



Field-Theory Amplitudes as Resurgent Functions

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Abstract

A writeup of a series of informal seminars, intended for graduate students, on the subject of coupling dependence in quantum field theory, with an elementary introduction to the notion of resurgent function that forms the appropriate framework for the coupling dependence of strictly renormalizable theories. While most of the discussion is pedagogical, there is also something for the expert: we demonstrate, by studying a model spectral integral, that an amplitude may possess both the t'Hooft set of singularities in the coupling-constant plane, and Borel-plane singularities of the infrared-renormalon type in its perturbative part, and yet be uniquely reconstructible from its resurgent symbol, the appropriate generalization of the semiconvergent perturbation expansion. In the same model we demonstrate the virtues of a quasi-perturbative expansion, obtained by resummation of the resurgent symbol in its nonperturbative direction, and which in contrast to the perturbative one is Borel summable. On the basis of this expansion, we discuss a systematic approximation method for the reconstruction of correlation functions with the resurgent coupling dependence typical of asymptotically free theories.

Contents

0	Generalities on Coupling Dependence	1
1	Superrenormalizable Theories: Summability	9
1.1	Semiconvergent or Asymptotic Series	9
1.2	Nevanlinna-Sokal Analyticity and Application to QFT	15
1.3	The Borel-Laplace Summation Process	22
2	Strictly Renormalizable Theories: Resurgence	33
2.1	Coupling Renormalization and Lambda Scale	33
2.2	Operator-Product Expansion and Lambda	40
2.3	Resurgent Symbols and Functions	47
2.4	A Resurgent Model Amplitude	64
3	Construction of Resurgent Amplitudes	88
3.1	The Approximating Sequence	88
3.2	Self-Consistency of the Generalized Feynman Rules	102
3.3	Things done and not done	122

0 Generalities on Coupling Dependence

We know everything that is knowable about a quantum field theory (QFT) once we know the full set of its Euclidean correlation functions,

$$G_N(x_1 \dots x_N) = \langle 0 | \varphi(x_1) \varphi(x_2) \dots \varphi(x_N) | 0 \rangle, \quad (0.1)$$

for all configurations N of elementary fields, or equivalently their Fourier transforms defined by

$$\begin{aligned} & \int d^D x_1 \dots d^D x_N e^{i(k_1 \cdot x_1 + \dots + k_N \cdot x_N)} G_N(x_1 \dots x_N) \\ &= (2\pi)^D \delta^D(k_1 + k_2 + \dots + k_N) \tilde{G}_N(k_1 \dots k_N) \end{aligned} \quad (0.2)$$

at Euclidean momenta k_i . Here we denote by D the Euclidean spacetime dimension, which we will allow to deviate from the realistic value of $D = 4$, both for the purpose of looking at lower-dimensional model theories ($D = 3, 2, 1$) and for dealing with ultraviolet divergence in $D = 4$ through dimensional regularization ($D = 4 - 2\varepsilon$). For simplicity we mostly consider, instead of (0.2), the momentum-space, Euclidean, proper vertex functions Γ_N , that is, the connected, amputated, and one-particle irreducible functions, since these are the simplest building blocks from which the full \tilde{G}_N can be pieced together purely algebraically. Their full set of functional dependences is

$$\Gamma_N = \Gamma_N(k_1, k_2, \dots, k_N; g^2(\mu), m(\mu), \xi(\mu) \dots; \mu)_R, \quad (0.3)$$

where the momenta obey the constraint $\Sigma k_i = 0$ of total-momentum conservation, and μ denotes the arbitrary renormalization mass scale introduced by renormalization in a scheme R not usually indicated explicitly – here, a dimensional-regularization-plus-minimal-subtraction scheme. Moreover, $g^2(\mu)$, $m(\mu)$, $\xi(\mu) \dots$ are the renormalized coupling constant, renormalized elementary masses, renormalized gauge-fixing parameter ... of the theory, defined at the "sliding scale" μ [WEI96] and therefore "running" with that scale, since upon changing the arbitrary μ they all need to be readjusted to keep the observable quantities constant.

In these seminars our focus will not, as usual, be on the momentum dependence of the functions (0.3), but rather on their *dependence on the coupling constant* $g^2(\mu)$. (Renormalization-Group (RG) analysis shows that these two are in fact related, and we are going to use this relation in places, but the link is really tight and restrictive only for two-point functions depending on a single squared Euclidean momentum k^2 , since for higher correlation functions the RG governs only the behavior under uniform scaling

of *all* momentum arguments but says little about individual momentum dependences.) We are less interested here in the dependence on other parameters such as $m(\mu)$, $\xi(\mu)$, and therefore will mostly get rid of those by considering massless theories ($m = 0$), adopting special choices of gauge fixing (such as $\xi = 0$), etc. This, of course, is not to say that the other dependences may not be of paramount interest in dealing with specific problems, but the interaction strength $g^2(\mu)$, which makes for a nontrivial theory in the first place, does play a distinguished role, of which these talks are intended to provide a glimpse.

To anyone with a basic knowledge of QFT, the full $g^2(\mu)$ dependence of amplitudes generated by a realistic, interacting, renormalized QFT would at first sight appear to be of unfathomable complexity, nothing short of a horrible mess. It is therefore surprising that for a class of interacting theories with mild ultraviolet problems, the so-called superrenormalizable theories (essentially, theories in which the mass dimension $[g^2]$ of $g^2(\mu)$ is positive, and which therefore mostly live in spacetime dimensions $D \leq 3$) this full coupling dependence can *in principle* be recovered from (though it is not given explicitly by) the most straightforward and unsophisticated ansatz conceivable, that of a "perturbative" power-series expansion about the free-field situation $g^2 = 0$:

$$\Gamma_N^{\text{pert}}(\{k_i\}; \alpha(\mu); \mu) = \sum_{p=0}^{\infty} \alpha^p \Gamma_N^{(p)\text{pert}}(\{k_i\}; \mu), \quad \alpha = \frac{g^2(\mu)}{4\pi}, \quad (0.4)$$

where the zeroth-order terms, the "bare vertices" $\Gamma_N^{(0)\text{pert}}$, are just the ordinary Feynman rules to be read off from the action of the theory. This situation will be briefly sketched and commented upon in chapter 1 below. We recall that an expansion of type (0.4) is straightforward to generate formally, either by expanding the path-integral representation of correlation functions (see (1.25) below) in powers of the interaction term of the Lagrangian ("integral" approach), or alternatively by iterating the coupled equations of motion of the vertex functions around their zeroth-order terms in α , which again are the $\Gamma_N^{(0)\text{pert}}$ of (0.4) ("differential" approach). Either route leads to the familiar representation of the p-th order term $\Gamma_N^{(p)\text{pert}}$ as a sum of Feynman diagrams with the external-legs configuration N and p successive momentum integrations ("loops"). The main practical obstacle at present to actually evaluating terms of higher perturbative order p ($p \geq 3$, say) is the explosive growth with p of the number of diagrams contributing that will soon come to haunt us in the form of eq. (0.5) below, but for the superrenormalizable theories, which enjoy the property that overall convergence of loop diagrams in the ultraviolet *improves* with increasing p , the time is probably not far away when computer programs, following one of the above two routes, will furnish us with $\Gamma_N^{(p)\text{pert}}$'s for all

configurations N of practical interest at the orders p (in the range of 10^1 to 10^2) that are nowadays routinely computed in series expansions for statistical-mechanics models.

That expansion (0.4) is at best a first step in analyzing coupling dependence has to do with the fact that, in the superrenormalizable and in more complex theories alike, the series turns out to be badly divergent. Dyson [DYS52] as early as 1952 was the first to show, by an ingenious physical argument, that in the case of quantum electrodynamics (QED), the correlation functions must possess some kind of non-analyticity at real negative α , and that consequently the radius of convergence in the complex α plane for expansion (0.4) is zero. Later work by many authors, work which received a particular boost from the instanton results of Lipatov [LIP72], established techniques for calculating the leading large- p behavior of the coefficients $\Gamma_N^{(p)\text{pert}}$, with the results always fitting into the generic form

$$\Gamma_N^{(p)\text{pert}} \longrightarrow c p^B A^{-p} p! \left[1 + \mathcal{O}\left(\frac{1}{p}\right) \right] \quad (p \rightarrow \infty), \quad (0.5)$$

where c carries the momentum dependence, and A, b are real constants. (The large- p calculations form an interesting subject in themselves, into which these seminars cannot really enter, so I must refer you to the editor's introduction and the reprinted primary sources in the edited volume [LGZ90] if you wish to pursue this special topic.) In particular, the *instanton contribution* to large- p behavior, which is calculated along the "integral" route mentioned above by expanding the path-integral representation semi-classically around a finite-action solution to the Euclidean classical equations of motion ("instanton"), has the constant A equal to that finite action. That this instanton saturation does catch the full large- p behavior is, as far as I know, currently a hope. Indeed in the strictly renormalizable theories additional contributions have been identified in perturbation theory, which we shall have occasion to mention in sect. 2.4; they also follow the pattern (0.5) but with different constants A, b, c . In the absence of a complete classification of possible contributions to large- p behavior, the mathematically minded person would not regard the divergence of the α expansion as proven, but since the pattern of eq. (0.5) has resurfaced in all large- p studies performed so far, we shall assume here pragmatically that up to linear combination with possibly different sets of constants, *but with the $p!$ factor common to all terms*, this relation does give the large- p behavior. The factorial growth, which in the superrenormalizable theories simply reflects the fast increase in the number of combinatorial possibilities of drawing a p -loop Feynman diagram, precludes convergence of the series (0.5) at any α .

It is then obvious that for coupling dependence, expansion (0.5) is not simply "the" answer. To make some use of the information contained in it, one essentially has three

options. If a regime exists where the running coupling $\alpha(\mu)$ is small, one may exploit the fact that (0.4) with (0.5), its divergent character notwithstanding, possesses the property of *semiconvergence* (section 1.1) to halt the expansion at a finite $p = p_{\max}$ and use the resulting polynomial approximant directly as an estimate for the function Γ_N – an estimate whose accuracy, while limited in principle, may in practice be quite satisfactory. This is what Poincaré, in chapter VIII of his celebrated *Nouvelles Méthodes de la Mécanique Céleste* [PO893] where he introduced the modern concept of semiconvergent series, referred to as *la règle des astronomes* – and "les astronomes" were, of course, the perturbative physicists of his day, the practitioners of celestial mechanics calculating planetary-orbit perturbations. This is also what is routinely done in the electroweak theory, where the gauge couplings are indeed small in the currently accessible range of scales, $\mu \lesssim v \approx 250$ GeV (the electroweak scale).

In regimes where $\alpha(\mu)$ is not small enough to provide good semiconvergence, one may try to adduce physical arguments to justify, for particular kinematical situations, an estimation of Γ_N by "partial resummation", i. e., the *exact summation to all orders of certain subseries* of (0.4) whose terms one has reason to believe will dominate the particular phenomenon. In most cases the series summed are subseries of nonzero convergence radius – geometric, exponential, or Bessel-function series – which stay clear of the $p!$ growth problem. Examples are the "leading-logarithms" geometric resummations one performs (as discussed in sect. 2.1 below) in order to trade the coupling $\alpha(\mu)$ for a coupling $\bar{\alpha}(Q^2)$ running with some momentum variable Q^2 (one of the k_i^2 or $k_i \cdot k_j$ or a combination of these) of Γ_N , or calculations of bound states like positronium by solving a Bethe-Salpeter equation with low-order irreducible kernel, which amounts to summing an integral-kernel-valued geometric series in α . Such summations are useful and (as in the bound-state problem) sometimes necessary to extend the range of applicability of low-order perturbative information, and to the extent that the resummed function often has properties qualitatively different from those of the finite polynomial approximants, they may be called "nonperturbative". One should however keep in mind, particularly when looking beyond the superrenormalizable theories, that they will not recover terms that are invisible in principle in the semi-convergent perturbative expansion, and in the context of these seminars I will therefore refer to them as "weakly" nonperturbative.

On the other hand if one is interested in the more theoretical question of whether, or to what extent, the series determines the "true", exact function Γ_N , one must look for uniqueness properties justifying the application of what mathematicians call limiting procedures, or summability methods – the fine art of sensibly assigning a value to a divergent series. Such uniqueness results exist for the superrenormalizable theories and

will be sketched in sect. 1.2; sect. 1.3 will then go on to discuss the most frequently used technique, the Borel-Laplace transform, of actually performing the reconstruction of Γ_N . In this context, sect. 1.3 will also take a look at the analytic-continuation step required for that transform – apparently a purely technical problem, but one that will guide us in more central, analogous problems arising further down the road in sect. 3.1. – All of the material of chapter 1 is "old", and most of it in fact was old already when physicists became interested in it in the late 1960's, dating back as it does to the work of mathematicians of the late 19th and early 20th centuries, but I believe it is of timeless interest and value, and deserves to be retaught on occasion.

The situation is markedly different in the realistic field theories in $D = 4$ – in particular, the asymptotically free gauge theories central to present-day particle physics – that are "strictly" or "marginally" renormalizable but not superrenormalizable (chapter 2). There, the very definition of the theory involves a crucial new element, a *divergent coupling renormalization*. (More precisely, the tri- and quadrilinear *interaction monomials* in the Lagrangian of the theory, which in the superrenormalizable cases underwent at most finite rescalings, now demand their own ultraviolet-divergent renormalizations, but here we will continue to somewhat loosely refer to just the coupling renormalization that is at the core of these). If $\bar{g}_0(\varepsilon)$ denotes the bare coupling originally appearing in the interaction terms of the Lagrangian, with mass dimension $[\bar{g}_0] = \varepsilon = \frac{1}{2}(4 - D)$ in $D = 4 - 2\varepsilon$ dimensions, then its relation to the dimensionless and renormalized coupling $g(\mu)$ that parametrizes the functions (0.3) must in these theories be written as

$$\bar{g}_0^2(\varepsilon) = g^2(\mu) \mu^{2\varepsilon} Z_\alpha(g^2(\mu), \varepsilon), \quad (0.6)$$

where the coupling-renormalization constant, Z_α , is ultraviolet divergent as $\varepsilon \rightarrow 0$ at any finite order in $g^2(\mu)$ except for the zeroth order, $Z_\alpha(0, \varepsilon) = 1$. This is in contrast to the superrenormalizable theories where Z_α is ultraviolet finite term by term and can always be transformed to 1 by a *finite* rescaling of the elementary fields $\varphi(x)$ (and, possibly, of $m(\mu)$). You have become accustomed (or so I presume) to the idea that the renormalization process is every bit as important in determining the dynamics of a QFT as the essentially classical Lagrangian, and you will therefore be prepared to hear that this new ingredient draws a fine but fateful line between the superrenormalizable and the strictly renormalizable field theories, a line signalling in fact a deep and fundamental divide. (I confess that doubts sometimes beset me as to whether the authors of the myriad scholarly papers studying superrenormalizable models in order to learn something for the realistic theory have always fully realized what they were up against.) As we shall recall in sect. 2.1, this new feature directly translates into the emergence of an

”RG-invariant mass scale”, Λ , that is non-analytic at $\alpha = 0$ in such a way as to permit no power-series expansion in α at all, not even in the semiconvergent sense [GNV]. In such a theory, the very act of writing expansion (0.4) already implies that one has lost an important class of terms in the correlation or vertex functions, and although certain traces of the loss remain, it is essentially irretrievable. (The terms lost will become important as soon as one or more of the Lorentz-invariant momentum arguments $k_i \cdot k_j$ in our functions are not much larger than Λ^2 .) To specify the minimal information required for a reconstruction of such a function one then needs, in place of (0.4), a more detailed formal construct called a general resurgent symbol, or a resurgent double expansion, which still has a ”perturbative direction” proceeding in powers of α , but now also features a ”nonperturbative direction” providing for a systematic approximation of the new, strongly nonanalytic α dependence. As discussed in sect. 2.3, the reconstruction (through an extension of the Borel-Laplace process) then takes place in the framework of what might be called, in relation to the summable semi-convergent series, the next larger, mathematically well-delineated class of functions, known to mathematicians as the *resurgent functions*. In sect. 2.2 we sketch how an old acquaintance, the operator-product expansion (OPE) of Wilson [WIL72] in the case of an asymptotically free theory, can be recognized as a resurgent double expansion, and provides hints as to the special form this expansion will take in an asymptotically free situation. Section 2.4, in which I hope you will find something new, examines a simple model of a coupling-resurgent function, close enough in form to simple field-theory amplitudes to claim relevance for our subject, but still fully tractable by analytic means. This example will (I hope) convince you of the importance of treating a resurgent function as a whole: you will see how its analytic properties seem to sabotage the Laplace reconstruction from the perturbative expansion (0.4), but how in fact the reconstruction is entirely well defined when the strongly nonperturbative parts, whose separation is structurally averse to the function’s resurgent character, are fully taken into account.

The final chapter 3 is more special. It attempts to explain the basic ideas (while glossing over much technical detail) of a particular strategy for dealing with the one question to which the present mathematics of resurgent functions does not seem to contain a ready-made answer, and on which the physicist, for the time being, is therefore left to his own heuristic devices. This is the question of how to obtain some systematic formal representation of a resurgent amplitude (from which resurgent analysis proper may start) when the double formal series is not satisfactory for this purpose but needs a certain amount of analytic continuation to begin with. This need arises from the simple fact that the basic equations of motion generating the resurgent amplitudes of QFT are not, as in almost all cases studied in mathematical theory, ordinary differential, integro-

differential, or difference equations *with respect to the coupling*, but a rather complex, hierarchically coupled, and renormalizably divergent system of integral equations *with respect to the momentum (or coordinate) arguments* of correlation functions, equations into which the coupling enters only as a parameter. Actually, the emergence of functions resurgent with respect to a parameter – rather than the variable acted upon by a differential operator – from differential equations has been studied in mathematics under the name of *quantum resurgence*, and properly speaking it is this kind of resurgence that will concern us below. But there is an extra difficulty here, arising from the fact that the nonperturbative coupling dependence enters as a *dimensionful* parameter, the mass scale Λ , and is therefore inextricably linked with the dependence on the momentum variables k_i : in the integral equations, the loop integrals extend down to the smallest momenta $|k_i|$, and in order to get started at all, we need to have something to insert for the integrands that is valid also at $|k_i| \ll \Lambda$, whereas resurgent symbols of the OPE variety only furnish asymptotic representations for $|k_i| \gg \Lambda$. We are therefore going to need, from the very start, at least some partial continuation (namely, with respect to the Λ variable) of the symbols. Section 3.1 outlines one practical form of such a partial continuation, drawing on what we may have learned in sect. 1.3, which can be cast in the suggestive form of *generalized Feynman rules*, objects of zeroth order ($p = 0$) in the "perturbative direction" but providing a systematic continuation-through-resummation in the "non-perturbative direction". Section 3.2 sketches the special self-consistency mechanism by which the generalized Feynman rules can establish themselves in the hierarchical vertex equations, and mentions some nontrivial consequences of the fact that this mechanism is tied to the renormalizable loop divergences of those equations. Section 3.3 is a brief outlook on some of the features of the coupling-resurgent iterative solution that can then in principle be generated from the generalized Feynman rules – an outlook certain to be incomplete, as many other such features probably remain to be explored. My goal here will have been attained if you come away from this with a realization that the *two-step nature* of this reconstruction process is not due to the shortcomings of the physicist's heuristic patchwork, but is firmly rooted in the mathematical nature of resurgent functions.

Despite their title, these talks are not lectures on mathematics. Although we are going to quote a few mathematical theorems without proofs and indulge in a certain amount of name-dropping, we will mostly proceed, as physicists do more often than not, by plausibility, formal calculation, and simple examples. The reason is lack of expertise, certainly not lack of enthusiasm, on the part of your speaker. I do feel obliged, however, to draw your attention to the fact that there already exists a substantial body of mathematical research on the fascinating objects called resurgent functions. (I learned

what little I hope to have understood from selected chapters of the books [CNP93] and [STS96].) A large part of this research builds on the fundamental contribution of one person, the French mathematician Jean Écalle, who in *Les fonctions réurgentes* (1981 - 85), a three-volume work totalling almost 600 pages [ECA81], first singled out resurgent functions as a remarkable and unexplored phenomenon within classical complex analysis, studied comprehensively their structural aspects – the algebras and subalgebras, auto- and homomorphisms, dual-space and fiber-bundle structures engendered by them – that are of premier interest to the mathematician, introduced, in the form of the so-called alien derivatives (*dérivations étrangères*) and of the associated alien differential equations, a whole array of powerful new tools and concepts for characterizing the peculiar continuation behavior of resurgent functions around singularities of their Borel transforms, and developed on this basis a new classification of ordinary differential equations of truly remarkable scope and depth, where large classes of equations related by quite general analytic mappings can now be characterized by a single "complete set of holomorphic invariants". In these seminars, because of the limitations imposed by time and by my insufficient mathematical preparation, I am going to use only a few of the very simplest and most qualitative concepts from this beautiful edifice, but even these few will, I am confident, demonstrate to you one hallmark property of Écalle's theory: its power to structure, to clarify, and to provide larger perspective on, what previously may have looked like heaps of isolated and poorly understood results.

1 Superrenormalizable Theories: Summability

1.1 Semiconvergent or Asymptotic Series

We are accustomed to dealing with functions $f(z)$ of a complex variable z analytic around $z = 0$ and therefore having a power-series expansion in z within a circle of convergence around the origin. Semiconvergent, or asymptotic, expansions at the origin usually arise in situations where $z = 0$ is not only a point of non-analyticity of $f(z)$, but is in fact a member of a whole set of non-analyticity points – an endpoint of a branch cut, say, or an accumulation point of a sequence of poles, etc. – that reach up to the origin from somewhere in the complex plane, and the analyticity situation, not only globally but already locally around $z = 0$, is therefore "anisotropic" to begin with: the residual domain of analyticity, starting at $z = 0$, is usually concentrated along some direction, or sector of directions, in the z plane. In our context, where z is the coupling g^2 or α , the positive real axis is distinguished among possible directions of this kind: it represents the *physical* values of the coupling, and if a field-theory amplitude were singular or two-valued at points on this axis, the underlying QFT would quite simply be nonexistent as a physically acceptable theory. We will therefore always assume that our functions $f(z)$ are analytic in at least some narrow region \mathcal{R} along the positive real axis (mathematical theory, to be sure, deals with much more general situations). This, for example, may simply be a strip of constant width along the axis (Fig. 1.1 (a)), or a sector of opening angle γ with apex at the origin (Fig. 1.1 (b)), or, to mention a seemingly exotic case that is nevertheless of relevance for QFT, a region bounded by circular arcs tangent to the positive real axis at $z = 0$ (Fig. 1.1 (c)). The "sectorial" analyticity of Fig. 1.1 (b) is the standard situation postulated in practically all mathematical treatments, and indeed often made part of the very definition of asymptotic series, but we should keep in mind that physics may force other kinds of domains on us. However, they all have in common that they do not allow for a finite circle of analyticity around the point $z = 0$.

A formal power series in z , i. e., an object of the form

$$\sum_{n=0}^{\infty} a_n z^n \tag{1.1}$$

with real or complex coefficients a_n and with nothing being said as yet about its possible value, is called a *semiconvergent or asymptotic expansion* for our function $f(z)$ if for all z in the analyticity region \mathcal{R} the error estimate

$$\left| f(z) - \sum_{n=0}^N a_n z^n \right| \leq C_{N+1} |z|^{N+1} \tag{1.2}$$

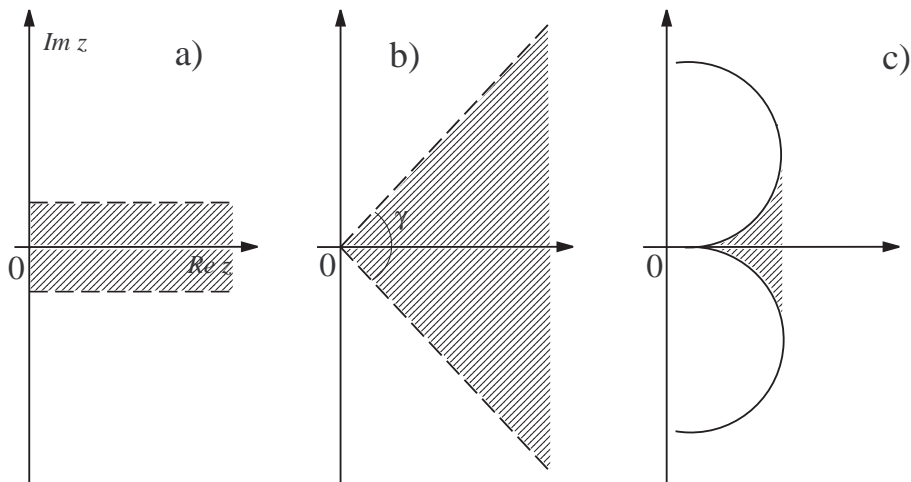


Fig. 1.1 Examples of regions \mathcal{R} of analyticity including the positive real axis

is valid for all nonnegative integers N , with constants C_{N+1} such that the error bound eventually diverges for large N everywhere in \mathcal{R} :

$$C_{N+1} |z|^{N+1} \longrightarrow \infty \quad (N \rightarrow \infty). \quad (1.3)$$

One expresses this situation by writing

$$f(z) \sim \sum_{n=0}^{\infty} a_n z^n, \quad (1.4)$$

with the \sim sign being read as "has the asymptotic expansion". In contrast to the case of a convergent series, (1.2) does not guarantee that the error on its l.h.s. can be made arbitrarily small at all points in \mathcal{R} by going to sufficiently large N . The expansion may nevertheless be useful for obtaining estimates of $f(z)$ at small z in \mathcal{R} : for $|z| \ll 1$, the exponential decrease of $|z|^N$ with N may, for the first few N 's, win over the growth of the C_{N+1} , with the divergence (1.3) making itself felt only at larger N 's, and if one halts the expansion "in time", one may come away with an error that is tolerable for the purpose at hand. Such an error pattern, illustrated in Fig. 1.2, is behind all "pragmatic" applications of asymptotic expansions, where one strictly avoids asking questions about a possible meaning or value of the series (1.1) to all orders.

