X is defined as

$$\mathcal{M}_{1,1}(X) := \mathrm{Diff}(X)/\mathrm{Diff}_0(X). \tag{367}$$

A reference on mapping class groups is [Bi74].

6.4.1 Generators of $\mathcal{M}_{1,1}(X)$

$\mathcal{M}_{1,1}(X)$ is generated by Dehn twists. On the torus, we have two kinds of Dehn twists, T_α and T_β. T_α is defined as follows. Consider the annulus $\{z \in X \,|\, r + \epsilon \le |z - w_1| \le r + 2\epsilon\}$ with $\epsilon > 0$ small. We define a map of this annulus to itself by $T_\alpha(w_1 + z) := w_1 + e^{i\varphi(|z|)} z$ with φ a smooth function interpolating between $\varphi(r+\epsilon) = 0$ and $\varphi(r + 2\epsilon) = 2\pi$. We say that T_α is the Dehn twist associated with the loop $t \mapsto w_1 - (r + 2\epsilon)e^{2\pi i t}$, $t \in [0,1]$. The Dehn twist T_β is associated with the loop $t \mapsto (w_1 + r)(1 - t) + (w_2 - r)t$. See figure (?). The orientations we use are shown by arrows. T_α and T_β leave the base point $*_r$ invariant. (Recall that $*_r$ is a configuration on ∂X.) Thus we have a map from $\mathcal{M}_{1,1}(X)$ to Aut$(\pi_1(X_r, *_r))$, the automorphisms of $\pi_1(X_r, *_r)$, $r \ge 1$.

6.4.2 Compatibility of local systems

We define a representation $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$ to be compatible with $\mathcal{M}_{1,1}(X)$ if $\rho \circ T \simeq \rho$ for all $T \in \mathcal{M}_{1,1}(X)$. This means that for every T there exists a matrix $D(T) \in \mathrm{GL}(V)$ such that

$$\rho \circ T(\sigma) = D(T)\rho(\sigma)D(T)^{-1}. \tag{368}$$

If a representation ρ_r is compatible with $\mathcal{M}_{1,1}(X)$, then we have an action of $\mathcal{M}_{1,1}(X)$ on L_r given by

$$L_r(T) : L_r \to L_r, \quad [x, v] \mapsto [T(x), D(T)v], \tag{369}$$

which is well defined due to equation (368).

The family of local systems $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$, $r \geq 1$, is indeed compatible with $\mathcal{M}_{1,1}(X)$. For the generators T_α and T_β the compatibility is proved by defining

$$D(T_\alpha) := \sum_{l=0}^{2p-1} q^{l(l-2)/2} A^l, \quad D(T_\beta) := \sum_{l=0}^{2p-1} q^{-l(l+2)/2} B^l. \tag{370}$$

and by noting that the action of T_α has the form $T_\alpha(\alpha) = \alpha$ and $T_\alpha(\beta) = \alpha \circ \beta$, while that of T_β has the form $T_\beta(\alpha) = \alpha \circ \beta$ and $T_\beta(\beta) = \beta$. Note also that $D(T_\alpha)$ and $D(T_\beta)$ have matrix elements

$$D(T_\alpha)_{m,n} = \left\{ \sum_{l=0}^{2p-1} q^{l^2/2} \right\} q^{-m^2/2} \delta_{m,n}, \tag{371}$$

$$D(T_\beta)_{m,n} = \sum_{k \in \mathbb{Z}} \sum_{l=0}^{2p-1} q^{-l(l+2)/2} \delta_{m,n+2pk+l}. \tag{372}$$

6.4.3 Action of $\mathcal{M}_{1,1}$ on $A_r(X_r, X_r^0; L_r)$

The mapping class group $\mathcal{M}_{1,1}$ acts on A_r as follows: $T \in \mathcal{M}_{1,1}$ acts by

$$L_r(T)[[\gamma_1, \ldots, \gamma_r], v] = [T \circ [\gamma_1, \ldots, \gamma_r], D(T)v]. \tag{373}$$

We compute the action of T_α and T_β on the basis elements $[[\beta^j, \alpha^k], e_n]$ of the linear space $A_r(X_r, X_r^0; L_r)$.

Action of T_α. Let us define $\delta := \alpha \circ \beta$. Let $r = j + k$. The action of T_α on $[[\beta^j, \alpha^k], e_n]$ is seen to have the form

$$L_r(T_\alpha)\Big([[\beta^j, \alpha^k], e_n]\Big) = [[\delta^j, \alpha^k], D(T_\alpha)e_n]. \tag{374}$$

The first problem is to decompose (374) again in terms of the basis of $A_r(X_r, X_r^0; L_r)$. The decomposition is performed with the help of

$$
\begin{aligned}
&[[\beta^j, \delta^{l+1}, \alpha^k], e_n] = \\
&\quad [[\beta^{(0)}, \dots, \beta^{(j-1)}, \delta^{(j)}, \dots, \delta^{(j+l)}, \alpha^{(j+l+1)}, \dots, \alpha^{(j+l+k)}], e_n] = \\
&\quad [[\beta^{(0)}, \dots, \beta^{(j)}, \delta^{(j+1)}, \dots, \delta^{(j+l)}, \alpha^{(j+l+1)}, \dots, \alpha^{(j+l+k)}], e_n] + \\
&\qquad [[\beta^{(0)}, \dots, \beta^{(j-1)}, \delta^{(j+1)}, \dots, \delta^{(j+l)}, \alpha^{(j+l+1)}, \dots, \alpha^{(j+l+k)}, \alpha^{(j)}]\sigma, e_n] = \\
&\quad [[\beta^{j+1}, \delta^l, \alpha^k], e_n] + (-1)^{l+k}[[\beta^j, \delta^l, \alpha^{k+1}], \rho_{j+l+k+1}(\sigma)e_n] = \\
&\quad [[\beta^{j+1}, \delta^l, \alpha^k], e_n] + q^{2(l+k)}[[\beta^j, \delta^l, \alpha^{k+1}], e_{n+2}].
\end{aligned} \tag{375}
$$

We use $\rho_{j+k+l+1}(\sigma) = (-q)^{2(k+l)}B^2$ and absorb the factor $(-1)^{l+k}$ by an isometry of $Q_{j+l+k+1}$. The ordering of loops according to deformation and the assignement of the components of $(t_0, \dots, t_{j+k+l}) \in Q_{j+l+k+1}$ to the individual loops indicated by superscripts should be clear from the notation used in (375). Iterate (375) to obtain

$$
[[\delta^j, \alpha^k], e_n] = \sum_{s=0}^{\min\{j,p-k-1\}} c_s(j,k)[[\beta^{j-s}, \alpha^{k+s}], e_{n+2s}] \tag{376}
$$

with coefficients

$$
c_s(j,k) = \sum_{0 \le i_1 \le \dots \le i_s \le j-s} \prod_{l=1}^{s} q^{2(k+j-i_l-1)} = q^{s(j+2k+s-2)} \begin{bmatrix} j \\ s \end{bmatrix}_q \tag{377}
$$

using Gauß's formula

$$
\sum_{0 \le i_1 \le \dots \le i_s \le j-s} q^{-2\sum_{l=1}^{s} i_l} = q^{s(s-j)} \begin{bmatrix} j \\ s \end{bmatrix}_q . \tag{378}
$$

The q-binomial coefficient is defined as

$$
\begin{bmatrix} j \\ s \end{bmatrix}_q := \frac{[j]_q!}{[s]_q![j-s]_q!}. \tag{379}
$$

Putting (372), (374), and (376) together, it follows that

$$
\begin{aligned}
&L_r(T_\alpha)\Big([[\beta^j, \alpha^k], e_n]\Big) = \\
&dq^{-n^2/2} \sum_{s=0}^{\min\{j,p-k-1\}} q^{s(j+2k+s-2)} \begin{bmatrix} j \\ s \end{bmatrix}_q [[\beta^{j-s}, \alpha^{k+s}], e_{n+2s}]
\end{aligned} \tag{380}
$$

with

$$
d = \sum_{l=0}^{2p-1} q^{l^2/2}. \tag{381}
$$

Thus (380) gives the matrix elements of $L_r(T_\alpha)$ in terms of the basis with elements $[[\beta^j, \alpha^k], e_n]$.

Action of T_β. Let $r = j + k$. The action of T_β has the form

$$L_r(T_\beta)\Big([[\beta^j, \alpha^k], e_n]\Big) = [[\beta^j, \delta^k], D(T_\beta)e_n], \tag{382}$$

which we again will express in terms of the basis of $A_r(X_r, X_r^0; L_r)$. Using (375) it follows that

$$[[\beta^j, \delta^k], e_n] = \sum_{s=\max\{0,j+k-p+1\}}^{k} b_s(j,k)[[\beta^{j+k-s}, \alpha^s], e_{n+2s}] \tag{383}$$

with coefficients

$$b_s(j,k) = \sum_{0 \le i_1 \le \cdots \le i_s \le k-s} \prod_{l=1}^{s} q^{2(k-i_l-1)} = q^{s(k+s-2)} \begin{bmatrix} k \\ s \end{bmatrix}_q . \tag{384}$$

Note that $b_s(j,k) = c_s(k,0)$. (382), (383), and (384) yield

$$L_r(T_\beta)\Big([\beta^j, \alpha^k], e_n]\Big) =$$
$$\sum_{l=0}^{2p-1} \sum_{s=\max\{0,j+k-p+1\}}^{k} q^{-l(l+2)/2+s(k+s-2)} \begin{bmatrix} k \\ s \end{bmatrix}_q [[\beta^{j+k-s}, \alpha^s], e_{n+l+2s}], \tag{385}$$

completing the calculation of the action of T_β on $A_r(X_r, X_r^0; L_r)$.

6.4.4 Action of $\mathcal{M}_{1,1}(X)$ on $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$

We have identified $A_r(X_r, X_r^0; L_r)$ as a $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ module as $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ with the adjoint action. The action of $\mathcal{M}_{1,1}(X)$ on $A_r(X_r, X_r^0; L_r)$ commutes with the topological action of $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$. In this section, we identify the action of $\mathcal{M}_{1,1}(X)$ on $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ defined by its action on $A_r(X_r, X_r^0; L_r)$. By construction it commutes with the adjoint action.

Action of T_α. Let $\phi_r : A_r(X_r, X_r^0; L_r) \to U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ be the restriction of the map defined above. Then

$$\phi_r\left\{L_r(T_\alpha)\Big([[\beta^j, \alpha^k], e_n]\Big)\right\} = dq^{-n^2/2} \sum_{s=0}^{\min\{j,p-k+1\}} q^{s(j+2k+s-2)} \begin{bmatrix} j \\ s \end{bmatrix}_q$$
$$\times N(j-s, k+s, n+2s)\; F^{k+s}\; T_{n+2s-1}\; E^{p-j+s-1}, \tag{386}$$

using (380). We define $U(T_\alpha) : U_q^{red}(\mathfrak{sl}_2(\mathbb{C})) \to U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ by

$$U(T_\alpha) \circ \phi_r = \phi_r \circ L_r(T_\alpha). \tag{387}$$

Thus, what is left to compute $U(T_\alpha)$, is the ratio of normalization constants

$$\frac{N(j-s,k+s,n+2s)}{N(j,k,n)} = (-1)^s q^{-s(j+2k+3s/2+n-3/2)}[1]_q^s \frac{[j-s]_q!}{[j]_q!}. \tag{388}$$

We conclude that

$$U(T_\alpha)(F^k\,T_{n-1}\,E^{p-j-1}) = dq^{-n^2/2} \sum_{s=0}^{\min\{j,p-1-k\}} (-1)^s q^{-s(s+1)/2-sn} \frac{[1]_q^s}{[s]_q!}\,F^{k+s}\,T_{n+2s-1}\,E^{p-j+s-1}, \tag{389}$$

giving the action of $U(T_\alpha)$ on $U_q^{red}(\mathfrak{s}l_2(\mathbb{C}))$.

Action of T_β. Let $r=j+k$. Using (385), we obtain

$$\phi_r\left\{L_r(T_\beta)\Big([[\beta^j,\alpha^k],e_n]\Big)\right\} = \sum_{l=0}^{2p-1}\sum_{s=\max\{0,j+k-p+1\}}^{k} q^{-\frac{1}{2}l(l+2)+s(k+s-2)}\begin{bmatrix}k\\s\end{bmatrix}_q \times N(j+k-s,s,n+l+2s)\,F^s\,T_{n+l+2s-1}\,E^{p-j-k+s-1}. \tag{390}$$

Insert

$$\frac{N(j+k-s,s,n+l+2s)}{N(j,k,n)} = \frac{(-1)^{k-s}}{[1]_q^{k-s}} q^{(k-s)(k-s+1)/2+(k-s)(j+n+2)+(2s+l)(2-s)} \frac{[j+k-s]_q!}{[j]_q!} \tag{391}$$

to conclude that

$$U(T_\beta)(F^kT_{n-1}E^{p-j-1}) = \sum_{l=0}^{2p-1}\sum_{s=\max\{0,j+k-p+1\}}^{k} \frac{(-1)^{k-s}}{[1]_q^{k-s}} q^{(k-s)(k-s+1)/2+(k-s)(j+n+s+2)+(l+2s)(1-l)/2} \times [k-s]_q! \begin{bmatrix}j+k-s\\j\end{bmatrix}_q \begin{bmatrix}k\\s\end{bmatrix}_q F^s\,T_{n+l+2s-1}\,E^{p-j-k+s-1} \tag{392}$$

completing the calculation of $U(T_\beta)$.

6.4.5 Action of $S_{\alpha\beta}$

We define $S_{\alpha\beta} := T_\beta(T_\alpha)^{-1}T_\beta$. The action of $S_{\alpha\beta}$ on $\pi_1(X,p_0)$ is given by $S_{\alpha\beta}(\alpha)=\beta$ and $S_{\alpha\beta}(\beta)=\alpha'$. Recall that $\alpha'=\beta^{-1}\circ\alpha^{-1}\circ\beta$. That is, $S_{\alpha\beta}$ maps α to β and β to

α^{-1} conjugated by β^{-1}. This transformation is known as the S-transformation. We put

$$D(S_{\alpha\beta}) := D(T_\beta)D(T_\alpha)^{-1}D(T_\beta), \tag{393}$$

$$D(S_{\alpha\beta})_{m,n} = \frac{2pq}{d^2}q^{(m+1)(n-1)}. \tag{394}$$

$D(S_{\alpha\beta})$ performs a discrete Fourier transformation on V. $\rho_r : \pi_1(X_r, *_r) \to \mathrm{GL}(V)$ is compatible with $S_{\alpha\beta}$, the equivalence being given by $D(S_{\alpha\beta})$. We thus have an action $L_r(S_{\alpha\beta})$ on $A_r(X_r, X_r^0; L_r)$. It has the form

$$L_r(S_{\alpha\beta})\Big([[\beta^j, \alpha^k], e_n]\Big) = [[\alpha'^j, \beta^k], D(S_{\alpha\beta})e_n], \tag{395}$$

$r = j+k$. As before, we deduce from (395) an action $U(S_{\alpha\beta})$ on $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$. However, we do not expand (395) in terms of the basis of $A_r(X_r, X_r^0; L_r)$ as we did in the case of $L_r(T_\alpha)$ and $L_r(T_\beta)$, although this could be done. Instead we compute directly the image of (395) under ϕ_r. Using

$$[[\alpha'^j, \beta^k], e_n] = q^{j(j+2k+1)}\,(F_R)^j\,[[\beta^k], e_n] \tag{396}$$

it follows that

$$\begin{aligned} \phi_r\Big([[\alpha'^j, \beta^k], e_n]\Big) &= (-1)^k[1]_q^{j-k}\frac{[k]_q!}{[j]_q!} \\ q^{j(j+1)/2+k(k+1)/2+2k(j+1)+n(j+k)}&N(j,k,n)\,T_{n-1}\,E^{p-k-1}\,F^j. \end{aligned} \tag{397}$$

By re-expressing (397) in terms of the basis $F^jT_{k-1}E^{p-l-1}$ and applying $(\phi_r)^{-1}$, the expansion of (395) in terms of the basis of $A_r(X_r, X_r^0; L_r)$ could be obtained. We conclude that

$$\begin{aligned} \phi_r\left\{L_r(S_{\alpha\beta})\Big([[\beta^j, \alpha^k], e_n]\Big)\right\} &= \frac{2pq}{d^2}(-1)^k[1]_q^{j-k}\frac{[k]_q!}{[j]_q!} \\ q^{j(j+3)/2+k(k+3)/2+k(n+1+2j)}&N(j,k,n)\,K^{2(j+n+1)}\,E^{p-k-1}\,F^j, \end{aligned} \tag{398}$$

using

$$\sum_{l=0}^{2p-1} q^{(n+l-1)(j+n-1)}T_{n+l-1} = K^{2(j+n+1)}. \tag{399}$$

The final result is

$$\begin{aligned} &U(S_{\alpha\beta})(F^k\,T_{n-1}\,E^{p-j-1}) = \\ &\frac{2pq}{d^2}(-1)^k[1]_q^{j-k}\frac{[k]_q!}{[j]_q!}q^{j(j+3)/2+k(k+3)/2+k(n+1+2j)}K^{2(j+n+1)}\,E^{p-k-1}\,F^j. \end{aligned} \tag{400}$$

$U(S_{\alpha\beta})$ is the algebraic version of the S-transformation. It is a mapping of $U_q^{red}(\mathfrak{sl}_2(\mathbb{C}))$ to itself, one-to-one and onto, which commutes with the adjoint action.

6.5 Identification of the S- and the T-transformation

We identify the operations $U(T_\alpha)$ and $U(S_{\alpha\beta})$ in terms of the quasitriangular structure of $U_q^{red}(\mathfrak{s}l_2(\mathbb{C}))$. Let us first adjust the normalization of $D(T_\alpha)$ and $D(S_{\alpha\beta})$ as follows:

$$D(T_\alpha) \to \frac{N_\alpha}{d} D(T_\alpha), \quad D(S_{\alpha\beta}) \to \frac{d^2 N_{\alpha\beta}}{2pq} D(S_{\alpha\beta}). \tag{401}$$

With this change of normalization, $U(T_\alpha)$ and $U(S_{\alpha\beta})$ act on $U_q^{red}(\mathfrak{s}l_2(\mathbb{C}))$ by

$$\begin{gathered} U(T_\alpha)(F^k T_{n-1} E^{p-j-1}) = \\ N_\alpha q^{-n^2/2} \sum_{s=0}^{\min\{j,p-1-k\}} (-1)^s q^{-s(s+1)/2-sn} \frac{[1]_q^s}{[s]_q!} \; F^{k+s} \; T_{n+2s-1} \; E^{p-j+s-1}, \end{gathered} \tag{402}$$

and

$$\begin{gathered} U(S_{\alpha\beta})(F^k T_{n-1} E^{p-j-1}) = \\ N_{\alpha\beta} (-1)^k [1]_q^{j-k} \frac{[k]_q!}{[j]_q!} q^{j(j+3)/2+k(k+3)/2+k(n+1+2j)} \; K^{2(j+n+1)} \; E^{p-k-1} \; F^j. \end{gathered} \tag{403}$$

6.5.1 Universal elements of $U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C}))$

Let us consider the K generated version of $U_q^{red}(\mathfrak{s}l_2(\mathbb{C}))$. It is known to be a ribbon Hopf algebra. The universal R-matrix is

$$R = \frac{1}{4p} \left(\sum_{n=0}^{p-1} (-1)^n \frac{[1]_q^n}{[n]_q!} q^{-n(n-1)/2} \; E^n \otimes F^n \right) \left(\sum_{m,n=0}^{4p-1} q^{nm/2} \; K^n \otimes K^m \right). \tag{404}$$

The associated central element V is

$$V = \sum_{n=0}^{p-1} \sum_{m=0}^{4p-1} (-1)^n \frac{[1]_q^n}{[n]_q!} q^{n(n+1)/2+n(m+1)+m(m+2)/2} \; F^n \; H_{m+2n} \; E^n, \tag{405}$$

$$H_n = \frac{1}{4p} \sum_{m=0}^{4p-1} q^{-nm/2} K^m. \tag{406}$$

6.5.2 Identification of $U(T_\alpha)$

Let $N_\alpha = q^{1/2}$. Then

$$\begin{array}{ccc} U_q^{red}(\mathfrak{s}l_2(\mathbb{C})) & \xrightarrow[U(T_\alpha)]{} & U_q^{red}(\mathfrak{s}l_2(\mathbb{C})) \\ \downarrow & & \downarrow \\ U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C})) & \xrightarrow{\lambda_{V^{-1}}} & U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C})) \end{array} \tag{407}$$

commutes. That is, $U(T_\alpha)$ is identified with the multiplication by the inverse of the central element V of (404). This is shown by

$$V^{-1}\; F^k\; T_{n-1}\; E^{p-j-1} =$$
$$\sum_{s=0}^{p-1}\sum_{l=0}^{4p-1}(-1)^s\frac{[1]_q^s}{[s]_q!}q^{-s(s+1)/2-sl-l^2/2+1/2}\; F^{k+s}\; H_{l+2s-1}\; T_{n+2s-1}\; E^{p-1-j+s} \tag{408}$$

with

$$\begin{aligned} H_{l+2s-1}\; T_{n+2s-1} &= H_{l+2s-1}\;(H_{n+2s-1}+H_{n+2p+2s-1}) = \\ &\delta_{l,n}H_{n+2s-1}+\delta_{l,n+2p}H_{n+2p+2s-1}, \end{aligned} \tag{409}$$

comparing the result with (402).