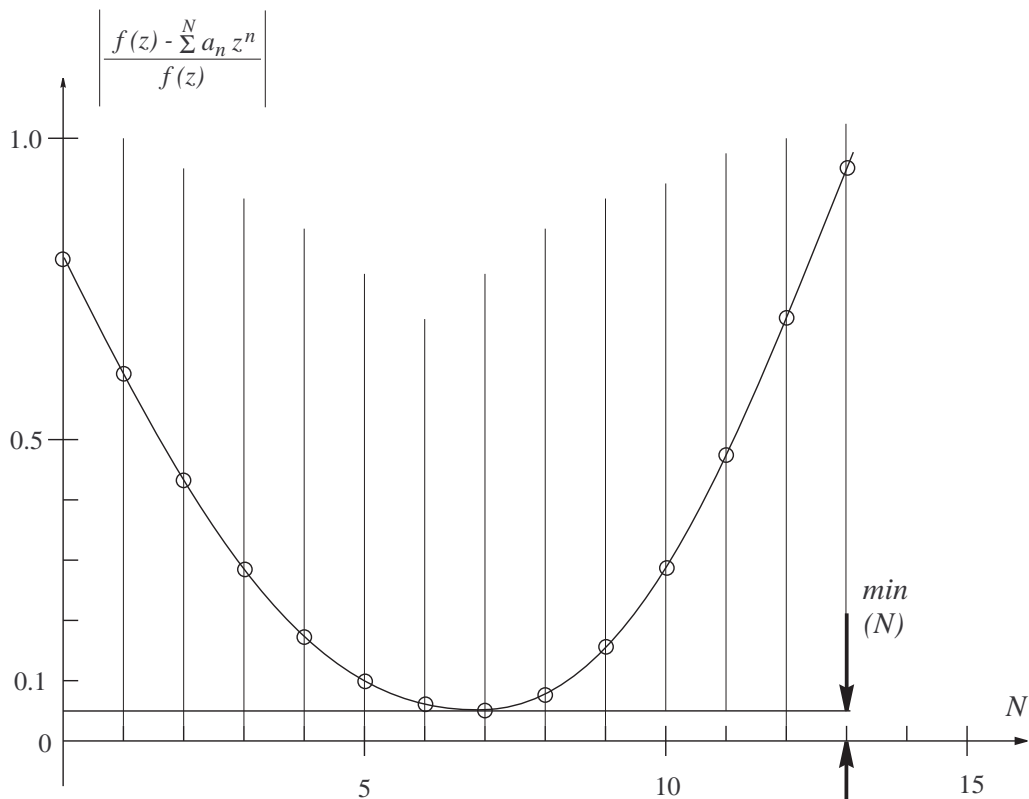


Fig. 1.2 Typical error behavior of semiconvergent expansion

This pragmatic handling of asymptotic series is facilitated by some simple formal properties: two formal series semiconvergent to functions $f_1(z)$ and $f_2(z)$ in the same region \mathcal{R} can be linearly combined, with complex coefficients ζ_1 and ζ_2 , and multiplied to give formal series that are semiconvergent to the corresponding linear-combination and product functions, $\zeta_1 f_1 + \zeta_2 f_2$ and $f_1 \cdot f_2$, respectively. A series (1.1) asymptotic to $f(z)$ may be integrated termwise from $z = 0$ out to some $z \in \mathcal{R}$ to give a series asymptotic to a primitive of f ,

$$\int_0^z f(w) dw \sim \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^n . \quad (1.5)$$

On the other hand the series produced by termwise differentiation,

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} z^n , \quad (1.6)$$

can be trusted to be semiconvergent to the derivative function $f'(z)$ in \mathcal{R} only if one knows in advance that this derivative exists, is continuous, and does possess an asymptotic expansion.

Can a function have more than one asymptotic expansion? It is easy to see that the answer is no [WIG77]: divide the estimate (1.2) by $|z|^N$ so that one factor of $|z|$ remains on the r. h. s., and let $z \rightarrow 0$ from within \mathcal{R} or, as we will put it in view of the special nature of our regions \mathcal{R} , "from the right". You get

$$\frac{f(z) - \sum_{n=0}^N a_n z^n}{z^N} \rightarrow 0 \quad \text{as } z \rightarrow 0+ \quad (\text{all } N), \quad (1.7)$$

a property often used as an alternative *definition* of semiconvergence. Spelling this out for successive N 's one has

$$\begin{aligned} \left[f(z) - a_0 \right]_{z \rightarrow 0+} &= 0 & \Rightarrow a_0 &= f(0+), \\ \left[\frac{f(z) - a_0}{z} - a_1 \right]_{z \rightarrow 0+} &= 0 & \Rightarrow a_1 &= f'(0+), \\ \left[\frac{\frac{f(z) - a_0}{z} - a_1}{z} - a_2 \right]_{z \rightarrow 0+} &= 0 & \Rightarrow a_2 &= \frac{f''(0+)}{2!}, \end{aligned} \quad (1.8)$$

etc. That is, if a function $f(z)$, analytic in a region \mathcal{R} as specified above, has an asymptotic expansion at all, then that expansion is identical with its Taylor series "from the right" at $z = 0$:

$$a_n = \frac{1}{n!} \left(\frac{d^n f}{dz^n} \right)_{z \rightarrow 0+} \quad (n = 0, 1, 2, \dots). \quad (1.9)$$

Moreover, by writing the Taylor expansion of $f(z)$ with remainder term at $z \in \mathcal{R}$, one obtains the integral representation

$$f(z) - \sum_{n=0}^N a_n z^n = \left[\frac{1}{N!} \int_0^1 dt (1-t)^N f^{(N+1)}(tz) \right] z^{N+1} \quad (1.10)$$

of the error term. If the derivatives $|f^{(N+1)}|$ are all bounded throughout \mathcal{R} by N -dependent constants, then so is the bracketed integral, and the bound gives back the constants C_{N+1} of (1.2). This point is important – for the bound of (1.2) to hold, it is

not sufficient to know the large- n behavior of the Taylor coefficients at $z = 0+$ (such as relation (0.5)). We also need some boundedness properties of the function f and its derivatives *away from zero*, which the formal series alone does not supply.

Importantly, the statement converse to this result is not true: a given formal series (1.1) can be a semiconvergent expansion for more than one function $f(z)$ analytic in \mathcal{R} , and in fact to infinitely many such functions. Indeed if $f_1(z)$ and $f_2(z)$ differ by a function $g(z)$ analytic in \mathcal{R} and having all its derivatives vanishing as $z \rightarrow 0+$,

$$f_2(z) - f_1(z) = g(z); \quad \left(\frac{d^n g}{dz^n} \right)_{z \rightarrow 0+} = 0 \quad (\text{all } n), \quad (1.11)$$

then both have the same asymptotic expansion. Such a $g(z)$ would vanish identically if it were to be analytic in a region around $z = 0$, but with our much weaker condition of "anisotropic" analyticity in a region \mathcal{R} , it is quite possible for $g(z)$ to be nonzero. From (1.2) we can immediately deduce an alternative characterization of such functions,

$$|g(z)| \leq D_{N+1} |z|^{N+1} \quad (z \in \mathcal{R}, \text{ all } N), \quad (1.12)$$

with constants D_N . That is, they "vanish faster than any power of z " as $z \rightarrow 0+$ in \mathcal{R} . Of course, the two functions f_1 and f_2 differing by g have different constants C_N (namely, $C_{N+1}^{(2)} = C_{N+1}^{(1)} + D_{N+1}$) in their error estimates (1.2), so these constants are characteristic of the *individual functions*, not of their asymptotic expansion.

In order to pin down more precisely this class of functions which "fail to show up" in an asymptotic expansion, and also to estimate the minimal error observed in Fig. 1.2, we need to adopt more specific hypotheses about the constants C_N in (1.2). In view of what we said in the introduction about the large- p pattern (0.5) in QFT, it seems natural to assume from now on that

$$C_N^{(f)} = c_{(f)} N^B |A|^{-N} N! \quad (N \geq 1), \quad (1.13)$$

(although as already emphasized this does *not follow* from (0.5) alone). Here only the positive constant $c_{(f)}$ depends on the individual function $f(z)$, while B and A are to be the same for all functions having the same asymptotic expansion (this is not essential but simplifies calculations). This form clearly has the property (1.3). It would obtain, for example, if in (1.10) we had the estimate

$$|f^{(N)}(z)| \leq c_{(f)} N^B |A|^{-N} (N!)^2, \quad z \in \mathcal{R}. \quad (1.14)$$

The case $B = 0$ is known as *functions of Gevrey class 1* in the mathematical literature. With this assumption, a function $g(z)$ as in (1.11) satisfies

$$|g(z)| \leq c_{(g)} N^B \left| \frac{z}{A} \right|^N N! \quad (N \geq 1). \quad (1.15)$$

We can find the lowest among this infinite sequence of bounds by minimizing the r. h. s. with respect to N at fixed $|z|$. For this purpose, we temporarily treat N as a continuous variable, writing $N! = N \Gamma(N)$. The condition for (1.15) to be minimal at $N = N_m$ then turns out to be

$$\ln \left| \frac{z}{A} \right| + \psi(N_m) + \frac{B+1}{N_m} = 0, \quad (1.16)$$

where $\psi(x)$ is the logarithmic derivative of the gamma function. There is no closed-form solution to this equation, but we are primarily interested in small- z behavior, $|z| \ll |A|$, where we expect $N_m \gg 1$, so we may use $\psi(x) = \ln x + \mathcal{O}(x^{-2})$ [GRY94], the Stirling expansion applied to ψ , to write

$$\ln \left(N_m \left| \frac{z}{A} \right| \right) + \frac{B+1}{N_m} + \mathcal{O} \left(\frac{1}{N_m^2} \right) = 0. \quad (1.17)$$

The leading-order solution obviously is $N_m \approx \left| \frac{A}{z} \right|$, so we can write N_m as a Laurent expansion in $\frac{z}{A}$ and obtain

$$N_m = \left| \frac{A}{z} \right| - (B+1) + \mathcal{O} \left(\left| \frac{z}{A} \right| \right). \quad (1.18)$$

Plugging this into the r. h. s. of (1.15) while using for $N!$ once more the Stirling formula, we obtain the lowest bound on $g(z)$ in the form of a small- z expansion,

$$|g(z)| \leq \left(c_{(g)} \sqrt{2\pi} |A|^{B'} \right) |z|^{-B'} e^{-\frac{|A|}{|z|}} [1 + \mathcal{O}(|z|)], \quad (1.19)$$

where $B' = B + \frac{1}{2}$, and where the last factor in square brackets is a power series in $|z|$ arising from the higher corrections in both eq. (1.18) and the Stirling formula. (Of course, unless the N_m of (1.18) happens to be an integer, the actual minimum will occur at $[N_m]$ or $[N_m] + 1$, but in the case $N_m \gg 1$ of primary interest here, this leads only to a minor readjustment of the constants in (1.19) [STS96]). We will refer to functions satisfying a bound of this type as *exponentially suppressed* (remember that even if $B' > 0$, the function $\exp\left(-\frac{|A|}{|z|}\right)$ vanishes so fast as $|z| \rightarrow 0$ as to suppress arbitrary negative powers). Thus if a function $g(z)$, analytic in \mathcal{R} , has vanishing asymptotic expansion and in addition satisfies a bound of the special form (1.15), then our earlier statement "vanishes faster than any power" can be replaced by the more precise statement of exponential suppression. It is clear that the same result (1.19), with $c_{(g)}$ replaced by $c_{(f)}$, also holds as a minimum-error estimate for the semiconvergent approximation of a function $f(z)$ having a *nonvanishing* semiconvergent expansion, and in addition obeying the special condition (1.13) on its error-bounding constants C_N .

On the other hand if $g(z)$ is known to be of the form

$$g(z) = d_{(g)} z^{-B'} e^{-\frac{A'}{z}} [1 + \mathcal{O}(z)] \quad (A' > 0) \quad (1.20)$$

with real constants A' , B' , $d_{(g)}$, then all its derivatives contain the factor $\exp(-A'/z)$ and therefore vanish as $z \rightarrow 0$ from the right, no matter what B' and the bracketed power series are, and so it at least has vanishing asymptotic expansion. That in addition it also falls into the class of eq. (1.15) can be proved *generally* only for the "sectorial" analyticity situation of Fig. 1.1 (b) with opening angle $\gamma < \pi$, where one may write

$$\left| e^{-\frac{A'}{z}} \right| \leq e^{-\frac{|A|}{|z|}} \quad \text{with} \quad |A| = A' \cos\left(\frac{1}{2}\gamma\right), \quad (1.21)$$

but it may be true in individual other cases, and in any case the weaker statement is quite sufficient for our purposes.

A very interesting observation in this context is that the square-bracketed power series in (1.19), arising as it does from the semiconvergent Stirling approximations for $\psi(N)$ and $\Gamma(N)$, is itself semiconvergent, and that semiconvergent series are also admissible for the bracketed series in (1.20), without changing our conclusion. Thus within the class of functions characterized by (1.13), the subclass of "escaping" functions that are left undetermined by a semiconvergent series are typically of the form of an exponential-suppression factor (possibly preceded by an arbitrary power $z^{-B'}$ that does no harm) *times another power series which may itself be semiconvergent*. This observation is one of the possible heuristic starting points for the more general theory discussed in chapter 2.

Of course, life would be much simpler if we could get rid of the undetermined part altogether when trying to reconstruct a function from its asymptotic expansion. In such cases the series is referred to as *summable*: there is then a *unique* function $f(z)$ having this series as its small- z expansion. It is to such a situation that we turn next.

1.2 Nevanlinna-Sokal Analyticity and Application to QFT

A *sufficient* condition that has been widely invoked for unique reconstruction of a function from its asymptotic expansion, in both mathematics and physics contexts, is based on a theorem found by Phragmén and Lindelöf already at the end of the 19th century. It again refers to the "sectorial" analyticity situation of Fig. 1.1 (b), but this

time with an opening angle γ larger than π (see Fig. 1.3):

$\|$ **Phragmén-Lindelöf Theorem:** *A function $g(z)$ analytic in a sector S of radius R and angle $\gamma > \pi$ with apex at the origin, and fulfilling the condition of exponential suppression (1.19) throughout S , must vanish identically.* $\|$
(1.22)

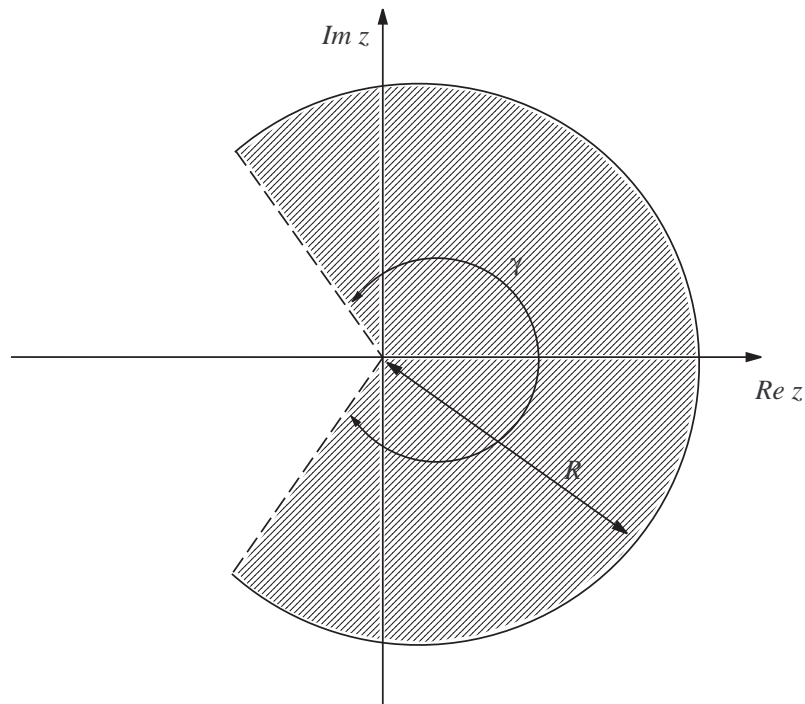


Fig.1.3 Sectorial region of analyticity of opening angle $> \pi$ in the Phragmén-Lindelöf theorem

I refer you to, for example, ref. [STS96] for a proof of this theorem, which is based on an application of the maximum principle of complex-function theory. Note that a finite radius R of the sector is quite sufficient for this theorem; the essential point is the large angle of the analyticity sector.

An immediate application of this theorem is to a function $f(z)$ analytic in such a sector of wide opening and having an asymptotic expansion (1.2) there, with error-bounding constants C_N given by (1.13). We have seen in the last section that for such a function the "escaping" functions $g(z)$, by which it differs from all other functions

having the same asymptotic expansion in the same analyticity domain, obey the bound (1.19). We therefore have the

(1.23)

Uniqueness Theorem I: *If a function $f(z)$ is analytic in a sector \mathcal{S} of opening angle $\gamma > \pi$, and possesses in this sector an asymptotic expansion (1.1) with error-bounding constants given by (1.13), then it is the only function with this asymptotic expansion in \mathcal{S} .*

This application was made in 1912 by G. N. Watson [WAT12], who combined it with a proof that one particular method of reconstructing the unique $f(z)$ from the series – the Borel-Laplace process we are going to discuss in the next section – works under the same hypothesis. His result, known as *Watson’s theorem*, became the favorite summability provider for more than half a century, right through to the first summability proofs for (superrenormalizable) QFT’s in the mid-1970’s. It is a quirk of mathematical history that nobody seems to have been irritated by the fact that the Laplace-transform representation established by Watson – our eq. (1.45) below – did not recover $f(z)$ in its full Phragmén-Lindelöf analyticity domain of Fig. 1.3 used as input to the proof, but only in a much smaller subdomain – the hatched circle of Fig. 1.4. Didn’t this look suspiciously as if analyticity in that smaller domain should have been sufficient in the first place?

Sokal [SOK80] in 1979 proved precisely this, and also redressed an injustice of history by pointing out that the result had in fact been given as early as 1918 by the Finnish mathematician F. Nevanlinna [NEV18], though in a journal perhaps not readily accessible to many researchers. We may state a first segment of this result as our

(1.24)

Uniqueness Theorem II: *If a function $f(z)$ is (i) analytic in the circular region \mathcal{C}_R given by*

$$\operatorname{Re} \frac{1}{z} > \frac{1}{R} \quad \Leftrightarrow \quad \left| z - \frac{1}{2} R \right| < \frac{1}{2} R$$

(R real and positive), and (ii) possesses in this region the semiconvergent expansion (1.1) with "factorial" error bound (1.13), then it is the only function with this asymptotic expansion in \mathcal{C}_R . In particular, with $R \rightarrow \infty$, analyticity in the half-plane $|\arg z| < \frac{\pi}{2}$ is a sufficient form of (i).

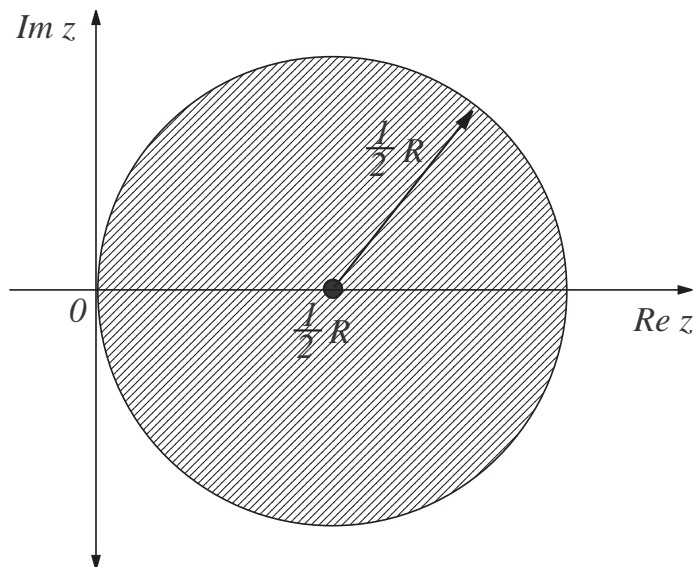


Fig. 1.4 The circular region \mathcal{C}_R in theorem (1.24).

The second, more technical segment of the Nevanlinna-Sokal result, which again concerns the Borel-Laplace reconstruction, will be stated and discussed as theorem (1.41) below.

Our two examples of sufficient summability conditions, theorems (1.23) and (1.24), create (when applicable) a situation which, if you think about it, is quite remarkable: under their hypotheses a *divergent* power series, initially a purely formal object, is nevertheless found to be associated with, and thus in a sense to determine, a unique analytic function, in much the same way as a convergent expansion does. They are therefore sometimes [STS96] referred to as *quasi-analyticity principles*. What makes them work is the potent mixture of analyticity and strong error bounding they postulate for the associated function in suitable domains – I only need to drop the catchword of Liouville’s theorem in complex analysis in order to remind you of how powerful that kind of combination can be. Yet it is prudent to remember that both theorems spell out *sufficient*, not necessary, conditions for unique reconstruction. Uniqueness may still hold in special cases not fitting into their assumptions, and section 2.4 will indeed treat you to an entirely nontrivial example of this.

For the physicist interested in QFT, the most important applications of these theorems undoubtedly have been the *summability results for the coupling dependence of correlation functions in superrenormalizable theories*, beginning in 1975 with a proof

[EMS75] for the ϕ^4 theory – a neutral scalar field with quartic self-interaction – in $D = 2$ Euclidean dimensions, and later extended along similar lines to $\bar{\psi}\phi\psi$ (Yukawa) theory in $D = 2$ and, most significantly, to the superrenormalizable ϕ^4 theory in $D = 3$, the workhorse of the modern theory of phase transitions and critical exponents [MAG77]. (It is widely believed that analogous summability properties also hold for other superrenormalizable theories, and in particular for abelian and non-abelian gauge theories in $D = 2$ and $D = 3$, though for some reason these do not seem to have been analyzed with the same degree of mathematical rigor.) These proofs involve a good deal of heavy mathematical technicality, and I cannot even begin to do them justice here, but let me try to give you at least a superficial glance at their main ideas. The starting point is always the Euclidean path-integral representation of correlation functions: taking the ϕ^4 theories as an example, we have

$$G_N(x_1 \dots x_N; g^2) = \frac{\int \mathcal{D}\mu[\phi] \phi(x_1) \dots \phi(x_N) e^{-S_I[\phi]}}{\int \mathcal{D}\mu[\phi] e^{-S_I[\phi]}}, \quad (1.25)$$

where

$$S_I[\phi] = \frac{g^2}{4!} \int d^D x \phi(x)^4 \quad (D = 2, 3), \quad (1.26)$$

and where the Gaussian functional measure, $\mathcal{D}\mu[\phi]$, incorporates the bilinear part of the Euclidean action. If the integrals were ordinary multi-dimensional integrals, it would be straightforward to conclude they are analytic functions of g^2 in at least the half-plane $|\arg(g^2)| < \frac{\pi}{2}$, i. e. $Re(g^2) > 0$, since the integrands are obviously analytic and the integrations strongly convergent in that region. In fact the integrations are infinite-dimensional and, for the axiomatic field theorist, therefore call for an exact definition of the Gaussian measures $\mathcal{D}\mu[\phi]$, and since ultraviolet renormalization in these theories can be dealt with at the level of one- and two-point functions, that definition will also have to account for renormalization. Conveniently, in the years before 1975, the measures $\mathcal{D}\mu[\phi]$ for the prototype superrenormalizable theories had already been studied exhaustively, and it had been found that the (small finite number of) *renormalization counterterms suggested by perturbation theory* were in fact sufficient for such a construction. Therefore these measures were already under good enough control mathematically so the analyticity argument could be carried over without difficulty.

To establish the second pillar of summability, the factorial error bound of eq. (1.13) in the analyticity domain $Re(g^2) > 0$, the successive derivatives with respect to g^2 of expression (1.25) were analyzed with the help of the cluster-expansion technique of statistical mechanics, which in particular allows division of the denominators into numerators – the “removal of vacuum bubbles” in physics language – to be made explicit,

and were estimated using suitable correlation-function inequalities also established by earlier constructive work. The result typically was an estimate of the form of eq. (1.14) above,

$$\left| \frac{\partial^n}{\partial (g^2)^n} G_N(\dots; g^2) \right| \leq C_1 (C_2)^n (n!)^2 \quad (\text{Re } g^2 > 0), \quad (1.27)$$

which when used in the remainder term (1.10) of the Taylor expansion around $g^2 = 0$ yields the desired bound.

An inconspicuous but nevertheless crucial precondition for the success of these proofs is that in eq. (1.26) we did not need to distinguish between a bare coupling g_0 that appears in the interaction Lagrangian, and a renormalized coupling g that serves as the perturbative expansion parameter. Indeed as remarked above, in the superrenormalizable theories the two differ at most by a *finite* rescaling, so one may expand directly in the g^2 of eq. (1.26) without loss of generality, since a finite rescaling merely changes the meaning of the constant $|A|$ in eq. (1.13). In the strictly renormalizable theories this simple connection is lost: between the Lagrangian g_0 and the expansion parameter of the perturbation series, there is now the abyss of an ultraviolet-divergent renormalization (0.6), which can no more be absorbed into a rigorous construction of a *Gaussian* measure $\mathcal{D}\mu[\phi]$. That this simplicity is lost is presumably the main reason for the continuing lack, after more than 25 years, of comparable results for realistic theories.

The superrenormalizable QFT's are as close as one can come to a perturbation theorist's paradise: in these theories the perturbation series, while divergent, nevertheless "contains the whole truth" in that it has associated with it *unique* correlation functions without escaping remainders. Conversely, in such theories there is in principle nothing that cannot be represented by, and recovered from, a semiconvergent expansion in the coupling. The term "nonperturbative" in these models can at best have a rather restricted meaning: it can refer only to what in the introduction we dubbed *weakly nonperturbative* phenomena, whose calculation requires some all-orders resummation of their perturbation series. This makes both for the special charm of these theories and for their limited usefulness when trying to learn something for the realistic, marginally renormalizable cases.