6.5.3 Trace on $U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C}))$

Let $\tau : U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C})) \to \mathbb{C}$ be the linear map such that

$$\begin{aligned} \tau(E^{p-1}F^{p-1}H_{n-1}) &:= 1, \qquad 1\le n\le 4p, \\ \tau(E^jF^kH_{n-1}) &:= 0, \qquad \text{else.} \end{aligned} \tag{410}$$

τ is a trace on $U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C}))$: For $X = E, F, K^{\pm 1}$ and $Y \in U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C}))$ we have $\tau(XY)=\tau(YX)$.

6.5.4 S-transformation

Let $R=\sum_i \alpha_i\otimes\beta_i$ the universal R-matrix (404). Define a linear map $S : U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C})) \to U_{q,K}^{red}(\mathfrak{s}l_2(\mathbb{C}))$ by

$$S(X) := \sum_{j,l}\beta_j\eta(\alpha_l)\tau\Big(\eta(\beta_l)K^{-2}\alpha_jX\Big) \tag{411}$$

with τ the trace of (410). A short computation reveals that

$$S(T_{n-1}E^{p-1-r}F^{p-1-s}) =$$
$$\frac{2(-1)^{r+s}}{[r]_q![s]_q!}\; q^{-r(r+1)/2-s(s+1)/2+(s-1)(2r+n-1)}\; F^r\; K^{2(1-n-r)}\; E^s. \tag{412}$$

We thus obtain a map $S : U_q^{red}(\mathfrak{s}l_2(\mathbb{C})) \to U_q^{red}(\mathfrak{s}l_2(\mathbb{C}))$ by restriction.

6.5.5 Identification of $S_{\alpha\beta}$

Put the normalization constant in (401) to be

$$N_{\alpha\beta} := \frac{[p-1]_q!}{2p[1]_q^{p-1}}q^{-(p-1)(p+2)/2}. \tag{413}$$

The transformation (403) is identified as

$$U(S_{\alpha\beta})^{-1}X = S(X) \tag{414}$$

with S the transformation (412). To verify (414), we compute the inverse transformation of (403). It is seen to be

$$\begin{aligned} U(S_{\alpha\beta})^{-1}(T_{n-1}E^{p-1-k}F^{j}) &= \frac{1}{pN_{\alpha\beta}}(-1)^{k}[1]_{q}^{k-j}\frac{[j]_{q}!}{[k]_{q}!} \\ q^{-j(j+3)/2-k(k+3)/2-jk-(n+k-1)(j+2)}\ F^{k}\ K^{-2(n+k-1)}\ E^{p-j-1}. \end{aligned} \tag{415}$$

Setting $k = r$ and $j = p-1-s$ and comparing (412) with (415), the result (414) follows.

7 The Knizhnik-Zamolodchikov-Bernard Equation on the Torus

In this chapter we give a description of conformal blocks of the Wess-Zumino-Witten model based on $\mathfrak{sl}_2(\mathbb{C})$ (in the sense of [TUY89]) in terms of solutions of the Knizhnik-Zamolodchikov-Bernard equation. We discuss the role of a doubly affine version of the the Weyl group.

7.1 Introduction

Two dimensional conformal field theory associates to a punctured Riemann surface a complex vector space of conformal blocks. It also tells how the vector spaces are identified when the punctures and the complex structure are varied. For the case of Wess-Zumino-Witten (WZW) models this construction can be done in mathematically well defined terms [TUY89]. In the case of the sphere the description can be made very explicit, in terms of the Knizhnik-Zamolodchikov differential equation. The aim of this chapter is to show that such a description in terms of differential equations is also possible in the genus one case. The Knizhnik-Zamolodchikov equation is replaced by a genus one generalization due to Bernard [Be88a, Be88b].

The construction is similar to the case of the sphere (see, e.g., [F93]) but requires a twist. The reason for this is the following: conformal blocks are defined as linear forms on a certain infinite dimensional space, obeying some invariance condition. Thanks to this invariance, in the case of the sphere, a conformal block is uniquely determined by its restriction on a finite dimensional subspace (of "primary states") isomorphic to a tensor product of finite dimensional representations of a finite dimensional Lie algebra. On higher genus surfaces this is not the case. The solution to this problem is to introduce additional parameters, and consider parametric families of conformal blocks, that are invariant under a Lie algebra depending on the parameters. Spaces of conformal blocks corresponding to different values of the parameters are then identified thanks to a flat connection. The point is that horizontal conformal blocks are determined, as functions of these parameters, by their values on primary states.

In this chapter we consider the WZW model based on the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on the torus. We construct conformal blocks for the torus with given family of punctures and modular parameter as certain multilinear forms invariant under a Lie algebra of twisted Lie algebra valued meromorphic functions. Under variations of the punctures they satisfy the Knizhnik-Zamolodchikov-Bernard (KZB) equation. We derive the KZB equation from the condition that conformal blocks be flat with respect to the Friedan-Shenker (FS) connection. We then introduce a doubly affine version of the Weyl group and formulate an invariance condition on conformal blocks as solutions to the KZB equation.

Some open ends of this construction will be left untouched, including the construction of integral representations [SV89, Ch91], the connection with quantum groups [FW91, SV91, CFW93], monodromy properties [TK88], and the role of integrability [TUY89].

Two dimensional conformal field theory has its origin in the work of [BPZ84]. Its formulation in terms of complex geometry can be found in [FS87]. The basic references to the WZW model are [W84, KZ84]. The notion of conformal blocks on a Riemann surfaces was formulated in mathematical terms in [TUY89], and by Beilinson and Feigin. The KZB equation on the torus was first derived in [Be88a]. It has also appeared recently in [FG92].

7.2 Lie algebra valued meromorphic functions

The infinite symmetry of the WZW model on the torus can be formulated in terms of a Lie algebra of twisted meromorphic Lie algebra valued functions.

Let $g = \mathfrak{sl}_2(\mathbb{C})$ with Cartan generators E, F, and H, and invariant bilinear form $(X, Y) = \mathrm{tr}(XY)$, normalized such that $(H, H) = 2$.

In our construction we will use the following kinds of configuration spaces. Fix $\tau \in \mathrm{H}_+ = \{\tau \in \mathbb{C} | \Im m(\tau) > 0\}$, and let $L(\tau) = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$. We then define

$$\triangle^n_{i,j}(\tau) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \,|\, z_i = z_j \bmod L(\tau)\} \tag{416}$$

and

$$\mathcal{C}^{[n]}(\tau) = \mathbb{C}^n \setminus (\cup_{i<j} \triangle^n_{i,j}(\tau)). \tag{417}$$

The space $\mathcal{C}^{[n]}(\tau)$ is a covering of the nth configuration space over the torus $\Sigma(\tau) = \mathbb{C}/L(\tau)$. Let us also introduce

$$\mathcal{D}^{[n]}(\tau) = \{(z_1, \ldots, z_n, \lambda) \in \mathcal{C}^{[n]}(\tau) \times \mathbb{C} | \, \lambda \in L(\tau)\} \tag{418}$$

and

$$\Sigma^{[n]}(\tau) = (\mathcal{C}^{[n]}(\tau) \times \mathbb{C}) \setminus \mathcal{D}^{[n]}(\tau). \tag{419}$$

The Lie algebra is now defined as follows. For $(z_1,\dots,z_n,\lambda)\in\Sigma^{[n]}(\tau)$, let $g(z_1,\dots,z_n,\lambda)$ be the Lie algebra of meromorphic functions $X:\mathbb{C}\to g$, holomorphic on $\mathbb{C}\backslash L(z_1,\dots,z_n)$, such that

$$X(z+1)=X(z),\quad X(z+\tau)=\exp(2\pi i\lambda\,\mathrm{ad}_H)X(z). \tag{420}$$

Here $L(z_1,\dots,z_n)=\cup_j(z_j+L(\tau))$ denotes the union of the punctures and their translates by the lattice $L(\tau)$. The Lie algebra consists of meromorphic functions on the plane, periodic along the A-period and twisted along the B-period, with possible poles on $L(z_1,\dots,z_n)$ and no others.

Let $Lg=g\otimes\mathbb{C}((t))$ and $Lg^\wedge_k=Lg\oplus\mathbb{C}$, the central extension of Lg associated to the two-cocycle

$$\omega(X\otimes f,Y\otimes g)=k\,(X,Y)\,\mathrm{res}_{t=0}(f'(t)g(t)) \tag{421}$$

with k a positive integer. With a highest weight g-module V one can associate a highest weight $Lg^\wedge_k$-module $V^\wedge$. The g-action is first extended to $b_+=g\otimes\mathbb{C}[[t]]\oplus\mathbb{C}$ by letting $g\otimes t\mathbb{C}[[t]]$ act by zero and $\mathbb{C}$ by multiplication. Then $V^\wedge$ is given by $U(Lg^\wedge_k)\otimes_{U(b_+)}V$. We will always view g as the Lie subalgebra $g\otimes t^0$ of $Lg^\wedge_k$, and V as the g-submodule $1\otimes V$ of $V^\wedge$. This construction as well as properties of $V^\wedge$ can be found in [K90].

As an input we require a family of finite-dimensional irreducible highest weight g-modules $V_1,\dots,V_n$. We can think of the module V_j as being attached to the point z_j. The next step in our construction is an action of the Lie algebra $g(z_1,\dots,z_n,\lambda)$ on the tensor product $V_1^\wedge\otimes\cdots\otimes V_n^\wedge$. Let $\pi_j(X)=X(z_j+t)$ be the Laurent expansion of X at z_j viewed as a formal power series in t. As an element of Lg it is included in $Lg^\wedge_k$. But $Lg^\wedge_k$ acts on V_j. Therefore, we can define

$$\pi(X)=\pi_1(X)\oplus\cdots\oplus\pi_n(X). \tag{422}$$

A computation reveals that, in $\mathrm{End}_{\mathbb{C}}(V_1^\wedge\otimes\cdots\otimes V_n^\wedge)$, we have the equation

$$\pi([X,Y])=[\pi(X),\pi(Y)]+k\sum_j\mathrm{res}_{z=z_j}((X'(z),Y(z))). \tag{423}$$

But (X',Y) is doubly periodic so that the sum of residues vanishes. We therefore obtain an action π of $g(z_1,\dots,z_n,\lambda)$ on $V_1^\wedge\otimes\cdots\otimes V_n^\wedge$.

7.3 Meromorphic vector fields

Analogous to the action of Lie algebra valued meromorphic functions, there is an action of doubly periodic meromorphic vector fields on $V_1^\wedge\otimes\cdots\otimes V_n^\wedge$.

Let us first consider any highest weight g-module V. Sugawara's construction yields a projective representation of $\mathrm{Vect}(S^1)=\mathbb{C}((t))\partial_t$ on $V^\wedge$. Let T^1, T^2, and T^3 form an orthogonal basis of g, normalized such that $(T^a,T^b)=\frac{1}{2}\delta_{a,b}$. For example, we may put $T^1=\frac{1}{2}(E+F)$, $T^2=\frac{1}{2i}(E-F)$, and $T^3=\frac{1}{2}H$. Then one defines, for $n\in\mathbb{Z}$,

$$L_n=\frac{1}{k+2}\sum_a\sum_m :T^a_{n-m}T^a_m: \tag{424}$$

with $T^a_m = T^a \otimes t^m$. As elements of $\mathrm{End}_{\mathbb{C}}(V^\wedge)$ the L_n obey the Virasoro relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1) \tag{425}$$

with central charge $c = \frac{3k}{k+2}$. The projective representation is now given by

$$\sum_n \xi_n t^{n+1}\partial_t \mapsto -\sum_n \xi_n\, L_n. \tag{426}$$

$\mathrm{Vect}(S^1)$ also acts by derivations on Lg. The explicit formula is

$$(\xi(t)\partial_t,\, X(t)) \mapsto \xi(t)\, X'(t). \tag{427}$$

It extends to an action on $Lg^\wedge_k$ by letting vector fields act by zero on the center.

For $z_1, \ldots, z_n \in \mathcal{C}^{[n]}(\tau)$, let $\mathrm{Vect}(z_1, \ldots, z_n)$ be the Lie algebra of meromorphic vector fields on $\mathbb{C}$, holomorphic on $\mathbb{C} \setminus L(z_1, \ldots, z_n)$, such that $\xi(z+1) = \xi(z)$ and $\xi(z+\tau) = \xi(z)$.

For $\xi \in \mathrm{Vect}(z_1, \ldots, z_n)$, let $\pi_j(\xi) = \xi(z_j + t)\partial_t$ denote the Laurent expansion of ξ at z_j viewed as an element of $\mathrm{Vect}(S^1)$. Through Sugawara's construction it is mapped to an element of $\mathrm{End}(V_j^\wedge)$. We then define as above

$$\pi(\xi) = \pi_1(\xi) \oplus \cdots \oplus \pi_n(\xi). \tag{428}$$

It turns out that, in $\mathrm{End}_{\mathbb{C}}(V_1^\wedge \otimes \cdots \otimes V_n^\wedge)$, we have the equation

$$\pi([\xi, \zeta]) = [\pi(\xi), \pi(\zeta)] + \frac{c}{12}\sum_j \mathrm{res}_{z=z_j}(\xi'''(z)\zeta(z)). \tag{429}$$

But the sum of residues is again zero so that we obtain an action of the Lie algebra $\mathrm{Vect}(z_1, \ldots, z_n)$ on $V_1^\wedge \otimes \cdots \otimes V_n^\wedge$. Moreover, it can be shown that

$$\pi(\xi(X)) = [\pi(\xi), \pi(X)], \tag{430}$$

for $\xi \in \mathrm{Vect}(z_1, \ldots, z_n)$ and $X \in g(z_1, \ldots, z_n, \lambda)$. That is, this action intertwines the natural action of vector fields on functions.

7.4 Conformal blocks

For $(z_1, \ldots, z_n, \lambda) \in \Sigma^{[n]}(\tau)$, we define the space $E(z_1, \ldots, z_n, \lambda)$ of conformal blocks as the vector space of linear forms $G : V_1^\wedge \otimes \cdots \otimes V_n^\wedge \to \mathbb{C}$ such that, for $X \in g(z_1, \ldots, z_n, \lambda)$ and $v \in V_1^\wedge \otimes \cdots \otimes V_n^\wedge$, G satisfies $\langle G, \pi(X)v\rangle = 0$. That is, G is required to be invariant under the action of $g(z_1, \ldots, z_n, \lambda)$.

We will also want to vary the parameters $(z_1, \ldots, z_n, \lambda)$. The behavior under variations of these parameters is determined by a further condition. This condition tells that conformal blocks are flat with respect to the FS connection.

In the following, we let $(z_1, \ldots, z_n, \lambda)$ take values in an open subset $U^{[n]} \subset \Sigma^{[n]}(\tau)$. We denote by $P^{[n]}(\tau)$ the set

$$P^{[n]}(\tau) = \{(z_1, \ldots, z_n, \lambda, z) \in U^{[n]} \times \mathbb{C} | \, z \in L(z_1, \ldots, z_n)\}. \tag{431}$$

Then we define $g(U^{[n]})$ to be the Lie algebra of meromorphic functions $X : U^{[n]} \times \mathbb{C} \to g$, holomorphic on $(U^{[n]} \times \mathbb{C}) \setminus P^{[n]}(\tau)$, such that, for $(z_1, \ldots, z_n, \lambda) \in U^{[n]}$, $X(z_1, \ldots, z_n, \lambda, \cdot) \in g(z_1, \ldots, z_n, \lambda)$. It generalizes our previous definition.

To introduce the FS connection, we require the following auxiliary function. Let $\rho : U^{[n]} \times \mathbb{C} \to \mathbb{C}$ be a meromorphic function, holomorphic on $(U^{[n]} \times \mathbb{C}) \setminus P^{[n]}(\tau)$, such that, for $(z_1, \ldots, z_n, \lambda) \in U^{[n]}$, $\rho(z) = \rho(z_1, \ldots, z_n, \lambda, z)$ satisfies $\rho(z+1) = \rho(z)$ and $\rho(z+\tau) = \rho(z) - 2\pi i$. The example which we will have in mind is

$$\rho(z) = \frac{\Theta_1'(z - z_j, \tau)}{\Theta_1(z - z_j, \tau)}, \tag{432}$$

where $\Theta_1(z, \tau) = \vartheta_1(\pi z, \tau)$ is a Jacobi theta function, see [GR80]. But the following considerations will be independent of the particular choice of ρ.

We define first order differential operators on $g(U^{[n]})$ by the formulas

$$D_{z_j} X = \partial_{z_j} X, \quad D_\lambda X = \partial_\lambda X + [H \otimes \rho, X]. \tag{433}$$

We leave it as an exercise to show that they map $g(U^{[n]})$ to itself.

Let us now introduce first order differential operators on the space of holomorphic functions $G : U^{[n]} \to (V_1^\wedge \otimes \cdots \otimes V_n^\wedge)'$, the space of multilinear formas on $V_1^\wedge \otimes \cdots \otimes V_n^\wedge$, as follows

$$\nabla_{z_j} G = \partial_{z_j} G - G \, L_{-1}^{(j)}, \quad \nabla_\lambda G = \partial_\lambda G - G \, \pi(H \otimes \rho). \tag{434}$$

Here $\pi(H \otimes \rho)$ is given by (422) and $\pi(\rho \partial_z)$ by (428).

The FS connection is the connection

$$\nabla = \sum_j dz_j \, \nabla z_j + d\lambda \, \nabla_\lambda \tag{435}$$

on the infinite rank trivial vector bundle $U^{[n]} \times (V_1^\wedge \otimes \cdots \otimes V_n^\wedge)'$. For $G : U^{[n]} \to (V_1^\wedge \otimes \cdots \otimes V_n^\wedge)'$ and $X \in g(U^{[n]})$, it obeys

$$\nabla(G \, \pi(X)) = (\nabla G) \, \pi(X) + G \, \pi(DX), \tag{436}$$

where D denotes the expression

$$D = \sum_j dz_j \, D_{z_j} + d\lambda D_\lambda. \tag{437}$$

This property ensures that the connection restricts properly to the subbundle of conformal blocks. The restriction of ∇ to conformal blocks is flat.

Let $E(U^{[n]})$ be the space of holomorphic functions $G : U^{[n]} \to (V_1^\wedge \otimes \cdots \otimes V_n^\wedge)'$ such that, for $(z_1, \ldots, z_n, \lambda) \in U^{[n]}$, $G(z_1, \ldots, z_n, \lambda)$ is an element of $E(z_1, \ldots, z_n, \lambda)$. Due to (436), the FS connection leaves $E(U^{[n]})$ invariant. Elements of $E(U^{[n]})$ are called holomorphic conformal blocks.

The horizontality condition is that $G \in E(U^{[n]})$ is required to satisfy the equations

$$\nabla_{z_j} G = 0, \quad \nabla_\lambda G = 0, \tag{438}$$

for fixed modular parameter τ. Eq. (438) completes the construction of holomorphic conformal blocks for the purpose of these notes. It is also possible to vary consistently the parameter τ, but we will not need this here.

7.5 Knizhnik-Zamolodchikov-Bernard equation

In this section we will consider holomorphic conformal blocks G restricted to the subspace $V_1 \otimes \cdots \otimes V_n$ of $V_1^\wedge \otimes \cdots \otimes V_n^\wedge$. The value of a conformal block on general vectors in $V_1^\wedge \otimes \cdots \otimes V_n^\wedge$ can be computed as the image of certain differential operators in λ with coefficients depending on $(z_1, \ldots, z_n, \lambda)$ acting on G, evaluated on certain vectors in $V_1 \otimes \cdots \otimes V_n$.

The condition (438) can be written in a more explicit form, when the conformal blocks are restricted to the subspace $V_1 \otimes \cdots \otimes V_n$ of $V_1^\wedge \otimes \cdots \otimes V_n^\wedge$. For $v \in V_1 \otimes \cdots \otimes V_n$, the horizontality condition $\nabla_{z_j} G = 0$ reads

$$(k+2)\partial_{z_j}\langle G, v\rangle = \langle G, (\frac{1}{2}H_{-1}H_0 + E_{-1}F_0 + F_{-1}E_0)^{(j)} v\rangle. \tag{439}$$

Recall that V_j is annihilated by E_n, F_n, and H_n with $n > 0$.

To proceed we require another auxiliary function $h : U^{[n]} \times \mathbb{C} \to \mathbb{C}$. It is the meromorphic function $h(\lambda, z) = h(z_1, \ldots, z_n, \lambda, z)$ given by the ratio

$$h(z) = \frac{\Theta_1(z - 2\lambda, \tau)\, \Theta_1'(0, \tau)}{\Theta_1(z, \tau)\, \Theta_1(-2\lambda, \tau)} \tag{440}$$

of Jacobi theta functions.