Before moving on to the technicalities of the most widely used reconstruction procedure, it may be useful to emphasize that once you have verified some sufficient condition for uniqueness of $f(z)$, it is in principle a secondary matter what method you use to get hold of that function. You may simply guess it if you are clever. Among more systematic methods, the choice depends largely on the pattern of divergence in your given series, but even for the factorially divergent series of type (0.5) one usually faces in

perturbative QFT, the Borel-Laplace procedure discussed below has no monopoly. To mention just one alternative, many functions having asymptotic expansions with this divergence pattern can be reconstructed by the method of Padé approximants [BAK70]: one approximates $f(z)$ by a rational function with numerator and denominator degrees M and N respectively,

$$f(z) \approx f_{M,N}(z) = \frac{b_0 + b_1 z + b_2 z^2 + \dots + b_M z^M}{1 + c_1 z + c_2 z^2 + \dots + c_N z^N}, \quad (1.28)$$

and adjusts the $M + N + 1$ coefficients of this function so as to make its first $M + N + 1$ Taylor coefficients at $z = 0$ agree with the coefficients $a_0 \dots a_{M+N}$ of the given divergent series (1.1). Of course, this method is easiest to apply if you have some advance information about the large- z behaviour of $f(z)$ to guide you in the choice of the relative degree $M - N$. For example, if $f(z)$ is known to fit into the general form

$$f(z) = \int_0^\infty dx \frac{\sigma(x)}{z+x}, \quad \sigma(x) \geq 0, \quad (1.29)$$

i. e. to admit a spectral representation with a cut on the negative real z axis (so that in fact we have the luxury of Phragmén-Lindelöf analyticity in a "sector" with $R = \infty$ and angle just infinitesimally smaller than 2π), and if the coefficients in its formal small- z expansion

$$a_n = (-)^n \int_0^\infty dx \frac{\sigma(x)}{x^{n+1}}, \quad n = 0, 1, 2, \dots \quad (1.30)$$

while forming a factorially divergent sequence, are individually finite, then the "diagonal" Padé approximants with $N = M$, $f_{(N,N)}(z)$, are known [BAK70] to converge to f uniformly on compact domains in the cut plane. (If (1.30) exists also for $n = -1$, so that $f(z) \rightarrow -a_{-1}/z$ for $|z| \rightarrow \infty$ outside the cut, then the $N = M + 1$ approximants, of course, will converge better). Such functions are known as *Stieltjes functions*, and their Padé reconstruction has been used successfully to sum perturbation series in the ϕ^4 theory at $D = 1$ (the theory of the anharmonic oscillator) [LOE69]. However, even in the absence of such detailed information, the method is often applied heuristically because of its simplicity, generality, and relative robustness. It does have its drawbacks – when applied directly to a factorially divergent series, its convergence in low orders $M + N$ may be erratic, and for this reason it is more frequently used as an analytic-continuation engine *within* the Borel-Laplace method, to which we turn in our next section. However, in principle it can stand on its own feet.

1.3 The Borel-Laplace Summation Process

In the case of a summable formal power series

$$f_f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (1.31)$$

the most widely used method of recovering from the series the function $f(z)$ uniquely associated with it is the venerable Borel-Laplace (BL) process [BOR28]. It requires some mild additional restriction on that function, but it is capable of handling the "factorial" coefficient growth

$$a_n \longrightarrow c A^{-n} \Gamma(n + B + 1) \quad (n \rightarrow \infty), \quad (1.32)$$

which is essentially the same as that of (0.5) since $\Gamma(n + B + 1) \rightarrow (n + 1)^B \Gamma(n + 1)$ for $n \rightarrow \infty$ [GRY94]. It is therefore suitable for summing the perturbative expansions of superrenormalizable QFT.

The first step is to map $f(z)$ by an invertible integral transformation \mathcal{B} onto a function $b(s)$ of another complex variable s , the *Borel transform* of f :

$$b(s) = \mathcal{B}[f; s] := \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} d\zeta e^{s\zeta} \tilde{f}(\zeta), \quad (1.33)$$

where

$$\tilde{f}(\zeta) = \frac{1}{\zeta} f\left(\frac{1}{\zeta}\right). \quad (1.34)$$

(This peculiar-looking prescription is adopted here to conform with the convention used below in eq. (1.45) for the Laplace transform.) The notation $\mathcal{B}[f; s]$ is used for a quantity that is both a functional of the function f and an ordinary function of the variable s . The path of integration in the ζ plane, $Re \zeta = \rho$, is chosen such that all singularities of $\tilde{f}(\zeta)$ remain to the left of it.

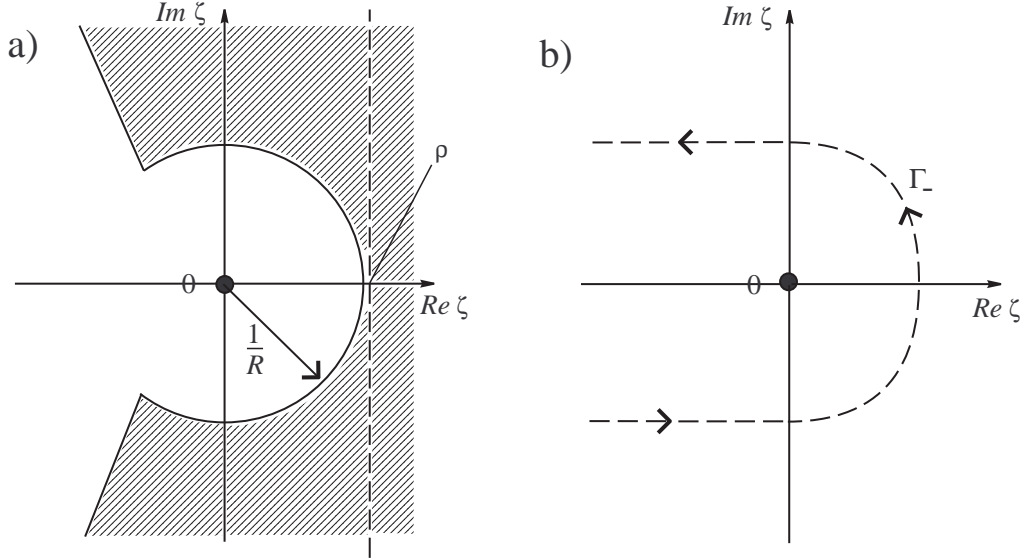


Fig.1.5 (a) Region of analyticity under Phragmén-Lindelöf conditions and integration contour in the Borel transform.

(b) Deformation of integration contour into Γ_-

In the situation where uniqueness of f is guaranteed by its fulfilling the conditions of theorem (1.23), the domain of analyticity of $\tilde{f}(\zeta)$ is the image of the "wide-angle-sector" of Fig. 1.3 under the inversion $\zeta = 1/z$, and is therefore the shaded region in Fig. 1.5 (a), while under the hypothesis of uniqueness theorem II, (1.24), it is simply the half-plane $Re \zeta > \frac{1}{R}$, so in both cases we should choose

$$\frac{1}{R} < \rho < \infty . \quad (1.35)$$

From eq. (1.33), it is clear that the growth restriction we must impose on our function $f(z)$ is modest: $f(z)$ should be such that its associated $\tilde{f}(\zeta)$, eq. (1.34), possesses a Fourier transform with respect to the imaginary part $\sigma = Im \zeta$ of its argument.

The transformation can next be applied, termwise, to the series (1.31), giving a series called the *formal* Borel transform,

$$b_f(s) = \mathcal{B}[f_f; s] = \sum_{n=0}^{\infty} a_n \left\{ \frac{1}{2\pi i} \int_{s(\rho-i\infty)}^{s(\rho+i\infty)} d\eta e^{\eta} \eta^{-(n+1)} \right\} s^n , \quad (1.36)$$

where $\eta = s \rho$, and if the integrals in both (1.33) and (1.36) exist, this series is in turn semiconvergent to $b(s)$. The integration path is $Re \eta = s \rho$ for real and positive s and is "tilted" for complex s , but since the only singularity of the integrand is now a pole at $\eta = 0$, it can always be deformed, by bending its ends over to the left, into the contour Γ_- of Fig. 1.5 (b). The resulting integral is Hankel's integral representation [GRY94] of the inverse gamma function,

$$\frac{1}{2\pi i} \int_{\Gamma_-} d\eta e^{\eta} \eta^{-\alpha} = \frac{1}{\Gamma(\alpha)}, \quad (1.37)$$

at integer $\alpha = n + 1$. (If this looks unfamiliar to you, take α integer and work out the residue of the integrand at $\eta = 0$.) Therefore

$$b_f(s) = \sum_{n=0}^{\infty} \frac{a_n}{n!} s^n. \quad (1.38)$$

We could have simply *defined* b_f by this series, but since we are going to need the concept of Borel transform for objects more general than the series (1.31), we have chosen to derive it here from the more general integral transform (1.33).

Thus we have manufactured a series with much faster-decreasing coefficients than (1.31), and if the a_n coefficients grow no faster than (1.32) at large n , then b_f is actually something much better than a mere asymptotic expansion of $b(s)$, namely a truly convergent expansion: it is given by the generalized geometric series

$$c \sum_{n=0}^{\infty} \frac{\Gamma(n+1+B)}{\Gamma(n+1)} \left(\frac{s}{A}\right)^n = c \Gamma(B+1) \left(1 - \frac{s}{A}\right)^{-(B+1)}, \quad (1.39)$$

plus a faster-convergent remainder series having a least the same convergence radius, and is therefore convergent in the circle

$$|s| < |A|, \quad (1.40)$$

where it defines $b(s)$ *constructively*. This is not yet all of $b(s)$; we shall need to perform an *analytic-continuation process* \mathcal{A} to continue b from the circle (1.40) into as much as possible of the s plane, and in particular along the real positive s axis if we are to recover the original $f(z)$ uniquely at the real positive values of z in which we are primarily interested. It is no surprise that the conditions of theorem (1.24) also guarantee that

this can be done – a statement known as the [SOK80]

Nevanlinna-Sokal theorem: If $f(z)$ fulfills the conditions of theorem (1.24), then its Borel transform $b(s)$ is analytic beyond the circle (1.40) in the striplike region $|s - \mathbb{R}^+| < |A|$ on both sides of the real s axis \mathbb{R}^+ , as illustrated in Fig. 1.6. Moreover, $b(s)$ is exponentially bounded there by $\text{const.} \exp\left(\frac{|s|}{R}\right)$, and $f(z)$ is uniquely given by the Laplace integral (1.45) below for all z in its region of analyticity $\text{Re} \frac{1}{z} > \frac{1}{R}$.

(1.41)

We can see from (1.39) that (at least for $B > -1$) the sum of the b_f series is singular at $s = +|A|$ for $A > 0$, and at $s = -|A|$ for $A < 0$. Thus under the conditions of the Nevanlinna-Sokal theorem, the A in (1.32) can only be negative: the large- n terms of (1.31) must have *alternating* signs. We emphasize that (1.41) again gives *sufficient* conditions; the desirable behavior of $b(s)$ on the positive s axis may still obtain in other cases with suitable modified hypotheses.

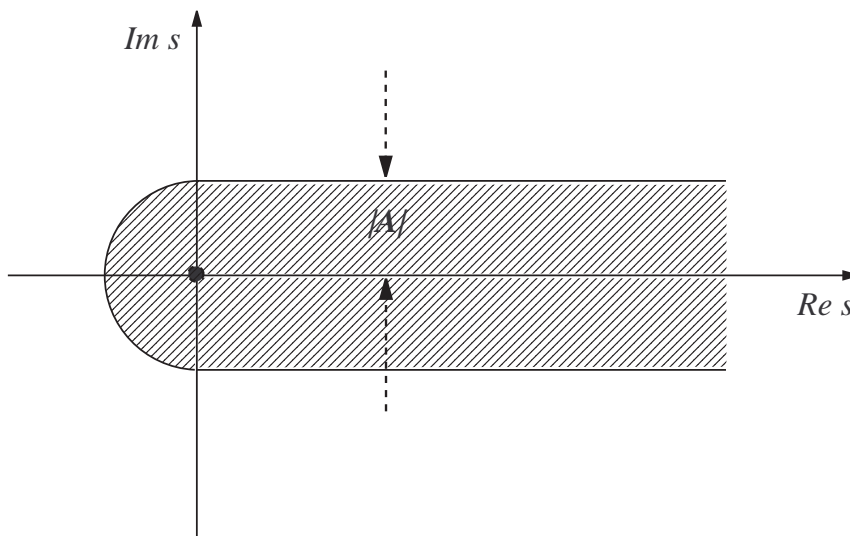


Fig. 1.6 Region of analyticity in the s plane of the Borel transform $b(s)$ in the Nevanlinna-Sokal theorem

With such behavior of $b(s)$ secured, we can finally reconstruct from it the function $f(z)$ in its region of analyticity by inverting the transformation \mathcal{B} of (1.33): the inversion

theorem in the theory of Laplace transforms [GRY94] states that

$$\tilde{f}(\zeta) = \int_0^{\infty} ds e^{-\zeta s} b(s) . \quad (1.42)$$

[If you feel you need proof of this, write eq. (1.33), with ζ parameterized as $\rho + i\sigma$, in the form of a Fourier transform,

$$e^{-\rho s} b(s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\sigma e^{i s \sigma} \tilde{f}(\rho + i\sigma) , \quad (1.43)$$

and evaluate the Fourier transform of the product

$$\Theta(s) [e^{-\rho s} b(s)] , \quad (1.44)$$

by the Fourier convolution theorem plus contour closing, using $[i(\omega - i\varepsilon)]^{-1}$ as the Fourier transform of the unit-step function $\Theta(s)$. Therefore we recover our function $f(z)$ through what for our purposes we shall call the *Laplace transform*,

$$f(z) = \mathcal{L}[b; z] := \frac{1}{z} \int_0^{\infty} ds e^{-\frac{s}{z}} b(s) , \quad (1.45)$$

for values of z such that the integral converges. Under the conditions of the Nevanlinna-Sokal theorem (1.41) above, this is obviously the case for $\operatorname{Re} \frac{1}{z} > \frac{1}{R}$, i. e. in the entire original region of analyticity of f . Under the stronger Phragmén-Lindelöf or Watson conditions, *that region does not become any larger*, but $b(s)$ then has in addition a "sectorial" region of analyticity in the s plane, with opening $\gamma - \pi$, that may be useful for some purposes. For example, the integration path in the variable $t = \frac{s}{z}$, namely, the ray

$$t = \frac{1}{z} \cdot (0 \dots \infty) , \quad (1.46)$$

can in this case be rotated back to the real axis without changing the value of the integral provided $|\arg z| < \frac{1}{2}(\gamma - \pi)$, and (1.45) is then equivalent to

$$f(z) = \int_0^{\infty} dt e^{-t} b(tz) , \quad (1.47)$$

which otherwise can be justified only for real, positive z . This form lends itself more readily to generalization to more than one variable [STS96], and incidentally is best

suitable to performing the final check on our reconstruction procedure: from (1.47), the Taylor coefficients of f from the right follow by differentiations under the integral sign and use of (1.38),

$$\frac{1}{n!} \left(\frac{d^n f}{dz^n} \right)_{z \rightarrow 0^+} = \int_0^\infty dt e^{-t} \left[\frac{1}{n!} \left(t \frac{d}{ds} \right)^n b(s) \right]_{s=0} = \frac{a_n}{n!} \int_0^\infty dt e^{-t} t^n, \quad (1.48)$$

and since

$$\int_0^\infty dt e^{-t} t^\beta = \Gamma(\beta + 1), \quad (1.49)$$

this becomes equal to a_n , so (1.45) indeed gives the function $f(z)$ that has the series (1.31) as its asymptotic expansion.

A remark on conventions: our definition of the Laplace transform, eq. (1.45), is adapted to the study of semiconvergent expansions *at small* z , and has been chosen so as to agree with the convention used in the majority of the field-theory textbooks that discuss the subject [WEI96] [ZNJ89] [ITZ80]. It is also the convention used by Watson [WAT12] and by Sokal [SOK80]. Most of the mathematical literature on resurgent analysis (such as references [ECA81], [CNP93], and parts of [STS96]) studies expansions at a *large* argument ζ and therefore needs to be translated to our context through $z = \frac{1}{\zeta}$; moreover it uses a different definition, namely eq. (1.42), for the Laplace transform. This makes for simpler appearance of a number of formulas (for example, the convolution theorem, line 10 in Table 1 below, gets rid of the extra z factor on its l. h. s.) but on the other hand requires the separate introduction of a unit element, the delta distribution $\delta(s)$, for the algebra of formal Borel transforms $b_f(s)$, since (1.42) fails to map $b \equiv 1$ onto $\tilde{f} \equiv 1$. To further complicate matters, a recent QFT textbook [SIE99] also uses this "mathematical" definition of the Laplace transform, while reference [STS96] occasionally switches between small-argument and large-argument considerations. So be prepared to spend some time climbing around between the various conventions.

To summarize, the BL reconstruction proceeds in three steps:

(1) "Borelize" the given asymptotic series:

$$f_f(z) \text{ (eq.(1.31))} \xrightarrow{\mathcal{B}} b_f(s) \text{ (eq.(1.38))} . \quad (1.50)$$

(2) Continue b_f to b analytically along the positive real s axis:

$$b_f(s) \xrightarrow{\mathcal{A}} b(s) . \quad (1.51)$$

(3) Obtain $f(z)$ from $b(s)$ through Laplace transform:

$$b(s) \xrightarrow{\mathcal{L}} f(z) . \quad (1.52)$$

The importance of the second step, the analytic continuation \mathcal{A} , can hardly be overemphasized: without it we would make no progress at all. Indeed if we forget that step and simply plug expansion (1.38) into the integral of (1.47) – i. e. if we use (1.38) beyond its radius of convergence – then obviously we do nothing but repeat the calculation of eqs. (1.48) and (1.49), and are led back to our original semiconvergent series (1.31) without having come any closer to recovering the "true" function $f(z)$. (Actually, one standard way in which semiconvergence arises is as a punishment for your having used a series in an integration outside its circle of convergence.) Needless to say the simplest method of continuing (1.38) analytically, namely to sum the series in closed form as we did in (1.39), is hardly ever applicable in realistic cases, where the a_n coefficients result from complicated perturbative calculations, and often are only available numerically for a limited number of orders n . The practical methods that have been used – particularly in studies of critical phenomena based on the superrenormalizable ϕ^4 theory in $D = 3$ – are surveyed in chapter 37 of [ZNJ89]. Here I would like to mention just one method which, while no more the most refined one for the specific purpose of calculating universal quantities in phase-transition theory, is still widely used in statistical mechanics because of its simplicity and flexibility. This is again the method of Padé approximants, whose direct application to the original asymptotic series (1.31) we already mentioned in sect. 1.1, but which is usually applied more profitably to the better-converging "borelized" series (1.38), where its convergence in the lower orders as a rule is less erratic. Thus one approximates

$$b(s) \approx b_{(K,L)}(s) = \frac{P_K(s)}{Q_L(s)} \quad (1.53)$$

with polynomials P_k and Q_L of orders K and L respectively, and with $Q_L(0)$ normalized to 1 without loss of generality. The $K + L + 1$ coefficients of this fraction are then determined either by matching the coefficients of $s^0, s^1, s^2, \dots, s^{K+L}$ in its expansion around $s = 0$ to those of $b_f(s)$ (Padé-I method), or by matching the entire fraction at $K + L + 1$ points s_i within the circle of convergence of b_f to the numerically summed b_f (Padé-II method). The L roots s_l of the denominator polynomial,

$$Q_L(s) = \prod_{l=1}^L (s - s_l) , \quad (1.54)$$

then approximate the singularities of $b(s)$ – it is an advantage of the method over, e. g., conformal-mapping methods that one does not have to know the positions of these

singularities in advance. Nor, in principle, does one need to know much about the large- $|s|$ behavior of $b(s)$, provided one is willing to work through the entire "Padé table", i. e. the entire set of possible (K, L) pairs at given $K + L + 1$. If the true $b(s)$ bears some resemblance to (1.39) at large s , then the approximants with $K < L$ will converge best, and one may write a partial-fraction decomposition,

$$b_{(K,L)}(s) = \sum_{l=1}^L \frac{r_l}{s - s_l}, \quad r_l = \frac{P_K(s_l)}{Q'_L(s_l)} \quad (1.55)$$

which allows the Laplace transform (1.45) to be carried out analytically according to line 11 of table 1 below:

$$f(z) \approx f_{(K,L)} = -\frac{1}{z} \sum_{l=0}^L r_l e^{-\frac{s_l}{z}} Ei\left(\frac{s_l}{z}\right). \quad (1.56)$$

If f is Borel summable and f_f has real coefficients, the s_l must all be off the positive real s axis, and thus come either at negative real s or in complex-conjugate pairs, so all the Ei functions in (1.56) are unambiguous.

For the last step (1.52) of the reconstruction scheme, it is often convenient to have a little toolkit of basic BL transforms available, so Table 1 offers such a list. All entries in this table are straightforward to verify by elementary calculation. (Many lines carry some clauses or restrictions that are usually obvious and not mentioned explicitly.) The first line, for $\nu = n$ an integer, repeats what we have already used to derive (1.38), but it also shows that it is just as easy to "Borelize" more general formal series with some noninteger power in front:

$$\mathcal{B} \left[z^\beta \cdot \sum_{n=0}^{\infty} a_n z^n; s \right] = s^\beta \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n + \beta + 1)} s^n \quad (\beta > -1). \quad (1.57)$$

The fourth line is interesting because it states that \mathcal{B} and \mathcal{L} commute with the infinitesimal generator of dilatations, so that $f(\lambda z)$ and $b(\lambda s)$ form a transform pair if $f(z)$ and $b(s)$ do. The tenth line expresses the well-known *convolution theorem* of Laplace-transform theory, with the convolution $b_1 * b_2$ of two functions being defined by the r. h. s.

Lines 11 and 12 feature a simple higher transcendental function studied already by Euler in the 18th century, the exponential-integral function $Ei(z)$ defined [GRY94] by the integral

$$Ei(z) = \int_{-\infty}^z du \frac{e^u}{u} \quad (z \neq 0). \quad (1.58)$$

Table 1: Some Borel-Laplace transform pairs

	$f(z) = \mathcal{L}[b; z]$	$b(s) = \mathcal{B}[f; s]$
1.	$z^\nu \quad (\nu > -1)$	$\frac{1}{\Gamma(\nu + 1)} s^\nu$
2.	$z \cdot f(z)$	$\int_0^s dt b(t)$
3.	$\frac{1}{z} \cdot f(z)$	$\left[\delta(s) + \frac{d}{ds} \right] b(s)$
4.	$z \cdot \frac{df(z)}{dz}$	$s \cdot \frac{db(s)}{ds}$
5.	$\frac{df(z)}{dz}$	$\left(\frac{d}{ds} s \frac{d}{ds} \right) b(s)$
6.	$z \frac{d}{dz} [z f(z)]$	$s \cdot b(s)$
7.	$\frac{1}{z} \int_0^z dw f(w)$	$\frac{1}{s} \int_0^s dt b(t)$
8.	$\frac{f(z) - f(0)}{z}$	$\frac{db(s)}{ds}$
9.	$\frac{1}{1 - \omega z} f\left(\frac{z}{1 - \omega z}\right)$	$e^{\omega s} b(s)$
10.	$z \cdot f_1(z) \cdot f_2(z)$	$(b_1 * b_2)(s) = \int_0^s dt b_1(t) b_2(s - t)$
11.	$-\frac{1}{z} e^{-\frac{s_0}{z}} Ei\left(\frac{s_0}{z}\right)$	$\frac{1}{s - s_0}$
12.	$-\frac{1}{z} Ei\left(-\frac{s_0}{z}\right)$	$\frac{1}{s} \Theta(s - s_0)$
13.	$e^{-\frac{s_0}{z}} f(z)$	$\Theta(s - s_0) b(s - s_0)$

The equivalent spectral representation ($w = z - u$)

$$Ei(z) = e^z \int_0^\infty dw \frac{e^{-w}}{z - w} \quad (1.59)$$

shows that the function is analytic in the z plane except for a cut along the positive real axis, where its discontinuity is

$$Ei(x + i0) - Ei(x - i0) = -2\pi i \Theta(x) \quad (x \text{ real}) . \quad (1.60)$$

The cut is thus logarithmic, as shown explicitly by the alternative representation

$$Ei(z) = \gamma_E + \ln(-z) + \int_0^z du \frac{e^u - 1}{u} , \quad (1.61)$$

in which the integral is an entire function. Formal expansion of the denominator in (1.59) and integration beyond the convergence circle results in the divergent series

$$Ei(z) = \frac{e^z}{z} \sum_{p=0}^{\infty} p! \frac{1}{z^p} , \quad (1.62)$$

so that

$$\frac{a}{z} e^{-\frac{a}{z}} Ei\left(\frac{a}{z}\right) = \sum_{p=0}^{\infty} (a^{-p} p!) z^p \quad (1.63)$$

is a simple example of a factorially divergent series of type (0.5); line 11 of Table 1 just describes the BL reconstruction from this series. Line 12, on the other hand, follows from the application of line 3 to the Laplace-transform representation

$$-Ei\left(-\frac{a}{z}\right) = \frac{1}{z} \int_0^\infty ds e^{-\frac{s}{z}} \left[\Theta(s - a) \ln\left(\frac{s}{a}\right) \right] , \quad (1.64)$$

where a is real and positive. This representation embodies the modern-day significance of the Ei function as the simplest among a family of elementary building blocks for resurgent functions, the so-called resurgence monomials [ECA81].

The last two lines of Table 1 actually go beyond the framework we discussed so far: by considering Borel-transform functions with jumps, they remind us of the fact that for the Laplace integral (1.45) to be well-defined, it is quite sufficient that $b(s)$ should only be *piecewise continuous* on the real s axis, rather than fully analytic in a region along it. In this they point ahead to chapter 2, where we will exploit this greater generality.

As a matter of etiquette, we must acknowledge that the three-step scheme of (1.50) to (1.52) should actually be preceded by a zeroth step:

$$(0) \text{ Check conditions for summability, i. e., uniqueness of } f(z) . \quad (1.65)$$

But you know that real life isn't like that – outside the arcadian realm of the superrenormalizable theories, physicists hardly ever have enough information to do such a check. So as a rule they will proceed heuristically, apply the BL machinery to the limited information they have, and expect to be punished by some singularities on the real Borel axis should they have committed some invisible offense. If you feel bad about this state of things, you may find solace in the words of the great Poincaré, who clearly had a heart for the poor physicists when (on the limits of "pragmatism" in dealing with asymptotic series) he wrote [PO893]: "Les astronomes ne les connaissent pas toujours d'une façon bien précise, mais ils les franchissent rarement, ... d'ailleurs leur instinct les guide et, s'il les trompait, le contrôle de l'observation les avertirait promptement de leur erreur".

2 Strictly Renormalizable Theories: Resurgence

2.1 Coupling Renormalization and Lambda Scale

When passing from the superrenormalizable theories discussed in sect. 1.2 to a "realistic", strictly renormalizable theory, such as $(\phi^4)_{D=4}$, $(\text{Yukawa})_{D=4}$, or a nonabelian gauge theory in $D = 4$, we encounter no changes in the form of the corresponding Lagrangians, or in the symmetries displayed by these Lagrangians. Also, the renormalizations related to the *bilinear* parts of the Lagrangians – field renormalizations and, if bare-mass terms are present, mass renormalizations – remain ultraviolet divergent, as they were already in the superrenormalizable cases, the only change being that they are now divergent in *all* orders of perturbation theory, rather than in a few low orders only. The only *fundamentally* new element in the definition of the theory is the divergent coupling renormalization (0.6). It becomes necessary because the three-point and four-point vertex functions that evolve through quantum effects from the trilinear and quadrilinear Lagrangian interaction terms now become superficially divergent, which amounts to saying that the bare coupling \bar{g}_0 becomes dimensionless as $D \rightarrow 4$.

The information about this new constitutive element, which will turn out to have a strong effect on all the correlation functions of the theory, is originally contained in the coupling-renormalization constant Z_α of (0.6), which perturbatively takes the form of a Laurent expansion in the dimensional regulator $\varepsilon = (4 - D)/2$,

$$Z_\alpha(g^2, \varepsilon) = 1 + \zeta_1(g^2) \frac{1}{\varepsilon} + \zeta_2(g^2) \frac{1}{\varepsilon^2} + \dots \quad (2.1)$$

Each ζ_n is in turn a (semiconvergent) series in g^2 starting at order $(g^2)^n$, since ultraviolet divergence of type ε^{-n} only comes from diagrams with at least n loops:

$$\zeta_n(g^2) = \zeta_{n,n} \left(\frac{g}{4\pi}\right)^{2n} + \zeta_{n,n+1} \left(\frac{g}{4\pi}\right)^{2n+2} \dots \quad (n \geq 1) \quad (2.2)$$

The set of all coefficients $\zeta_{n,k}$ ($n \geq 1, k \geq n$) perturbatively represents the complete information about the coupling renormalization. However there is a much more compact and convenient way of encoding the same information, suggested by renormalization-group (RG) analysis, namely, in terms of the RG beta function,

$$\beta(g(\mu)) = \mu \frac{dg(\mu)}{d\mu} \quad (2.3)$$

This function is UV finite (the renormalized coupling $g(\mu)$ being UV finite by definition), and its dependence on the regulator is therefore much simpler. Since this is an important

point in all that follows, let me briefly repeat how these two encodings are connected [THO73]. We can in principle determine $\beta(g)$ from Z_α by applying the operation $\mu d/d\mu$ on both sides of (0.6) and using the fact that \bar{g}_0 , a bare quantity, knows nothing about the sliding scale μ , so $d\bar{g}_0/d\mu = 0$. We find

$$\beta(g, \varepsilon) = \frac{-\varepsilon g}{1 + g^2 \frac{\partial}{\partial g^2} (\ln Z_\alpha)} . \quad (2.4)$$

Upon inserting expansion (2.1) this becomes

$$\beta(g) = -\varepsilon g + g^3 \zeta'_1 + \frac{g^3}{\varepsilon} \left[\zeta'_2 - \zeta_1 \zeta'_1 - g^2 (\zeta'_1)^2 \right] + \mathcal{O} \left(\frac{1}{\varepsilon^2} \right) \quad (2.5)$$

Here the primes denote differentiation by g^2 . Since $\beta(g)$ must be finite as $\varepsilon \rightarrow 0$ we conclude, first, that all terms with powers of $\frac{1}{\varepsilon}$ on the r. h. s. must vanish separately. This gives a sequence of conditions on the ζ_n which generally determine ζ'_n for $n \geq 2$ in terms of lower ζ 's. For example, the $\frac{1}{\varepsilon}$ term displayed in (2.5) gives

$$\zeta'_2(g^2) = \left[\zeta_1(g^2) + g^2 \zeta'_1(g^2) \right] \zeta'_1(g^2) . \quad (2.6)$$

These relations are interesting in themselves; they may be exploited e.g. to derive a succession of partially resummed forms for the perturbative expansion of Z_α , but since such forms are more easily derived from the closed-form result of eq. (2.12) below, we do not dwell on them here, noting only that all the ζ_n with $n \geq 2$ are in principle determined by the function $\zeta_1(g^2)$ alone, the residue of Z_α at $\varepsilon = 0$. Thus (2.1) is actually a highly redundant form of conveying the information on coupling renormalization.

Second, comparison of the finite parts in eq. (2.5) now gives

$$\beta(g, \varepsilon) = -\varepsilon g + g^3 \frac{d\zeta_1(g^2)}{dg^2} \quad (2.7)$$

and therefore a (semiconvergent) perturbative expansion

$$\beta(g, \varepsilon) = -g \left[\varepsilon + \beta_0 \left(\frac{g}{4\pi} \right)^2 + \beta_1 \left(\frac{g}{4\pi} \right)^4 + \dots \right] , \quad (2.8)$$

$$\beta_k = -(k+1) \zeta_{1, k+1} \quad (k = 0, 1, 2, \dots) . \quad (2.9)$$

You will remember the particular significance of β_0 , the leading beta-function coefficient at $\varepsilon = 0$ – the theory is asymptotically free, i. e. its coupling decreases like an inverse logarithm at large scales μ , if and only if $\beta_0 > 0$. Also, let me recall without comment

that the first two coefficients, β_0 and β_1 , are independent of the renormalization scheme R in a large class of schemes that includes the dimensional schemes [WEI96].

Having expressed, in eq. (2.7), the beta function in terms of the residue at $\varepsilon = 0$ of Z_α , we may now invert the procedure and express Z_α in terms of the beta function. From (2.4), we have

$$(4\pi)^2 \frac{\partial}{\partial g^2} \ln Z_\alpha = \frac{1}{\kappa + \varepsilon \chi(\kappa)}, \quad (2.10)$$

where we abbreviated $(g/4\pi)^2 = \kappa$, and where

$$\chi(\kappa) = -\frac{1}{(4\pi)^2 \zeta_1'(g^2)} = \frac{1}{\beta_0 + \beta_1 \kappa + \beta_2 \kappa^2 + \dots} \quad (2.11)$$

contains the information of the beta function at $\varepsilon = 0$. Integrating with respect to g^2 under the initial condition $Z_\alpha(0, \varepsilon) = 1$ then gives the integral representation [THO73]

$$Z_\alpha(g^2, \varepsilon) = \exp \left\{ - \int_0^{(g/4\pi)^2} \frac{d\kappa'}{\kappa' + \varepsilon \chi(\kappa')} \right\}. \quad (2.12)$$

This exact expression shows the well-known double face of renormalization constants: when expanding the integrand "perturbatively", i. e. in powers of κ with β_0/ε as a leading term, and integrating termwise, you get back the perturbation expansion of (2.1) and (2.2) in which every term is ultraviolet divergent as $\varepsilon \rightarrow 0$ – though with all coefficients now expressed in terms of those of $\beta(g)$ or $\zeta_1(g^2)$, and with the relations for the higher ζ_n 's, like (2.6), therefore built in:

$$Z_\alpha(g, \varepsilon) = 1 - \beta_0 \left(\frac{g}{4\pi} \right)^2 \frac{1}{\varepsilon} + \mathcal{O}(g^4). \quad (2.13)$$

On the other hand if you ask about the behavior as $\varepsilon \rightarrow 0$ of the full, resummed expression (2.12), you find $\exp(-\infty) = 0!$ To see exactly how the function vanishes requires a minor detour in order to get rid of the lower-limit divergence of the κ integral at $\varepsilon = 0$: write

$$\frac{1}{\kappa + \varepsilon \chi(\kappa)} = \frac{1}{1 + \varepsilon \chi'(\kappa)} \frac{d}{d\kappa} \ln[\kappa + \varepsilon \chi(\kappa)], \quad (2.14)$$

where the prime is now a differentiation with respect to κ , and perform a partial integration. You get

$$Z_\alpha(g, \varepsilon) = \frac{\varepsilon}{\beta_0 \left[\left(\frac{g}{4\pi} \right)^2 + \varepsilon \chi \left(\left(\frac{g}{4\pi} \right)^2 \right) \right]} \times$$

$$\begin{aligned} & \times \exp \left\{ \varepsilon \left[\frac{\chi'(\kappa) \ln(\kappa + \varepsilon \chi(\kappa))}{1 + \varepsilon \chi'(\kappa)} - \frac{\chi'(0) \ln\left(\frac{\varepsilon}{\beta_0}\right)}{1 + \varepsilon \chi'(0)} \right. \right. \\ & \left. \left. - \int_0^\kappa d\kappa' \frac{\chi''(\kappa')}{[1 + \varepsilon \chi'(\kappa')]^2} \ln[\kappa' + \varepsilon \chi(\kappa')] \right] \right\}_{\kappa = \left(\frac{g}{4\pi}\right)^2}. \end{aligned} \quad (2.15)$$

The remaining integral is finite at $\varepsilon = 0$; the exponent has terms of order ε and $\varepsilon \ln \varepsilon$ only in that limit; so

$$Z_\alpha(g^2, \varepsilon) = \frac{\varepsilon}{\beta_0 \left(\frac{g}{4\pi}\right)^2} [1 + \mathcal{O}(\varepsilon, \varepsilon \ln \varepsilon)(g^2(\mu))] \quad (2.16)$$

is an exact statement. (The writing emphasizes that the additional terms still depend on $g^2(\mu)$ in a complicated way). This will not surprise lattice theorists, since when inserted into (0.6) it says that in the removal-of-regulator limit of a *nonperturbative* calculation, the bare coupling should be tuned to zero according to

$$\left(\frac{\bar{g}_0(\varepsilon)}{4\pi}\right)^2 = \frac{\varepsilon}{\beta_0} \mu^{2\varepsilon} [1 + \mathcal{O}(\varepsilon, \varepsilon \ln \varepsilon)(g^2(\mu))], \quad (2.17)$$

and for them this is entirely familiar if they replace ε with something proportional to their regulator a , the lattice constant.

There is yet a third way of encoding the information on coupling renormalization, which in a sense is the most physical since it relates to the representation of dimensionful observables. In the strictly renormalizable theories the coupling becomes dimensionless as $D \rightarrow 4$, and if the theory has no explicit mass terms in its Lagrangian (or, as in QCD with only light quarks, masses that are orders of magnitude too small to explain anything observed), then there seems to be at first sight no dimensionful quantity available to set the scale for dimensionful observables. For example, a pure Yang-Mills theory can still generate nonzero glueball masses, or nonzero scattering lengths (low-energy scattering amplitudes) between its gauge bosons. The solution to the paradox [GNV] is that the theory creates its own mass scale "spontaneously" through quantum effects – and essentially the same quantum effects (i. e., loops) that are responsible for the necessity of coupling renormalization. Since observable quantities are to be expressed as multiples of powers of this scale, it cannot simply be the arbitrary sliding scale μ , nor can it depend on arbitrary gauge-fixing parameters. In an otherwise massless theory, this scale, called Λ , must therefore depend on μ and on $g(\mu)$ in such a way as to be an *RG invariant*, i. e. it must be a solution of

$$\mu \frac{d}{d\mu} \Lambda(g(\mu), \mu) = \left[\mu \frac{\partial}{\partial \mu} + \beta(g(\mu), \varepsilon) \frac{\partial}{\partial g} \right] \Lambda(g, \mu) = 0, \quad (2.18)$$

the simplest RG equation conceivable. Mass dimension requires Λ to be of the form $\mu \cdot \mathcal{E}_\varepsilon(g(\mu))$, with the dimensionless function $\mathcal{E}_\varepsilon(g)$ fulfilling

$$\left[1 + \beta(g, \varepsilon) \frac{\partial}{\partial g}\right] \mathcal{E}_\varepsilon(g) = 0, \quad (2.19)$$

so separation of variables gives the representation

$$\Lambda_\varepsilon(g(\mu), \mu) = \mu \exp \left\{ - \int_{g_1}^{g(\mu)} \frac{dg'}{\beta(g', \varepsilon)} \right\} \quad (2.20)$$

with an integration constant g_1 , a trivial RG invariant. A simpler but somewhat more formal way of seeing the existence of Λ is to write eq. (2.3) as

$$\frac{1}{\mu} d\mu = \frac{1}{\beta(g, \varepsilon)} dg \quad (2.21)$$

and integrate between any two values μ_1, μ_2 of the sliding scale, which gives

$$\mu_1 e^{-\Psi(g(\mu_1), \varepsilon)} = \mu_2 e^{-\Psi(g(\mu_2), \varepsilon)}, \quad (2.22)$$

with $\Psi(g, \varepsilon)$ denoting a primitive of $1/\beta(g, \varepsilon)$; this says that the combination (2.20) is μ -independent. Thus a divergent coupling renormalization, as expressed in the RG beta function (2.7), *directly* translates into the "spontaneous" emergence of a Λ scale.

The integration constant g_1 , like the higher beta-function coefficients (2.9) with $k \geq 2$, depends on the renormalization scheme adopted, and in fact is part of its specification. Changing g_1 to g'_1 while keeping the higher β_k 's unchanged (a special change of scheme) gives another scale Λ' differing from Λ by a finite dimensionless factor:

$$\frac{\Lambda'_\varepsilon}{\Lambda_\varepsilon} = \exp \left[\Psi(g'_1) - \Psi(g_1) \right]. \quad (2.23)$$

Thus the Λ scale, while an invariant within one renormalization scheme, is still different for different schemes, with the Λ 's for two

schemes differing by a pure-number factor. In that sense it is not yet on the same level of theoretical nobility as a physical mass, which must in addition be scheme-independent, but it is a necessary and important intermediary between the arbitrary μ and the measurable masses and lengths.

From the point of view of coupling dependence, the essential point about the Λ scale emerges in the limit $\varepsilon \rightarrow 0$: by expanding perturbatively,

$$\frac{1}{\beta(g, 0)} = -\frac{(4\pi)^2}{\beta_0 g^3} \left[1 - \left(\frac{\beta_1}{\beta_0}\right) \left(\frac{g}{4\pi}\right)^2 + \mathcal{O}(g^4) \right], \quad (2.24)$$

where the terms $\mathcal{O}(g^4)$ again contain the scheme-dependent higher beta coefficients, we have

$$\Lambda_0 = \mu e^{-\frac{(4\pi)^2}{2\beta_0 g(\mu)^2}} \left[\beta_0 \left(\frac{g}{4\pi}\right)^2 \right]^{-\frac{\beta_1}{2\beta_0^2}} \left\{ 1 - \left(\frac{\beta_1^2 - \beta_0 \beta_2}{2\beta_0^3}\right) \left(\frac{g}{4\pi}\right)^2 + \mathcal{O}(g^4) \right\}_R \quad (2.25)$$

where the (semiconvergent) series in curly brackets depends on the renormalization scheme R , and where for the class of schemes we use we have chosen g_1 such that, in the notation of (2.22),

$$\Psi(g_1, 0) = -\frac{\beta_1}{2\beta_0^2} \ln \left[\frac{\beta_0}{(4\pi)^2} \right]. \quad (2.26)$$

Thus Λ at $D = 4$ is the perfect example on an "escaping" function of the form of (1.20), a quantity that will never show up in a perturbation expansion in g^2 . In making this identification, it is clearly essential that $\beta_0 > 0$, i. e. that *the theory should be asymptotically free*. In asymptotically non-free theories the scale Λ_0 , rather than being exponentially suppressed, would be *exploding* as $g^2 \rightarrow 0+$, and none of the things we are going to discuss in the following would be applicable.

Moreover, one should not lose sight of the fact that when talking about "behavior as $g^2 \rightarrow 0+$ " here, we are actually talking about a rather fictitious, unphysical limit – a limit in which we imagine g^2 being tuned to zero while μ is being kept constant. Only in this purely formal limit can we speak of Λ_0 being "exponentially suppressed" – in the physical world there is no way of getting Λ_0 to vanish, since the only parameter there which we can "tune" at will is μ , and Λ_0 by construction just does *not* change with μ .

Now eq. (2.25) would not create an essentially new situation if the only way Λ_0 occurred in correlation functions were the trivial one: one may of course invert eq. (2.20) to express g in terms of $\Lambda_\varepsilon / \mu$, obtaining at $\varepsilon = 0$ the well-known expression ([WEI96])

$$\frac{\alpha(\mu)}{4\pi} = \frac{1}{\beta_0 \lambda_0} \left\{ 1 - \frac{2\beta_1}{\beta_0^2} \left(\frac{\ln \lambda_0}{\lambda_0}\right) + \mathcal{O}\left(\frac{1}{\lambda_0^2}, \frac{\ln \lambda_0}{\lambda_0^2}\right) \right\} \quad (2.27)$$

where

$$\lambda_0 = \ln \left(\frac{\mu^2}{\Lambda_0^2} \right). \quad (2.28)$$

One may then parameterize the perturbation series in terms of the inverse of this logarithm rather than in terms of α , but this will not introduce any new information beyond the perturbative one, and this remains true even if we go one step further and invoke the RG scaling properties of vertex functions to rewrite the expansion in terms of a coupling $\bar{\alpha}$ that "runs" with Q^2 , some representative combination of the function's momenta, rather than with μ^2 [WEI96]. (This would be the *same* function as (2.27) with μ^2 simply replaced by Q^2 .) Since

$$\bar{\alpha}(Q^2) = \alpha \left\{ 1 + \left[\text{power series in } \frac{\alpha}{4\pi} \beta_0 \ln \left(\frac{Q^2}{\mu^2} \right) \right] \right\} \quad (2.29)$$

(as follows by combining eqs. (2.20) at scales μ and $\sqrt{Q^2}$), this amounts to a resummation of the perturbation series that is useful for many purposes, but again creates no new content beyond that series. But then if your QFT text mentions no other application of the lambda scale than eq. (2.27), you should protest – they are hiding the best from you! For example, an s -wave scattering amplitude in a massless theory at low center-of-mass energies \sqrt{s} would look like

$$f_0(s) = \frac{1}{d_0 \Lambda_0 + d_1 \sqrt{s} + d_2 \frac{s}{\Lambda_0} + \dots}, \quad (2.30)$$

with $d_0 \Lambda_0$ the inverse scattering length, and with the dimensionless coefficients d_i possibly having some perturbative, power-series dependence on the coupling (2.29). Or a bound-state pole in a four-point amplitude T of a massless theory would appear as

$$T(p', P, p; \alpha) = \frac{\alpha \Phi(p' / \Lambda_0; \alpha) \cdot \alpha \bar{\Phi}(p / \Lambda_0; \alpha)}{P^2 + c \Lambda_0^2} + [\text{regular terms}], \quad (2.31)$$

(where the notation indicates that the factorizing residue, in contrast to the purely non-perturbative denominator, is a mixed perturbative-nonperturbative quantity beginning with at least an overall factor of α^2). In other words, the truly interesting quantities are powers and inverse powers, polynomials and inverse polynomials, series and inverse series in Λ_0 , and in these, unlike in (2.27), Λ_0 appears in its full, non-analytic nastiness, eq. (2.25). We will, for the purpose of these seminars, refer to such quantities as *strongly nonanalytic* at $\alpha = 0$ or *strongly nonperturbative*, as opposed to mere resummations of a perturbation series, where Λ_0 only appears in inverse logarithms.

2.2 Operator-Product Expansion and Lambda

Our digression into the Operator-Product Expansion (OPE) will be brief and highly selective, serving only the narrow purpose of reminding you of a long established way [WIL72] of asymptotically representing the strongly nonperturbative coupling dependence. For a systematic development of the OPE, refer to [WEI96]; this reference also contains, in the special context of renormalons, a discussion equivalent to the one below. For a terse account of Zimmermann's perturbative renormalization [ZIM70] of the OPE, you may consult [KUG97].

Again we only look at one simple example that will exhibit all features of interest – the Euclidean vacuum expectation of a single product of two generic elementary fields with naive mass dimension one at variable separation x ,

$$D(x) = \langle 0 | \varphi \left(\frac{1}{2} x \right) \varphi \left(-\frac{1}{2} x \right) | 0 \rangle , \quad (2.32)$$

in the context of an asymptotically free theory ($\beta_0 > 0$) without Lagrangian masses ($m = m_0 = 0$). (General OPE treatments would consider correlation functions with this pair of operators inserted into an arbitrary product $\varphi(y_1) \varphi(y_2) \dots \varphi(y_n)$ of fields at fixed, distinct, nonzero points y_i). Loosely stated, the OPE is the postulate that the field product permits a small- x expansion of the form

$$\varphi \left(\frac{1}{2} x \right) \varphi \left(-\frac{1}{2} x \right) = \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} W_{n,i}(x) \mathcal{O}_{2n,i}(0) , \quad (2.33)$$

where the $\mathcal{O}_{2n,i}(z)$ are the full set of local operators of mass dimension $2n$, elementary or composite, that bear the conserved quantum numbers of the field product. Thus at the core of the OPE is a *completeness hypothesis* which, to my knowledge, has nowhere been proven with full mathematical rigor but is nevertheless highly plausible: the full set of local operators at x forms a basis for the expansion of operator products clustering around x .

These operators are enumerated here according to their naive mass dimensions $2n = 0, 2, 4, \dots$ and to a counting label i ranging over the finite number $l(n)$ of such operators with given n , the enumeration beginning at $n = 0$ with $l(0) = 1$ and the dimensionless unit operator,

$$\mathcal{O}_{0,1} = \mathbb{1} . \quad (2.34)$$

To make up for the increasing mass dimensions, the c -number functions $W_{n,i}(x)$, the "Wilson coefficients", consist of powers $(x^2)^{n-1}$ times a perturbatively calculable series in the coupling that contains powers of $\ln x^2$.

More precisely, what is meant [WIL72] by (2.33) is that this expansion should hold in the sense of "weak operator convergence", i. e., not with respect to any operator norm, but only when both sides are sandwiched between normalizable states $|\Psi_1\rangle$ and $\langle\Psi_2|$ of the quantized-field system. The two-point function (2.32) is then the simplest sandwich of this kind, where $|\Psi_1\rangle = |\Psi_2\rangle = |0\rangle$, the physical vacuum state. In a renormalizable theory, the resulting matrix elements, the "vacuum condensates"

$$\langle 0|\mathcal{O}_{2n,i}(0)|0\rangle \quad (n = 0, 1, 2, \dots, \quad i = 1, 2, \dots l(n)), \quad (2.35)$$

should finally appear in a renormalized form, referring to a definite renormalization scheme R and to a scale μ within that scheme, but since the detailed way the renormalization works in an expansion like (2.33) has some aspects of interest for our later discussion, we will begin by considering a not-yet-completely-renormalized version, parameterized by $\alpha(\mu)$ but still at a positive value of the dimensional regulator $\varepsilon = (4 - D)/2$. When taking Fourier transforms, such as

$$W_{n,i}(x) = \int d^{4-2\varepsilon}x \tilde{W}_{n,i}(k^2; \alpha; \frac{1}{\varepsilon}) e^{ik \cdot x}, \quad (2.36)$$

the above mentioned structure of the Wilson coefficients is mirrored in the momentum-space structure

$$\tilde{W}_{n,i} = \left(\frac{1}{k^2}\right)^{n+1} \times \{ \text{perturbatively calculable series in } \alpha \}, \quad (2.37)$$

with the coefficients of the series containing, in general, both $\frac{1}{\varepsilon}$ ultraviolet divergences and the typical logarithmic modifications $\ln(k^2/\mu^2)$. Therefore when passing to the two-point vertex function $\Gamma_2 = -D^{-1}$ the expansion will look like

$$-\Gamma_2(k^2; \alpha; \varepsilon) = k^2 \sum_{n=0}^{\infty} \sum_{i=1}^{l(n)} V_{n,i} \left(\alpha; \ln \frac{k^2}{\mu^2}; \frac{1}{\varepsilon} \right) \frac{\langle 0|\mathcal{O}_{2n,i}|0\rangle_{\alpha,\varepsilon}}{(k^2)^n}, \quad (2.38)$$

with perturbatively calculable coefficients,

$$V_{n,i} \left(\alpha; \ln \frac{k^2}{\mu^2}; \frac{1}{\varepsilon} \right) = \sum_{p=0}^{\infty} V_{n,i,p} \left(\ln \frac{k^2}{\mu^2}; \frac{1}{\varepsilon} \right) \alpha(\mu)^p. \quad (2.39)$$

As for the vacuum condensates, which at this stage have their own ultraviolet divergences at $\varepsilon \rightarrow 0$, let us start by considering the expansion purely within the context of perturbation theory. There, we have no known way of "renormalizing" composite operators; i. e. redefining them so as to have finite matrix elements, other than defining

them as free-field normal products $:\mathcal{O}_{2n,i}:$, but since those all have zero expectation values in the perturbative vacuum, all condensate terms with $n \geq 1$ in (2.38) would then "renormalize" to zero. Thus we conclude that the $n = 0$ term, involving the trivial operator (2.34), is the perturbative portion of D^{-1} ,

$$-\Gamma_2^{pert}(k^2; \alpha; \varepsilon) = k^2 V_{01} \left(\alpha; \ln \frac{k^2}{\mu^2}; \frac{1}{\varepsilon} \right), \quad (2.40)$$

and, since the zeroth-order perturbative part is simply k^2 ,

$$V_{01} \left(\alpha; \ln \frac{k^2}{\mu^2}; \frac{1}{\varepsilon} \right) = 1 + \mathcal{O}(\alpha). \quad (2.41)$$

The situation would be different, of course, for the more general correlation-with-insertion functions on which most discussions of the OPE focus: there one encounters matrix elements of the free-field normal product between non-vacuum states, which in general have nonvanishing perturbative limits. Pure vacuum expectations like (2.32), which in a sense are more vulnerable to perturbative mutilation, are particularly suited to our purpose because they compel us to make a crucial point: as originally envisaged by its inventor [WIL72], expansion (2.38) is to be understood here in a more general, nonperturbative sense, where vacuum condensates for $2n \geq 2$ can be nonzero even though the OPE by itself offers no immediate recipe for calculating them in a given renormalization scheme. This indeed is the standpoint taken in the successful semi-empirical applications of the OPE through QCD sum rules [SVZ79], another wide and interesting field on which these seminars will be unable to enter [NAR89].