The functions $E \otimes h(\lambda, z - z_j)$ and $F \otimes h(-\lambda, z - z_j)$ are elements of $g(U^{[n]})$. As functions of z they have simple poles at $z = z_j$. For $w \in V_1 \otimes \cdots \otimes V_n$, we then compute

$$\begin{gathered}\langle G, \pi(E \otimes h(\lambda, z - z_j))w\rangle = \\ \langle G, \{(E_{-1} + \frac{\Theta_1'(-2\lambda, \tau)}{\Theta_1(-2\lambda, \tau)}\, E_0)^{(j)} + \sum_{k \neq j} h(-\lambda, z_k - z_j)\, E_0^{(k)}\} w\rangle\end{gathered} \tag{441}$$

and

$$\begin{gathered}\langle G, \pi(F \otimes h(-\lambda, z - z_j))w\rangle = \\ \langle G, \{(F_{-1} + \frac{\Theta_1'(2\lambda, \tau)}{\Theta_1(2\lambda, \tau)}\, F_0)^{(j)} + \sum_{k \neq j} h(-\lambda, z_k - z_j)\, F_0^{(k)}\} w\rangle.\end{gathered} \tag{442}$$

But due to $g(U^{[n]})$-invariance both right hand sides are zero. As a result we can express $\langle G, E_{-1}^{(j)} w\rangle$ and $\langle G, F_{-1}^{(j)} w\rangle$ in terms of G evaluated on certain vectors in $V_1 \otimes \cdots \otimes V_n$.

For $w \in V_1 \otimes \cdots \otimes V_n$, the second horizontality condition $\nabla_\lambda G = 0$ takes the form

$$\partial_\lambda \langle G, w\rangle = \langle G, \{H_{-1}^{(j)} + \sum_{k \neq j} \frac{\Theta_1'(z_k - z_j, \tau)}{\Theta_1(z_k - z_j, \tau)} H_0^{(k)}\} w\rangle. \tag{443}$$

From it we obtain an expression for $\langle G, H_{-1}^{(j)} w\rangle$.

Both horizontality conditions together in this explicit form yield the differential equation

$$(k+2)\partial_{z_j} G = (\frac{1}{2}\partial_\lambda + \frac{\Theta_1'(2\lambda, \tau)}{\Theta_1(2\lambda, \tau)}) G\, H_0^{(j)} - \sum_{k \neq j} G\, \Omega^{(k,j)}, \tag{444}$$

with

$$\begin{aligned}\Omega^{(k,j)} = {} & h(\lambda, z_k - z_j)\, E_0^{(k)}\, F_0^{(j)} + h(-\lambda, z_k - z_j)\, F_0^{(k)}\, E_0^{(j)} + \\ & \frac{1}{2} \frac{\Theta_1'(z_k - z_j, \tau)}{\Theta_1(z_k - z_j, \tau)}\, H_0^{(k)}\, H_0^{(j)}.\end{aligned} \tag{445}$$

This equation is called the KZB equation. The tensor $\Omega^{(k,j)}$ is a unitary solution of a genus one generalization of the classical Yang-Baxter equation. The Yang-Baxter equation is a consistency condition reflecting the fact that the KZB equation comes from an integrable connection.

The calculation of this section shows one part of the following theorem.

Theorem 7.1 *Let $U^{[n]}$ be any sufficiently small neighborhood of a point in $\Sigma^{[n]}(\tau)$. Let $E_0(U^{[n]})$ be the space of holomorphic functions on $U^{[n]}$ with values in the linear forms on $V_1 \otimes \cdots \otimes V_n$ invariant under the action of the Cartan subalgebra $\mathbb{C}H$. Then the natural map*

$$E(U^{[n]}) \to E_0(U^{[n]}) \tag{446}$$

restricts to an isomorphism between the space of conformal blocks obeying the horizontality condition (438) and the space of local solutions in $E_0(U^{[n]})$ of the KZB equation.

7.6 Weyl invariance

In this section we introduce the notion of solutions invariant under a doubly affine version of the Weyl group of $\mathfrak{sl}_2(\mathbb{C})$.

For $\tau \in \mathrm{H}_+$, let $W(\tau)$ be the group of transformations of $\mathbb{C}$ given by $(\sigma, \omega)(\lambda) = \sigma\lambda + \omega$, where $\sigma \in \{\pm 1\}$ and $\omega \in L(\tau)$. That is, $W(\tau)$ is a semidirect product of $\mathbb{Z}_2$ with $\mathbb{Z} \times \mathbb{Z}$.

Let us fix $(z_1,\dots,z_n)\in\mathcal{C}^{[n]}(\tau)$. For $\gamma\in W(\tau)$, the meromorphic Lie algebras $g(z_1,\dots,z_n,\lambda)$ and $g(z_1,\dots,z_n,\gamma(\lambda))$ are isomorphic. We have explicit formulas for isomorphisms

$$\phi(\gamma): g(z_1,\dots,z_n,\lambda)\to g(z_1,\dots,z_n,\gamma(\lambda)) \tag{447}$$

for the generators $(-1,0)$, $(1,1)$, and $(1,\tau)$ of $W(\tau)$. They are given by

$$\phi(-1,0)X(z)=\exp(\frac{\pi i}{2}\mathrm{ad}_H)\exp(\frac{\pi i}{2}\mathrm{ad}_{E+F})\,X(z) \tag{448}$$

$$\phi(1,1)X(z)=X(z)\quad \phi(1,\tau)X(z)=\exp(2\pi i\mathrm{ad}_H)X(z). \tag{449}$$

We have seen that $g(z_1,\dots,z_n,\lambda)$ acts on $V_1^\wedge\otimes\cdots\otimes V_n^\wedge$. This action has been denoted by π. There exists an action Φ of $W(\tau)$ on $V_1^\wedge\otimes\cdots\otimes V_n^\wedge$ such that, for $\gamma\in W(\tau)$,

$$\pi(\phi(\gamma)X)\Phi(\gamma)=\Phi(\gamma)\pi(X). \tag{450}$$

For the generators of $W(\tau)$, this action can be written explicitly in the form

$$\Phi(-1,0)=\otimes_j(\exp(\frac{\pi i}{2}H_0)\exp(\frac{\pi i}{2}(E_0+F_0)))^{(j)}, \tag{451}$$

$$\Phi(1,1)=\mathrm{id},\quad \Phi(1,\tau)=(\exp(2\pi i(z_j\,H_0+H_1)))^{(j)}. \tag{452}$$

That is, the action os $W(\tau)$ is intertwined by isomorphisms of $V_1^\wedge\otimes\cdots\otimes V_n^\wedge$.

For $G\in E(z_1,\dots,z_n,\gamma(\lambda))$, it follows that $G\,\Phi(\gamma)\in E(z_1,\dots,z_n,\lambda)$. At this point we want to vary the parameters again. For the generators of $W(\tau)$, we define functions

$$g_{(-1,0)}(z_1,\dots,z_n,\lambda)=\exp(2\pi\,k\,\lambda), \tag{453}$$

$$g_{(1,1)}(z_1,\dots,z_n,\lambda)=g_{(1,\tau)}(z_1,\dots,z_n,\lambda)=1. \tag{454}$$

Let γ be a generator of $W(\tau)$. We then define transformed conformal blocks by

$$G_\gamma(z_1,\dots,z_n,\lambda)=g_\gamma(z_1,\dots,z_n,\lambda)G(z_1,\dots,z_n,\gamma(\lambda))\Phi(\gamma). \tag{455}$$

It can be shown that G_γ is a holomorphic conformal block provided that G is holomorphic conformal block. G_γ satisfies the KZB equation if G does. A solution is then called invariant under $W(\tau)$ if it obeys

$$G_\gamma(z_1,\dots,z_n,\lambda)=G(z_1,\dots,z_n,\lambda). \tag{456}$$

The condition of invariance under the doubly affine Weyl group, and regularity of G as a function of $\lambda\in\mathbb{C}$ selects a finite dimensional space of solutions to the KZB equation.

If the level is a positive integer, and the highest weights are integrable (see [K84]) the space of conformal blocks contains, according to [TUY89] (for $\lambda=0$, but the proof in [TUY89] extends to the case of general λ) a finite dimensional subspace of conformal blocks that are defined on the tensor product of irreducible quotients of Verma modules. The horizontal sections in this subspace are conjecturally contained in the above space of solutions, (see [FG92], where this is proven in a slightly different setting). Moreover in this case one can write down integral representations of solutions.

8 Conformal blocks on elliptic curves and the Knizhnik–Zamolodchikov–Bernard equations

We give an explicit description of the vector bundle of WZW conformal blocks on elliptic curves with marked points as subbundle of a vector bundle of Weyl group invariant vector valued theta functions on a Cartan subalgebra. We give a partly conjectural characterization of this subbundle in terms of certain vanishing conditions on affine hyperplanes. In some cases, explicit calculation are possible and confirm the conjecture. The Friedan–Shenker flat connection is calculated, and it is shown that horizontal sections are solutions of Bernard's generalization of the Knizhnik–Zamolodchikov equation.

8.1 Introduction

The aim of this chapter is to give a description of conformal blocks of the Wess–Zumino–Witten model on genus one curves as explicit as on the Riemann sphere.

Let us recall the well-known situation on the sphere. One fixes a simple finite dimensional complex Lie algebra $\mathfrak{g}$, with invariant bilinear form $(\ , \ )$, normalized so that the longest roots have length squared 2, and a positive integer k called level. One then considers the corresponding affine Kac–Moody Lie algebra, the one dimensional central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}((t))$ associated to the 2-cocycle $c(X \otimes f, Y \otimes g) = (X, Y)\,\mathrm{res}\, df g$. Its irreducible highest weight integrable representations of level (= value of central generator) k are in one to one correspondence with a certain finite set I_k of finite dimensional irreducible representations of $\mathfrak{g}$. These representations extend, by the Sugawara construction, to representations of the affine algebra to which an element L_{-1} is adjoined, such that $[L_{-1}, X \otimes f] = -X \otimes \frac{d}{dt} f$. Then to each n-tuple of distinct points $z_1, \ldots, z_n$ on the complex plane, and of representations $V_1, \ldots, V_n$ in I_k one associates the space of conformal blocks $E(z_1, \ldots, z_n)$. It is the space of linear forms on the tensor product $\otimes_1^n V_i^\wedge$ of the corresponding level k representations of the affine algebra, which are annihilated by the Lie algebra $\mathcal{L}(z_1, \ldots, z_n)$ of $\mathfrak{g}$-valued meromorphic functions with poles in $\{z_1, \ldots, z_n\}$ and regular at infinity. The latter algebra acts on $\otimes V_i^\wedge$ by viewing $\mathcal{L}(z_1, \ldots, z_n)$ as a Lie subalgebra of the direct sum of n copies of the loop algebra via Laurent expansion at the poles. The central extension does not cause problems as the corresponding cocycle vanishes on $\mathcal{L}(z_1, \ldots, z_n)$ in virtue of the residue theorem.

It turns out that the spaces $E(z_1, \ldots, z_n)$ are finite dimensional and are the fibers of a holomorphic vector bundle over the configuration space $\mathbb{C}^n -$ diagonals, carrying the flat connection $d - \sum_i dz_i L_{-1}^{(i)}$ ($L_{-1}^{(i)}$ acts on the right of a linear form) given in terms of the Sugawara construction. We use the notation $X^{(i)} = \cdots \otimes \mathrm{Id} \otimes X \otimes \mathrm{Id} \cdots$ to denote the action of on the ith factor of a tensor product.

This part of the construction generalizes to surfaces of arbitrary genus (see [TUY89]). What is new is that one has to also specify local coordinates around the points z_i to give a meaning to the Laurent expansion, and that the connection is in general only projectively flat (i.e., the curvature is a multiple of the identity).

To give a more explicit description of the vector bundle of conformal blocks on the sphere, and in particular to compute the holonomy of the connection, one observes that the map $E(z_1,\dots,z_n)\to(\otimes_i V_i)^*$ given by restriction to $V_i\subset V_i^\wedge$ is injective. Thus we can view E as a subbundle of a trivial vector bundle of finite rank. This subbundle can be described by an explicit algebraic condition [FSV]. After this identification the connection can be given in explicit terms and the equation for horizontal sections reduces to the famous Knizhnik–Zamolodchikov equations

$$(k+h^\vee)\partial_{z_i}\omega(z_1,\dots,z_n)=\sum_{j:j\neq i}\sum_a\frac{T_a^{(i)}T_a^{(j)}}{z_i-z_j}\omega(z_1,\dots,z_n). \tag{457}$$

In this equation, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and T_a, $a=1,\dots,\dim(\mathfrak{g})$ is any orthonormal basis of $\mathfrak{g}$. We view here the dual spaces V_i^* as contragradient representations. Let us now consider the situation on genus one curves, which we view as $\mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$ for τ in the upper half plane. Let us denote by $E(z_1,\dots,z_n,\tau)$ the space of conformal blocks. Again, by [TUY89], this is the finite dimensional fiber of a holomorphic vector bundle with flat connection on the elliptic configuration space $C^{[n]}$ of $n+1$-tuples $(z_1,\dots,z_n,\tau)$ with $\mathrm{Im}(\tau)>0$ and $z_i\neq z_j \bmod \mathbb{Z}+\tau\mathbb{Z}$ if $i\neq j$.

The trouble is that the restriction to $\otimes_i V_i$ is no longer injective, the reason being that there are no meromorphic functions on elliptic curves with one simple pole only. The way out is the following construction which brings the moduli space of flat G-bundles into the game. Consider the Lie algebras $\mathcal{L}(z_1,\dots,z_n,\tau,\lambda)$, parametrized by λ in a Cartan subalgebra $\mathfrak{h}$, of $\mathbb{Z}$-periodic meromorphic functions $X:\mathbb{C}\to\mathfrak{g}$ with poles at $z_1,\ \dots,\ z_n$ modulo $\mathbb{Z}+\tau\mathbb{Z}$, such that $X(t+\tau)=\exp(2\pi i\,\mathrm{ad}\,\lambda)X(t)$.

These algebras act on $\otimes_i V_i^\wedge$ and we can define a space of (twisted) conformal blocks $E_{\mathfrak{h}}(z,\tau,\lambda)$ as space of invariant linear forms (see 8.2.3). The original space of conformal blocks is recovered by setting $\lambda=0$.

It turns out that $E_{\mathfrak{h}}(z,\tau,\lambda)$ is again the fiber over (z,τ,λ) of a holomorphic vector bundle $E_{\mathfrak{h}}$ over $C^{[n]}\times\mathfrak{h}$ with flat connection, whose restriction to $C^{[n]}\times\{0\}$ is E. Thus we can by parallel transport in the direction of $\mathfrak{h}$ identify the space of sections $E(U)$ of E over an open set $U\subset C^{[n]}$ with the space of sections of $E_{\mathfrak{h}}$ which are horizontal in the direction of $\mathfrak{h}$:

$$E(U)\simeq E_{\mathfrak{h}}(U\times\mathfrak{h})^{\mathrm{hor}}. \tag{458}$$

The point is now that the restriction map

$$E_{\mathfrak{h}}(U\times\mathfrak{h})^{\mathrm{hor}}\to(\otimes_i V_i)^*\otimes\mathcal{O}(\mathcal{U}\times\mathfrak{h}), \tag{459}$$

to $V^{[n]}$ is injective (Proposition 8.9). Composing these two maps we may view the vector bundle of conformal blocks as a subbundle of an explicitly given vector bundle on $C^{[n]}$ of finite rank. Indeed we show that the image is contained in the space of functions on $U\times\mathfrak{h}$ which have definite transformation properties (of theta function type) under translations of λ by $Q^\vee + \tau Q^\vee$ where $Q^\vee$ denotes the coroot lattice. Moreover the theta functions in the image are invariant under a natural action of the Weyl group, and obey a certain vanishing condition as the argument approaches affine root hyperplanes. We conjecture that these conditions characterize completely the image. This conjecture is confirmed in some cases, including a special case which arises [EK94] in the theory of quantum integrable many body problems (see 8.4.1): we describe explicitly the space of conformal blocks in the case of sl_N, $n = 1$, where the representation is any symmetric power of the defining N-dimensional representation.

The characterization of conformal blocks in terms of invariant theta functions obeying vanishing conditions was first given (in the sl_2 case) by Falceto and Gawędzki [FG92], who define conformal blocks as Chern–Simons states in geometric quantization.

After the identification of conformal blocks as subbundle of the "invariant theta function" bundle, we describe the connection in explicit terms (Theorem 8.24), and get a generalization of the Knizhnik–Zamolodchikov equations. These equations were essentially written by Bernard [Be88a, Be88b] in a in a slightly different context, and were recently reconsidered from a more geometrical point of view in [FG92]. They have the form (see Sect. 8.4)

$$\begin{aligned} \kappa\partial_{z_j}\tilde{\omega} &= -\sum_\nu h_\nu^{(j)}\partial_{\lambda_\nu}\tilde{\omega} + \sum_{l:l\neq j}\Omega^{(j,l)}(z_j - z_l, \tau, \lambda)\tilde{\omega}, && (460)\\ 4\pi i\kappa\partial_\tau\tilde{\omega} &= \sum_\nu \partial^2_{\lambda_\nu}\tilde{\omega} + \sum_{j,l}\mathrm{H}^{(j,l)}(z_j - z_l, \tau, \lambda)\tilde{\omega}, && (461) \end{aligned}$$

for some tensors Ω, $\mathrm{H} \in \mathfrak{g}\otimes\mathfrak{g}$, given in terms of Jacobi theta functions. Here $\tilde{\omega}$ is related to ω by multiplication by the Weyl–Kac denominator. Thus, the right way to look at these equations is to view u as a section of a subbundle of the vector bundle over the elliptic configuration space of $n+1$ tuples $(z_1, \ldots, z_n, \tau)$, whose fiber is a finite dimensional space of invariant theta functions.

In this chapter we do not discuss an alternative approach to conformal blocks on elliptic curves, which is in terms of traces of products of vertex operators. Bernard [Be88a] showed that such traces obey his differential equations. Using this formulation, integral representation of solutions were given in the sl_2 case in [BF90]. To complete the picture, one should show that those solutions are indeed theta functions with vanishing condition.

Let us also point out the paper [EFK94] that shows that the same space of invariant theta functions with vanishing condition can be identified with a space of equivariant functions on the corresponding Kac–Moody group.

8.2 Conformal blocks on elliptic curves

8.2.1 Elliptic configuration spaces

Let $\mathrm{H}_+ = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}\,\tau > 0\}$ be the upper half plane and for $\tau \in \mathrm{H}_+$ denote by $L(\tau)$ the lattice $\mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$. Let n be a positive integer. We define the elliptic configuration space to be the subset of $\mathbb{C}^n \times \mathrm{H}_+$ consisting of points $(z_1, \ldots, z_n, \tau)$ so that $z_i \neq z_j$ mod $L(\tau)$ if $i \neq j$.

The space of points $(z, \tau) \in C^{[n]}$ with fixed τ is a covering of the configuration space of n ordered points on the elliptic curve $\mathbb{C}/L(\tau)$.

8.2.2 Meromorphic Lie algebras

Let $\mathfrak{g}$ be a complex simple Lie algebra with dual Coxeter number $h^\vee$ and k be a positive integer. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ be the corresponding root space decomposition. The invariant bilinear form is normalized in such a way that $(\alpha^\vee, \alpha^\vee) = 2$ for long roots α (see [H80]). We choose an orthonormal basis (h_ν) of $\mathfrak{h}$. The symmetric invariant tensor $C \in \mathfrak{g} \otimes \mathfrak{g}$ dual to $(\ ,\ )$ admits then a decomposition $C = \sum_{\alpha \in \Delta \cup \{0\}} C_\alpha$, with $C_0 = \Sigma\, h_\nu \otimes h_\nu \in \mathfrak{h} \otimes \mathfrak{h}$ and $C_\alpha \in \mathfrak{g}_\alpha \otimes \mathfrak{g}_{-\alpha}$, if $\alpha \in \Delta$.

We define a family of Lie algebras of meromorphic functions with values in $\mathfrak{g}$ parametrized by $C^{[n]} \times \mathfrak{h}$

Definition: For $(z, \tau) = (z_1, \ldots, z_n, \tau) \in C^{[n]}$ and $\lambda \in \mathfrak{h}$, let $\mathcal{L}(z, \tau, \lambda)$ be the Lie algebra of meromorphic functions $t \mapsto X(t)$ on the complex plane with values in $\mathfrak{g}$ such that

$$X(t+1) = X(t), \qquad X(t+\tau) = \exp(2\pi i \operatorname{ad} \lambda) X(t), \tag{462}$$

and whose poles belong to $\cup_{i=1}^n z_i + L(\tau)$. More generally, for any open set $U \subset C^{[n]} \times \mathfrak{h}$ let $\mathcal{L}_{\mathfrak{h}}(U)$ be the Lie algebra of meromorphic functions $(t, z, \tau, \lambda) \mapsto X(t, z, \tau, \lambda)$ on $\mathbb{C} \times U$ with values in $\mathfrak{g}$, whose poles are on the hyperplanes $t = z_i + r + s\tau$, $1 \leq i \leq n$, $r, s \in \mathbb{Z}$, and such that for all $(z, \tau, \lambda) \in U$, the function $t \to X(t, z, \tau, \lambda)$ belongs to $\mathcal{L}(z, \tau, \lambda)$. Similarly, define $\mathcal{L}(U)$ for an open subset U of $C^{[n]}$ to be the Lie algebra of meromorphic functions $(t, z, \tau) \mapsto X(t, z, \tau)$ on $\mathbb{C} \times U$ with values in $\mathfrak{g}$, whose poles are on the same hyperplanes, and such that for all $(z, \tau) \in U$, the function $t \to X(t, z, \tau)$ belongs to $\mathcal{L}(z, \tau, 0)$.

An explicit description of these Lie algebras is given in Appendix 8.5.1. An important property is that they have a filtration by locally free finitely generated sheaves: Let $\mathcal{O}(\mathcal{U})$ be the algebra of holomorphic functions on an open set $U \subset C^{[n]} \times \mathfrak{h}$, and for any non-negative integer j let $\mathcal{L}_{\mathfrak{h}}^{\leq j}(U)$ be the $\mathcal{O}(\mathcal{U})$-submodule of $\mathcal{L}_{\mathfrak{h}}(U)$ consisting of functions whose poles have order at most j. Similarly we define $\mathcal{L}^{\leq j}(U)$ for open sets $U \in C^{[n]}$. The assignments $U \to \mathcal{L}^{\leq j}(U)$, $U \to \mathcal{L}_{\mathfrak{h}}^{\leq j}(U)$ are sheaves of $\mathcal{O}$-modules.

Proposition 8.1 *$\mathcal{L}_{\mathfrak{h}}^{\leq j}$ is a locally free, locally finitely generated sheaf of $\mathcal{O}$-modules. In other words, every point in $C^{[n]} \times \mathfrak{h}$ has a neighborhood U such that $\mathcal{L}_{\mathfrak{h}}^{\leq j}(U) \simeq \mathbb{C}^{n_j} \otimes \mathcal{O}(\mathcal{U})$ as an $\mathcal{O}(\mathcal{U})$-module, for some n_j. Moreover for each $x \in C^{[n]} \times \mathfrak{h}$, every $X \in \mathcal{L}(x)$ extends to a function in $\mathcal{L}_{\mathfrak{h}}^{\leq j}(U)$ for some j and $U \ni x$. The same results hold for $\mathcal{L}^{\leq j}$.*

The proof is contained in Appendix 8.5.1 (see Corollary 8.27).

8.2.3 Tensor product of affine Kac–Moody algebra modules

Let $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}((t))$ be the loop algebra of $\mathfrak{g}$. Fix a positive integer $k \in \mathbb{N}$. Let $L\mathfrak{g}\,\hat{} = L\mathfrak{g} \oplus \mathbb{C}K$ be the central extension of $L\mathfrak{g}$ associated with the 2-cocycle

$$c(X \otimes f, Y \otimes g) = (X, Y)\,\mathrm{res}(f' g \mathrm{d}t), \tag{463}$$

where the residue of a formal Laurent series is given by $\mathrm{res}(\sum_n a_n t^n \mathrm{d}t) = a_{-1}$. Thus the Lie bracket in $L\mathfrak{g}\,\hat{}$ has the form

$$[X \otimes f \oplus \zeta K, Y \otimes g \oplus \xi K] = [X, Y] \otimes fg \oplus c(X, Y)K. \tag{464}$$

With every irreducible highest weight $\mathfrak{g}$-module V is associated an irreducible highest weight $L\mathfrak{g}\,\hat{}$-module $V\hat{}$ of level k. Its construction goes as follows. The action of $\mathfrak{g}$ is first extended to the Lie subalgebra $\mathfrak{b}_+ = \mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}K$ of $L\mathfrak{g}\,\hat{}$, by letting $\mathfrak{g} \otimes t\mathbb{C}[[t]]$ act by zero and the central element K by k. Then a generalized Verma module $\tilde{V} = U(L\mathfrak{g}\,\hat{}) \otimes_{U(\mathfrak{b}_+)} V$ is induced. It is freely generated by (any basis of) V as a $\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$-module. The polynomial subalgebra $L\mathfrak{g}_P = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$ of $L\mathfrak{g}\,\hat{}$ is $\mathbb{Z}$-graded with $\deg(X \otimes t^j) = -j$. Since $\tilde{V} \simeq U(L\mathfrak{g}\,\hat{}_P) \otimes_{U(\mathfrak{b}_+ \cap L\mathfrak{g}\,\hat{}_P)} V$, the generalized Verma module is naturally graded. By definition the irreducible module $V\hat{}$ is the quotient of the generalized Verma module by its maximal proper graded submodule.

We will consider integrable modules, $V\hat{}$ of fixed level $k = 0, 1, 2, \ldots$. If we fix a set of simple roots $\alpha_1, \ldots, \alpha_l \in \Delta$, and denote by θ the corresponding highest root, $V\hat{}$ is integrable of level k if the irreducible $\mathfrak{g}$-module V has highest weight μ in the the subset

$$I_k = \{\mu \in P | (\mu, \alpha_i) \geq 0, \quad i = 1, \ldots, l, \quad (\mu, \theta) \leq k\}, \tag{465}$$

of the weight lattice P. Let v be the highest weight vector of V and e_θ a generator of $\mathfrak{g}_\theta$. Then the maximal proper submodule is generated by $(e_\theta \otimes t^{-1})^{k-(\mu,\theta)+1} v$.

The grading extends to $V\hat{}^{[n]}$ by setting $\deg(v_1 \otimes \cdots \otimes v_n) = \Sigma \deg(v_i)$. With our convention all homogeneous vectors have non-negative degree.

Fix n highest weight $\mathfrak{g}$-modules V_j, $1 \leq j \leq n$ such that the corresponding $L\mathfrak{g}\,\hat{}$-modules $V_j\hat{}$ are integrable of level k, and let $\tau \in \mathrm{H}_+$ and $z_1, \ldots, z_n$ complex numbers with $z_i \neq z_j \bmod L(\tau)$ if $i \neq j$. We think of $V_j\hat{}$ as an $L\mathfrak{g}\,\hat{}$-module which is attached to the point z_j.

In the following we will use the abbreviations $V^{[n]} = V_1 \otimes \cdots \otimes V_n$ and $V\hat{}^{[n]} = V_1\hat{} \otimes \cdots \otimes V_n\hat{}$.

We now construct an action of $\mathcal{L}(z, \tau, \lambda)$ on $V\hat{}^{[n]}$. For $X \in \mathcal{L}(z, \tau, \lambda)$ let $\delta_j(X) = X(z_j + t) \in \mathfrak{g} \otimes \mathbb{C}((t))$ be the Laurent expansion of X at z_j viewed as a formal Laurent series in t. Then

$$\delta(X) = \delta_1(X) \oplus \cdots \oplus \delta_n(X), \tag{466}$$

defines a Lie algebra embedding of $\mathcal{L}(z, \tau, \lambda)$ into $L\mathfrak{g} \oplus \cdots \oplus L\mathfrak{g}$. As a vector space $L\mathfrak{g} \oplus \cdots \oplus L\mathfrak{g}$ is embedded in $L\mathfrak{g}\,\hat{} \oplus \cdots \oplus L\mathfrak{g}\,\hat{}$. The embedding is of course not a Lie algebra homomorphism. Since $L\mathfrak{g}^\wedge \oplus \cdots \oplus L\mathfrak{g}^\wedge$ acts on $V^{\wedge[n]}$ we obtain a map from $\mathcal{L}(z,\tau,\lambda)$ to $\mathrm{End}_{\mathbb{C}}(V^{\wedge[n]})$. This map will also be denoted by δ. Thanks to the residue theorem it turns out to be a Lie algebra homomorphism.

Proposition 8.2 *For X, $Y \in \mathcal{L}(z,\tau,\lambda)$,*

$$\delta([X,Y]) = [\delta(X), \delta(Y)]. \tag{467}$$

Proof: In $\mathrm{End}_{\mathbb{C}}(V^{\wedge[n]})$ we have the equation

$$\delta([X,Y]) = [\delta(X), \delta(Y)] + k \sum_{j=1}^{n} \mathrm{res}_{t=z_j} \left((X'(t), Y(t))\, dt \right). \tag{468}$$

But $(X'(t), Y(t))$ is doubly periodic (by ad-invariance of $(\ ,\ )$) so that the sum of residues vanishes. □

8.2.4 Vector fields

The Lie algebra $\mathrm{Vect}(S^1) = \mathbb{C}((t))\frac{\mathrm{d}}{\mathrm{d}t}$ of formal vector fields on the circle acts by derivations on $L\mathfrak{g}$. Let us denote this action simply by $(\xi(t)\frac{d}{dt}, X(t)) \mapsto \xi(t)\frac{d}{dt}X(t)$. It extends to an action on $L\mathfrak{g}^\wedge$ by letting vector fields act trivially on the center. The Sugawara construction yields a projective representation of $\mathrm{Vect}(S^1)$ on $V^\wedge$, for any finite dimensional $\mathfrak{g}$-module V. The Sugawara operators $L_n \in \mathrm{End}(V^\wedge)$ are defined by choosing any basis $\{B_1, \ldots, B_d\}$ of $\mathfrak{g}$, with dual basis $\{B^1, \ldots, B^d\}$ of $\mathfrak{g}$ so that $(B^a, B_b) = \delta_{ab}$, and setting

$$L_n = \frac{1}{2(k+h^\vee)} \sum_{m\in\mathbb{Z}} \sum_{a=1}^{d} (B^a \otimes t^{n-m})(B_a \otimes t^m), \qquad n \neq 0 \tag{469}$$

$$L_0 = \frac{1}{2}[L_1, L_{-1}]. \tag{470}$$

These operators are independent of the choice of basis and obey the commutations relations of the Virasoro algebra $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)$ with central charge $c = k\dim(\mathfrak{g})/(k+h^\vee)$. Then

$$\sum_n \xi_n t^{n+1} \frac{d}{dt} \mapsto -\sum_n \xi_n L_n \in \mathrm{End}(V^\wedge), \tag{471}$$

defines a projective representation of $\mathrm{Vect}(S^1)$ on $V^\wedge$, with the intertwining property $[\xi(t)\frac{d}{dt}, X(t)] = \xi(t)\frac{d}{dt}X(t)$, for any $X(t) \in L\mathfrak{g}^\wedge$. Note that all infinite sums are actually finite when acting on any vector in $V^\wedge$.

8.2.5 Conformal blocks

For a Lie algebra module V we denote by V^* the dual vector space with natural (right) action of the Lie algebra. The notation $\langle\omega, v\rangle$ will be used to denote the evaluation of a linear form ω on a vector v.

Definition: The space of *twisted conformal blocks* associated to data $\mathfrak{g}$, k, $V_1, \ldots, V_n$ as above, is the space $E_{\mathfrak{h}}(z, \tau, \lambda)$ of linear functionals on $V^{\wedge[n]}$ annihilated by $\mathcal{L}(z_1, \ldots, z_n, \tau, \lambda)$. If $\lambda = 0$, then $E(z, \tau) = E_{\mathfrak{h}}(z, \tau, 0)$ is called the space of *conformal blocks* at (z, τ).

Let us vary the parameters: let U be an open subset of $C^{[n]} \times \mathfrak{h}$. Then the space of holomorphic functions $\omega : U \to V^{\wedge[n]*}$, (i.e., of functions ω whose evaluation $\langle\omega, u\rangle$ on any fixed vector $u \in V^{\wedge[n]}$ is holomorphic on U), is a right $\mathcal{L}_{\mathfrak{h}}(U)$-module.

Definition: The space $E_{\mathfrak{h}}(U)$ of *holomorphic twisted conformal blocks on* $U \subset C^{[n]} \times \mathfrak{h}$ is the space of $V^{\wedge[n]*}$-valued holomorphic functions ω, so that for all open subsets U' of U, the restriction of ω to U' is annihilated by $\mathcal{L}_{\mathfrak{h}}(U')$. We also define the space $E(U)$ of *holomorphic conformal blocks on* $U \subset C^{[n]}$ by replacing $\mathcal{L}_{\mathfrak{h}}$ by $\mathcal{L}$.

With this definition, the assignments $U \mapsto E_{\mathfrak{h}}(U)$, $U \mapsto E(U)$ are sheaves of $\mathcal{O}$-modules.

Lemma 8.3 *Let U be an open subset of $C^{[n]} \times \mathfrak{h}$ (resp. of $C^{[n]}$). Then $\omega \in E_{\mathfrak{h}}(U)$ (resp. $E(U)$) if and only if ω is holomorphic on U and $\omega(x) \in E_{\mathfrak{h}}(x)$ (resp. $E(x)$) for all $x \in U$.*

Proof: It is obvious that if ω is holomorphic and if $\omega(x) \in E_{\mathfrak{h}}(x)$ for all $x \in U$, then $\omega \in E_{\mathfrak{h}}(U)$. Let $\omega \in E_{\mathfrak{h}}(U)$, and $x \in U$. To show that $\omega(x) \in E_{\mathfrak{h}}(x)$, we have to show that every element X of $\mathcal{L}(x)$ is the restriction of an element of $\mathcal{L}_{\mathfrak{h}}(U')$ for some neighborhood U' of x. But this follows from Prop. 8.1. The same applies in the untwisted case. $\square$

8.3 Flat connections, theta functions

8.3.1 The flat connection

For each open subset U of $C^{[n]} \times \mathfrak{h}$ we have defined a Lie algebra $\mathcal{L}_{\mathfrak{h}}(U)$ acting on $V^{\wedge[n]}(U)$, the space of holomorphic functions on U with values in $V^{\wedge[n]}$. It is convenient to extend this definition. Let G be the simply connected complex Lie group whose Lie algebra is $\mathfrak{g}$, and for $(z, \tau, g) \in C^{[n]} \times G$, let $\mathcal{L}(z, \tau, g)$ be the Lie algebra of meromorphic $\mathfrak{g}$-valued functions $X(t)$, on the complex plane whose poles modulo L_τ belong to $\{z_1, \ldots, z_n\}$, and with multipliers

$$X(t+1) = X(t), \quad X(t+\tau) = \mathrm{Ad}(g)X(t). \tag{472}$$

If U is an open subset of $C^{[n]} \times G$, define $\mathcal{L}_G(U)$ to be the Lie algebra of meromorphic functions on $U \times \mathbb{C} \ni (z, \tau, g, t)$ whose poles are on the hyperplanes $t = z_i + n + m\tau$,

$n, m \in \mathbb{Z}$, and restricting to functions in $\mathcal{L}(z,\tau,g)$ for fixed $(z,\tau,g) \in U$. As above, we introduce the space $E_G(z,\tau,g)$ of $\mathcal{L}(z,\tau,g)$-invariant linear forms on $V^{\wedge[n]}$, and the sheaf $U \to E_G(U)$ of $\mathcal{L}_G(U)$ invariant holomorphic $V^{\wedge[n]*}$-valued functions.

Let $\eta(z,\tau,t)$ be a meromorphic function on $C^{[n]} \times \mathbb{C}$ whose poles belong to the hyperplanes $t = z_i + n + m\tau$ and such that, as function of $t \in \mathbb{C}$,

$$\eta(z,\tau,t+1) = \eta(z,\tau,t), \qquad \eta(z,\tau,t+\tau) = \eta(z,\tau,t) - 2\pi i. \tag{473}$$

Although the construction does not depend on which η we chose, we will always set

$$\eta(z,\tau,t) = \rho(t - z_1,\tau), \tag{474}$$

$$\rho(t,\tau) = \frac{\partial}{\partial t}\log\theta_1(t|\tau), \tag{475}$$

$$\theta_1(t|\tau) = -\sum_{j=-\infty}^{\infty} e^{\pi i(j+\frac{1}{2})^2\tau + 2\pi i(j+\frac{1}{2})(t+\frac{1}{2})}, \tag{476}$$

for definiteness.

Let $A_Y(z,\tau,g,t)$ be a meromorphic function on $C^{[n]} \times G \times \mathbb{C}$, depending linearly on $Y \in \mathfrak{g}$, whose poles as a function of t belong to $\{z_1,\dots,z_n\}$, and such that

$$A_Y(z,\tau,g,t+1) = A_Y(z,\tau,g,t), \tag{477}$$

$$A_Y(z,\tau,g,t+\tau) = \mathrm{Ad}(g)(A_Y(z,\tau,g,t) - Y). \tag{478}$$

If $\psi(A) = \sum_{j\in\mathbb{Z}} e^{i\pi(j+\frac{1}{2})^2\tau}(-A)^j$, we may take A_Y to be

$$A_Y(z,\tau,g,t) = \frac{1}{1-\mathrm{Ad}(g^{-1})}\Big(1 - \frac{\psi(e^{2\pi i(t-z_1)}\mathrm{Ad}(g^{-1}))}{\psi(e^{2\pi i(t-z_1)})}\Big)Y. \tag{479}$$

Note that for fixed z,τ,t and Y, A_Y extends to a regular function of $g \in G$.

Denote by ∂_Y the derivative in the direction of the left invariant vector field on G associated to $Y \in \mathfrak{g}$: $\partial_Y f(g) = \lim_{\epsilon\to 0}\frac{d}{d\epsilon}f(g\exp\epsilon Y)$, and by ∂_{z_i}, ∂_τ, ∂_t the partial derivatives with respect to the coordinates z_i,τ,t of $C^{[n]} \times \mathbb{C}$.

The properties (473), (477) imply the

Proposition 8.4 *Let U be an open subset of $C^{[n]} \times G$. The differential operators*

$$D_{z_i}X(x,t) = \partial_{z_i}X(x,t), \tag{480}$$

$$D_\tau X(x,t) = \partial_\tau X(x,t) - \frac{1}{2\pi i}\eta(x,t)\partial_t X(x,t), \tag{481}$$

$$D_Y X(x,t) = \partial_Y X(x,t) + [A_Y(x,t), X(x,t)], \qquad x = (z,\tau,g) \in U \times \mathbb{C}, \tag{482}$$

map $\mathcal{L}_G(U)$ to itself.

Therefore, we have a connection $D : \mathcal{L}_G(U) \to \Omega^1(U) \otimes \mathcal{L}_G(U)$, defined by $D = \sum dz_j \otimes D_{z_j} + d\tau \otimes D_\tau + \sum \theta_a \otimes D_{\theta^a}$, for any basis of left invariant vector fields θ^a on G with dual basis θ_a.

We proceed to define a connection on E_G. Consider first the following differential operators on the space $V^{\wedge[n]}(U)$ of $V^{\wedge[n]}$-valued holomorphic functions on the open set $U \subset C^{[n]} \times G$.

$$\nabla_{z_i} v(x) = \partial_{z_i} v(x) + L_{-1}^{(i)} v(x), \tag{483}$$

$$\nabla_\tau v(x) = \partial_\tau v(x) - \frac{1}{2\pi i}\delta(\eta(x)\partial_t) v(x), \tag{484}$$

$$\nabla_Y v(x) = \partial_Y v(x) + \delta(A_Y(x)) v(x), \qquad x \in U. \tag{485}$$

In this formula the definition of the operator δ taking the Laurent expansion at the points z_i (see (466)) is extended to general meromorphic $\mathfrak{g}$-valued functions and vector fields considered as a function of the variable $t \in \mathbb{C}$. For a meromorphic vector field $\xi = \xi(t)\frac{d}{dt}$ on the complex plane we set $\delta(\xi) = \Sigma\delta_i(\xi)$, with $\delta_i(\xi) = \xi(z_i + t)\frac{d}{dt} \in \mathbb{C}((t))\frac{d}{dt}$. Let $\nabla : V^{\wedge[n]}(U) \to \Omega^1(U) \otimes V^{\wedge[n]}(U)$ denote the connection $\sum dz_j \otimes \nabla_{z_j} + d\tau \otimes \nabla_\tau + \sum \theta_a \otimes \nabla_{\theta^a}$,

Proposition 8.5 *The connections D, ∇ obey the compatibility condition*

$$\nabla(Xv) = (DX)v + X\nabla v, \qquad X \in \mathcal{L}_G(U), \quad v \in V^{\wedge[n]}. \tag{486}$$

Proof: This is verified by explicit calculation. □

This has the following consequence. Define ∇ on holomorphic functions ω on U with values in the dual $V^{\wedge[n]*}$ (i.e., such that $\langle\omega(x), v\rangle$ is holomorphic on U for all $v \in V^{\wedge[n]}$) by the formula $\langle\nabla\omega(x), v(x)\rangle = d\langle\omega(x), v(x)\rangle - \langle\omega(x), \nabla v(x)\rangle$.

Corollary 8.6 *The connection ∇ preserves twisted holomorphic conformal blocks, i.e., it maps $E_G(U)$ to $\Omega^1(U) \otimes E_G(U)$.*

Proposition 8.7 *The connection ∇ on $E_G(U)$ is flat.*

Proof: For $X, Y \in \mathfrak{g}$, the curvature $F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is given by the expression

$$F(X,Y) = \partial_X \delta(A_Y) - \partial_Y \delta(A_X) + [\delta(A_X), \delta(A_Y)] - \delta(A_{[X,Y]}). \tag{487}$$

Note that the cocycle

$$\int_\gamma (\frac{d}{dt} A_X, A_Y) dt, \tag{488}$$

vanishes: indeed, the integrand $I(t)$ is $\mathbb{Z}$-periodic and obeys $I(t+\tau) = I(t) + \frac{d}{dt}g(t)$ for some $\mathbb{Z}$-periodic function $g(t)$ and the integration cycle γ can be decomposed into a sum of contours bounding some fundamental domains. The contributions of the four edges cancel by periodicity, except for a term $\int_x^{x+1} g'(t)dt = 0$.