Even without calculating the condensates we may, to some extent, discuss their renormalization formally to see more precisely what is "incalculable" about them. Under renormalization, composite operators of the same mass dimension are known to mix linearly:

$$\langle 0 | \mathcal{O}_{2n,i}(0) | 0 \rangle_{\alpha, \varepsilon} = \sum_{j=1}^{l(n)} Z_{ij}^{(2n)} \left(\alpha, \frac{1}{\varepsilon} \right) \langle \mathcal{O}_{2n,j} \rangle_{R, \mu}. \quad (2.42)$$

The renormalization matrices $Z^{(2n)}$ are of the general form

$$Z_{ij}^{(2n)} \left(\alpha, \frac{1}{\varepsilon} \right) = \delta_{ij} + \mathcal{O}(\alpha), \quad (2.43)$$

with all terms $\mathcal{O}(\alpha)$ ultraviolet divergent, while the renormalized condensates $\langle \mathcal{O}_{2n,j} \rangle_{R, \mu}$ are by definition ultraviolet finite but in general dependent on μ and on the the scheme

R. The right-hand side of (2.38) becomes,

$$k^2 V_{0,1} \left(\alpha; \ln \frac{k^2}{\mu^2}; \frac{1}{\varepsilon} \right) + k^2 \sum_{n=1}^{\infty} \sum_{j=1}^{l(n)} \left\{ \sum_{i=1}^{l(n)} V_{n,i} Z_{ij}^{(2n)} \right\} \frac{\langle \mathcal{O}_{2n,j} \rangle_{R,\mu}}{(k^2)^n}. \quad (2.44)$$

For such an expansion to remain within the realm of renormalizable QFT in the presently accepted sense of the term, one of the minimal prerequisites is that nonlocal ultraviolet divergences of the form

$$\left(\frac{1}{\varepsilon} \right)^m \left(\frac{1}{k^2} \right)^n, \quad (2.45)$$

which would require counterterms of the same form in the Lagrangian, should be absent. Therefore the curly-bracketed quantities in (2.44) must all be finite for at least $n \geq 2$; the Wilson coefficients must be rendered finite by the composite-operator renormalizations *separately* for each n :

$$\sum_{i=1}^{l(n)} V_{n,i} \left(\alpha; \ln \frac{k^2}{\mu^2}; \frac{1}{\varepsilon} \right) Z_{ij}^{(2n)} \left(\alpha, \frac{1}{\varepsilon} \right) = V_{n,j}^{(R)} \left(\alpha; \ln \frac{k^2}{\mu^2} \right), \text{ finite as } \varepsilon \rightarrow 0, \quad (2.46)$$

where, because of (2.43),

$$V_{n,j}^{(R)} = V_{n,j}(\alpha = 0) + \mathcal{O}(\alpha). \quad (2.47)$$

This "internal" renormalization at each $n \geq 2$ is indeed what happens in the purely perturbative calculations alluded to before, for quantities where our condensates are replaced by more general matrix elements having nonzero perturbative limits. It is also implicit (though, curiously, never discussed explicitly) in all cases where one- or two-loop Wilson coefficients have been calculated under nonperturbative assumptions allowing for nonvanishing vacuum condensates [NAR89] [RRY85] [LAV92]. This is analogous to the well-known result of renormalization theory that nonlocal divergences of type

$$\left(\frac{1}{\varepsilon} \right)^m \left(\ln \frac{k^2}{\mu^2} \right)^p \quad (2.48)$$

cancel separately in each order of the *perturbative* expansion. Here, since we lack the time to delve into explicit examples, we will simply accept this general statement as plausible. (Incidentally, one of the uses of the techniques described in chapter 3 may be to provide a more precise framework for such demonstrations.)

In the $n = 1$ term, the counterterm required for an ultraviolet divergence would be constant in momentum, of the form of a mass renormalization, and therefore not

excluded by renormalizability, but it turns out to be really necessary only in the case of $m \neq 0$, a nonzero Lagrangian mass; for a massless theory, where this term represents a contribution to dynamical-mass formation not visible in the Lagrangian, the finiteness of eq. (2.46) will persist at $n = 1$.

On the other hand for the perturbative term with $n = 0$, associated with the trivial operator (2.34), there is no internal renormalization to bring ultraviolet finiteness – the unit operator "doesn't volunteer" to tame its Wilson coefficient, and that coefficient alone – that is, the perturbative series – must therefore be supplied with the usual "external" renormalization through a field-renormalization factor Z_3 . Thus the general scheme, in the presence of the nonperturbative OPE terms, is no more that of multiplication with one overall Z_3 for the entire vertex function, but rather

$$(D^{-1})_{R,\mu} = \underbrace{Z_3 (D^{-1})_\varepsilon^{pert}}_{\text{finite}} + \sum_{n,j} \underbrace{V_{n,j}^{(R)}}_{\text{finite}} \frac{\langle \mathcal{O}_{2n,j} \rangle_{R,\mu}}{(k^2)^{n-1}}. \quad (2.49)$$

I have been dwelling on this simple but important point because the OPE literature, vast as it is, does not seem to state it clearly, and because it means that in the presence of the nonperturbative terms, multiplicative renormalizability in the usual simple sense is no more a property of QFT amplitudes. We will encounter this peculiar "two-track" renormalization pattern again, in the context of a more dynamical approach, in ch. 3. below.

In eq. (2.49) we have not yet separated the perturbative and strongly nonperturbative coupling dependences completely. To achieve such a separation, we need to note that in a massless theory, the renormalized condensates at $\varepsilon = 0$ must be of the form

$$\langle \mathcal{O}_{2n,j} \rangle_{R;\mu} = C_{n,j}(\alpha(\mu))_R (\Lambda_0^2)_R^n, \quad (2.50)$$

since Λ_0 is then the only invariant mass scale available for this purpose. The coefficients $C_{n,j}$, power series in $\alpha(\mu)$ and in general semiconvergent,

$$C_{n,j}(\alpha) = \sum_{q=0}^{\infty} c_{n,j}^{(q)} \alpha^q, \quad (2.51)$$

are what gives the vacuum condensates their residual μ dependence within the scheme R – "residual" because the terms without explicit powers of α ,

$$c_{n,j}^{(0)} (\Lambda_0^2)_R^n, \quad (2.52)$$

are RG invariants within R . It is these terms, with no perturbative power of α but with the strongly nonanalytic α dependence in Λ_0 still present, that form the "incalculable"

cores of the condensates in the context of the OPE. One of the exhilarating experiences to which the formalism of chapter 3 will invite you is that these cores become calculable there as multiples of powers of Λ_0 in a given (dimensional) scheme, with coefficients $c_{n,j}^{(0)}$ given by a systematic sequence of approximations, which in turn can be determined self-consistently from the equations of motion of the basic vertex functions.

Putting things together, and introducing modified Wilson coefficients

$$U_n^{(R)} \left(\ln \frac{k^2}{\mu^2}; \alpha(\mu) \right) = \sum_{j=1}^{l(n)} V_{n,j}^{(R)} \left(\ln \frac{k^2}{\mu^2}; \alpha \right) C_{n,j}(\alpha) \quad (2.53)$$

that refer only to the mass dimension $2n$ but no more to individual condensates, we then have a more transparent form of the OPE,

$$\begin{aligned} [D^{-1}(k^2; \alpha; \mu)]_R &= k^2 \left\{ U_0^{(R)} \left(\ln \frac{k^2}{\mu^2}; \alpha \right) \right. \\ &\quad \left. + \sum_{n=1}^{\infty} U_n^{(R)} \left(\ln \frac{k^2}{\mu^2}; \alpha \right) \left[\frac{(\Lambda_0^2)_R}{k^2} \right]^n \right\}, \end{aligned} \quad (2.54)$$

where

$$U_0^{(R)} \left(\ln \frac{k^2}{\mu^2}; \alpha \right) = \left\{ Z_3 \left(\alpha; \frac{1}{\varepsilon} \right) V_{0,1} \left(\ln \frac{k^2}{\mu^2}; \alpha; \frac{1}{\varepsilon} \right) \right\}_{\varepsilon=0} \quad (2.55)$$

denotes the renormalized perturbation series. Again, the terms without powers of α but with Λ_0 kept nonzero,

$$(D^{-1})_R^{(0)} = k^2 \left\{ 1 + \sum_{n=1}^{\infty} U_n^{(R)}(0, 0) \left(\frac{\Lambda_0^2}{k^2} \right)^n \right\}, \quad (2.56)$$

with

$$U_n^{(R)}(0, 0) (\Lambda_0^2)^n = \sum_{j=1}^{l(n)} V_{n,j}(\alpha=0) \left[c_{n,j}^{(0)} (\Lambda_0^2)^n \right], \quad (2.57)$$

form an RG-invariant "core" for the two-point function, to which the U_n power series add μ -dependent, perturbative corrections.

As rearranged in the form of eq. (2.54), the OPE tells us three things. First, the genuinely nonperturbative coupling dependence of a correlation function can be represented by a series in *integer* powers of Λ_0 , each power being accompanied by its own series of perturbation corrections with the typical logarithmic momentum dependences.

This structure has been confirmed in all cases where an OPE could be studied to some degree of explicitness in renormalizable model theories with a lambda scale [DAV82] [BBK98], and may nowadays be regarded as well established, but it was not always so clear in the early years when attempts to calculate condensates by instanton methods seemed to indicate the necessity of terms with noninteger powers of Λ [THO79]. Work by G. Münster [MUN82] later established that these noninteger powers were really an artefact of the "dilute-instanton-gas" approximation used in most instanton calculations, and that by taking account of the denseness of the instanton gas one is led back to integer powers of Λ (see t'Hooft's introduction for the 1994 reissue of his 1980 lectures, p. 457 of ref. [THO94]).

Second, since Λ_0 is dimensionful as opposed to the dimensionless α , such an expansion is *automatically* a large-momentum expansion in powers of Λ_0^2 / k^2 . This automatic linkage between coupling and momentum dependences is much tighter and more direct than in the perturbative series, where rearrangement in terms of a momentum-dependent running coupling $\bar{\alpha}(Q^2)$ is a useful option for physics but not a mathematical necessity.

Third, we have learned that each of the strongly nonperturbative ($n \geq 1$) terms must establish itself in an ultraviolet-*finite* manner – a requirement which obviously places nontrivial restrictions on any dynamical theory seeking to calculate those terms.

The OPE, since it does not by itself produce a prescription to calculate its own vacuum condensates, remains a deep *structural* statement but does not represent a completely known starting point for an *ab initio* dynamical theory. It is nevertheless – even apart from the successful applications already mentioned – of great value for clarifying the coupling dependence of amplitudes in the strictly renormalizable theories with a Λ scale. To see this dependence more explicitly, we may once more rewrite our result (2.54) by using the structure (2.25) of Λ_0 . Introducing the perturbative expansions

$$\begin{aligned} U_n^{(R)} \left(\ln \frac{k^2}{\mu^2}; \alpha \right) & \cdot \left\{ 1 - \left(\frac{\beta_1^2 - \beta_0 \beta_2}{2 \beta_0^3} \right) \left(\frac{g}{4\pi} \right)^2 + O(g^4) \right\}_R^{2n} \\ & = \sum_{p=0}^{\infty} \tilde{u}_{n,p}^{(R)} \left(\ln \frac{k^2}{\mu^2} \right) \alpha^p, \end{aligned} \quad (2.58)$$

we then have for our two-point function the form

$$(D^{-1})_R = k^2 \sum_{n=0}^{\infty} \left(\frac{\mu^2}{k^2} \right)^n \left(e^{-\frac{4\pi}{\beta_0 \alpha}} \right)^n \left\{ \left(\beta_0 \frac{\alpha}{4\pi} \right)^{-\frac{n\beta_1}{\beta_0^2}} \sum_{p=0}^{\infty} \tilde{u}_{n,p}^{(R)} \alpha^p \right\}, \quad (2.59)$$

a form that stands directly at the entrance door to the theory of resurgent functions.

2.3 Resurgent Symbols and Functions

You will remember that at the end of sect. 1.1 we made a point of the fact that the "escaping" functions of a semiconvergent expansion typically appear again as small- z expansions, eq. (1.20), whose power-series factors $\{1 + O(z)\}$ in general are again semiconvergent. This observation may be taken as a starting point for the following heuristic consideration.

Assume we are dealing with a function $f(z)$ having an asymptotic expansion

$$f(z) \sim \sum_{p=0}^{\infty} a_{0,p} z^p, \quad (2.60)$$

but not sufficient analyticity to be reconstructed uniquely from it as in chapter 1. We may take some representative $f_0(z)$ from the class of functions having this expansion and write

$$f(z) = f_0(z) + g_0(z), \quad (2.61)$$

with $g_0(z)$ the "escaping" remainder. Assume further that $f(z)$ has bounding constants of the form of eq. (1.13) so the bound of eq. (1.19) holds, and that $g_0(z)$ actually has terms, as e. g. in (1.20), that exhaust the bound:

$$g_0(z) = e^{-\frac{s_1}{z}} F_1(z); \quad F_1(z) \sim z^{-B_1} \sum_{p=0}^{\infty} a_{1,p} z^p \quad (Re s_1 > 0). \quad (2.62)$$

Of course, g_0 may have other terms, of faster exponential or even super-exponential decrease, but the series in the F_1 factor, an object as formal as the original series (2.60), has enough freedom of interpretation to accommodate all of these. In the simplest case, that series will turn out to have a finite radius of convergence and thus to determine uniquely an analytic function $F_1(z)$, so that the \sim sign in (2.62) turns into an equals sign within that convergence radius. In this case, the combination of eqs. (2.61) and (2.62) is a complete and exact representation of our function $f(z)$.

But a much more interesting case, and the one we concentrate on in all that follows, is for the situation of eq. (2.60) to repeat itself: assume $F_1(z)$ again has a small- z

expansion that is only semiconvergent, and "escaping" terms led by an exponentially suppressed one:

$$F_1(z) = f_1(z) + g_1(z), \quad g_1(z) = e^{-\frac{r_2}{z}} F_2(z) \quad (\operatorname{Re} r_2 \geq 0), \quad (2.63)$$

where f_1 is some representative of the class of functions having the second member of (2.62) as asymptotic expansion. Again $g_1(z)$ may have other terms, of faster exponential or even super-exponential decrease, but we are again free to stuff these into the "escaping" terms left undetermined by the formal series expansion

$$F_2(z) \sim z^{-B_2} \sum_{p=0}^{\infty} a_{2,p} z^p. \quad (2.64)$$

With that understanding we may then write

$$f(z) = f_0(z) + e^{-\frac{s_1}{z}} f_1(z) + e^{-\frac{s_2}{z}} F_2(z) \quad (s_2 = s_1 + r_2). \quad (2.65)$$

Imagine now that this pattern keeps repeating itself, or "resurging", so that at every stage the escaping functions turn out to be led by an exponentially suppressed term. Then upon putting conventionally

$$s_0 = B_0 = 0, \quad (2.66)$$

we arrive at a formal representation

$$f(z) \sim \sum_{n=0}^{\infty} e^{-\frac{s_n}{z}} f_n(z) \quad (2.67)$$

where

$$\begin{aligned} f_n(z) &:= \text{a representative of the class of functions} \\ &\text{having asymptotic expansion } z^{-B_n} \sum_{p=0}^{\infty} a_{n,p} z^p, \end{aligned} \quad (2.68)$$

and where

$$0 = \operatorname{Re} s_0 < \operatorname{Re} s_1 \leq \operatorname{Re} s_2 \leq \operatorname{Re} s_3 \dots \quad (2.69)$$

The representation may terminate if at some n the formal series for the function $F_n(z)$ turns out to have a finite radius of convergence and determines uniquely an analytic function $F_n(z)$, which is then equal to $f_n(z)$ without a nonanalytic remainder. (This case actually occurs [STS96] in some of the classical transcendental functions obtained as solutions to linear, second-order differential equations.)

However if the decomposition process does not stop and new, nonzero, suppression constants r_n (or $s_n = s_{n-1} + r_n$) keep emerging indefinitely, then the representation (2.67) remains highly formal in that it lacks proper definition at two fronts. First, we have said nothing as yet about the convergence or divergence of the n series, nor about its resummation in the case of divergence. Second, while it appears plausible that by going far enough in the expansion (2.67) we will eventually account for all the *exponentially* suppressed terms, we have completely lost track of the possible terms with super-exponential suppression, which at every step we swept under the rug of the next $F_{n+1}(z)$. Thus at every *finite* stage of the decomposition, such as eq. (2.65), they were still accounted for by the simple device of the last term having an F_n , instead of f_n , that still had room for such terms – just as in an ordinary semiconvergent series *all* suppressed terms were still present in the exact remainder term of the finite Taylor expansion (1.10). We lost them the moment we discarded this subtle difference and jumped ahead to the infinite formal series (2.67) – just as in earlier times we lost *all* terms with faster-than-powers suppression in jumping from (1.10) to the formal series (1.1). The first and second deficiencies are of course related – by carrying the analogy one step further, we can guess that the super-exponentially vanishing terms, if any, must be hiding behind a divergence of the n series, and that accounting for such terms must be tantamount to giving a prescription for resumming that series.

Making bad things worse, we now forget even about our list of specific choices of representatives $f_n(z)$ for the sums of the various semiconvergent p series, and simply write

$$f(z) \sim \sum_{n=0}^{\infty} e^{-\frac{s_n}{z}} \left\{ z^{-B_n} \sum_{p=0}^{\infty} a_{n,p} z^p \right\}, \quad (2.70)$$

a representation still more formal than (2.67) in that it now cries out for answers, not only to the above-mentioned and related two questions, but also to the question of the meaning of each of the p sums. This formal double expansion is called a general *resurgent symbol* in the mathematical literature [ECA81], and the successive terms of its n sum – each carrying a definite exponential-suppression factor $\exp(-s_n/z)$, except for the $n = 0$ term – are referred to as the *elementary resurgent symbols*. Because of its highly formal character, it would at first seem like an exotic way of making one's life complicated; yet it is precisely such a double expansion that arises in several areas of both mathematics and physics – the perturbation theory of quantum-mechanical energy levels, the JWKB semi-classical expansion, the thermodynamic functions of the ideal Fermi gas, the perturbation expansions of celestial mechanics, or, to mention a purely mathematical problem that forms the subject of vol. II of ref. [ECA81], the

classification of germs of analytic functions around the origin through a study of their iterations. Most important for us, a glimpse at eq. (2.59) tells us that the OPE, in a strictly renormalizable and asymptotically free theory, is also of this form: in a QFT with divergent coupling renormalization (0.6) and therefore a coupling-nonanalytic mass scale (2.20), the *operator-product expansion is a resurgent-symbol formal representation with respect to the coupling* of the correlation functions. This representation in terms of a formal double expansion is the new paradigm that replaces the formal perturbation series (0.4) in the realm of the strictly renormalizable theories.

You should realize that eq. (2.70) is a highly peculiar kind of expansion: it looks as if one and the same functional dependence on a single variable z were to be expanded in two sets of basis functions simultaneously. If somebody would suggest this to you in the context of convergent expansions, you would immediately protest the uncontrollable double counting. What makes the resurgent-symbol expansion possible is the specific, and matching, incompleteness in both types of "basis functions": the z^p expansion can never account for an $\exp(-\text{const.}/z)$, and $\exp(-\text{const.}/z)$, being an infinite Laurent series with only negative powers of z , can never account for positive powers z^p .

The possibilities of reconstructing analytic functions from such resurgent symbols form an important part of the theory of *resurgent functions*. As a preliminary definition – which will not satisfy the mathematician, but conveys enough of the agenda to be helpful to the consumer – we may thus formulate:

$$\left\| \begin{array}{l} \text{A resurgent function is an analytic function that can be} \\ \text{reconstructed from its formal small-}z \text{ expansion if the latter} \\ \text{is given at least "to exponential accuracy", i. e. in the form} \\ \text{of a resurgent symbol (2.70).} \end{array} \right\| \quad (2.71)$$

In dealing with such functions, the physicist must revise his or her ingrained habit of neglecting exponentially suppressed terms in the small- z expansion "because they are smaller than any power". While this may be a useful approximation *numerically* at sufficiently small z and under favorable circumstances (namely, if the leading coefficients $a_{n,0}$, $n \geq 1$, are not too large), it is completely irrelevant when it comes to knowing the "true" function: in reconstructing the latter, one either keeps the exponentially suppressed terms or gets nonsense, *tertium non datur*.

Our above heuristic argument emphasized the peculiar property that gave resurgent functions their name – the fact that semiconvergent expansions with the special properties of "factorial" error bounding in N , and of the ensuing exponential error suppression in z , "resurge" at every level of refinement of the small- z expansion. Yet it does not lead

to the most general type of resurgent symbol. The more systematic mathematical approach [ECA81] proceeds by firmly placing the problem from the outset in the framework of the most powerful and versatile reconstruction method, the Borel-Laplace process of sect. 1.3. The starting point there is a reformulation of the basic Laplace-transform representation (1.45): one introduces a modified Borel-transform function $B(s)$, called the *major* of f as opposed to the *minor* $b(s)$, such that $B(s)$ has as its sole singularity a branch cut along the positive real s axis with discontinuity equal to the minor $b(s)$:

$$B(s + i\varepsilon) - B(s - i\varepsilon) = \Theta(s) b(s). \quad (2.72)$$

For a $b(s)$ sufficiently decreasing as $s \rightarrow +\infty$, such a function may simply be constructed as a spectral integral,

$$B(s) = \frac{1}{2\pi i} \int_0^\infty ds' \frac{b(s')}{s' - s}, \quad (2.73)$$

which by virtue of the well-known distribution relation

$$\frac{1}{s' - (s + i\varepsilon)} - \frac{1}{s' - (s - i\varepsilon)} = 2i\pi \delta(s' - s) \quad (2.74)$$

fulfills condition (2.72). (Such functions have occasionally been used in physics work too [DAV82].) If $b(s')$ does not decrease fast enough, or even increases, so that the simple spectral integral fails to exist, one may use various "subtracted" or "divided" forms of the spectral representation: for example, if under Nevanlinna-Sokal conditions $b(s)$ turns out to be unfriendly enough to exhaust its exponential-growth restriction $|b(s)| \leq \text{const. exp}\left(\frac{s}{R}\right)$ at $s \geq 0$, one may set up a spectral representation for the function $\exp\left(-\frac{s}{R'}\right) \cdot B(s)$ with $R' < R$, which retains all analytic properties of B since the exponential is an entire function, and obtain

$$B_e(s) = e^{\left(\frac{s}{R'}\right)} \frac{1}{2\pi i} \int_0^\infty ds' \frac{e^{-\frac{s'}{R'}} b(s')}{s' - s}, \quad (2.75)$$

where the integral now converges. Of course, this superconvergence device may also be applied in cases where it is not strictly necessary; for example, when the pure-and-simple Cauchy integral (2.73) already exists, we obviously have

$$B_e(s) = B(s) + \frac{1}{2\pi i} \int_0^\infty ds' \frac{e^{\frac{s-s'}{R'}} - 1}{s' - s} b(s'), \quad (2.76)$$

where the second term is an entire function since it is now analytic both off and on the positive s axis. This example illustrates the general truth that an infinite family of functions with the prescribed right-hand discontinuity (2.72) can be constructed, which all differ from each other by the addition of entire functions. The notion of major therefore really stands for such a whole equivalence class of functions, known to mathematicians as a *hyperfunction*. In a QFT context we are usually dealing with minors $b(s)$ sufficiently decreasing as $s \rightarrow +\infty$ – the toy example of eq. (2.191) below being typical in that respect – and therefore will usually be able to get along with the "canonical" representatives of eq. (2.73) without really using this generality of hyperfunctions.

For an ordinary asymptotic series summable to a function $f(z)$ by the Borel-Laplace procedure, the Laplace representation (1.45), by virtue of eq. (2.72), can now be rewritten in terms of the major $B(s)$ as

$$f(z) = \frac{1}{z} \int_{\Gamma_+} ds e^{-\frac{s}{z}} B(s), \quad (2.77)$$

the integral being taken along the contour Γ_+ on both sides of the positive s axis as shown in Fig. 2.1. In view of the analyticity of $B(s)$, this contour can in turn be opened up, as indicated in Fig. 2.1, to turn into the contour Γ_0 parallel to the imaginary s axis. Thus as an alternative to (1.45) we have an integral transform along the path Γ_0 – that is, in essence a Fourier transform – of a major function $B(s)$ whose sole singularity is a branch point at $s_0 = 0$.