Thus we can write F as

$$F(X,Y) = \delta(\tilde{F}(X,Y)), \tag{489}$$

$$\tilde{F}(X,Y) = \partial_X A_Y - \partial_Y A_X + [A_X, A_Y] - A_{[X,Y]}. \tag{490}$$

Now, as a simple calculation shows, $\tilde{F}(X,Y)$, viewed as a function of $t \in \mathbb{C}$ with values is $\mathfrak{g}$, is $\mathbb{Z}$-periodic, and obeys $\tilde{F}(X,Y)(t+\tau) = \mathrm{Ad}(g)\tilde{F}(X,Y)(t)$, as a consequence of (477). Thus $F(X,Y)$ is in the image of $\mathcal{L}_G(U)$, and vanishes on invariant linear forms.

A similar reasoning applies to the commutators $[\nabla_{z_i}, \nabla_X]$, $[\nabla_\tau, \nabla_X]$, $X \in \mathfrak{g}$. These commutators are also in the image of $\mathcal{L}_G(U)$ and thus vanish on invariant forms. We are left with the commutator $[\nabla_\tau, \nabla_{z_i}]$, which vanishes except possibly for $i = 1$. The proof that it vanishes also for $i = 1$ will be given later on (see 8.4.1). $\square$

The group G acts on $V^{\wedge[n]}$, since the cocycle vanishes on $\mathfrak{g} \subset L\mathfrak{g}^\wedge$. Denote this action simply $G \times V^{\wedge[n]} \ni (h, v) \mapsto hv$.

Proposition 8.8 *For all $h \in G$, $X \to \mathrm{Ad}(h)X$ is a Lie algebra isomorphism from $\mathcal{L}(z,\tau,g)$ to $\mathcal{L}(z,\tau,hgh^{-1})$. Thus the map $X \mapsto \phi_h X$ with $\phi_h X(z,\tau,g) = \mathrm{Ad}(h)X(z,\tau,h^{-1}gh)$ is an isomorphism from $\mathcal{L}_G(U)$ to $\mathcal{L}_G(U')$ for any open $U \subset C^{[n]} \times G$, where $U' = \{(z,\tau,hgh^{-1}) | (z,\tau,g) \in U\}$. Moreover, for any $X \in \mathcal{L}_G(U)$, $\delta(\mathrm{Ad}(h)X) = h\delta(X)h^{-1}$, and thus $\rho_h\omega(z,\tau,g) = \omega(z,\tau,hgh^{-1})h$ defines an isomorphism $\rho_h : E_G(U') \to E_G(U)$. This isomorphism maps horizontal sections to horizontal sections.*

Proof: The first statement follows immediately from the definitions. The fact that $\delta(\mathrm{Ad}(h)X) = h\delta(X)h^{-1}$ is also clear, once one notices that the 2-cocycle defining the central extension vanishes if one of the arguments is a constant Lie algebra element. Finally h commutes with ∇_{z_i} and ∇_τ, and we have $\nabla_X \rho_h = \rho_h \nabla_{\mathrm{Ad}(h)X}$, $X \in \mathfrak{g}$. The latter identity follows from the equality (see (479))

$$\mathrm{Ad}(h)A_X(z,\tau,g,t) = A_{\mathrm{Ad}(h)X}(z,\tau,hgh^{-1},t). \tag{491}$$

Thus ρ_h preserves horizontality. $\square$

The existence of a connection implies, as in [TUY89], that the sheaf $U \mapsto E_G(U)$ is (the sheaf of holomorphic sections of) a holomorphic vector bundle whose fiber over x is $E_G(x)$. This follows once one notices that $E_G(U)$ is actually a subsheaf of a locally free finitely generated sheaf carrying a connection whose restriction to E_G is ∇. Details on this point are in Appendix 8.5.2.

To make connection with the previous sections, consider the pull back of E_G by the map $\lambda \mapsto \exp(2\pi i\lambda)$, from $\mathfrak{h}$ to G. It is the vector bundle $E_\mathfrak{h}$ on $C^{[n]} \times \mathfrak{h}$. Let us introduce coordinates λ_ν on $\mathfrak{h}$ with respect to some orthonormal basis (h_ν). Then the pull-back connection on $E_\mathfrak{h}$ is given by (483), (484), and, in the direction of λ,

$$\nabla_{\lambda_\nu} = \partial_{\lambda_\nu} - \delta(h_\nu\rho(\cdot - z_1, \tau)). \tag{492}$$

Moreover, we can use the connection to identify by parallel translation the space of conformal blocks $E(U)$ with the space of twisted conformal blocks ω in $E_G(U\times G)$ (or in $E_{\mathfrak{h}}(U\times \mathfrak{h}))$ such that $\nabla_X\omega=0$, $X\in\mathfrak{g}$ (or $\nabla_{\lambda_\nu}\omega=0$, respectively). Here we use the fact that G and $\mathfrak{h}$ are simply connected.

The point of this construction is given by the following result. Let $V^{[n]*}(U)$ be the space of holomorphic functions on an open set U with values in the finite dimensional space $V^{[n]*}$. We also set, for any open subset U of $C^{[n]}\times G$, or $C^{[n]}\times\mathfrak{h}$, respectively,

$$E_G(U)^{\mathrm{hor}}=\{\omega\in E_G(U)|\nabla_X\omega=0,\quad \forall X\in\mathfrak{g}\},\tag{493}$$
$$E_{\mathfrak{h}}(U)^{\mathrm{hor}}=\{\omega\in E_{\mathfrak{h}}(U)|\nabla_X\omega=0,\quad \forall X\in\mathfrak{h}\}.\tag{494}$$

Proposition 8.9 *The compositions*

$$\iota_G:E(U)\to E_G(U\times G)^{\mathrm{hor}}\to V^{[n]*}(U\times G),\tag{495}$$
$$\iota_{\mathfrak{h}}:E(U)\to E_{\mathfrak{h}}(U\times \mathfrak{h})^{\mathrm{hor}}\to V^{[n]*}(U\times \mathfrak{h}),\tag{496}$$

where the first map sends a holomorphic conformal block ω to the unique twisted holomorphic conformal block horizontal in the G (resp. $\mathfrak{h}$) direction, which coincides with ω on $U\times\{1\}$ (resp. $U\times\{0\}$), and the second map is the restriction to $V^{[n]}$, are injective.

Proof: The first map is an isomorphism to the space of twisted holomorphic conformal blocks horizontal in the G (resp. $\mathfrak{h}$) direction. The fact that the second map is injective follows from the fact that using the invariance and the equation $\nabla_X\omega=0$ (resp. $\nabla_{\lambda_\nu}\omega=0$), one can express $\langle\omega,v\rangle$ for any $v\in V^{\wedge[n]}$ linearly in terms of the restriction of ω to $V^{[n]}$. $\square$

We may (and will) thus view the sheaf E of sections of the vector bundle on $C^{[n]}$ of conformal blocks as a subsheaf of $V^{[n]}(U\times\mathfrak{h})$. The next steps are a characterization of this subsheaf and a formula for the connection after this identification.

8.3.2 Theta functions

Let $Q^\vee=\{q\in\mathfrak{h}\,|\,\exp(2\pi iq)=1\in G\}$ be the coroot lattice of $\mathfrak{g}$.

Definition: Let $(z,\tau)\in C^{[n]}$, and $V_1,\dots,V_n$ be finite dimensional $\mathfrak{g}$-modules, and k a non-negative integer. The space $\Theta_k(z,\lambda)$ of theta functions of level k is the space of holomorphic functions $f:\mathfrak{h}\to V^{[n]*}$ such that

(i) $\sum_{i=1}^n f(\lambda)h^{(i)}=0$.

(ii) One has the following transformation properties unter the lattice $Q^\vee+\tau Q^\vee\subset\mathfrak{h}$:

$$f(\lambda+q)=f(\lambda)\tag{497}$$
$$f(\lambda+q\tau)=f(\lambda)\exp\Big(-\pi ik(q,q)\tau-2\pi ik(q,\lambda)-2\pi i\sum_{j=1}^n z_jq^{(j)}\Big)\tag{498}$$

The space of such theta functions is finite dimensional, as can be easily seen by Fourier series theory. Denote by W be the Weyl group of $\mathfrak{g}$, generated by reflection with respect to root hyperplanes. It is known that this group is isomorphic to $N(H)/H$, $N(H) \subset G$ being the normalizer of $H = \exp(\mathfrak{h})$. For $w \in W$, let $\hat{w} \in N(H)$ be any representative of the class of w in $N(H)/H$. The Weyl group acts on the space of theta functions. Indeed, if $f \in \Theta_k(z,\tau)$, then $(wf)(\lambda) = f(w^{-1}\lambda)\hat{w}^{-1}$ also in $\Theta_k(z,\tau)$, (the coroot lattice and the invariant bilinear form are preserved by the Weyl group), and is independent of the choice of representative $\hat{w}$ by (i). Let $\Theta_k(z,\tau)^W$ denote the space of W-invariant theta functions.

Theorem 8.10 *Let $\mathfrak{g} = A_l$, $l \geq 2$, D_l, $l \geq 4$, E_6, E_7, E_8, F_4, or G_2. Then the image of $\iota_{\mathfrak{h}}$ is contained in the space of holomorphic functions $\omega \in V^{[n]*}(U \times \mathfrak{h})$ such that for all $(z,\tau) \in C^{[n]}$, $\omega(z,\tau,\cdot)$ belongs to $\Theta_k(z,\tau)^W$, and such that for all roots α, $X \in \mathfrak{g}_\alpha$ and nonnegative integers p,*

$$\omega(z,\tau,\lambda)X^p = O(\alpha(\lambda)^p), \tag{499}$$

as $\alpha(\lambda) \to 0$.

In the remaining cases, we have

Theorem 8.11 *Let $\mathfrak{g} = A_1$, B_l or C_l, $l \geq 2$. Then the image of $\iota_{\mathfrak{h}}$ is contained in the space of holomorphic functions $\omega \in V^{[n]*}(U \times \mathfrak{h})$ such that for all $(z,\tau) \in C^{[n]}$, $\omega(z,\tau,\cdot)$ belongs to $\Theta_k(z,\tau)^W$, and such that for all $\alpha \in \Delta$, $r,s \in \{0,1\}$, $X \in \mathfrak{g}_\alpha$ and nonnegative integers p,*

$$\omega(z,\tau,\lambda)\exp\left(2\pi i\, c_{r,s}\sum_j z_j\lambda^{(j)}\right)X^p = O((\alpha(\lambda) - r - s\tau)^p), \tag{500}$$

as $\alpha(\lambda) \to r + s\tau$, with $c_{r,0} = 0$, $c_{r,1} = (r+\tau)^{-1}$.

The proof of these theorems will be completed in 8.3.7. We conjecture that the space of functions described in Theorems 8.10, 8.11 actually coincides with the image of $\iota_{\mathfrak{h}}$. This conjecture is verified in a simple class of examples in 8.3.8 below.

The fact that the formulation of the result is simpler for certain Lie algebras is due to the following property shared by the Lie algebras of Theorem 8.10: for each root α and integer m there exist an element q in the coroot lattice with $\alpha(q) = m$. For the other simple Lie algebras this is true only if m is assumed to be even. More on this in 8.3.6.

8.3.3 Affine Weyl group

Since H acts trivially on $E_{\mathfrak{h}}$, the Weyl group acts (on the right) on the values of $E_{\mathfrak{h}}$: w acts as $\hat{w}$, and this is independent of the choice of representative.

Proposition 8.12 *Let $\omega \in E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}}$. Then, for all $q \in Q^\vee$,*

$$\omega(z,\tau,\lambda+q) = \omega(z,\tau,\lambda). \tag{501}$$

For all $w \in W$,

$$\omega(z,\tau,w\lambda) = \omega(z,\tau,\lambda)\hat{w}^{-1}. \tag{502}$$

Proof: The value of ω at $(z,\tau,\lambda+q)$ is obtained from $\omega(z,\tau,\lambda)$ by parallel transport along some path from λ to $\lambda+q$. Recall that ω is the pull back of a section of $E(U\times G)^{\mathrm{hor}}$ to $U\times\mathfrak{h}$. The image of the path in G is closed, and contractible (G is simply connected), which proves the first claim.

From Prop. 8.8 and the fact that $w\cdot\lambda = \mathrm{Ad}(\hat{w})\lambda$, we see that if ω is horizontal then also $\rho_{\hat{w}}\omega$ is horizontal. But these horizontal sections coincide at $\lambda=0$, and thus everywhere. □

8.3.4 Modular transformations

The group $\mathrm{SL}(2,\mathbb{Z})$ acts as follows on $C^{[n]}\times\mathfrak{h}$: if $A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$,

$$A\cdot(z,\tau,\lambda) = \left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}, \frac{\lambda}{c\tau+d}\right). \tag{503}$$

Lemma 8.13 *Introduce the linear functions $\ell_\lambda(t) = 2\pi i\lambda t$ for $\lambda\in\mathfrak{h}$. For all $x \in C^{[n]}\times\mathfrak{h}$, $A\in\mathrm{SL}(2,\mathbb{Z})$, the map $X\to\phi_A(x)X$ with*

$$(\phi_A(x)X)(t) = \exp(-\,\mathrm{ad}\, c\,\ell_\lambda(t))X((c\tau+d)t), \qquad A = \begin{pmatrix} a & b \\ c & d\end{pmatrix}, \tag{504}$$

is a Lie algebra isomorphism from $\mathcal{L}(x)$ to $\mathcal{L}(Ax)$.

Proof: This follows directly from the definitions. □

We have defined an action $\mathcal{L}(x)\otimes V^{\wedge[n]}\to V^{\wedge[n]}$, for all $x\in C^{[n]}\times\mathfrak{h}$. Let us denote it as $X\otimes v\mapsto\delta_x(X)v$ (see (466)) to emphasize the x-dependence.

Lemma 8.14 *Define a linear isomorphism $v\to\rho_A(x)v$ from $V^{\wedge[n]}$ viewed as $\mathcal{L}(x)$-module to $V^{\wedge[n]}$ viewed as $\mathcal{L}(Ax)$-module: if $x=(z,\tau,\lambda)$,*

$$\rho_A(x) = \eta_A(x)(c\tau+d)^{-\sum_{j=1}^n L_0^{(j)}}\exp\left(-\frac{c\,\delta_x(\ell_\lambda)}{c\tau+d}\right), \qquad \eta_A(x) = \exp\left(-\frac{\pi i ck(\lambda,\lambda)}{c\tau+d}\right). \tag{505}$$

This map has the intertwining property

$$\rho_A(x)\delta_x(X) = \delta_{Ax}(\phi_A(x)X)\rho_A(x), \tag{506}$$

for all $X\in\mathcal{L}(x)$.

Note that the choice of coefficient η_A is irrelevant for the validity of the Lemma. However, it is important for compatibility with the connection, see below.

We should also add a remark about the power of $(c\tau + d)$. The exponent $\Sigma L_0^{(i)}$ is diagonalizable with finite dimensional eigenspaces. However the eigenvalues are fractional in general, and the power is defined for a choice of branch of the logarithm for each $A \in \mathrm{SL}(2,\mathbb{Z})$. This is made more systematic in the next subsection.

Proof: This is again straightforward. The only subtlety is that, a priori, there could be a contribution from the central extension in the computation of the intertwining property. However the central term appearing in this computation is proportional to the sum of the residues of the component of $\partial_t X$ along λ, which is doubly periodic. By the residue theorem, this sum vanishes. $\square$

We can thus define linear maps $\omega \mapsto \rho_A^*\omega$ by $\rho_A^*\omega(x) = \omega(Ax)\rho_A(x)$. Lemma 8.14 implies then that ρ_A^* is an isomorphism from $E_{\mathfrak{h}}(U)$ to $E_{\mathfrak{h}}(A^{-1}(U))$.

Lemma 8.15 *Let A^* be the pull back on one-forms of the map $x \mapsto Ax$ defined on some open $U \subset C^{[n]} \times \mathfrak{h}$, and $\nabla : E_{\mathfrak{h}}(U) \to \Omega^1(U) \otimes E_{\mathfrak{h}}(U)$ be the connection defined in 8.3.1. We have*

$$\nabla \rho_A^* = A^* \otimes \rho_A^* \nabla. \tag{507}$$

This fact can be derived from a straightforward but unfortunately lengthy calculation. The main identity one uses is

$$\rho\left(\frac{t}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)\rho(t, \tau) + 2\pi i c t. \tag{508}$$

Lemma 8.15 ensures that ρ_A^* maps horizontal sections to horizontal sections. Moreover, since $A_*\partial_{\lambda_\nu} = (c\tau + d)^{-1}\partial_{\lambda_\nu}$ does not have components in z or τ direction, ρ_A^* maps sections which are horizontal in the $\mathfrak{h}$ direction to sections with the same property:

$$\rho_A^* : E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}} \to E_{\mathfrak{h}}(A^{-1}U \times \mathfrak{h})^{\mathrm{hor}}. \tag{509}$$

Let us apply this in the special case $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Proposition 8.16 *Let $\omega \in V^{[n]*}(U)$ be in the image of $\iota_{\mathfrak{h}}$. Then for all $q \in Q^\vee$,*

$$\omega(z, \tau, \lambda + \tau q) = \omega(z, \tau, \lambda) \exp(-2\pi i(q, \lambda)k - \pi i(q, q)k\tau - 2\pi i \sum_{j=1}^n z_j q^{(j)}). \tag{510}$$

Proof: We have $\rho_A^*\omega \in E_{\mathfrak{h}}(A^{-1}U \times \mathfrak{h})^{\mathrm{hor}}$. Thus,

$$\rho_A^*\omega(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\lambda}{\tau} + q) = \rho_A^*\omega(\frac{z}{\tau}, -\frac{1}{\tau}, \frac{\lambda}{\tau}), \tag{511}$$

for all coroots q, by Lemma 8.12, and $(z,\tau)\in U$. Explicitly,

$$\omega(z,\tau,\lambda+q\tau)\rho_A(\frac{z}{\tau},-\frac{1}{\tau},\frac{\lambda}{\tau}+q)=\omega(z,\tau,\lambda)\rho_A(\frac{z}{\tau},-\frac{1}{\tau},\frac{\lambda}{\tau}). \tag{512}$$

Inserting the formula for ρ_A we obtain

$$\omega(z,\tau,\lambda+q\tau)\tau^{\Sigma_j L_0^{(j)}}=\omega(z,\tau,\lambda)\tau^{\Sigma_j L_0^{(j)}}e^{-2\pi i(q,\lambda)k-\pi i(q,q)k\tau}\exp(-\tau\delta_x(\ell_q)), \tag{513}$$

with $x=(z/\tau,-1/\tau,\lambda/\tau)$. Now we use the fact that, on $V^{[n]}$, $\delta_x(\ell_q)$ acts as $\Sigma_j 2\pi i(z_j/\tau)q^{(j)}$, and that $L_0^{(j)}$ acts as a multiple of the identity, to conclude the proof. $\square$

8.3.5 Monodromy (projective) representations of $\mathrm{SL}(2,\mathbb{Z})$

In this subsection, we assume that $n=1$, set $z_1=0$, and show that a central extension of $\mathrm{SL}(2,\mathbb{Z})$ acts on the space of horizontal sections of the bundle of conformal blocks.

The fact that we have a central extension comes from the necessity to choose a branch of the logarithm to define the expression $(c\tau+d)^{L_0}$. In fact L_0 is diagonalizable with finite dimensional eigenspaces, and any two eigenvalues differ by an integer. Moreover, L_0 acts by a non-negative rational multiple $(\mathrm{Cas}(V)/(k+h^\vee))$ of the identity on $V\subset V^\wedge$ for any integrable $V^\wedge$ of level k. Let $L_0|_V=\frac{r}{s}\,\mathrm{id}_V$, $r,s\in\mathbb{N}$. We introduce a central extension

$$0\to\mathbb{Z}/s\mathbb{Z}\to\Gamma_s\to\mathrm{SL}(2,\mathbb{Z})\to 1, \tag{514}$$

of $\mathrm{SL}(2,\mathbb{Z})$ by the cyclic group of order s. The group Γ_s consists of pairs (A,ϕ) where $A\in SL(2,\mathbb{Z})$ has matrix elements a,b,c,d and ϕ is a holomorphic function on the upper half plane such that $\phi(\tau)^s=c\tau+d$. The product is $(A,\phi)(B,\psi)=(AB,\phi\circ A\cdot\psi)$. Then this group acts on V^* valued functions on $\mathrm{H}_+\times\mathfrak{h}$ as above, but keeping track of the choice of branch:

$$(A,\phi)^{-1}\omega(\lambda,\tau)=\omega(A\cdot(\lambda,\tau))\eta_A(\lambda,\tau)\phi(\tau)^{-r}. \tag{515}$$

This action preserves the connection. (The inversion here is to correct for the "wrong" order $\rho_A^*\rho_B^*=\rho_{BA}^*$, up to ambiguity in the choice of branch). Thus we conclude that Γ_s acts on the space of global horizontal sections on $\mathrm{H}_+\times\mathfrak{h}$ of $E_{\mathfrak{h}}$. This monodromy representation restricts to the character $[m]\mapsto\exp(2\pi imr/s)$ of $\mathbb{Z}/s\mathbb{Z}$.