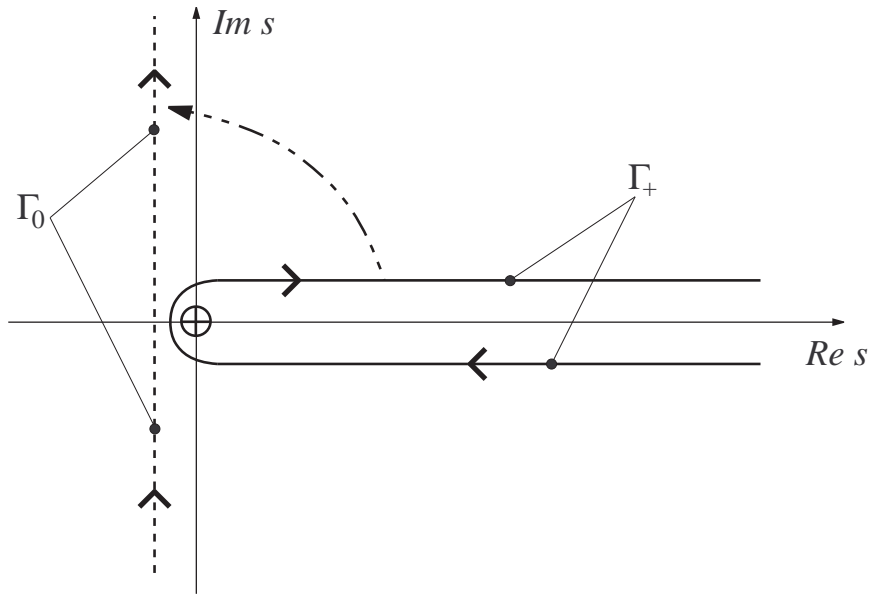


Fig. 2.1 Integration contour Γ_+ in major-based BL transform and its deformation into Γ_0

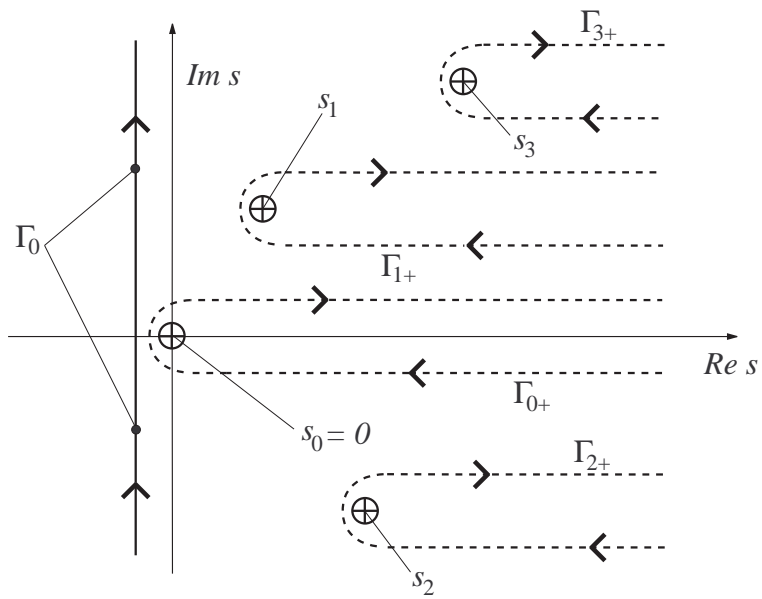


Fig. 2.2 Integration contours in BL integral for resurgent function with support points s_n

It is this alternative Laplace integral that forms a convenient point of departure for the generalization to resurgent functions. Assume we are now given a function $B(s)$ which instead of a single branch point at $s_0 = 0$ has a denumerable set of singularity points s_n , either branch points or poles, in the right half plane $Re s > 0$, possibly including one at $s_0 = 0$, and ordered as in eq. (2.69). See Fig. 2.2. We label $B(s)$, somewhat summarily, by this set $\{s_0, s_1, s_2, \dots\}$ of singularities, which in the mathematical literature is referred to as the *support* of a resurgent function or symbol (or both). Assume that $B(s)$ is such that the Laplace integral with contour Γ_0 ,

$$f(z) = \frac{1}{z} \int_{\Gamma_0} ds e^{-\frac{s}{z}} B_{\{s_0, s_1, s_2, \dots\}}(s), \quad (2.78)$$

exists in at least some "anisotropic", right-hand z -plane domain. This is the basic representation of a general resurgent function, *provided* one further condition is fulfilled: in order to obtain a formula suitable for calculating $f(z)$ at small, positive z , we must be able to bend the contour Γ_0 back to the right as in Fig. 2.2 so as to turn it into a union of horizontal back-and-forth contours Γ_{n+} , $n = 0, 1, 2, \dots$, each slung around one of the singularity points s_n , so that

$$f(z) = \sum_n \frac{1}{z} \int_{\Gamma_{n+}} ds e^{-\frac{s}{z}} B_{\{s_0, s_1, s_2, \dots\}}(s). \quad (2.79)$$

By using integration variables $s' = s - s_n$ in the individual terms, this can be written as

$$f(z) = \sum_n e^{-\frac{s_n}{z}} \left\{ \frac{1}{z} \int_0^\infty ds' e^{-\frac{s'}{z}} b_n(s') \right\} \quad (2.80)$$

with a set of discontinuity functions, or Borel minors, given by

$$b_n(s') = B(s_n + s' + i0) - B(s_n + s' - i0), \quad n = 0, 1, 2, \dots, \quad (2.81)$$

where s' is real and positive. The representation (2.80) is precisely of the form encountered in eq. (2.67) above, but with the $f_n(z)$ factors now being given a more concrete meaning through the bracketed Laplace transforms. The final step towards a small- z expansion is then to perform Taylor expansions at $z = 0+ -$ in general semiconvergent - of these transforms. For the more common types of singularities of the major $B(s)$, this step is straightforward: in particular, if $B(s)$, locally around each of the s_n , is of the logarithmically singular form

$$\left[-\frac{1}{2i\pi} \ln(s_n - s) \right] \phi_n(s - s_n) + \{\text{regular at } s_n\}, \quad (2.82)$$

with factors ϕ_n analytic at zero argument and thus having expansions

$$\phi_n(v) = \sum_{q=0}^{\infty} b_{n,q} v^q \quad (2.83)$$

with finite convergence radii, then since

$$\ln(-s' - i0) - \ln(-s' + i0) = -2i\pi \Theta(s') \quad (2.84)$$

at real s' , the n -th Borel minor simply becomes

$$b_n(s') = \phi_n(s') = \sum_{q=0}^{\infty} b_{n,q} s'^q \quad (s' > 0), \quad (2.85)$$

and the small- z expansion is obtained in the form of eq. (2.70) above, with all B_n 's equal to zero. A function $f(z)$ with such an "all-logarithmic" Borel major is called a "resurgent function with simple singularities", and is the kind of function most frequently considered in the mathematical literature on resurgence. In the QFT context, where in view of the form (2.59) of the Operator-Product Expansion we expect an expansion with non-integer B_n exponents, only a minor extension of this concept is necessary as long as $0 < B_n < 1$: we assume $B(s)$ to be locally of the form (2.82) only near $s = s_0 = 0$, but of the form

$$\frac{(s_n - s)^{-B_n} - 1}{2i \sin(B_n \pi)} \phi_n(s - s_n) + \{\text{regular at } s_n\} \quad (n = 1, 2, \dots) \quad (2.86)$$

near the other s_n , with B_n non-integer. (The numerator term of minus unity serves no purpose other than to make the limit $B_n \rightarrow 0$ exist and agree with (2.82)). Then

$$b_n(s') = s'^{-B_n} \phi_n(s') = \sum_{q=0}^{\infty} b_{n,q} (s')^{q-B_n} \quad (0 < B_n < 1), \quad (2.87)$$

and the general form of eq. (2.70) is obtained. We see how a resurgent symbol arises as a small- z expansion "to exponential accuracy" of a resurgent function given by eq. (2.78).

It has been essential, in deriving the resurgent symbol, that the original contour Γ_0 could be deformed into the set of Γ_{n+} 's without changing the value of the integral. For this it is necessary, of course, that the growth (if any) of the major $B(s)$, or at least of its discontinuities (2.81), as $Re s \rightarrow +\infty$ be sufficiently limited to allow the Laplace transforms of eq. (2.80) to converge. But there is also a more subtle precondition, which

in the presence of an *infinite* set of singularities s_n as allowed by our definition is no more trivial: we tacitly assumed that the arcs-at-infinity that are invisible in Fig. 2.2, but necessary to turn the union of (minus) the old Γ_0 and the new Γ_{n+} 's into a closed total contour and thus to establish the equivalence, can always be supplied even if the singularities occur at arbitrarily large distances $|s_n|$. In other words the set $\{s_n\}$, while possibly spreading out to infinity in the right half of the s plane, must not be allowed to accumulate anywhere in such a way as to completely obstruct, for some direction or sector of directions, analytic continuation of $B(s)$ to infinity along paths between them. The mathematical literature on resurgence calls such functions *endlessly continuable*. We are thus in a position to state the mathematical definition of resurgent functions:

$$\left\| \begin{array}{l} \text{A resurgent function is an analytic function } f(z) \text{ permitting} \\ \text{a Laplace-transform representation (2.78) / (2.79) with an} \\ \text{endlessly continuable Borel major } B(s). \end{array} \right\| \quad (2.88)$$

It is clear that this definition is extremely general – it places almost no restrictions on the detailed nature of the singularities of the major at the points s_n , or on the detailed structure of the analytic factors $\phi_n(s - s_n)$ associated with them, except that they should not destroy the possibility of a Laplace-transform representation, and that they should remain sufficiently isolated to permit endless continuation in between them. It is no more necessary, as our introductory heuristic argument seemed to suggest, that each $r_n = s_n - s_{n-1}$ be identifiable as the exponential-suppression constant in the series of the preceding, $(n - 1)$ -th term. That argument, leading to eqs. (2.67) and (2.70), really dealt with a special case, where one not only encountered the same type of singularity, namely (2.86), for all n , but also the same type of semiconvergent series $F_n(z)$ (namely, with factorial error bounding), and therefore the same type of convergence in the analytic factors ϕ_n , so that in fact the complete local behavior of the major near any s_n gets revived near all the others. It is this special but also especially interesting case which gave the entire class of resurgent functions their name. In Écalle's words [ECA81], these are functions "possessing a built-in mechanism that counteracts the usual "erosion" of local properties under analytic continuation and provides instead for their spotwise resurgence".

Even within this special class, you should still be prepared for all sorts of nasty little complications, when actually trying to reconstruct a resurgent function through, say, the series (2.80) of partial Laplace transforms. Our definitions required the expansions (2.83) of the analytic factors $\phi_n(v)$ to have finite convergence radii, but this does not prevent them from having the singularities that limit their convergence circles sit maliciously at real positive v . If these singularities are poles, then the corresponding partial Borel

minors, obtained by analytic continuation of (2.87), have poles at real positive s' , and the associated Laplace integrals in (2.80) have no unique meanings any more – they depend on the way one evades the pole when integrating, and therefore become two-valued functions. The complete resurgent function is then, in general, multi-valued.

The mathematical definition (2.88), apart from the weak restriction of endless continuability, allows for very general forms of the support $\{s_n\}$, the set of singularities of the Borel major, which via eq. (2.80) become the set of exponential-suppression constants. In many cases – particularly when considering nonlinear equations where products of a function with itself or its derivatives must be dealt with – one may want to move within sets of resurgent symbols that are closed under formal series multiplication (form a multiplicative algebra), and mathematical theory therefore prefers to deal with support sets that form *additive semigroups*. Of special importance among these is one particular kind of support, where all s_n sit equidistantly on the nonnegative real axis,

$$s_n = n s_1 ; \quad s_1 \text{ real and } > 0 ; \quad n = 0, 1, 2, \dots , \quad (2.89)$$

and which not only plays a prominent role in resurgence theory (in particular, it is ubiquitous in Écalle's treatment of resurgent solutions to nonlinear differential equations) but, as it happens, is also present in the resurgent symbols of OPE type relevant to QFT coupling dependence. These, as illustrated by our two-point example of eq. (2.59) above, feature the constants

$$s_n = n s_1 ; \quad s_1 = \frac{4\pi}{\beta_0} > 0 ; \quad n = 0, 1, 2, \dots \quad (2.90)$$

$$B_n = n B_1 ; \quad B_1 = \frac{\beta_1}{\beta_0^2} ; \quad n = 0, 1, 2, \dots \quad (2.91)$$

In this case, the resurgent symbol (2.70) assumes the special form of a double formal power series:

$$f(\alpha) \sim \sum_{n=0}^{\infty} u^n \left(\alpha^{-B_1} e^{-\frac{s_1}{\alpha}} \right)^n \sum_{p=0}^{\infty} a_{n,p} \alpha^p . \quad (2.92)$$

The occurrence of an n -th power of a dimensionless "counting parameter" u for the n summation, where in our case

$$u = \frac{\mu^2}{k^2} , \quad (2.93)$$

is a typical feature of such symbols. Since all s_n are now on the real s axis, the representation of eq. (2.79) can alternatively be written as a single Laplace integral,

$$f(\alpha) = \frac{1}{\alpha} \int_0^{\infty} ds e^{-\frac{s}{\alpha}} b(s) , \quad (2.94)$$

with a *combined Borel minor* given, according to line 13 of Table 1, by

$$b(s) = \sum_{n=0}^{\infty} \Theta(s - n s_1) b_n(s - n s_1). \quad (2.95)$$

It is no more analytic along the positive s axis, but has finite discontinuities in the support points. For the individual minors $b_n(s)$, the class of functions (2.87) considered up to now must be enlarged here to include distribution-valued terms, since B_n , by eq. (2.91), becomes larger than one as n increases so the first line of Table 1 can no more be used. (Recall typical values such as $B_1 = \frac{32}{81}$ or $B_1 = \frac{13}{49}$ for $SU(3)$ gauge theory with 3 or 6 fermion species respectively, for which $B_n > 1$ at $n \geq 3$ or $n \geq 4$.) For the terms in eq. (2.92) where $p - n B_1 = -k$ is a negative integer, the Borel transform is a delta distribution $\delta(s)$ when $k = 1$ and a derivative of a delta distribution, $\delta^{(k-1)}(s - s_n)$, when $k \geq 2$:

$$\alpha^{-k} e^{-\frac{s n}{\alpha}} = \mathcal{L}[\delta^{(k-1)}(s - s_n); \alpha] \quad (k = 1, 2, 3, \dots). \quad (2.96)$$

This is verified immediately by using the definition

$$\int ds \delta^{(k-1)}(s) g(s) = (-)^{k-1} \left(\frac{d^{k-1} g}{d s^{k-1}} \right)_{s=0} \quad (2.97)$$

in the Laplace integral (2.94). For the terms with noninteger $p - n B_1 = -\beta < -1$, one even needs to introduce fractional derivatives of delta distributions, functionals that perform the minor miracle of the mapping

$$\frac{1}{z} \int_0^{\infty} ds e^{-\frac{s}{z}} \delta^{(\beta-1)}(s - s_n) = z^{-\beta} e^{-\frac{s n}{z}} \quad (\beta \text{ non-integer}). \quad (2.98)$$

I can hear your oohs – that is quite some machinery to deal with an object as simple as $\alpha^{-n B_1}$! Indeed the combined minor of eq. (2.95), having now not only the finite step-function jumps but also all kinds of distribution-valued terms at the support points $s = s_n$, becomes complicated in a way that appears somewhat out of proportion for the relatively simple structure of the symbol (2.92). While the Borel-Laplace framework remains essential for the *definition* and basic theory of resurgence, it may therefore well turn out to be a clumsy tool when it comes to the practical recovery of a resurgent $f(z)$ from its symbol, and it may become legitimate, for that purpose, to look around for simpler alternatives – we emphasized earlier that Borelization, for all its power, has no monopoly.

On the other hand these observations show clearly that distribution-valued terms of delta and derivative-of-delta type are natural ingredients for Borel transforms of resurgent functions – as indeed we might have guessed already from line 3 of our Table 1. They do not create new problems, in the sense that they do not infringe on either the existence or the univaluedness of the Laplace transform. This is the main reason why in the following we will not be concerned with the so-called "instanton singularities" of Borel transforms that arise in the semiclassical treatment of QFT's with instanton solutions. These, as discussed by t'Hooft [THO79], are simply integer-order derivatives of $\delta(s - s_n)$, which produce discrete additions to the Laplace transform by the pattern of eq. (2.96) above.

When does a resurgent symbol (2.70), or the more special double-power-series form (2.92), determine uniquely an analytic function $f(\alpha)$ whose small- α expansion it represents? (In this case the symbol is again referred to as *summable*.) Here comes a big disappointment: to the best of my knowledge, no sufficient conditions are presently known that would come in as handy for a realistic, asymptotically free QFT as the Nevanlinna-Sokal criterion (1.24) does for the superrenormalizable theories. Such conditions, if they were to be of practical value, would as a minimum have to allow for the special analyticity situation in the complex α plane governed by the so-called t'Hooft singularities [THO79], at which we had a glance in Fig. 1.1 c) and whose origins we will explain in more detail in sect. 2.4 below – singularities that preclude even the minimalistic assumption of a small-angle "sectorial" analyticity of $f(\alpha)$ that is routinely made in the mathematical treatments of resurgence. The modest contribution of sect. 2.4 will be to prove, by explicit counterexample, that these singularities by themselves need not sabotage unique reconstruction. But a compact and convenient positive criterion accommodating them seems to be lacking. It is clear that if *all* the Borel minors obtained by analytic continuation of the series (2.87) have sufficient smoothness and limited growth to make the partial Laplace transforms in (2.80) well-defined, *and* if the n series in (2.80), with these transforms as weights, converges on at least some segment of the positive α axis including the point $\alpha = 0$, then at least on that segment the resurgent function $f(\alpha)$ is uniquely recoverable. But this set of conditions is so restrictive as to be practically useless. For example it has been plausibly conjectured for realistic QFT amplitudes [DAV82], and will be strikingly apparent again in sect. 2.4, that the interesting case is for all b_n 's individually to have positive- s poles, but for these poles to cancel in the combined minor $b(s)$ of eq. (2.95) so the total $f(\alpha)$ is nevertheless univalued. (This scenario seems also to be known [CNP93] in mathematical theory.) The resurgent symbol is then obviously an awkward way of decomposing such an amplitude, but we still need to rack our brains about it because physics makes us run straight into

such awkward decompositions.

In this (for the time being) unsatisfactory situation, the best one can do perhaps is to ask what the "escaping" parts, the parts undetermined by the resurgent symbol, would look like if $f(\alpha)$ were *not* to be uniquely recoverable. We already noted in passing that just as the escaping terms for an ordinary semiconvergent expansion must vanish faster than any power as $\alpha \rightarrow 0+$, the escaping parts for the more comprehensive resurgent symbol must be what resurgence theory calls *rapidly decreasing functions*, functions vanishing faster than any exponential $\exp(-\text{const.}/\alpha)$. To be more precise we must, as earlier, adopt more specific hypotheses: let us therefore focus on the special case of (2.92) relevant for QFT, and let us assume a behavior of the $a_{n,p}$ coefficients with respect to n such that the n sum at fixed p ,

$$\sum_{n=0}^{\infty} a_{n,p} X^n, \quad \text{where} \quad X = u \alpha^{-B_1} e^{-\frac{s_1}{\alpha}}, \quad (2.99)$$

is no worse than semiconvergent with factorial error bounding, analogous to eq. (1.13). We may then repeat verbatim the error-minimization argument with respect to N of sect. 1.1, and conclude that the sum of (2.99) is determined up to escaping terms with the small- α expansion

$$e^{-\frac{K_p}{X}} X^{-L_p} \{1 + \mathcal{O}(X)\}_p, \quad (2.100)$$

with constants $K_p (> 0)$ and L_p (real), and with the bracketed power series in X possibly being again semiconvergent. This formula is interesting in that it shows that – under these more special but still entirely reasonable assumptions – the super-exponential suppression of the "rapidly decreasing" functions turns out to really be as fast as $\exp(-\exp 1/\alpha)$, or in other words, faster than $\exp(-c/\alpha^k)$ for any c and k . For the semiconvergent bracket, arguing heuristically, we may repeat our chain of argument at the beginning of this section, and conclude that the escaping terms form the nontrivial part of a resurgent symbol in X ,

$$\sum_{q=0}^{\infty} e^{-\frac{K_{p,q}}{X}} X^{-L_{p,q}} \{1 + \mathcal{O}(X)\}_{p,q}. \quad (2.101)$$

By resupplying the p summation omitted up to now and reordering the entire expansion by increasing K 's (here of course, is another tacit assumption – that there is a smallest K), we find that the terms left undetermined by the symbol (2.92), if any, should themselves be representable as a small- α expansion of the form

$$\sum_{m=0}^{\infty} e^{-K_m (\alpha^{B_1} e^{\frac{s_1}{\alpha}})} (\alpha^{B_1} e^{\frac{s_1}{\alpha}})^{L_m} R_m(\alpha), \quad (2.102)$$

where $R_m(\alpha)$ is some resurgent symbol in α .

In the physics context, clarity may be helped by rewriting the symbol (2.92) as a double expansion in powers of Λ_0 and α as in (2.54), i. e. by undoing the series multiplication of (2.58). Call d_Γ the mass dimension, and Q^2 some typical, Euclidean squared momentum of the vertex function Γ under consideration; all other invariant momentum arguments of Γ may then be replaced by their dimensionless ratios, x_i , with Q^2 . The heuristic argument just given may then be summed up in the small- α expansion (we mostly omit dependences on the uninteresting set $\{x_i\}$ for leaner writing):

$$\begin{aligned} \Gamma(Q^2, \{x_i\}; \mu^2; \alpha) = & (Q^2)^{\frac{1}{2}d_\Gamma} \left\{ \sum_{p=0}^{\infty} a_{0,0,p} \left(\frac{Q^2}{\mu^2}\right) \alpha^p \right. \\ & + \sum_{n=1}^{\infty} \left(\frac{\Lambda_0^2}{Q^2}\right)^n \left[\sum_{p=0}^{\infty} a_{0,n,p} \left(\frac{Q^2}{\mu^2}\right) \alpha^p \right] \\ & + \sum_{m=1}^{\infty} e^{-K_m \left(\frac{Q^2}{\Lambda_0^2}\right)} \left[\sum_{n=1}^{\infty} \sum_{p=1}^{\infty} a_{m,n,p} \left(\frac{Q^2}{\mu^2}\right) \left(\frac{\Lambda_0^2}{Q^2}\right)^n \alpha^p \right] \\ & + \dots \left. \right\}. \end{aligned} \tag{2.103}$$

The new escaping terms are characterized by a *Gaussian* decay at large momentum, with Λ_0 providing the decay length. In view of their exotic coupling dependence displayed in eq. (2.102), they are probably outside the range of what can usefully be treated by a Borel-Laplace process with respect to α .

We should now pause and reflect on the fact that here, at a point where we have hardly done more about resurgence theory than quote a few general concepts and basic formulas, we have already been touched by its great organizing power: where in the matter of coupling dependence we originally expected a horrible mess, we now see an orderly *hierarchy of small- α suppressions* emerging, with progressively stronger suppression at small coupling accompanied by progressively faster decrease at large momenta. This insight remains valuable even while we have no complete certainty about the presence or absence of the third line (and possibly fourth and further lines) in eq. (2.103). If we find ways of treating the dynamical effects leading to the second line of (2.103), i. e. of constructing vertex functions whose small- α expansions are resurgent symbols rather than perturbation series, we will have made definite progress and omitted at most terms with a definite, rather exotic large-momentum behavior. Moreover, we will have gained a framework for adequately treating an impressive array of nontrivial dynamical

problems. Take nonperturbative mass generation and dynamical symmetry breaking: the conversion of a massless perturbative into a massive nonperturbative propagator, described schematically by

$$k^2 \rightarrow k^2 + c\Lambda_0^2 + \dots = k^2 \left[1 + c \left(\frac{\Lambda_0^2}{k^2} \right) + \dots \right] \quad (2.104)$$

involves power, not exponential, corrections in Λ_0^2/k^2 . The breaking of dilatation symmetries by renormalization involves trace anomalies that are powers, not exponentials, of Λ_0^2 .

In my opinion, the view is even brighter, since everything in present knowledge favors the assumption that for the strictly renormalizable and asymptotically free theories, eq. (2.103) is complete after its second line, just as for the superrenormalizable theories it was complete after the first line. (Here "complete" refers to unique recoverability from a small- α expansion.) Let me list a few arguments (there may be more) in favor of this assumption.

- A purely qualitative but nevertheless strong argument is that the completeness postulate underlying the OPE is very probably correct. In a theory based on a finite number of local fields as basic degrees of freedom, it is difficult to see what basic operators should be available for expanding products near a point other than the set of all local composites at that point. Therefore the OPE, very probably, does *not* miss further escaping terms as in the third line of eq. (2.103), and represents a *summable* resurgent-symbol expansion.
- Although the question of convergence for the OPE is not settled, the n summation at fixed p in the OPE, eq. (2.99), in contrast to the perturbative p sum at fixed n , is very probably *not* semiconvergent but has a finite radius of convergence, so that in fact it leaves no room for escaping terms of type (2.100). This is because the growth of the coefficients $a_{n,p}$ with n is controlled by the growth in the number $l(n)$ of local operators with mass dimension n and vacuum quantum numbers, and this number, in contrast to the factorial growth of the number of Feynman diagrams in the p direction, increases only exponentially. Consider a theory with B bosonic fields $\phi_b(x)$ ($b = 1 \dots B$) of unit mass dimension in D spacetime dimensions, and regard a local operator product of mass dimension n as an array of n slots. Each slot can be filled, either with one of the ϕ_b , or (apart from the rightmost slot) by one of the D spacetime derivatives ∂^μ . (We adopt the convention of having each ∂^μ acting only on the next ϕ field to the right, since the action of a ∂^μ on two or

more fields gives terms like

$$\partial^\mu (\phi_a \phi_b) = (\partial^\mu \phi_a) \phi_b + \phi_a (\partial^\mu \phi_b) \quad (2.105)$$

and therefore nothing new). Thus the number of local composites of mass dimension n is

$$B \cdot (B + D)^{n-1}. \quad (2.106)$$

The number we called $l(n)$ above is distinctly lower, since only the operators contracted to vacuum quantum numbers will give nonzero vacuum condensates, but in any case we have the exponential estimate

$$l(n) < (B + D)^n. \quad (2.107)$$

It is true that coincidence of the n' ϕ -field factors, where $n' \leq n$, at the same spacetime point x leads in momentum space to n' loop integrations in a vacuum condensate, but this increase in the number of loops is already present in the perturbation sum in the p direction, where it is known to contribute only to the power-and-exponential growth in eq. (0.5), not to the factorial one (the p -loop integrals can usually be bounded from above by a $p - th$ power of some one-loop integral). Thus

$$a_{n,p} \rightarrow (\text{const.})_p n^V u^n \left[1 + \mathcal{O}\left(\frac{1}{n}\right) \right] \quad (n \rightarrow \infty) \quad (2.108)$$

is the likely behavior. (Well, yes, as I hear some initiates among you saying, there are still the "renormalon" type sequences of p -bubble diagrams that we will also meet in sect. 2.4 and that generate their own $p!$ growth in the p direction in the manner exemplified by eq. (2.139) below – but remember we are dealing here with behavior *at fixed* p . In the n direction, what we have diagrammatically is growth in the number of lines entering and leaving a single composite vertex, at constant number of interaction vertices in the remainder diagram, and this does not generate bubble chains). Thus the series of (2.99) is likely to have the finite convergence circle (in X) of a generalized geometric series, such as (1.39), and the question of escaping terms (2.100) does not arise.

- Dimensionful observables in lattice-QFT calculations, in the removal-of-regulator limit, scale like a power of Λ_0 as functions of the coupling, but not like

$$(\text{power of } \Lambda_0) \times \left\{ c_0 + c_1 e^{-\frac{\text{const.} \cdot 1}{\Lambda_0^2}} + \dots \right\} \quad (2.109)$$

(I am, of course, simplifying a little: lattice calculations are parametrized in terms of a bare coupling $g_0(a)$ at lattice spacing a rather than a renormalized one, but the dependence of $(\Lambda_0)_{lattice} \cdot a$ on g_0 is still the same as for our Λ / μ in eq. (2.25) on g). Indeed this behavior is routinely used by lattice theorists as an indicator that they are sufficiently close to the continuum limit. This is a special case of the general statement that momentum or coupling dependences as in the third line of (2.103) have nowhere been encountered in any of the nonperturbative approaches applied so far to QFT.

There are therefore sound reasons for believing that strictly renormalizable and asymptotically free field theories will be about summable resurgent symbols, just as the superrenormalizable theories are about summable perturbation series. The methods described in Chapter 3 are tuned to this expectation.

2.4 A Resurgent Model Amplitude

Several features of amplitudes having resurgent coupling dependence through the lambda scale, such as their peculiar singularity structure with respect to the coupling, and the tricky things that can happen when such an amplitude is approached through decomposition into its semiconvergent perturbation series and a nonperturbative remainder, are well illustrated by a simple model function that can be analyzed completely in both the α and Borel planes. This model depends, other than on the coupling α , on a single scalar Euclidean momentum variable k^2 , as does a two-point function in QFT, and explicitly on the sliding scale μ , but since it is taken to be dimensionless, it can depend on the latter two arguments only through the dimensionless ratio k^2 / μ^2 . It is given by the spectral representation

$$\Gamma\left(\frac{k^2}{\mu^2}; \alpha\right) = 1 - \int_0^\infty d(q^2) \frac{\rho\left(\frac{q^2}{\mu^2}; \alpha\right)}{k^2 + q^2} \quad (2.110)$$

with the deceptively simple spectral function,

$$\rho\left(\frac{q^2}{\mu^2}; \alpha\right) = \Theta(q^2 - \Lambda^2(\mu, \alpha)) \frac{\alpha}{1 + \alpha \frac{\eta}{2} \ln\left(\frac{q^2}{\mu^2}\right)}, \quad (2.111)$$

where as a caricature of a QFT lambda scale we adopt

$$\Lambda^2(\mu, \alpha) = \mu^2 e^{-\frac{1}{\alpha}} \Leftrightarrow \alpha(\mu) = \left[\ln\left(\frac{\mu^2}{\Lambda^2}\right) \right]^{-1}. \quad (2.112)$$

The first, "nonperturbative" factor in (2.111), which simply limits the spectral integration to $q^2 \geq \Lambda^2$, schematically represents the assumption that the intermediate states contributing to Γ have through nonperturbative mass generation acquired invariant masses of at least Λ . The second, "perturbative" factor mimicks a typical resummation of a perturbative subseries; it displays the logarithmic q^2 dependence typical of perturbative amplitudes, and contains a positive constant η characteristic of its (imagined) building-block diagram. We will always restrict η to

$$0 < \eta < 2, \quad (2.113)$$

since at $\eta = 2$ the amplitude would cease to exist, even in its renormalized form of eq. (2.115) below, due to a divergence of the integral at the lower limit $q^2 = \Lambda^2$. (This is merely a consequence of the fact that in order to minimize kinematical complications, we have avoided building correct threshold behavior into the spectral weight (2.111). If we were more fussy about threshold behavior, we would include in the ρ of eq. (2.111) an extra factor

$$\frac{\sqrt{q^2(q^2 - \Lambda^2)}}{q^2 + c\Lambda^2} \quad (2.114)$$

that would preserve integrability at $q^2 = \Lambda^2$ even for $\eta = 2$, but this would also burden us with an extra parameter, and would deprive the model of the useful generating-function property mentioned after eq. (2.222) below, thus rendering the whole exercise much less transparent and instructive).

Since (2.111) decreases too slowly at large q^2 to make the spectral integral convergence at the upper limit, we apply a caricature of renormalization, rewriting the amplitude as

$$\Gamma\left(\frac{k^2}{\mu^2}; \alpha\right) = \Gamma(1; \alpha) + \alpha \int_{\mu^2}^{k^2} \frac{d(p^2)}{p^2} F\left(\frac{p^2}{\mu^2}; \alpha\right) \quad (2.115)$$

in terms of the derivative function

$$F\left(\frac{k^2}{\mu^2}; \alpha\right) = \frac{k^2}{\alpha} \frac{\partial}{\partial(k^2)} \Gamma\left(\frac{k^2}{\mu^2}; \alpha\right) = \frac{k^2}{\alpha} \int_0^{\infty} d(q^2) \frac{\rho\left(\frac{q^2}{\mu^2}; \alpha\right)}{(k^2 + q^2)^2}, \quad (2.116)$$

while absorbing the ultraviolet divergence in the constant

$$\Gamma(1; \alpha) = 1 - \int_0^{\infty} d(q^2) \frac{\rho\left(\frac{q^2}{\mu^2}; \alpha\right)}{q^2 + \mu^2}. \quad (2.117)$$

It is the finite amplitude F of eq. (2.116), rather than the original quantity of eq. (2.115) which serves more or less as a pretext, that will be the main object of our study.

We first take a look at the *singularity structure in α* of (2.110) or (2.116) with the spectral weight (2.111). From the "perturbative" factor of the latter, the amplitude receives a k^2 -independent singularity at

$$1 + \alpha \frac{\eta}{2} \ln \left(\frac{q^2}{\mu^2} \right) = 0, \quad (2.118)$$

while from the spectral denominator we have a k^2 -dependent singularity at

$$q^2 + k^2 = 0, \quad (2.119)$$

which when varying k^2 produces a branch cut in the k^2 plane at "Minkowskian" values $k^2 = -\Lambda^2 \dots -\infty$, the standard pattern in QFT two-point functions. Here, however, we are after the α plane: in both conditions q^2 ranges from Λ^2 to $+\infty$ so we may parameterize

$$q^2 = \Lambda^2 e^t = \mu^2 e^{(t - \frac{1}{\alpha})}, \quad t = 0 \dots \infty. \quad (2.120)$$

Then since $\alpha \ln(q^2 / \mu^2) = \alpha t - 1$, condition (2.118) gives a "perturbative" coupling-plane singularity line at

$$\alpha = -\frac{\left(\frac{2}{\eta}\right) - 1}{t}, \quad t = 0 \dots \infty, \quad (2.121)$$

i. e., a branch cut on the negative real axis at

$$\alpha = -\infty \dots 0. \quad (2.122)$$

If this were the only singularity with respect to the coupling, we would again be in the enviable situation of having Phragmén-Lindelöf analyticity in a sector of infinite radius and opening angle $2(\pi - \delta)$ with infinitesimal δ . The physics behind this kind of branch cut – which, incidentally, is also inherent in Dyson's QED argument [DYS52] – is generally that the QFT's underlying an amplitude such as (2.110) become unstable at real negative α and produce complex, two-valued Euclidean correlation functions, whose imaginary parts are related to the decay width of the unstable vacuum state.

The spoiler of this simple picture is the singularity arising from (2.119), which upon using (2.120) reads

$$e^{(t - \frac{1}{\alpha})} = -\frac{k^2}{\mu^2} \quad (2.123)$$

or, since k^2 is Euclidean and therefore positive,

$$t - \frac{1}{\alpha} - \ln \left(\frac{k^2}{\mu^2} \right) = \ln(-1) = \pm i\pi(2n + 1); \quad n = 0, 1, 2, \dots \quad (2.124)$$

At a given Euclidean k^2 , we therefore have a whole infinite sequence of singularity curves in the α plane,

$$\alpha(t) = \frac{1}{\left(t - \ln \left(\frac{k^2}{\mu^2} \right) \mp i\pi(2n + 1) \right)}; \quad t = 0 \dots \infty; \quad n = 0, 1, 2, \dots \quad (2.125)$$

By taking the real and imaginary parts of this equation and eliminating the parameter t between the two, one finds

$$(Re \alpha)^2 + (Im \alpha \mp r_n)^2 = r_n^2, \quad (2.126)$$

i. e., the equations of two sequences of nested circular arcs in the α plane, with radii

$$r_n = \frac{1}{2(2n + 1)\pi} = \frac{1}{2\pi}, \frac{1}{6\pi}, \frac{1}{10\pi}, \dots, \quad (2.127)$$

and tangent to the real axis from above and below at $\alpha = 0+$ (see Fig. 2.3). The k^2 dependence is in the lengths of these arcs, which are more or less than those of semicircles according to whether k^2 is more or less than μ^2 . (The situation depicted in Fig. 2.3 corresponds to the first, that of Fig. 1.1 (c) to the second case).

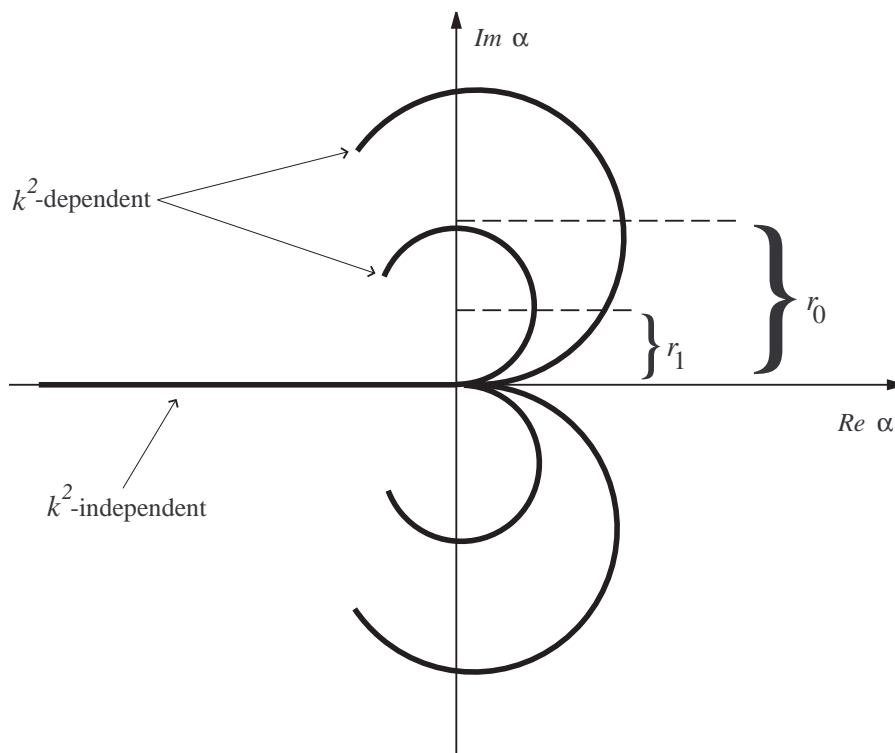


Fig. 2.3 Singularity lines of the model amplitude in the complex α plane at Euclidean k^2 . In addition to the k^2 -independent branch line on the real negative α axis, only the first two pairs from the infinite sequence of nested, k^2 -dependent branch lines are shown.

Obviously, this "nonperturbative" kind of singularity is very general: it arises solely from the fact that the scale of the spectral integration, or of the k^2 branch cut in the Minkowskian, is set by a threshold proportional to the strongly nonperturbative Λ^2 . This is why t'Hooft, in discovering this peculiar family of singularities [THO79], did not even need an explicit amplitude function. It is clear that this set of singularities precludes not only the use of theorem (1.24) to ascertain uniqueness of reconstruction from the perturbative expansion, but also any form of sectorial analyticity – the "horn-shaped" region of analyticity defined by the two largest arcs allows for no finite sectorial opening angle along the positive real axis. Although we have emphasized that the conditions of theorem (1.24) are sufficient but not necessary conditions, this has generally been taken as a plausibility argument against the possibility of unique reconstruction, and the t'Hooft singularities ever since have hovered as a kind of vague threat over the entire

subject of coupling dependence. This still should not deter us from proceeding with the analysis of (2.116), since after all at physical α , which are real and positive, that amplitude is *manifestly* well-defined and single-valued, and since we know by now that there is more to physics than the perturbative expansion.

To examine the perturbation series of the amplitude (2.116), it is convenient to use

$$x = \frac{1}{2} \ln \left(\frac{q^2}{\mu^2} \right), \quad \lambda(k^2) = \frac{1}{2} \ln \left(\frac{k^2}{\mu^2} \right) \quad (2.128)$$

as integration and external variables respectively. Then

$$F \left(\frac{k^2}{\mu^2}; \alpha \right) = \frac{1}{2} \int_{-\frac{1}{2\alpha}}^{\infty} dx \frac{1}{\cosh^2(x - \lambda)} \frac{1}{1 + \alpha \eta x}. \quad (2.129)$$

Look first at the Taylor coefficients of F with respect to α at $\alpha = 0+$: you quickly realize that the nonperturbative α dependence through the lower integration limit does not contribute here. For example,

$$\begin{aligned} \frac{\partial F}{\partial \alpha} &= \frac{1}{2} \int_{-\frac{1}{2\alpha}}^{\infty} dx \frac{1}{\cosh^2(x - \lambda)} \left[\frac{\partial}{\partial \alpha} \left(\frac{1}{1 + \alpha \eta x} \right) \right] \\ &\quad - \frac{1}{2} \left[\frac{d}{d\alpha} \left(-\frac{1}{2\alpha} \right) \right] \left\{ \frac{1}{\cosh^2(x - \lambda)} \frac{1}{1 + \alpha \eta x} \right\}_{x = -\frac{1}{2\alpha}}. \end{aligned} \quad (2.130)$$

As $\alpha \rightarrow 0+$, the second term vanishes exponentially, and so do all terms in the higher derivatives $\partial^n F / \partial \alpha^n$ except those in which all n derivations act only on the "perturbative" factor of the integrand: the lower limit in (2.129) only creates the exponentially suppressed, and therefore "escaping", α dependence we described in (1.20). Thus the perturbation series is identical with the formal expansion around $\alpha = 0$ of the integral

$$F^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right) = \frac{1}{2} \int_{-\infty}^{+\infty} dx \frac{1}{\cosh^2(x - \lambda)} \frac{1}{1 + \alpha \eta x}. \quad (2.131)$$

Accordingly, we would call the remainder integral

$$F^{nonp} \left(\frac{k^2}{\mu^2}; \alpha \right) = -\frac{1}{2} \int_{-\infty}^{-\frac{1}{2\alpha}} dx \frac{1}{\cosh^2(x - \lambda)} \frac{1}{1 + \alpha \eta x} \quad (2.132)$$

the nonperturbative part, so that

$$F\left(\frac{k^2}{\mu^2}; \alpha\right) = F^{pert} + F^{nonp} . \quad (2.133)$$

But something weird has happened here: the prescription to let $\alpha \rightarrow 0+$ in the Taylor coefficients – that is, the very *definition* of perturbation theory – has forced us to take the lower integration limit in (2.131) to minus infinity, so that now we not only integrate beyond the circle of convergence, $|x| < \frac{1}{\eta\alpha}$, of the perturbative integrand – a sin for which we expect punishment in the form of a semiconvergent perturbation series –, but inevitably also over its pole at $x = -1/\eta\alpha$, thus creating a branch cut and the associated multi-valuedness in α on the *entire* real α axis in the F^{pert} and the *positive* α axis in the F^{nonp} portion. (On the other hand it is obvious that in F^{pert} , t’Hooft’s ”circular” α -plane singularities of Fig. 2.2 have disappeared: with Λ^2 no more setting the scale of the q^2 integration as in (2.120), condition (2.119) no more translates into a singularity condition in the α -plane.) Clearly this problem is entirely spurious, since in the full amplitude (2.129) at physical positive α , the integration starts only at $x = -\frac{1}{2\alpha}$ and never touches that pole. Thus if we want to interpret (2.131) as the sum of perturbation theory, we will have to specify a prescription of how to approach the spurious branch cut, and be careful to use the same prescription in the nonperturbative piece (2.132) so the two nuisances can ultimately cancel. Looking back at (2.128), we see that $x = -\infty$ corresponds to having $q^2 = 0$, rather than the ”true” $q^2 = \Lambda^2$, as the lower limit in the spectral integral of (2.110) or (2.116): perturbation theory has created for us a *spurious infrared problem* by forcing us to integrate over massless intermediate states that are plainly absent in the full theory. Again it is immediately clear that this problem is very general, and of significance far beyond the present simple model: the full theory has the spontaneous Λ scale in almost all of its thresholds (the only exception being electroweak amplitudes involving the massless photon), but perturbation theory, by ”losing” that scale, gets us involved in false infrared problems that will manifest themselves in false nonsummability.

For the moment, we may ignore this problem, since we are only interested in the *formal* power-series expansion of (2.131),

$$F_f^{pert}\left(\frac{k^2}{\mu^2}; \alpha\right) = \sum_{p=0}^{\infty} \alpha^p F^{(p)pert}(\lambda) , \quad (2.134)$$

whose individual Taylor coefficients are well defined:

$$F^{(p)pert}(\lambda) = \frac{1}{2} (-\eta)^p \int_{-\infty}^{+\infty} dx' \frac{(x' + \lambda)^p}{\cosh^2 x'} . \quad (2.135)$$

Here we used $x' = x - \lambda$. The evaluation is facilitated by the observation that F^{pert} possesses a scaling property of the renormalization-group type: by writing in (2.131)

$$\frac{1}{1 + \alpha \eta (x' + \lambda)} = \left[\frac{\left(\frac{\alpha}{1 + \alpha \eta \lambda} \right)}{\alpha} \right] \frac{1}{1 + \left(\frac{\alpha}{1 + \alpha \eta \lambda} \right) \eta x'} , \quad (2.136)$$

one sees that apart from the square-bracketed prefactor, F^{pert} at general k^2 looks like the one at $k^2 = \mu^2$, i. e. at $\lambda = 0$, but with a k^2 -dependent effective coupling:

$$F^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right) = \left[\frac{\bar{\alpha}(k^2)}{\alpha} \right] F^{pert} (1; \bar{\alpha}(k^2)) ; \quad (2.137)$$

$$\bar{\alpha}(k^2) = \frac{\alpha}{1 + \alpha \frac{\eta}{2} \ln \left(\frac{k^2}{\mu^2} \right)} . \quad (2.138)$$

It is therefore sufficient to evaluate the coefficients (2.135) at $\lambda = 0$. This can be done with the help of the formula [GRY94]

$$\int_0^\infty dx \frac{x^p}{\cosh^2 x} = \frac{1}{2^{p-1}} \left(1 - \frac{1}{2^{p-1}} \right) \zeta(p) \Gamma(p+1) \quad (p \neq 1) \quad (2.139)$$

and of the symmetry properties of the integrand, giving

$$F^{(p)pert}(\lambda = 0) = [(-)^p + 1] \left(1 - \frac{1}{2^{p-1}} \right) \zeta(p) \left(\frac{\eta}{2} \right)^p p! \quad (2.140)$$

Since $\zeta(p) \rightarrow 1$ as $p \rightarrow \infty$, this indeed displays the factorial-growth pattern of (0.5), with constants $c_0 = 1$, $B_0 = 0$, and $A_0 = \frac{2}{\eta}$. Due to the symmetric range of the x integration, "alternating" and "monotonic" contributions are present in equal mixture so only even- p coefficients are nonzero – an oversimplified feature of the model. This feature will disappear anyway when calculating the full expansion (2.134) at $k^2 \neq \mu^2$ from the scaling relation (2.137): by expanding the last factor in each term of

$$F_f^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right) = \sum_{p=0}^\infty F^{(p)pert}(\lambda = 0) \alpha^p [1 + \alpha(\eta \lambda)]^{-(p+1)} \quad (2.141)$$

in powers of α , and collecting powers of α , one finds the series (2.134) with

$$F^{(p)pert}(\lambda) = (-)^p \left(\frac{\eta}{2} \right)^p \left[\sum_{q=0}^p f_q^{(p)} (2 \lambda)^q \right] p! , \quad (2.142)$$

$$f_q^{(p)} = [1 + (-)^{p-q}] \frac{1}{q!} \left(1 - \frac{1}{2^{p-q-1}} \right) \zeta(p-q). \quad (2.143)$$

These coefficients, for all the simplicity of the model, again display typical properties of QFT perturbation series: the k^2 dependence through logarithms, which in this case are infrared singular at $k^2 = 0$ due to the spurious infrared problem we produced, and the fact that the p -th term contains a p -th degree polynomial in these logarithms. The "leading-logarithms" (LL) subseries, comprising the terms with $q = p$, has $f_p^{(p)} \cdot p! = 1$ (use $\zeta(0) = -\frac{1}{2}$), and is therefore, as usual, a purely geometric series,

$$F_{(LL)}^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right) = \sum_{p=0}^{\infty} (-\alpha \eta \lambda)^p = \frac{1}{1 + \alpha (\eta \lambda)}, \quad (2.144)$$

which is identical with the leading term in $\bar{\alpha}$ (the prefactor) of relation (2.137) and, moreover, with the complete perturbative factor of our spectral function (2.111) taken at k^2 instead of q^2 – in LL approximation, that factor seems to sail unchanged through the folding operation of (2.131). It is clear from a look at (2.142), however, that such an approximation barely scratches the surface of the resummation problem: as mentioned already in chapter 0, the summation of subseries with finite convergence radius usually evades the difficulty of the $p!$ growth that resides here in the $q = 0$ (i. e. $\lambda = 0$) terms.

For later use we note a different, still *formal* representation of F^{pert} that follows upon using the property

$$\left(1 - \frac{1}{2^{p-1}} \right) \zeta(p) = \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{n^p} \quad (2.145)$$

of the zeta function [GRY94]. In the ensuing, double formal series

$$F_f^{pert}(1; \alpha) = - \sum_{n=1}^{\infty} (-)^n \left\{ \sum_{p=0}^{\infty} [(-)^p + 1] p! \left(\frac{\eta \alpha}{2n} \right)^p \right\}, \quad (2.146)$$

we would be tempted to identify the inner series as the asymptotic expansion of a combination of Ei functions according to (1.63):

$$F_f^{pert}(1; \alpha) = -\frac{1}{\eta \alpha} \sum_{n=1}^{\infty} (-)^n 2n \left[e^{-\frac{2n}{\eta \alpha}} Ei \left(\frac{2n}{\eta \alpha} \right) - e^{\frac{2n}{\eta \alpha}} Ei \left(-\frac{2n}{\eta \alpha} \right) \right] \quad (2.147)$$

but even this partial resummation would confront us with the necessity of supplying a prescription for approaching the spurious branch cut, which at real and positive values

of α would reside in the second *Ei* function of (2.147) at each n . (The function at $k^2/\mu^2 \neq 1$ would then again follow from the scaling relation (2.137), i. e. by making the substitution

$$\frac{1}{\alpha} \longrightarrow \frac{1}{\alpha} + \eta \lambda \quad (2.148)$$

and supplying the prefactor). Exploiting a representation of type (2.146) will therefore be much easier for the Borel transform.

Borel-Laplace analysis of the F^{pert} amplitude begins with the observation that the right-hand side of the scaling relation (2.137) is precisely of the form appearing in the 9th line of our Table 1 (page 30) on the left-hand side, with $\omega = -\eta \lambda$. Thus

$$b^{pert} \left(\frac{k^2}{\mu^2}; s \right) = e^{-(\eta \lambda) s} b^{pert} (1; s) \quad (2.149)$$

is the Borel image of the scaling relation (2.137). The "Borelized" perturbation expansion at $k^2/\mu^2 = 1$ has coefficients (2.140) without the $p!$ factor:

$$b_f^{pert} (1; s) = \sum_{p=0}^{\infty} \left\{ [(-)^p + 1] \left(1 - \frac{1}{2^{p-1}} \right) \zeta(p) \left(\frac{\eta}{2} \right)^p \right\} s^p. \quad (2.150)$$

Its circle of convergence is $|s| < 2/\eta$. Analytic continuation beyond the circle is, in this case, possible by summing the series in closed form. One may, for example, apply relation (2.139) in reverse after having divided it by $p!$:

$$\begin{aligned} b^{pert} (1; s) &= \frac{1}{2} \int_{-\infty}^{+\infty} \frac{dx}{\cosh^2 x} \left[\sum_{p=0}^{\infty} \frac{1}{p!} (\eta x s)^p \right] \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} dx \frac{e^{\eta s x}}{\cosh^2 x} = \frac{1}{2} \int_{-\infty}^{+\infty} dx \frac{\cosh(\eta s x)}{\cosh^2 x}, \end{aligned} \quad (2.151)$$

where we used the symmetry of the integrand. Use of the integral formula [GRY94]

$$\int_0^{\infty} dx \frac{\cosh(yx)}{\cosh^2 x} = B \left(1 + \frac{y}{2}, 1 - \frac{y}{2} \right) = \frac{\Gamma \left(1 + \frac{y}{2} \right) \Gamma \left(1 - \frac{y}{2} \right)}{\Gamma(2)} \quad (2.152)$$

and of a well-known property of the gamma function [MOS66],

$$\Gamma(1+z) \Gamma(1-z) = \frac{\pi z}{\sin(\pi z)}, \quad (2.153)$$

gives finally

$$b^{pert}(1; s) = \frac{\frac{\pi}{2} \eta s}{\sin\left(\frac{\pi}{2} \eta s\right)} \quad (2.154)$$

and, therefore,

$$b^{pert}\left(\frac{k^2}{\mu^2}; s\right) = \left(\frac{\mu^2}{k^2}\right)^{\frac{\eta}{2}s} \frac{\frac{\pi}{2} \eta s}{\sin\left(\frac{\pi}{2} \eta s\right)}. \quad (2.155)$$

The consequences of our having spuriously integrated over the perturbative pole in (2.131) are now manifestly on us: this Borel transform has poles on the real s axis at

$$s = \pm \frac{2}{\eta} n, \quad n = 1, 2, 3, \dots; \quad (2.156)$$

and while those at negative s do not obstruct unique Laplace reconstruction (1.45) of F^{pert} at physical values of α , those at positive s do. The perturbative expansion of F is not Borel summable. We already pointed out that this property has at best indirectly to do with the presence of t'Hooft's set of circular-arc singularities in Fig. 2.3, since in F^{pert} these are altogether absent; instead it arises from our having created, through the very definition of the perturbative expansion, a spurious branch cut in F^{pert} on the real positive α axis – that is, from our having *lost* the Λ^2 threshold that gave rise to the t'Hooft arcs in the first place. I am emphasizing this in order to bring more precision to the vague notion that the perturbative nonsummability in, for example, QCD amplitudes is somehow connected to the t'Hooft singularities.