In the case of $V=$ trivial representation, this monodromy representation is just the representation of $SL(2,\mathbb{Z})$ on characters of affine Lie algebras (see [K90]). It would be interesting to calculate this monodromy representation explicitly for general V. Some progress in the sl_2 case was made in [CFW93], where a connection with the adjoint representation of the corresponding quantum group was established.

8.3.6 The vanishing condition

Let G be a simple complex Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h}$ a Cartan subalgebra, $\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ a Cartan decomposition and $H = \exp\mathfrak{h}$. Suppose that $\rho : G \to \mathrm{End}(V)$ is a finite dimensional representation of G. Thus V is also a $\mathfrak{g}$-module. For $K = G$ or H let $I(K,V)$ be the space of holomorphic functions on K with values in V such that $\forall g,h \in K$, $u(ghg^{-1}) = \rho(g)u(h)$. The Weyl group W acts on $I(H,V)$: let $\hat{w}$ be a representative of $w \in W$ in $N(H)$. Then $(wf)(h) = \rho(\hat{w})f(w^{-1}\cdot h)$ is well defined for $f \in I(H,V)$, since H acts trivially on the image of functions in $I(H,V)$. We denote by $I(H,V)^W$ the space of Weyl-invariant functions in $I(H,V)$.

Lemma 8.17 *The restriction map $I(G,V) \to I(H,V)$ is injective. Its image is the space $I_0(H,V)^W$ of functions u in $I(H,V)^W$ such that for all positive roots α, $X \in \mathfrak{g}_\alpha$ $p \in \mathbb{N}$, and $m \in \mathbb{Z}$*

$$X^p u(\exp 2\pi i\lambda) = O((\alpha(\lambda) - m)^p), \tag{516}$$

as $\alpha(\lambda) \to m$.

Proof: The behavior of functions in $I(G,V)$ under conjugation by $N(H)$ implies Weyl invariance.

Let $X \in \mathfrak{g}_\alpha$ and $\lambda \in \mathfrak{h}$. Then

$$\mathrm{Ad}(\exp(2\pi i\lambda))\, X = e^{2\pi i\alpha(\lambda)}X. \tag{517}$$

If $u \in I(G,V)$, $u(\exp(X)\exp(2\pi i\lambda))$ is a holomorphic $Q^\vee$-periodic function of $\lambda \in \mathfrak{h}$, (thus a holomorphic function on H). On the other hand, by (517)

$$\begin{aligned} u(\exp(X)\exp(2\pi i\lambda)) &= u\left(\exp\left(\frac{X}{1-e^{2\pi i\alpha(\lambda)}}\right)\exp(2\pi i\lambda)\exp\left(-\frac{X}{1-e^{2\pi i\alpha(\lambda)}}\right)\right) \\ &= \rho(\exp((1-e^{2\pi i\alpha(\lambda)})^{-1}X))u(\exp(2\pi i\lambda)) \\ &= \sum_{p=0}^{M}\frac{1}{p!}(1-e^{2\pi i\alpha(\lambda)})^{-p}X^p u(\exp(2\pi i\lambda)), \end{aligned} \tag{518}$$

for some M. We see that the latter expression is holomorphic on the affine hyperplanes $\alpha(\lambda) = m$, if and only if, for all p, $X^p u$ vanishes there to order at least p.

To conclude the proof, we use some facts about conjugacy classes in algebraic groups (see, e.g., [S74], Chapter 3). Let, for each root α and integer m, $H_{\alpha,m} \subset H$ be the set of elements of the form $\exp(2\pi i\lambda)$ such that $\alpha(\lambda) = m$. The conjugacy classes containing elements in $H_{ss} = H - \cup H_{\alpha,m}$ form the dense open subset G_{ss} of regular semisimple elements in G. Its complement contains the set H_1 consisting of conjugacy classes of elements of the form $\exp(X)\exp(2\pi i\lambda)$, where λ lies on precisely one of the distinct $H_{\alpha,m}$. This elements are regular, as they are regular in the identity component of the stabilizer of $\exp(2\pi i\lambda)$, (see [S74], 3.5), which is the direct product of a torus of dimension rank-1 times the SL(2) subgroup associated with α. By the above reasoning, a Weyl invariant function on H extends uniquely to an equivariant holomorphic function on G_{ss}, and the vanishing conditions imply that it extends to a holomorphic functions on $G_{ss} \cup H_1$. The complement of $G_{ss} \cup H_1$ consists of higher codimension classes whose closure intersects H_1, and of classes whose closure do not intersect $H_{ss} \cup H_1$. Counting dimensions shows that this complement is of codimension at least two, so by Hartogs' theorem, our vanishing conditions are sufficient to have an extension to all of G. $\square$

By Weyl invariance, we may replace the set of positive roots in the formulation of the lemma to a subset of roots consisting of one root for each Weyl group orbit. Also, we may restrict the values of m, by $Q^\vee$ periodicity of $u(\exp 2\pi i\lambda)$. Indeed, if the vanishing condition holds at $\alpha(\lambda) = m$, it also holds at $\alpha(\lambda) = m - \alpha(q)$ for all $q \in Q^\vee$.

We thus have the following result. The action of the affine Weyl group $W^\wedge = W \tilde{\times} Q^\vee$ on $\Delta \times \mathbb{Z}$ is defined by

$$(w, q)(\alpha, m) = (w\alpha, m - \alpha(q)). \tag{519}$$

Lemma 8.18 *The subspace $I_0(H, V)^W \subset I(H, V)^W$ is characterized by the vanishing condition (516), for (α, m) in any fundamental domain for the action of $W^\wedge$ on $\Delta \times \mathbb{Z}$.*

From [Bou81], we see that in the cases A_l, $l \geq 2$, D_l, $l \geq 4$, E_6, E_7, E_8, F_4, G_2 a fundamental domain is $\{(\alpha, 0), \alpha \in F\}$, where α runs over a fundamental domain F (consisting of one or two elements) of W. If $\mathfrak{g} = A_1$, B_l, C_l, then we have to add $(\alpha, 1)$ where α is a long root.

As a corollary we obtain a more precise characterization of the image of $i_{\mathfrak{h}}$. Let us identify functions on H with $Q^\vee$-periodic functions on $\mathfrak{h}$ via the map $\lambda \mapsto \exp(2\pi i\lambda)$, and view $V^{[n]*}$ as a representation of G by $\langle \rho(g)u, v\rangle = \langle u, g^{-1}v\rangle$.

Corollary 8.19 *The image of $E(U)$ by $\iota_{\mathfrak{h}}$ is contained in the space of functions $\omega \in V^{[n]*}(U \times \mathfrak{h})$ such that for all $(z, \tau) \in C^{[n]}$, $\omega(z, \tau, \cdot)$ belongs to $I_0(H, V^{[n]*})^W$.*

Moreover, if $\omega \in E_{\mathfrak{h}}(U \times \mathfrak{h})^{\mathrm{hor}}$, then $\rho_A^*\omega \in E_{\mathfrak{h}}(A^{-1}U \times \mathfrak{h})^{\mathrm{hor}}$, implying further vanishing conditions: let $x = (z, \tau, \lambda)$ and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. On $V^{[n]}$ $L_0^{(j)}$ acts by a scalar Δ_j. The restriction of $\rho_A^*\omega$ to $V^{[n]}$ is

$$(\rho_A^*\omega)(x) = \omega(Ax)(c\tau + d)^{-\Sigma_j \Delta_j} \eta_A(x) e^{-\frac{c}{c\tau+d}\sum_j z_j \lambda^{(j)}}.$$

It follows that, for all p,

$$\omega(Ax) \exp(-\frac{c}{c\tau + d} \sum_j z_j \lambda^{(j)}) X^p = O((\alpha(\lambda) - m)^p),$$

if $\alpha(\lambda) \to m$. Changing variables, this implies that

$$\omega(z, \tau, \lambda) \exp\left(-2\pi i \frac{c}{a - c\tau} \sum_j z_j \lambda^{(j)}\right) X^p = O((\alpha(\lambda) - m(a - c\tau))^p).$$

Since any pair a, c of relatively prime integers appear in the first column of some SL(2,$\mathbb{Z}$) matrix, we obtain the result

Corollary 8.20 *The image of $E(U)$ by $\iota_{\mathfrak{h}}$ is contained in the space of functions $\omega \in V^{[n]*}(U \times \mathfrak{h})$ such that for all $(z, \tau, \lambda) \in C^{[n]}$, r, s, $p \in \mathbb{Z}$, $p \geq 1$, $(r, s) \neq (0, 0)$,*

$$\omega(z, \tau, \lambda) \exp(2\pi i \frac{s}{r + s\tau} \sum_j z_j \lambda^{(j)}) X^p = O((\alpha(\lambda) - r - s\tau)^p), \tag{520}$$

as $\alpha(\lambda) \to r + s\tau$.

8.3.7 Proof of Theorems 8.10, 8.11

Theorems 8.10, 8.11 follow from Propositions 8.12, 8.16, and Corollaries 8.19, 8.20 together with the fact that twisted conformal blocks are annihilated by $\mathfrak{h} \subset \mathcal{L}_{\mathfrak{h}}(U)$.

8.3.8 Examples

Here we give an explicit description of the space of conformal blocks in some special cases. The discussion parallels the constructions in [FG92], where Chern–Simons states in the case of sl_2 are studied. First of all consider the case of one point z_1 with the trivial representation. Then the vanishing condition is vacuous, and we are left to classify scalar Weyl invariant theta functions of level k. This space coincides with the space spanned by characters of irreducible highest weight $L\mathfrak{g}\hat{}$-modules, in accordance with the Verlinde formula.

Next, we consider the case of one point z_1, with a symmetric tensor power of the defining representation $\mathbb{C}^N$ of sl_N.

If $N \geq 3$, the problem is reduced to describing the space of Weyl invariant theta functions ω of level k, with the property that

$$e_\alpha^p u(\alpha(\lambda)) = O(\alpha(\lambda)^p), \quad \alpha(\lambda) \to 0, \tag{521}$$

for all $p = 1, 2, \ldots$ and root vectors $e_\alpha \in \mathfrak{g}_\alpha$. Actually it is sufficient to consider one root α, since the Weyl group acts transitively on the set of roots of sl_N.

The symmetric power $S^j\mathbb{C}^N$ has a non-zero weight space if and only if j is a multiple of N. Let us set $j = lN$, and denote by ϵ_i the elements of the standard basis of $\mathbb{C}^N$. Then the weight zero subspace of $S^{lN}\mathbb{C}^N = (\mathbb{C}^N)^{\otimes l}/S_N$ is one-dimensional and is spanned by the class of $v = \epsilon_1^{\otimes l} \otimes \cdots \otimes \epsilon_N^{\otimes l}$. The following considerations apply also to the case $l = 0$, if we agree that $S^0\mathbb{C}^N$ is the trivial representation.

The Weyl group of sl_N is the symmetric group S_N and is generated by adjacent transpositions s_j, $j = 1, \ldots, N-1$. If we identify the Weyl group with $N(H)/H$, then a representative in $N(H)$ of s_j is given by $\hat{s}_j\epsilon_r = \epsilon_r$, if $r \neq j, j+1$, $\hat{s}_j\epsilon_j = \epsilon_{j+1}$, $\hat{s}_j\epsilon_{j+1} = -\epsilon_j$. It follows that S_N acts on the weight zero space by the lth power of the alternating representation: $\hat{w}v = \epsilon(w)^l v$.

The next remark is that $e_\alpha^{l+1}v = 0$ but $e_\alpha^l v \neq 0$. We thus see that $\omega(\alpha(\lambda)) = O(\alpha(\lambda)^l)$ as λ approaches the hyperplane $\alpha(\lambda) = 0$. If ω is a Weyl-invariant theta function, it then also vanishes to order l on all hyperplanes $\alpha(\lambda) = n + m\tau$, $n, m \in \mathbb{Z}$. Therefore the quotient of ω by the lth power of the Weyl–Kac denominator $\Pi(\lambda, \tau)$ is an entire function, as Π has simple zeroes on those hyperplanes. Moreover, Π is a (scalar) theta function of level N (the dual Coxeter number of sl_N), and $\Pi(w\lambda, \tau) = \epsilon(w)\Pi(\lambda, \tau)$.

We conclude that the space of conformal blocks at fixed τ is contained in the space of functions of the form

$$\omega(\lambda) = \Pi(\lambda, \tau)^l u(\lambda)\, v, \tag{522}$$

where u is an entire $Q^\vee$-periodic scalar function on $\mathfrak{h}$, such that $u(w\lambda) = u(\lambda)$, for all $w \in S_N$ and

$$\begin{aligned} u(\lambda + q\tau) &= \alpha(q, \lambda, \tau)^{k-Nl} u(\lambda), & (523)\\ \alpha(q, \lambda, \tau) &= \exp(-2\pi i(q, \lambda) - \pi i(q, q)\tau), \qquad q \in Q^\vee. & (524) \end{aligned}$$

We have assumed here that $N \geq 3$. In the $_2$ case, where the vanishing condition must be satisfied also at 3 other points on $\mathfrak{h}$, one can proceed in the same way, noticing that the Weyl denominator vanishes there too.

A basis of $Q^\vee$-periodic functions with multipliers (523) is easily given using Fourier series. The basis elements θ_μ are labeled by $\mu \in P/(k - Nl)Q^\vee$, where the weight lattice P is dual to $Q^\vee$ (if $k < N$ there are no non-zero conformal blocks). The Weyl group acts as $\theta_\mu(w^{-1}\lambda) = \theta_{w\mu}(\lambda)$.

Therefore the dimension of our space is the number of orbits of the Weyl group in $P/(k - Nl)Q^\vee$. This number is well-known: a fundamental domain in P for the action of the semidirect product of the Weyl group by the group of translations by $(k - Nl)Q^\vee$ is the set of weights in the (dilated) Weyl alcôve I_{k-Nl}, see (465).

More explicitly, if α_i are simple roots, ω_i fundamental weights with $(\omega_i, \alpha_j) = \delta_{ij}$, and $\mu = \Sigma_i n_i \omega_i$, then $\mu \in I_{k-Nl}$ if and only if the integers n_i satisfy the inequalities

$$n_i \geq 0, \quad i = 1, \ldots, N-1, \qquad \sum_{i=1}^{N-1} n_i \leq k - Nl. \tag{525}$$

The number of $N - 1$-tuples of integers with these properties is calculated to be

$$\binom{k - N(l-1) - 1}{N - 1}. \tag{526}$$

This is the formula for the dimension of the space of Weyl-invariant theta functions of level k extending to holomorphic functions on SL_N, with values in the $(l \cdot N)$th symmetric power of the defining representation of sl_N. We now show that this coincides with the Verlinde formula [V88], which according to [TUY89], [Fa93] give the dimension of the space of conformal blocks.

Let I_k be the set of integrable highest weights of level k. It consists of dominant integral weights μ with $(\mu, \theta) \leq k$. The dimension of the space of conformal blocks with one point, to which an irreducible representation of highest weight $\mu \in I_k$ is attached, is given by the formula

$$d_\mu = \sum_{\nu \in I_k} N_{\nu\mu}^{\nu}, \tag{527}$$

in terms of the structure constants N_{bc}^a of Verlinde's fusion ring. A convenient formula for these constants in terms of the classical fusion coefficients m_{bc}^a (= the multiplicity of a in the decomposition of the tensor product of b with c) was given in [GW90], and [K90], Exercise 13.35.

Let $W_k^\wedge \simeq W^\wedge$ be the group of affine transformations of $\mathfrak{h}^*$ generated by the Weyl group W and the reflection s_0 at the hyperplane $\{\lambda \in \mathfrak{h}^* | (\theta, \lambda) = k + h^\vee\}$ (θ is the highest root and $h^\vee$ the dual Coxeter number). Let ρ be half the sum of the positive roots of $\mathfrak{g}$ and define another action of $W_k^\wedge$ on $\mathfrak{h}^*$ by $w * \lambda = w(\lambda + \rho) - \rho$. Let $\epsilon : W_k^\wedge \to \{1, -1\}$ be the homomorphism taking reflections to -1.

Then, for all $a, b, c \in I_k$,

$$N_{bc}^a = \sum_{w \in W_k^\wedge} m_{bc}^{w*a}. \tag{528}$$

Actually, in Verlinde's formula the coefficients N_{bc}^a are given in terms of modular transformation properties of characters. They are uniquely determined by the equation

$$\frac{S_{db}}{S_{d0}} \frac{S_{dc}}{S_{d0}} = \sum_a N_{bc}^a \frac{S_{da}}{S_{d0}}, \tag{529}$$

where, according to [K90], (13.8.9)

$$\frac{S_{ab}}{S_{a0}} = \chi_b \left(\exp \left(-2\pi i \frac{a + \rho}{k + h^\vee} \right) \right). \tag{530}$$

Here, χ_a is the character of the representation of G with highest weight a.

Let us check that the two formulas agree (this is essentially the solution to Exercise 13.35 of [K90]). Let $w \in W$ and $q \in (k + h^\vee) Q^\vee$ and suppose that both a and $w * a + q$ are dominant integral weights. Then it is easy to see from the Weyl character formula (see [H80]) that if $\lambda \in (k + h^\vee)^{-1} P$,

$$\chi_{w*a+q}(\exp(2\pi i \lambda)) = \epsilon(w) \chi_a(\exp(2\pi i \lambda)).$$

There is a unique element in each affine Weyl group orbit in the shifted Weyl alcôve $I_k + \rho$. Using these facts and the formula for the multiplicities in the decomposition of tensor products $\chi_b \chi_c = \Sigma n_{bc}^a \chi_a$, we deduce (528).

Let us apply this to our example. Identify $\mathfrak{h}^*$ with $\mathbb{C}^N / \mathbb{C}(1, 1, \ldots, 1)$. Then integral weights are classes $a = [a_1, \ldots, a_n]$ of n-tuples of integers defined modulo $\mathbb{Z}(1, \ldots, 1)$.

The Weyl group S_N acts in the obvious way, and a weight is dominant if $a_j \geq a_{j+1}$. The affine reflection s_0 is

$$s_0[a_1, \dots, a_N] = [a_N + k + N, a_2, \dots, a_{N-1}, a_1 - k - N], \tag{531}$$

and $\rho = [N-1, N-2, \dots, 0]$. Let $c = [r, 0, \dots, 0]$ be the highest weight of $S^r\mathbb{C}^N$. Then the decomposition rules of tensor products say that $m^a_{bc} = 1$ if $a_j = b_j + l_j$ $(1 \leq j \leq N)$ for some integers l_j such that $0 \leq l_j \leq a_{j-1} - a_j$, $2 \leq j \leq N$ and $\Sigma_j l_j = r$. Otherwise, $m^a_{bc} = 0$. As $\theta = [1, 0, 0, \dots, 0, -1]$, a dominant weight a belongs to I_k if and only if $a_1 - a_N \leq k$.

We need two properties (see [GW90]) of the coefficients N^a_{bc}, valid for any a, b, $c \in I_k$: (i) $0 \leq N^a_{bc} \leq m^a_{bc}$, for all $a, b, c \in I_k$, and (ii) $N^{\sigma(a)}_{\sigma(b),c} = N^a_{bc}$ where $\sigma([a_1, \dots, a_N]) = [k + a_N, a_1, \dots, a_{N-1}]$. We will also use: (iii) Each orbit of $W^\wedge_k$, acting via $*$ on $\mathfrak{h}^*$, contains at most one point in I_k.

Let us now fix $c = [Nl, 0, \dots, 0]$, and do the classical calculation first.

Lemma 8.21 *Let $c = [Nl, 0, \dots, 0]$. Then $m^a_{ac} = 1$ iff $a_j - a_{j+1} \geq l$ for all $j \in \{1, \dots, N-1\}$.*

Proof: The coefficient m^a_{ac} is non-zero if and only if there exist non-negative integers $l_1, \dots, l_N$, summing up to Nl, such that $l_j \leq a_{j-1} - a_j$ if $j \geq 2$ and $[a_1 + l_1, \dots, a_N + l_N] = a$. It follows that $l_j = l$ for all j, and this solution obeys the inequality iff $a_{j-1} - a_j \geq l$ for all $j \geq 2$. $\square$

Lemma 8.22 *Let $c = [Nl, 0, \dots, 0]$, with $Nl \leq k$ and suppose $a \in I_k$. Then $N^a_{ac} = 0$ if $a_1 - a_N > k - l$.*

Proof: In this case $\sigma(a) = [k + a_N, a_1, \dots]$, and since $(k + a_N) - a_1 < l$, $m^{\sigma(a)}_{\sigma(a),c} = 0$, by Lemma 8.21. Therefore $N^a_{ac} = 0$, by properties (i), (ii). $\square$

Lemma 8.23 *Let $c = [Nl, 0, \dots, 0]$, with $Nl \leq k$. Then $N^a_{ac} = 1$ if and only if $a_j - a_{j+1} \geq l$, $1 \leq j \leq N-1$, and $a_1 - a_N \leq k - l$*

Proof: We need to prove only the "if" part. We do this by showing that only the first term in the sum (528) is non-zero. Let us suppose that a obeys the hypothesis of the Lemma, and that $m^a_{ac} = m^{w*a}_{ac} = 1$, with $w \neq 1$ and derive a contradiction. Since $b = w * a$ is dominant, and is not in I_k by (iii), we have $b_1 - b_N \geq k + 1$. Let us choose the representative in a with $a_N = 0$, and identify $a_1, \dots, a_{N-1}$ with the row lengths of a Young diagram. Then b is obtained by adding Nl boxes to this Young diagram, in such a way that $a_i \leq b_i \leq a_{i-1}$. Then w^{-1} with $w^r-1 * b = a$ is the unique element mapping b to I_k. This element is constructed as follows: (i) Add, for all j, $N - j$ boxes to the jth row of b (this adds ρ). (ii) Draw a vertical line at distance $k + N$ from the end of the Nth row to the right of it; the only boxes to the right of this line are in the first row, and their number is at most $Nl \leq k$. (iii) Take these boxes and add them to the Nth row (i.e., act by s_0, see (531)); permute the rows to get a Young diagram (i.e. act by an element of W). (iv) Subtract $N-j$ boxes from the jth row, $j=1,\dots,n$.

We obtain in this way a diagram which has Nl boxes more than the original diagram with row lengths a_i and whose first row has $b_N+k+1\geq k+1$ boxes. The two diagrams are equivalent, meaning that the latter is obtained from the former by adding the same number of boxes to each row. This number is at least $k+1-a_1$ which by hypothesis is strictly larger than l. We need thus more than Nl boxes, and this is a contradiction. □

The dimension of the space of conformal blocks can be now computed: note that $a\mapsto a-l\rho$ (i.e. $a_j\mapsto a_j-l(N-j)$) maps bijectively the set of weights obeying the conditions of Lemma 8.23 onto I_{k-lN} whose cardinality coincides with the dimension of the space of invariant theta functions with vanishing condition.

We conclude that the space of invariant theta functions satisfying our vanishing condition coincides with the space of conformal blocks, in accordance with our conjecture.

8.4 The Knizhnik–Zamolodchikov–Bernard equations, and generalized classical Yang–Baxter equation

8.4.1 The KZB equations

The Knizhnik–Zamolodchikov–Bernard (KZB) equations, first written in [Be88b], for a holomorphic conformal block $\omega\in E(U)$ are the horizontality conditions $\nabla\omega=0$, where ω is identified with its image by $\iota_{\mathfrak{h}}$. To write these equations explicitly, let us compute the expression of the connection ∇ on $E(U)$ viewed as a subsheaf of $V^{[n]}(U\times\mathfrak{h})$ via $\iota_{\mathfrak{h}}$.

It is convenient to introduce functions ρ, σ_w, $w\in\mathbb{C}$ expressed in terms of the function θ_1:

$$\begin{aligned}\rho(t,\tau) &= \partial_t\log\theta_1(t|\tau),\\ \sigma_w(t,\tau) &= \frac{\theta_1(w-t|\tau)\partial_t\theta_1(0|\tau)}{\theta_1(w|\tau)\theta_1(t|\tau)}.\end{aligned}$$

See Appendix 8.5.1 for details on these functions. We use the notation

$$\kappa=k+h^\vee, \tag{532}$$

and the abbreviation X_m for $X\otimes t^m$. We also identify $\mathfrak{g}$ as a Lie subalgebra of $L\mathfrak{g}\,\hat{}$: $X_0=X\in\mathfrak{g}$. Let $C_\alpha=e_\alpha\otimes e_{-\alpha}$ (see (8.2.2)). Then we can write L_{-1} as

$$L_{-1}=\frac{1}{\kappa}\sum_{n=0}^{\infty}\left(\sum_{\alpha\in\Delta}e_{\alpha,-n-1}e_{-\alpha,n}+\sum_{\nu}h_{\nu,-n-1}h_{\nu,n}\right). \tag{533}$$

Now let $U \subset C^{[n]}$, and $\omega \in E(U)$, which we identify via $\iota_{\mathfrak{h}}$ with a function on U with values in $V^{[n]}$. We then have, for fixed $u \in V^{[n]}$,

$$\kappa\langle\nabla_{z_j}\omega, u\rangle = \kappa\frac{\partial}{\partial z_j}\langle\omega, u\rangle - \langle\omega, (\sum_\nu h_{\nu,-1}h_\nu + \sum_\alpha e_{\alpha,-1}e_{-\alpha})u\rangle. \tag{534}$$

Recall that vectors in V are annihilated by X_n, with $X \in \mathfrak{g}$, $n > 0$. We now use the invariance of ω under the action of $\mathcal{L}$. The functions $t \mapsto e_\alpha\,\sigma_{\alpha(\lambda)}(t - z_j)$ are elements of $\mathcal{L}(z, \tau, \lambda)$. They have simple poles at $t = z_j$ with residue e_α. As a consequence of the invariance of ω, we have

$$\langle\omega, e^{(j)}_{\alpha,-1}u\rangle = \rho(\alpha(\lambda))\langle\omega, e^{(j)}_\alpha u\rangle - \sum_{k:k\neq j}\sigma_{\alpha(\lambda)}(z_k - z_j)\langle\omega, e^{(k)}_\alpha u\rangle, \tag{535}$$

for all $u \in \bigotimes_j V_j$. We can use this identity to compute the value of ω on vectors $e^{(j)}_{-1}u$. The flatness condition $\nabla_{\lambda_\nu}\omega = 0$ translates to

$$\langle\omega, h^{(j)}_{\nu,-1}u\rangle = \frac{\partial}{\partial\lambda_\nu}\langle\omega, u\rangle - \sum_{k:k\neq j}\rho(z_k - z_j)\langle\omega, h^{(k)}_\nu u\rangle. \tag{536}$$

To compute further we need the commutation relation $[e_\alpha, e_{-\alpha}] = \sum_\nu \alpha(h_\nu)h_\nu$, that follows from $([e_\alpha, e_{-\alpha}], h_\nu) = (e_\alpha, [e_{-\alpha}, h_\nu])$. We therefore obtain the formula

$$\begin{aligned}\kappa\langle\nabla_{z_j}\omega, u\rangle &= \kappa\frac{\partial}{\partial z_j}\langle\Pi\omega, u\rangle - \sum_\nu\frac{\partial}{\partial\lambda_\nu}\langle\Pi\omega, h^{(j)}_\nu u\rangle \\ &\quad - \sum_{k\neq j}\langle\Pi\omega, \Omega^{(k,j)}(z_k - z_j, \tau, \lambda)u\rangle\end{aligned} \tag{537}$$

where $\Pi = \Pi(\lambda, \tau)$ is (essentially) the “Weyl–Kac denominator” (for any choice of positive roots Δ_+)

$$q^{\frac{\dim\mathfrak{g}}{24}}\prod_{\alpha\in\Delta_+}(e^{i\pi\alpha(\lambda)} - e^{-\pi i\alpha(\lambda)})\prod_{n=1}^{\infty}[(1-q^n)^{\mathrm{rank}\,\mathfrak{g}}\prod_{\alpha\in\Delta}(1 - q^n e^{2\pi i\alpha(\lambda)})], \tag{538}$$

($q = e^{2\pi i\tau}$), and with the abbreviation

$$\Omega(t, \tau, \lambda) = \rho(t)C_0 + \sum_{\alpha\in\Delta}\sigma_{\alpha(\lambda)}(t)C_\alpha. \tag{539}$$

We also use the standard notation $\Omega^{(i,j)}$ to denote $\sum_s X^{(i)}_s Y^{(j)}_s$, if $\Omega = \sum_s X_s \otimes Y_s$. This notation will be used below also in the case $i = j$. The λ independent factors in Π do not play a role here, but will provide some simplifications later. In deriving (537), we have used that, by the classical product formula for Jacobi theta functions, Π is, up to a λ independent factor, the product $\Pi_{\alpha\in\Delta_+}\theta_1(\alpha(\lambda))$. Before continuing, we can use the formula (537) to complete the proof of Prop. 8.7.

End of the proof of Prop. 8.7. What is left to prove is that $[\nabla_\tau, \nabla_{z_1}]$ on $E(U)$. But from the above formula for ∇_{z_j} it follows that $\Sigma_j \nabla_{z_j}$ vanishes. Indeed we have $\omega \Sigma_j h_\nu^{(j)} = 0$ by $\mathfrak{h}$-invariance, and the other terms cancel by antisymmetry. As ∇_τ preserves conformal blocks, we have $[\nabla_\tau, \Sigma_j \nabla_{z_j}] = 0$, and the claim follows from the fact that ∇_τ commutes with ∇_{z_j} with $j \neq 1$. $\square$

A more involved but similar calculation gives a formula for ∇_τ, also essentially due to Bernard, which will be given here without full derivation,

One of the ingredients is Macdonald's (or denominator) identity (see [K90])

$$\Pi(\lambda, \tau) = \sum_{q \in Q^\vee} e^{i\pi\tau \frac{1}{2h^\vee}(\rho + h^\vee q, \rho + h^\vee q)} \sum_{w \in W} \epsilon(w) e^{2\pi i(\rho + h^\vee q, w\lambda)}, \tag{540}$$

implying (one form of) Fegan's heat kernel identity

$$4\pi i h^\vee \partial_\tau \Pi(\lambda, \tau) = \sum_\nu \partial_{\lambda_\nu}^2 \Pi(\lambda, \tau). \tag{541}$$

Here, ρ is half the sum of all positive roots of $\mathfrak{g}$, W is the Weyl group, and $\epsilon(w)$ is the sign of $w \in W$. The (complex) dimension of $\mathfrak{g}$ enters the game through the Freudenthal-de Vries strange formula $(\rho, \rho)/2h^\vee = \dim \mathfrak{g}/24$.

Let us summarize the results. We switch to the more familiar left action notation, by setting $\langle X\omega, v\rangle = -\langle \omega, Xv\rangle$ if X is in a Lie algebra and ω is in the dual space to $\mathfrak{g}$-module. We also need the following special functions of $t \in \mathbb{C}$, expressed in terms of $\sigma_w(t)$, $\rho(t)$ and Weierstrass' elliptic function $\wp$ with periods $1, \tau$.

$$\begin{aligned} I(t) &= \frac{1}{2}(\rho(t)^2 - \wp(t)), &(542)\\ J_w(t) &= \partial_t \sigma_w(t) + (\rho(t) + \rho(w))\sigma_w(t). &(543) \end{aligned}$$

These functions are regular at $t = 0$. Introduce the tensor

$$\mathrm{H}(t, \tau, \lambda) = I(t) C_0 + \sum_{\alpha \in \Delta} J_{\alpha(\lambda)}(t) C_\alpha. \tag{544}$$

Theorem 8.24 *The image ω by $\iota_{\mathfrak{h}}$ of a horizontal section of $E(U)$ obeys the KZB equations*

$$\begin{aligned} \kappa \partial_{z_j} \tilde{\omega} &= -\sum_\nu h_\nu^{(j)} \partial_{\lambda_\nu} \tilde{\omega} + \sum_{l: l \neq j} \Omega^{(j,l)}(z_j - z_l, \tau, \lambda) \tilde{\omega}, &(545)\\ 4\pi i \kappa \partial_\tau \tilde{\omega} &= \sum_\nu \partial_{\lambda_\nu}^2 \tilde{\omega} + \sum_{j,l} \mathrm{H}^{(j,l)}(z_j - z_l, \tau, \lambda) \tilde{\omega}, &(546) \end{aligned}$$

where $\tilde{\omega}(z, \tau, \lambda) = \Pi(\tau, \lambda)\omega(z, \tau, \lambda)$, and Ω, H are the tensors (539), (544), respectively.

Remark. For $n = 1$, these equations reduce to $\partial_{z_1} \tilde{\omega} = 1$, thus $\tilde{\omega}$ is a V^*-valued function of τ and λ only, and

$$4\pi i \kappa \frac{\partial}{\partial \tau} \tilde{\omega} = \sum_\nu \frac{\partial^2}{\partial \lambda_\nu^2} \tilde{\omega} - \eta_1(\tau) \,\mathrm{Cas(V)}\, \tilde{\omega} - \sum_{\alpha \in \Delta} \wp(\alpha(\lambda)) e_\alpha e_{-\alpha} \tilde{\omega}, \tag{547}$$

where $\rho(z) = z^{-1} - \eta_1 z + O(z^2)$, and $\mathrm{Cas}(V)$ is the value of the quadratic Casimir element $C^{(1,1)}$ in the representation V. This equation was considered recently by Etingof and Kirillov [EK94], who noticed that if $\mathfrak{g} = sl_N$ and V^* is the symmetric tensor product $S^{lN}\mathbb{C}^N$, $e_\alpha e_{-\alpha} = l(l+1)\,\mathrm{Id}$ on the one dimensional weight zero space of V^*, and the equation reduces to the heat equation associated to the elliptic Calogero–Moser–Sutherland–Olshanetsky–Perelomov integrable N-body system:

$$4\pi i\kappa\frac{\partial}{\partial\tau}\tilde{\omega} = \sum_\nu \frac{\partial^2}{\partial\lambda_\nu^2}\tilde{\omega} - \eta_1(\tau)l(l+1)N(N-1)\tilde{\omega} - l(l+1)\sum_{i\neq j}\wp(\lambda_i - \lambda_j)\tilde{\omega}. \tag{548}$$

See also 8.3.8 for a description of the space of conformal blocks in this case.

8.4.2 The classical Yang–Baxter equation

The tensor $\Omega^{(1,2)} = \Omega^{(1,2)}(z_1 - z_2, \tau, \lambda) \in \mathfrak{g}\otimes\mathfrak{g}$ obeys the "unitarity" condition

$$\Omega^{(1,2)} + \Omega^{(2,1)} = 0. \tag{549}$$

Let us remark that the fact that the connection is flat is then equivalent to the identity

$$\sum_\nu \partial_{\lambda_\nu}\Omega^{(1,2)}h_\nu^{(3)} + \sum_\nu \partial_{\lambda_\nu}\Omega^{(2,3)}h_\nu^{(1)} + \sum_\nu \partial_{\lambda_\nu}\Omega^{(3,1)}h_\nu^{(2)} \tag{550}$$

$$-[\Omega^{(1,2)}, \Omega^{(1,3)}] - [\Omega^{(1,2)}, \Omega^{(2,3)}] - [\Omega^{(1,3)}, \Omega^{(2,3)}] \quad = \quad 0 \tag{551}$$

in $\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$. This identity may be thought of as the genus one generalization of the classical Yang-Baxter equation. It admits an interesting "quantization" [F94].

8.5 Appendix

8.5.1 Lie algebras of meromorphic functions

We have the following explicit description of $\mathcal{L}(z_1, \ldots, z_n, \tau, \lambda)$. Let ρ, σ_w, $(w \in \mathbb{C})$ be meromorphic $\mathbb{Z}$-periodic functions on the complex plane, whose poles are simple and belong to $L(\tau)$, and such that

$$\rho(t+\tau) \;=\; \rho(t) - 2\pi i \tag{552}$$

$$\sigma_w(t+\tau) \;=\; e^{2\pi i w}\sigma_w(t) \tag{553}$$

$$\sigma_w(t) \;\sim\; \frac{1}{t}, \qquad t \to 0. \tag{554}$$

Such functions exist (for $w \in \mathbb{C} - L(\tau)$) and are unique, if we require that $\rho(-t) = -\rho(t)$. They can be expressed in terms of the Jacobi theta function θ_1:

$$\rho(t) \;=\; \frac{\partial}{\partial t}\log\theta_1(t|\tau), \tag{555}$$

$$\sigma_w(t) \;=\; \frac{\theta_1(t-w|\tau)\theta_1'(0|\tau)}{\theta_1(t|\tau)\theta_1(-w|\tau)}, \tag{556}$$

$$\theta_1(t|\tau) \;=\; -\sum_{n=-\infty}^{\infty} e^{2\pi i(t+\frac{1}{2})(n+\frac{1}{2})+\pi i\tau(n+\frac{1}{2})^2}. \tag{557}$$

Here a prime denotes a derivative with respect to the first argument.

Proposition 8.25 *For $\alpha \in \Delta \cup \{0\}$, $\lambda \in \mathfrak{h}$, and $(z,\tau) \in C^{[n]}$, the meromorphic functions of t (defined as limits at the removable singularities $\alpha(\lambda) \in \mathbb{Z}$)*

$$X(e^{2\pi i\alpha(\lambda)} - 1)\sigma_{\alpha(\lambda)}(t - z_1), \tag{558}$$

$$X(\sigma_{\alpha(\lambda)}(t - z_l) - \sigma_{\alpha(\lambda)}(t - z_1)), \qquad 2 \le l \le n, \tag{559}$$

$$X\frac{\partial^j}{\partial t^j}\sigma_{\alpha(\lambda)}(t - z_l), \qquad j \ge 1, \qquad 2 \le l \le n, \tag{560}$$

$X \in \mathfrak{g}_\alpha$ or $\mathfrak{h}$ if $\alpha = 0$, are well defined provided $|\operatorname{Im}\alpha(\lambda)| < \operatorname{Im}\tau$ and belong to $\mathcal{L}(z_1,\dots,z_n,\lambda,\tau)$. If α runs over $\Delta \cup \{0\}$ and X runs over a basis of $\mathfrak{g}_\alpha$ ($\mathfrak{h}$ if $\alpha = 0$), then these functions form a basis of $\mathcal{L}(z,\tau,\lambda)$.

Proof: It is easy to check that these functions belong to $\mathcal{L}(z,\tau,\lambda)$. Let $\mathcal{L}^{\le j}(z,\tau,\lambda)$ be given by the functions in $\mathcal{L}(z,\tau,\lambda)$ whose pole orders do not exceed j. By the Riemann–Roch theorem,

$$d(j) := \dim(\mathcal{L}^{\le j}(z,\tau,\lambda)) = \dim(\mathfrak{g})jn, \tag{561}$$

if $j \ge 1$. Indeed $\mathcal{L}^{\le j}(z,\tau,\lambda)$ is the space of holomorphic sections of the tensor product of a flat vector bundle on the elliptic curve by the line bundle associated to jD, where D is the positive divisor Σz_i.

The functions given here are linear independent, as can be easily checked by looking at their poles, and have the property that for $j \ge 1$, the first $d(j)$ functions belong to $\mathcal{L}^{\le j}(z,\tau,\lambda)$. $\square$

To obtain a basis outside the strip $|\operatorname{Im}\alpha(\lambda)| < \operatorname{Im}\tau$ we can transport our basis using the following isomorphisms.

Proposition 8.26 *Let $(z,\tau) \in C^{[n]}$, and $q, q' \in P^\vee$. Then the map sending $X \in \mathcal{L}(z,\tau,\lambda)$ to the function*

$$t \mapsto \exp(2\pi i t \operatorname{ad} q')X(t), \tag{562}$$

is a Lie algebra isomorphism from $\mathcal{L}(z,\tau,\lambda)$ to $\mathcal{L}(z,\tau,\lambda + q + q'\tau)$.

For any open subset U of $C^{[n]}$, $C^{[n]} \times \mathfrak{h}$ or $C^{[n]} \times G$ define $\mathcal{L}^{\le j}(U)$, $\mathcal{L}_{\mathfrak{h}}^{\le j}(U)$, $\mathcal{L}_G^{\le j}(U)$ to be the space of functions in $\mathcal{L}(U)$, $\mathcal{L}_{\mathfrak{h}}(U)$, $\mathcal{L}_G(U)$, respectively, whose pole orders do not exceed j.

Corollary 8.27 *The sheaves $\mathcal{L}^{\le j}$, $\mathcal{L}_{\mathfrak{h}}^{\le j}$ are locally free, finitely generated for all $j \ge 1$. Moreover for each $x \in C^{[n]} \times \mathfrak{h}$, every $X \in \mathcal{L}_{\mathfrak{h}}(x)$ extends to a function in $\mathcal{L}_{\mathfrak{h}}^{\le j}(U)$ for some j and $U \ni x$.*

The proof in the case of $\mathcal{L}^{\leq j}$ is obtained by setting simply $\lambda = 0$.

We wish to extend this result to $\mathcal{L}_G$. Let us first notice that the function $\sigma_w(t)$ is actually a meromorphic function of $e^{2\pi i w}$. Thus if $g = \exp(2\pi i\lambda)$, the functions in Prop. 8.25 can be written as $f(\mathrm{Ad}(g), t, z, \tau)X$, where the meromorphic function f is regular as a function of the first argument in the range corresponding to $|\,\mathrm{Im}\,\alpha(\lambda)| < \mathrm{Im}(\tau)$. Therefore we may extend the definition of the basis to give a basis of $\mathcal{L}_G(z, \tau, g)$ for g in some neighborhood of $g = \exp(2\pi i\lambda)u$, with $\mathrm{Ad}(u)$ unipotent commuting with $\mathrm{Ad}(g)$ (It is clear that the multipliers are correct if g is on some Cartan subalgebra, but such g's form a dense set in G). The pole structure does not change if the neighborhood is sufficiently small. In this way by choosing properly the Cartan subalgebra, we find local bases of $\mathcal{L}_G$ in the neighborhood of all points in G whose semisimple parts are of the form $\exp(2\pi i\lambda)$ with λ in some Cartan subalgebra and $|\,\mathrm{Im}\,\alpha(\lambda)| < \mathrm{Im}(\tau)$, for all $\alpha \in \Delta$.

Proposition 8.28 *Let $(z, \tau) \in C^{[n]}$, and $q, q' \in P^\vee$. Then the map sending $X \in \mathcal{L}_G(z, \tau, g)$ to the function*

$$t \mapsto \exp(2\pi i t\,\mathrm{ad}\;q')X(t), \tag{563}$$

is a Lie algebra isomorphism from $\mathcal{L}_G(z, \tau, g)$ to $\mathcal{L}_G(z, \tau, \exp(2\pi i(q + \tau q'))g)$.

With the Jordan decomposition theorem, we get a local basis around all points of G, and we obtain:

Proposition 8.29 *The sheaf $\mathcal{L}_G^{\leq j}$,is locally free, finitely generated for all $j \geq 1$. Moreover for each $x \in C^{[n]} \times G$, every $X \in \mathcal{L}(x)$ extends to a function in $\mathcal{L}_G^{\leq j}(U)$ for some j and $U \ni x$.*

8.5.2 Connections on filtered sheaves

Let S be a complex manifold, and denote by $\mathcal{O}$ the sheaf of germs of holomorphic sections on S. A sheaf of Lie algebras over S is a sheaf of $\mathcal{O}$-modules $\mathcal{L}$ with Lie bracket $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{L} \to \mathcal{L}$ a homomorphism of sheaves of $\mathcal{O}$-modules, obeying antisymmetry and Jacobi axioms. A sheaf of Lie algebras $\mathcal{L}$ over S is said to be locally free if it is locally free as $\mathcal{O}$-module, i.e., if every $x \in S$ has a neighborhood U such that, as $O(U)$-module, $\mathcal{L}(U) \simeq W \otimes \mathcal{O}(\mathcal{U})$ for some complex vector space W In this case, $\mathcal{L}(U)$ is freely generated over $\mathcal{O}(\mathcal{U})$ by a basis $e_1, e_2, \ldots$ with Lie brackets $[e_i, e_j] = \Sigma f_{ij}^k e_k$, (with finitely many non-zero summands) for some holomorphic functions f_{ij}^k on U.

We will consider the case in which the sheaf $\mathcal{L}$ of Lie algebra is filtered by locally free sheaves of $\mathcal{O}$-modules of finite type. In other words, $\mathcal{L}$ admits a filtration

$$\mathcal{L}^{\leq 0} \subset \cdots \subset \mathcal{L}^{\leq j} \subset \mathcal{L}^{\leq j+1} \subset \cdots \subset \mathcal{L} = \cup_{j=0}^{\infty} \mathcal{L}^{\leq j}, \tag{564}$$

with $\mathcal{L}^{\leq j}$ locally[4] isomorphic to some $\mathbb{C}^{n_j} \otimes \mathcal{O}$, inclusions induced from inclusions $\mathbb{C}^{n_j} \subset \mathbb{C}^{n_{j+1}}$, and such that $[\mathcal{L}^{\leq j}, \mathcal{L}^{\leq l}] \subset \mathcal{L}^{\leq j+l}$. In particular $\mathcal{L}$ is locally free.

[4]i.e., every point of S has a neighborhood such that the statement holds for the restriction of the sheaf to this neighborhood

A sheaf of $\mathcal{L}$-modules is a sheaf V of $\mathcal{O}$-modules with an action $\mathcal{L} \otimes_{\mathcal{O}} V \to V$ which is assumed to be a homomorphism of $\mathcal{O}$-modules. The image sheaf of this homomorphism is denoted by $\mathcal{L}V$. In the filtered situation it is assumed further that V is filtered by locally free, finitely generated $\mathcal{O}$-modules:

$$V^{\leq 0} \subset \cdots \subset V^{\leq j} \subset V^{\leq j+1} \subset \cdots \subset V = \cup_{j=0}^{\infty} V^{\leq j}, \tag{565}$$

and that the action is compatible with the filtration, i.e., $\mathcal{L}^{\leq j} V^{\leq l} \subset V^{\leq j+l}$. In particular V is locally free, and we can define a dual sheaf V^* locally as $V^*(U) = \mathrm{Hom}_{\mathcal{O}(\mathcal{U})}(V(U), \mathcal{O}(\mathcal{U}))$. If $V(U)$ is of the form $\bar{V} \otimes \mathcal{O}(\mathcal{U})$ for some vector space $\bar{V}$, then $V^*(U)$ is the space of functions u on U with values in the dual $\bar{V}^*$ such that $\langle u, w \rangle$ is holomorphic for all $w \in \bar{V}$. The dual sheaf V^* has a natural structure of a sheaf of right $\mathcal{L}$-modules and we have a natural pairing $\langle\ ,\ \rangle : V^* \times V \to \mathcal{O}$.

We can define the associated graded objects

$$\mathrm{Gr}\,\mathcal{L} = \oplus_{j=0}^{\infty} \mathcal{L}^{\leq j} / \mathcal{L}^{\leq j-1}, \tag{566}$$
$$\mathrm{Gr}\,V = \oplus_{j=0}^{\infty} V^{\leq j} / V^{\leq j-1}, \tag{567}$$

with the understanding that $V^{\leq -1} = 0 = \mathcal{L}^{\leq -1}$.

Then $\mathrm{Gr}\,\mathcal{L}$ is a graded sheaf of Lie algebras acting on the graded sheaf $\mathrm{Gr}\,V$ of $\mathcal{O}$-modules, and homogeneous components are locally free and finitely generated.

The sheaf of coinvariants is $V/\mathcal{L}V$, and the sheaf of invariant forms E is locally given by

$$U \mapsto E(U) = \{\omega \in V^*(U) | \omega X = 0\ \forall X \in \mathcal{L}(U)\}. \tag{568}$$

In the filtered situation, $\mathcal{L}V$ is filtered, with $(\mathcal{L}V)^{\leq j} = \Sigma_{r+s=j} \mathcal{L}_r V_s$ and we have induced homomorphisms

$$(V/\mathcal{L}V)^{\leq 0} \to \cdots \to (V/\mathcal{L}V)^{\leq j} \to (V/\mathcal{L}V)^{\leq j+1} \to \cdots \to V/\mathcal{L}V = \varinjlim (V/\mathcal{L}V)^{\leq j}. \tag{569}$$

Locally, $(V/\mathcal{L}V)^{\leq j}(U)$ is the quotient $V^{\leq j}(U)$ by the submodule of linear combinations of elements of the form Xv, $X \in \mathcal{L}^{\leq r}$, $v \in V^{\leq s}$ with $r + s \leq j$.

A connection ∇ on a sheaf of $\mathcal{O}$-modules V is a $\mathbb{C}$ linear map $V \to \Omega^1 \otimes_{\mathcal{O}} V$, where Ω^1 is the sheaf of holomorphic $(1,0)$-differential forms on S, such that for all open sets $U \subset S$,

$$\nabla(fv) = f\nabla v + df \otimes v, \tag{570}$$

for any $f \in \mathcal{O}(\mathcal{U})$, $v \in V(U)$. The notation ∇_ξ is used to denote the covariant derivative in the direction of a local holomorphic vector field ξ: if $\nabla v = \Sigma_i \alpha_i \otimes v_i$, $\nabla_\xi v = \Sigma \alpha_i(\xi) v_i$. A connection D on a sheaf of Lie algebras is furthermore assumed to have covariant derivatives being derivations for all local vector fields ξ:

$$D_\xi[X, Y] = [D_\xi X, Y] + [X, D_\xi Y], \qquad X, Y \in \mathcal{L}(U), \tag{571}$$

and a connection ∇ on a sheaf of $\mathcal{L}$-modules with connection D is assumed to be compatible with the action, i.e.,

$$\nabla_\xi(Xv) = (D_\xi X)v + X\nabla_\xi v, \qquad X \in \mathcal{L}(U), \quad v \in V(U). \tag{572}$$

Such a connection induces a unique connection, also called ∇ on V^* such that for all open $U \subset S$, $u \in V^*(U)$, $v \in V(U)$,

$$d\langle u, v\rangle = \langle \nabla u, v\rangle + \langle u, \nabla v\rangle, \tag{573}$$

Let ∇ be a connection on a sheaf V of $\mathcal{O}$-modules. If V is filtered by free, finitely generated $\mathcal{O}$-modules $V^{\leq j}$, we say that ∇ is *of finite depth* if there exists an integer d such that $\nabla V^{\leq j} \subset \Omega^1 \otimes V^{\leq j+d}$. The smallest non-negative such integer will be called depth of the connection.

Theorem 8.30 *Let $\mathcal{L}$ be a sheaf of Lie algebras and V a sheaf of $\mathcal{L}$-modules over a complex manifold S. Suppose that $\mathcal{L}$ and V have a filtration by locally free finitely generated $\mathcal{O}$-modules, and compatible connections D and ∇ of finite depth. If $\mathrm{Gr}\, V/\,\mathrm{Gr}\,\mathcal{L}\,\mathrm{Gr}\, V$ has only finitely many non zero homogeneous summands, then the sheaf of invariant forms E is locally free and finitely generated.*

Proof: Let $z_0 \in S$ and U be a neighborhood of z_0, such that the restriction of V to U is free. Thus there exist vector spaces $\bar{V}^{\leq j}$, $\bar{V}$, such that

$$V^{\leq j}(U) \simeq \bar{V}^{\leq j} \otimes \mathcal{O}(\mathcal{U}) \qquad \mathcal{V}(\mathcal{U}) \simeq \bar{\mathcal{V}} \otimes \mathcal{O}(\mathcal{U}). \tag{574}$$

The assumption that $\mathrm{Gr}\, V/\,\mathrm{Gr}\,\mathcal{L}\,\mathrm{Gr}\, V$ has vanishing components of degree $\geq N$ means that if $j \geq N$ and $v \in V^{\leq j}(U)$ we have a decomposition (not necessarily unique)

$$v = v' + Xv'', \tag{575}$$

for some $v' \in V^{\leq j-1}$ and $X \in \mathcal{L}(U)$. By iterating this we see that we can take $v' \in V^{\leq N}$.

The first consequence of this is that the restriction map $E(U) \to V^{\leq l*}(U)$ is injective for all sufficiently large l.

The second consequence is that we can replace the connection by a connection which preserves $V^{\leq l*}(U)$ for some large l, and coincides with the given one on the image of invariant forms. The construction goes as follows.

Let us choose a basis $e_1, e_2, \ldots$ of $\bar{V}$ with the property that, for all j, a basis of $\bar{V}^{\leq j}$ is obtained by taking the first $\dim(\bar{V}^{\leq j})$ elements of this sequence. View $\bar{V}$ as the subspace of constant functions in $V(U)$, and choose a decomposition (575) for all e_i:

$$e_i = e_i' + X_i e_i''. \tag{576}$$

with $e_i' \in V^{\leq N}(U)$. Define a new connection $\tilde{\nabla}$ by

$$\tilde{\nabla} e_i = \nabla e_i'. \tag{577}$$

This formula uniquely determines a connection $\tilde{\nabla}$ on the restriction of E to U. The dual connection on $V^*(U)$, also denoted $\tilde{\nabla}$ is defined as usual by $\langle\tilde{\nabla}\alpha, e_i\rangle = d\langle\alpha, e_i\rangle - \langle\alpha, \tilde{\nabla}e_i\rangle$. By construction, this dual connection coincides with ∇ on invariant forms, and, if d denotes the depth of the connection ∇, it maps $V^{\leq N+d\,*}(U)$ to itself.

If we introduce local coordinates $t_1, \dots, t_n$ around z_0, with z_0 at the origin, we see that we have to solve the following problem: given a subsheaf E of a finitely generated free sheaf F on an open neighborhood U of the origin in $\mathbb{C}^n$, with connection $\tilde{\nabla}$ on F preserving E, show that there exists an open set $U' \subset U$ containing z_0, such that $E(U')$ is a free $\mathcal{O}(\mathcal{U}')$-module. Write F as $F_0 \otimes \mathcal{O}$, for a vector space F_0. We may assume that U is a ball centered at the origin.

Lemma 8.31 *Let ξ be the vector field $\sum_i t_i\partial_{t_i}$ on $\mathbb{C}^n$, and $\tilde{\nabla}$ be a connection on a free, finitely generated sheaf of $\mathcal{O}$-modules $F = F_0 \otimes \mathcal{O}$ on a ball U centered at the origin of $\mathbb{C}^n$. For each $\phi \in F_0$ there is a unique $\hat{\phi} \in F(U)$ such that $\hat{\phi}(0) = \phi$, and $\nabla_\xi\phi = 0$.*

The proof is more or less standard: the F_0-valued holomorphic function $\hat{\phi}$ on U is a solution of the system of linear differential equations

$$\sum_{i=1}^{n} t_i \frac{\partial}{\partial t_i}\hat{\phi}(t) = \sum_{i=1}^{n} t_i A_i(t)\hat{\phi}(t), \tag{578}$$

for some holomorphic matrix-valued functions A_i, with initial condition $\hat{\phi}(0) = \phi$. It is convenient to rewrite this equation in the form

$$\frac{d}{dx}\hat{\phi}(xt) = B(x,t)\hat{\phi}(xt), \qquad B(x,t) = \sum t_i A_i(xt). \tag{579}$$

In this form we can apply the standard existence and uniqueness theorem: the unique solution with initial condition ϕ is given by the absolutely convergent Dyson series

$$\hat{\phi}(t) = \phi + \sum_{m=1}^{\infty} \int_{\Delta_m} B(x_1,t)\cdots B(x_m,t)\phi\, dx_1 \cdots dx_m, \tag{580}$$

The domain Δ_m of integration is the simplex $0 < x_1 < \cdots < x_m < 1$. It is clear from this formula that $\hat{\phi}$ is holomorphic on U. This concludes the proof of the lemma.

Let E_0 be the subspace of F_0 consisting of all values at 0 of sections of $E(U')$ where U' runs over all open balls contained in U and centered at the origin. Let $e_1, \dots, e_r$ be a basis of F_0 such that the first s e_i build a basis of E_0. The homomorphism of $\mathcal{O}(\mathcal{U}')$-modules

$$\tau : E_0 \otimes \mathcal{O}(U') \to F(U'), \qquad \phi \otimes h \mapsto \hat{\phi}h, \tag{581}$$

is injective since $\hat{\phi}$ vanishes if and only if ϕ vanishes. We claim that the image of τ is precisely $E(U')$, if U' is small enough. Let $\psi \in E(U')$. Then, we can write ψ as

$$\psi(t) = \sum_{j=1}^{r} a_j(t)\hat{e}_j(t), \tag{582}$$

for some holomorphic functions $a_j(t)$. By assumption, $a_j(0) = 0$ if $j > s$. Since $\tilde{\nabla}\hat{e}_i = 0$, we have

$$\tilde{\nabla}\psi(t) = \sum_{j=1}^{r}\sum_{i=1}^{n} t_i\partial_{t_i}a_j(t)\hat{e}_j(t). \tag{583}$$

But $\tilde{\nabla}$ preserves E and, therefore, $\Sigma_i\partial_{t_i}a_j(t) = 0$ if $j > s$. It follows that $a_j(t) = a_j(0) = 0$ if $j > s$. We have shown that $E(U')$ is contained in the image of the homomorphism τ. Now let, for $j = 1, \ldots, s$, $\psi_j(t)$ be sections of $E(U')$ such that $\psi_j(0) = e_j$. Such sections exist, by definition of E_0, for some neighborhood U'. Then, the construction above gives

$$\psi_j(t) = \sum_{l=1}^{s} a_{jl}(t)\hat{e}_l(t). \tag{584}$$

The holomorphic matrix-valued function $(a_{ij}(t))$ is the unit matrix at $t = 0$ and is thus invertible for $t \in U'$, if the ball U' is small enough. We conclude that $\hat{e}_j \in E(U')$, which completes the proof. $\square$

Let us see how this applies to our situation, following [TUY89]. For us S is either of $C^{[n]}$, $C^{[n]} \times \mathfrak{h}$, $C^{[n]} \times G$, and $\mathcal{L}$ is the corresponding sheaf of Lie algebras, which we denoted $\mathcal{L}$, $\mathcal{L}_{\mathfrak{h}}$, $\mathcal{L}_G$, respectively. The module V is the free graded $\mathcal{O}$-module $V^{\wedge[n]} \otimes \mathcal{O}$. The key observation is that $\mathrm{Gr}(\mathcal{L})_j$ consists of the degree j part of $(\mathfrak{g} \otimes \mathbb{C}[t^{-1}])^n \otimes \mathcal{O}$ for all sufficiently large j. Moreover $\mathrm{Gr}(V) = V$ canonically since V is graded, and the action of elements of sufficiently high degrees in $\mathrm{Gr}(\mathcal{L})$ on $\mathrm{Gr}(V)$ comes from the action of $\mathfrak{g} \otimes \mathbb{C}[t^{-1}]$ on the factors $V_j^{\wedge}$.

The fact that $\mathrm{Gr}\, V/\,\mathrm{Gr}\,\mathcal{L}\,\mathrm{Gr}\, V$ has only finitely many non trivial homogeneous components follows then from the fact that $V^{\wedge}/t^{-N}\mathfrak{g} \otimes \mathbb{C}[t^{-1}]$ is finite dimensional for all positive integers N, which is proved in [TUY89] using Gabber's theorem.

9 Summary

In this thesis we considered two classic animals in the rational conformal zoo, the minimal models [BPZ84] and the Wess–Zumino–Witten models [W84, KZ84, TK88]. We restricted our attention to correlation functions in the analytic chiral sector, the conformal blocks. In both cases integral representations for the conformal blocks are known from [DF84, ZF86, CF87, SV89, F89, BF90]. They are given by multiple contour integrals, which generalize Gauss' integral representation for the hypergeometric function. The integrals are many valued functions on configuration spaces. Their monodromy yield representations of braid groups. An amazing observation is that the monodromy of these integrals, computed by contour deformation [GN84, DF84, TK88, FFK89, La90], is given by quantum group data [FW91, GS90, PS90, AGS89]. Thus one has an explicit connection to another field of two dimensional symmetry, the theory of quantum groups [D86, Ji85, Lu88, FK93, FRT87] and integrable lattice models [PS90, Pa88, Ka89].

Chapter two was devoted to this connection. We introduced a topological quantum group action on an enlarged space of integration contours, which label conformal blocks. The physical conformal blocks were identified with singular vectors in the quantum group module. We identified the quantum group representation with a product of Verma modules. This topological quantum group representation was our main subject. The analysis was done in entirely in terms of integral representations for conformal blocks. It did not make use of the properties of field operators. In chapter six and chapter seven we furthermore used a scheme which defined conformal blocks without direct reference to field operators. Recall also chapter three for the operator origin of integral representations.

This point of view leaves aside some important issues like braid group statistics [Fr88], super–selection sectors [DHR71],and their quantum symmetry [MS90, FRS89, FK93]. To make contact with these subjects one should reconstruct the conformal field theory [FFK89]. Another important issue also left aside is BRST-invariance. Topological representations should also have a meaning in terms of the BRST complex [F89, BMP90]. We hope to return to this question in future work.

Conformal models exhibit very interesting structures when they are formulated on higher topologies [FS87]. In chapter four we investigated the topological representation on a toroidal space-time. There it was more intricate than on the sphere. We were led to integrate multi–component many valued differential forms to obtain conformal blocks. Again a fundamental question was the monodromy of the outcome. The quantum group provided an efficient and beautiful answer. The monodromy was again given by an R-matrix, but this time in the adjoint representation [CFW93] rather than in a Verma module. We mention that R–matrix representations are a wide subject themselves, with applications to link invariants [KR88], invariants of three manifolds [RT91], state sum invariants [TV89], monodromy representations [Ko87] of braid groups, and braid group statistics [Fr88].

Chapter five addressed the interplay between the topological representation and modular transformations of space-time. It turned out that the differential forms from conformal field theory gave rise to topological representations which worked nicely together with the modular group. The outcome was a representation of the modular group on the quantum group representation. It was given in terms of universal quantum group elements.

Chapter six and chapter seven were devoted to a mathematical construction of conformal blocks on the torus using methods of [TUY89]. This approach was purely representation theoretic and did not require the construction of field operators. Differing from the sphere we were led to consider twisted conformal blocks on the torus [Be88a]. We derived the Knizhnik–Zamolodchikov–Bernard equation in this twisted framework and discussed the role of the modular group. As a byproduct we arrived at a genus one generalization of the classical Yang–Baxter equation. We remark that the analogous construction on higher genus surfaces [Be88b] in this framework is still an open and important problem. Progress in this direction was recently made in [I94].

All this is one glimpse of the outstanding coherency and unity of two dimensional conformal field theory and its branches in the physics and mathematics of integrable systems and lattice models [Pa88, PS90, Ka89], quantum groups [D86, Ji85, FRT87, Lu90], infinite dimensional Lie algebras [K90], low dimensional topology [RT91, TV89], string theory [Po81, FMS86, DFMS87, GW86, GSW87], Riemann surfaces [FS87, Be88b, FG92], braid groups and braid statistics [Fr88, MS90], Chern–Simons theory [W89, GK90], and many more areas.

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