It is useful for later purposes to make the poles of b^{pert} more explicit, either by looking up the Mittag-Leffler expansion of the inverse sine in (2.155), or by using the representation analogous to (2.146), with α replaced by s and $p!$ omitted, and with its inner series therefore summing to

$$\frac{1}{1 + \left(\frac{\eta s}{2}\right)} + \frac{1}{1 - \left(\frac{\eta s}{2}\right)}. \quad (2.157)$$

Either route leads to the representation

$$b^{pert}\left(\frac{k^2}{\mu^2}; s\right) = e^{-(\eta\lambda)s} \frac{2}{\eta} \sum_{n=1}^{\infty} (-)^n \left[\frac{-n}{s + \left(\frac{2}{\eta}\right)n} + \frac{+n}{s - \left(\frac{2}{\eta}\right)n} \right]. \quad (2.158)$$

It consists of a meromorphic part,

$$\left[b^{pert}\left(\frac{k^2}{\mu^2}; s\right) \right]_{poles} = \sum_{n=1}^{\infty} \left[\frac{-\left(\frac{2}{\eta}\right)n (-e^{2\lambda})^n}{s + \left(\frac{2}{\eta}\right)n} + \frac{+\left(\frac{2}{\eta}\right)n (-e^{-2\lambda})^n}{s - \left(\frac{2}{\eta}\right)n} \right], \quad (2.159)$$

and an entire remainder function. (Remember $e^{2\lambda} = k^2/\mu^2$). In trying to give a meaning to the Laplace integral

$$F^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right) = \frac{1}{\alpha} \int_0^\infty ds e^{-\frac{s}{\alpha}} b^{pert} \left(\frac{k^2}{\mu^2}; s \right) \quad (2.160)$$

we are therefore forced [ECA81] to specify a path of integration around *each* of the infinite sequence of positive- s poles in the second term of (2.159): at each of these we may deform the path of integration along the positive real s axis to run slightly above ($s \rightarrow s + i\varepsilon$) or slightly below ($s \rightarrow s - i\varepsilon$) the pole. We thus face an infinitely multivalued function of α , whose branches may be indexed as

$$F_{\{\sigma_1, \sigma_2, \sigma_3, \dots\}}^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right); \quad \sigma_n = +1 \text{ or } -1 \quad (\text{all } n) \quad (2.161)$$

and are computed by replacing the second term in (2.159) by

$$\sum_{n=1}^{\infty} \left(-\frac{\mu^2}{k^2} \right)^n \frac{\left(\frac{2}{\eta} \right)_n}{s + i\varepsilon \sigma_n - \left(\frac{2}{\eta} \right)_n}. \quad (2.162)$$

Among these, the two branches where all σ_n 's are +1 or -1 may be characterized by a single overall shift, $s \rightarrow s + i\varepsilon$ or $s \rightarrow s - i\varepsilon$, of the Borel variable, which by virtue of (1.47) may in turn be transferred to the variable α :

$$b((t \pm i\varepsilon)\alpha) \longrightarrow b(t\alpha \pm i\varepsilon') \longrightarrow b(t(\alpha \pm i\varepsilon'')). \quad (2.163)$$

So these two branches meet at the branch cut on the positive real α axis of what by definition is the *physical sheet with respect to α* of the function F^{pert} :

$$F^{pert} \left(\frac{k^2}{\mu^2}; \alpha + i0 \right) = F_{\{+, +, +, \dots\}}^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right), \quad (2.164)$$

$$F^{pert} \left(\frac{k^2}{\mu^2}; \alpha - i0 \right) = F_{\{-, -, -, \dots\}}^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right). \quad (2.165)$$

With this clarification of branch-point structure, both the perturbative amplitude F^{pert} in its two main branches (2.164) and (2.165) and its Borel transform may now be given in well-defined closed forms involving known transcendental functions. The function [MOS66]

$$G(z) = \psi \left(\frac{z+1}{2} \right) - \psi \left(\frac{z}{2} \right) = 2 \sum_{n=0}^{\infty} \frac{(-)^n}{n+z} \quad (2.166)$$

and its property

$$G(z+1) = -G(z) + \frac{2}{z} \quad (2.167)$$

may be used to express

$$b^{pert} \left(\frac{k^2}{\mu^2}; s \pm i0 \right) = e^{-(\eta\lambda)s} \frac{\eta s}{4} \left[G \left(\frac{\eta}{2} s \right) + G \left(1 - \frac{\eta}{2} s \mp i0 \right) \right], \quad (2.168)$$

where the first term in the square bracket contains the contribution from "harmless", negative- s poles, while the second term brings in the undesirable poles at positive s . Analogously, the integral representation (2.131), now given a precise meaning as a representation of the two main branches through the $\alpha \rightarrow \alpha \pm i0$ prescription, may be written (at $k^2 = \mu^2$)

$$F^{pert}(1; \alpha \pm i0) = \frac{1}{2} \int_0^\infty \frac{dx}{\cosh^2 x} \left[\frac{1}{1 + \alpha \eta x} + \frac{1}{1 - (\alpha \pm i0) \eta x} \right] \quad (2.169)$$

and compared to an integral representation of the function $\psi'(z) = d^2 \ln \Gamma(z) / dz^2$ [MOS66] along its branch line $Re z = \frac{1}{2}$,

$$\frac{i}{\pi y} \psi' \left(\frac{1}{2} \pm 0 + \frac{i}{\pi y} \right) = 1 + \frac{1}{2} \int_0^\infty \frac{dx}{\cosh^2 x} \left[\frac{1}{1 + yx} + \frac{1}{1 - (yx \pm i0)} \right], \quad (2.170)$$

to give

$$F^{pert}(1; \alpha \pm i0) = \frac{i}{\pi \eta \alpha} \psi' \left(\frac{1}{2} \pm 0 + \frac{i}{\pi \eta \alpha} \right) - 1. \quad (2.171)$$

These are evidently unphysical constructs, since an Euclidean amplitude at real momenta and physical couplings must be real and single-valued.

Enough of the perturbative portion – there is more to life! We turn to the nonperturbative remainder (2.132), at first treated formally since again we wish to derive only a formal resurgent-symbol expansion. In the variable

$$y = - \left(2x + \frac{1}{\alpha} \right) = - \left(\frac{1}{\alpha} + \ln \frac{q^2}{\mu^2} \right) \quad (2.172)$$

we have

$$F^{nonp} \left(\frac{k^2}{\mu^2}; \alpha \right) = -\frac{1}{4} \int_0^\infty dy \frac{1}{\cosh^2 \left(\frac{1}{2} \left(y + \frac{1}{\alpha} \right) + \lambda \right)} \frac{1}{\left(1 - \frac{\eta}{2} \right) - \frac{\eta}{2} \alpha y}. \quad (2.173)$$

We expand

$$\frac{1}{4 \cosh^2 \left(\frac{1}{2} y + \left(\frac{1}{2\alpha} + \lambda \right) \right)} = \sum_{n=1}^{\infty} (-)^{n-1} \left[e^{-\left(\frac{1}{\alpha} + 2\lambda \right)} \right]^n e^{-n y} \quad (2.174)$$

and obtain F^{nonp} in the form (2.70) of a resurgent symbol,

$$F^{nonp} \left(\frac{k^2}{\mu^2}; \alpha \right) = \sum_{n=1}^{\infty} \left(\frac{\mu^2}{k^2} \right)^n e^{-\frac{n}{\alpha}} f_{n,f}(\alpha), \quad (2.175)$$

with support at the points $s_n = n$, $n = 1, 2, 3, \dots$, and with the formal series resulting from the expansion of

$$f_n(\alpha) = (-)^n n \int_0^{\infty} dy \frac{e^{-n y}}{\left(1 - \frac{\eta}{2} \right) - \left(\frac{\eta}{2} \right) \alpha y} \quad (2.176)$$

as coefficients. By using $t = n y$ as an integration variable, we obtain

$$f_n(\alpha) = (-)^{n-1} f_1 \left(\frac{\alpha}{n} \right); \quad n = 1, 2, 3, \dots, \quad (2.177)$$

where

$$f_1(\alpha) = - \left(\frac{2}{2-\eta} \right) \int_0^{\infty} dt \frac{e^{-t}}{1 - \left(\frac{\eta}{2-\eta} \right) \alpha t} \quad (2.178)$$

has the formal expansion

$$f_{1,f}(\alpha) = - \left(\frac{2}{2-\eta} \right) \sum_{p=0}^{\infty} \left(\frac{\eta}{2-\eta} \right)^p p! \alpha^p. \quad (2.179)$$

The series $f_{n,f}$ therefore all exhibit the factorial-divergence pattern of eq. (0.5), with constants

$$c_n = (-)^n \frac{2}{2-\eta}, \quad B_0 = 0, \quad A_n = n \left(\frac{2-\eta}{\eta} \right), \quad (2.180)$$

and, therefore, also a pattern similar to the perturbative expansion, whose coefficients (at $k^2 = \mu^2$) were given by (2.140). For higher n 's they all follow from the series $f_{1,f}$ by simple dilatations, $\alpha \rightarrow \frac{\alpha}{n}$. This is a special, simple form of the phenomenon of resurgence described in sect. 2.3. With a look back at eq. (1.59), f_n may alternatively be expressed as

$$f_n(\alpha) = (-)^n \frac{2}{\eta} \frac{n}{\alpha} e^{-\frac{(2-\eta)n}{\eta\alpha}} Ei \left(\frac{(2-\eta)n}{\eta\alpha} \right). \quad (2.181)$$

It is one of the oversimplified and unrealistic features of our model that at $k^2 \neq \mu^2$, the $f_{n,f}$ series for $n \geq 1$ do not develop the perturbative logarithms, $2\lambda = \ln(k^2/\mu^2)$, that were present in eq. (2.142) for the perturbative series

$$f_{0,f}(\alpha) := F_f^{pert} \left(\frac{k^2}{\mu^2}; \alpha \right) . \quad (2.182)$$

This is essentially a consequence of the simple factorizing structure of eq. (2.111). The only k^2 dependence in (2.175) is therefore the purely nonperturbative one of "power corrections" through the appearance of n -th powers of the dimensionless parameter,

$$u = e^{-2\lambda} = \left(\frac{\mu^2}{k^2} \right) , \quad (2.183)$$

just as in the generic form of eq. (2.92). In view of the lambda-scale definition (2.112), this gives to the complete resurgent-symbol representation of F ,

$$F_f \left(\frac{k^2}{\mu^2}; \alpha \right) = \sum_{n=0}^{\infty} \left(\frac{\Lambda^2}{k^2} \right)^n f_{n,f}(\lambda; \alpha) , \quad (2.184)$$

the form of an operator-product expansion as in (2.54).

It is easy to find the "Borelized" f_n series,

$$b_{n,f}(s) = (-)^n \left(\frac{2}{2-\eta} \right) \sum_{p=0}^{\infty} \left(\frac{\eta}{2-\eta} \right)^p \left(\frac{s}{n} \right)^p \quad (n \geq 1) , \quad (2.185)$$

and their analytic continuations through resummation,

$$b_n(s) = (-)^{n-1} n \left(\frac{2}{\eta} \right) \frac{1}{s + n - \left(\frac{2}{\eta} \right) n} . \quad (2.186)$$

Each has a pole on the positive real s axis, but the interesting observation is where these poles are when we form the combined Borel transform (1.19) of the nonperturbative part (2.175) by using the last line of Table 1:

$$b^{nonp} \left(\frac{k^2}{\mu^2}; s \right) = - \sum_{n=1}^{\infty} \Theta(s-n) \frac{\left(\frac{2}{\eta} \right) n (-e^{-2\lambda})^n}{s - \left(\frac{2}{\eta} \right) n} . \quad (2.187)$$

The poles, because of (2.113), are at points where the step functions are unity, so in order to isolate them we write

$$\Theta(s-n) = 1 - \Theta(n-s) \quad (2.188)$$

and have

$$b^{nonp} \left(\frac{k^2}{\mu^2}; s \right) = \sum_{n=1}^{\infty} \Theta(n-s) \frac{n (-e^{-2\lambda})^n}{\left(\frac{\eta}{2}\right) s - n} - \sum_{n=1}^{\infty} \frac{n (-e^{-2\lambda})^n}{\left(\frac{\eta}{2}\right) s - n}. \quad (2.189)$$

What we suspected all along is now manifest: the second term, which sums up the poles on the positive Borel axis, *exactly* cancels the corresponding second term in the meromorphic part (2.159) of b^{pert} . The full Borel transform

$$b \left(\frac{k^2}{\mu^2}; s \right) = b^{pert} + b^{nonp} \quad (2.190)$$

is only piecewise smooth on the positive s axis due to the first term in (2.189), but otherwise perfectly well-behaved in the Laplace integral, which therefore yields a well-defined function having the resurgent symbol (2.184) as its formal small- α expansion.

It only remains for us to check whether the Laplace reconstruction indeed gives back the original function (2.129). We use (2.168) and (2.189) to write

$$F \left(\frac{k^2}{\mu^2}; \alpha \right) = \frac{1}{\alpha} \int_0^{\infty} ds e^{-\frac{s}{\alpha}} \left\{ e^{-\eta \lambda s} \left[\frac{\eta s}{4} G \left(\frac{\eta s}{2} \right) - \frac{1}{2} \right] + R \left(\frac{\eta s}{2}, \lambda \right) \right\}, \quad (2.191)$$

where R collects the pieces with smoothed-out, positive- s poles:

$$R \left(\frac{\eta}{2} s, \lambda \right) = \sum_{n=1}^{\infty} n (-e^{-2\lambda})^n \frac{[e^{-\lambda(\eta s - 2n)} - 1] + \Theta(n-s)}{\frac{\eta}{2} s - n}. \quad (2.192)$$

For the first term in (2.191), use of relation (2.167) and of the integral representation [GRY94]

$$G \left(\frac{z+1}{2} \right) = 2 \int_0^{\infty} dx \frac{e^{-zx}}{\cosh x} \quad (2.193)$$

gives, after a few lines of calculation, the contribution

$$\frac{1}{2} \int_0^{\infty} \frac{dx}{\cosh^2 x} \frac{1}{1 + \eta \alpha (x + \lambda)} \quad (2.194)$$

to the Laplace integral. Note the lower limit of *zero*, rather than of minus infinity as in F^{pert} . In the second, R part of (2.191), the only minor trick you may need is to observe that in the term coming from the step functions, which may be written as

$$\frac{1}{\alpha} \int_0^1 dy \frac{1}{1 - \left(\frac{\eta}{2}\right) y} \left[\sum_{n=1}^{\infty} (-)^{n-1} n e^{-n \left(\frac{y}{\alpha} + 2\lambda\right)} \right] \quad (2.195)$$

with integration variables $y = s/n$, the convergent series in square brackets can be resummed in the manner of eq. (2.174) above. The rest is straightforward and the contribution from the R term becomes

$$\frac{1}{2} \int_{-\frac{1}{2\alpha} - \lambda}^0 \frac{dx}{\cosh^2 x} \frac{1}{1 + \eta \alpha (x + \lambda)}. \quad (2.196)$$

The two contributions together indeed reproduce the function (2.129). The Borel-Laplace process at physical values of the coupling α has worked perfectly, unobstructed by either the t'Hooft set of singularities, or by the spurious poles on the positive Borel axis introduced by the artificial perturbative-plus-nonperturbative decomposition. It has worked because we based it on the full resurgent symbol as the adequate asymptotic representation of a resurgent amplitude, rather than on the inadequate perturbation series.

The spurious right-hand poles in the *perturbative* Borel transform (second term of eq. (2.159)) are of course, as some of you already noted, a not-so-unrealistic model of the so-called *infrared renormalons*, just as those on the left-hand Borel axis (first term of eq. (2.159)), which are not spurious at all but remain a true and important feature of the full resurgent amplitude, represent the so-called *ultraviolet renormalons* [GNV] [WEI96] [ZNJ89]. (The completely symmetric manner in which both arise here from eq. (2.154) is presumably among the oversimplified features of our model.) The renormalons – another specialty on which we will be unable to dwell in any detail [BEN99] – generally arise through one of the perturbative partial resummations with finite

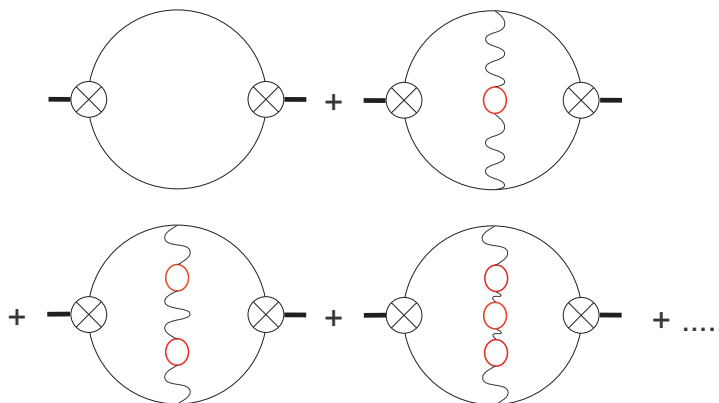


Fig. 2.4 Example of a family of diagrams whose summation in perturbation theory gives rise to renormalons. The diagrams represent loop contributions to the two-point function of an elementary or composite operator denoted by a cross.

convergence radius mentioned in the introduction: in this case, the geometric-series summation of chains of two-point or four-point bubble graphs inserted into a larger loop for some two-point correlation (of elementary or composite fields), as illustrated in Fig. 2.4. Although in such a diagram family there is only one chain diagram per loop order p – namely, the one with $p - 2$ bubbles –, their insertion into the larger loop generates $p!$ growth by the mathematics expressed in formula (2.139), so that here one has a mechanism producing factorial divergence independent of the "standard" mechanism of growth-in-number-of-diagrams. That these infrared renormalons must get cancelled, on the level of the combined Borel minor (2.95), by opposite contributions from what we would now call the higher elementary resurgent symbols, seems to have been inferred first, in the context of an OPE analysis of the nonlinear sigma model, by David [DAV82], who also conjectured the cancellation would persist in realistic theories. What the present model study can add to his (exact but somewhat indirect) argument is to show, with the greatest simplicity and explicitness, how the problem arises directly from the definition of perturbation theory, and how it disappears when the resurgent coupling dependence of an amplitude is viewed and treated as a whole.

What, then, are the morals to be drawn from this model? It seems to me there are at least two. First, there is absolutely no reason to be afraid of t'Hooft's α -plane singularities: in the perturbative portion, they are missing anyway; in the full resurgent function, they do not impede BL reconstruction. In my opinion they are actually a sign of health – they indicate that the nonperturbative thresholds proportional to Λ_0 have been left intact, and have not been perturbatively mutilated.

Second, as for the recoverability of the amplitude from its resurgent symbol, it is clearly imperative to keep together the perturbative and strongly nonperturbative portions as much as possible, since only the sum of both leads (at positive α) to an univalent function. *As much as possible?* Since we were dealing here with a simple model amplitude having an explicit integral representation, we were able to "sum the whole thing", i. e. to perform Laplace reconstruction from the combined Borel minor in the form of eqs. (2.194) and (2.196) – but that route is clearly not viable in the real world where all we may know about an amplitude is that it should be (part of) a coupling-resurgent solution to some complicated Dyson-Schwinger hierarchy. But here a simple idea almost

suggests itself: if we cannot hope to sum everything in one blow, can't we at least establish what one might call a *quasi-perturbative series*, i. e. an expansion in which the n summation in the "nonperturbative direction" has been performed, either completely or in some systematic approximation, at each fixed p ? In the physically relevant case of the real-and-equidistant support, eq. (2.92), we would then have something like

$$f(\alpha) \sim \sum_{p=0}^{\infty} c_p (\Lambda_0^2) \alpha^p, \quad (2.197)$$

a semiconvergent series in α in which the coefficients c_p carry the strongly nonperturbative α dependence through $\Lambda_0 \propto \exp(-s_1/\alpha)$. Since each c_p keeps the perturbative ($n = 0$) and nonperturbative ($n \geq 1$) terms together, and since we know the full function is ultimately univalued at real positive α , we may be bold and conjecture that *this series should be Borel summable as a formal series in α* , i. e. if we treat the Λ_0^2 dependence in the c_p as a dependence on an extra parameter unrelated to α . It will be interesting, and yield a little harvest of extra insight into the resurgent structure, to check this out for our model amplitude.

We write the complete resurgent symbol F_f of eq. (2.184) in the form

$$F_f = \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} a_{n,p} \left(\frac{\Lambda^2}{k^2} \right)^n \alpha^p, \quad (2.198)$$

where the coefficients $a_{n,p}$ can be assembled from eqs. (2.142) and (2.179):

$$a_{n,p} = \left\{ \delta_{n,0} (-)^p \left(\frac{\eta}{2} \right)^p \left[\sum_{q=0}^p f_q^{(p)} (2\lambda)^q \right] + (1 - \delta_{n,0}) (-)^n \frac{\left(\frac{\eta}{2} \right)^p}{\left(1 - \frac{\eta}{2} \right)^{p+1}} \frac{1}{n^p} \right\} p! \quad (2.199)$$

Performing the n sum at fixed p then gives the form of eq. (2.197), with coefficients

$$c_p (\Lambda^2) = a_{0,p} + \sum_{n=1}^{\infty} a_{n,p} \left(\frac{\Lambda^2}{k^2} \right)^n. \quad (2.200)$$

Here, the sum of the $n \geq 1$ terms,

$$\frac{\left(\frac{\eta}{2} \right)^p}{\left(1 - \frac{\eta}{2} \right)^{p+1}} p! \sum_{n=1}^{\infty} \frac{1}{n^p} \left(-\frac{\Lambda^2}{k^2} \right)^n, \quad (2.201)$$

converges in the domain

$$\left| \frac{\Lambda^2}{k^2} \right| < 1, \quad (2.202)$$

which illustrates our general expectation that the n summation should not be semiconvergent but have a finite radius of convergence. Within that radius, it can be performed in terms of a known class of transcendental functions, the polylogarithms,

$$Li_p(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^p} \quad (|z| < 1), \quad (2.203)$$

whose $p = 2$ member you may have met in calculations of two-loop (or three-point one-loop) Feynman graphs; the $p = 0$ and $p = 1$ members are actually elementary functions:

$$Li_0(z) = \frac{z}{1-z}; \quad Li_1(z) = -\ln(1-z). \quad (2.204)$$

The analytic continuation beyond (2.202) is provided by the integral representation

$$Li_p(z) = \frac{1}{p!} \int_0^{\infty} dx x^p \frac{z e^{-x}}{(1 - z e^{-x})^2}, \quad (2.205)$$

which again encompasses the two elementary cases (2.204). (We have rewritten the usual integral definition of Li_p found in tables [DEV84] through a change of variable and a couple of partial integrations.) Thus,

$$c_p(\Lambda^2) = p! \left(\frac{\eta}{2}\right)^p \left\{ (-)^p \sum_{q=0}^p f_q^{(p)} \left[\ln \left(\frac{k^2}{\mu^2} \right) \right]^q + \frac{1}{(1 - \frac{\eta}{2})^{p+1}} Li_p \left(-\frac{\Lambda^2}{k^2} \right) \right\}. \quad (2.206)$$

Now for Borel summability we do not seem to have made much progress – in particular, the telltale alternating signs that would indicate banishment of Borel singularities to the left-hand, real Borel axis are not visible. But wait! Up to now we failed to express both parts of eq. (2.206) in the same variable. Maybe in order to really fuse them into one quantity, what we should do is to reexpress the perturbative portion of F_f ,

$$F_f^{(pert)} = \sum_{p=0}^{\infty} a_{0,p} \left(\ln \frac{k^2}{\mu^2} \right) \cdot \alpha^p, \quad (2.207)$$

in terms of the variable k^2 / Λ^2 , rather than k^2 / μ^2 , by using

$$\ln \left(\frac{k^2}{\mu^2} \right) = \ln \left(\frac{k^2}{\Lambda^2} \right) - \frac{1}{\alpha} \quad (2.208)$$

(remember eq. (2.112)) and reshuffling the perturbative α series. This requires a page of algebra but leads to a simple result, expressible as another RG-type scaling relation:

$$F_f^{(pert)} \left(\frac{k^2}{\mu^2}; \alpha \right) = \frac{1}{1 - \frac{\eta}{2}} F_f^{(pert)} \left(\frac{k^2}{\Lambda^2}; \frac{\alpha}{1 - \frac{\eta}{2}} \right). \quad (2.209)$$

(Note that the μ dependence hidden in $\alpha = \alpha(\mu)$ is still present on the r. h. s.; we merely changed the scaling of the momentum variable k^2 .) In other words we now have for $F^{(pert)}$ a rearranged expansion

$$F_f^{(pert)} = \sum_{p=0}^{\infty} \bar{a}_{0,p} \left(\ln \frac{k^2}{\Lambda^2} \right) \alpha^p, \quad (2.210)$$

$$\bar{a}_{0,p} = (-)^p p! \frac{\left(\frac{\eta}{2}\right)^p}{\left(1 - \frac{\eta}{2}\right)^{p+1}} \sum_{r=0}^p f_r^{(p)} \left[\ln \left(\frac{k^2}{\Lambda^2} \right) \right]^r, \quad (2.211)$$

which is no more the perturbation expansion in the strict sense – the Taylor series of F at $\alpha = 0+$ – but has the formal advantage of being expressed in the same variables as the nonperturbative terms. (Let me once more emphasize it is an oversimplified feature of our model that this reshuffling of logarithms should be necessary only in the perturbative part. More realistic amplitudes always have dependence on $\ln(k^2/\mu^2)$ also in the modified Wilson coefficients $U_n^{(R)}$, $n \geq 1$, of eq. (2.54), and will need regrouping of logarithms in these quantities as well). It is then plausible that we should redefine our quasi-perturbative expansion to mean

$$F = \sum_{p=0}^{\infty} \bar{c}_p \left(\frac{\Lambda^2}{k^2} \right) \alpha^p = \bar{F} \left(\frac{k^2}{\Lambda^2}; \alpha \right), \quad (2.212)$$

with coefficients now given by

$$\bar{c}_p = \bar{a}_{0,p} + \sum_{n=1}^{\infty} a_{n,p} \left(\frac{\Lambda^2}{k^2} \right)^n, \quad (2.213)$$

rather than by eq. (2.200). Since the structure (2.143) of the $f_r^{(p)}$ coefficients enforces $(-)^r = (-)^p$, we have

$$\bar{c}_p \left(\frac{\Lambda^2}{k^2} \right) = \frac{(\eta/2)^p}{[1 - (\eta/2)]^{p+1}} p! \left\{ \sum_{r=0}^p f_r^{(p)} \left[\ln \left(\frac{\Lambda^2}{k^2} \right) \right]^r + Li_p \left(-\frac{\Lambda^2}{k^2} \right) \right\}. \quad (2.214)$$

This redefinition now pays off in a surprising way: what we have in the curly bracket is the r. h. s. of a known "duality relation" for the polylogarithms [DEV84], a relation

connecting Li_p 's at large and small arguments:

$$(-)^{p+1} Li_p(z) = \sum_{r=0}^p f_r^{(p)} \left[\ln \left(-\frac{1}{z} \right) \right]^r + Li_p \left(\frac{1}{z} \right). \quad (2.215)$$

Under this new definition we can therefore indeed merge the perturbative and nonperturbative parts at a fixed p into a new quasi-perturbative object given, up to factors, by a single polylogarithm:

$$\bar{c}_p \left(\frac{\Lambda^2}{k^2} \right) = (-)^p p! \frac{(\eta/2)^p}{[1 - (\eta/2)]^{p+1}} \left[-Li_p \left(-\frac{k^2}{\Lambda^2} \right) \right]. \quad (2.216)$$

From eq. (2.205), it is clear that $-Li_p(-z)$ at real, positive z is positive, so the alternating signs $(-)^p$ we were looking for are now present. That the series (2.212) with coefficients (2.216), and with Λ^2/k^2 treated as a parameter unrelated to α , is indeed Borel summable with respect to α can be seen by yet another rewriting of the integral representation (2.129), using as integration variable

$$t = 2x + \frac{1}{\alpha} = -y, \quad (2.217)$$

which is the negative of the one employed in (2.173) for the nonperturbative part alone. We get

$$F \left(\frac{k^2}{\mu^2}; \alpha \right) = \bar{F} \left(\frac{k^2}{\Lambda^2}; \alpha \right) = \frac{1}{2(2-\eta)} \int_0^\infty dt \frac{1}{\cosh^2 \left(\frac{t-\kappa}{2} \right) \left[1 + \left(\frac{\eta}{2-\eta} \alpha \right) t \right]}, \quad (2.218)$$

where

$$\kappa = 2\lambda + \frac{1}{\alpha} = \ln \left(\frac{k^2}{\Lambda^2} \right). \quad (2.219)$$

Due to the strong convergence provided by the \cosh^{-2} factor, the function $\bar{F}(e^\kappa; \alpha)$ at fixed κ is obviously analytic with respect to α in $Re \alpha > 0$. Note that the 't Hooft singularities do not now disturb this conclusion; they would arise from the "hidden" α dependence in $e^\kappa = k^2/\Lambda^2$ which for the present purpose we have eliminated by treating κ as a fixed parameter. Moreover, one easily evaluates

$$\frac{\partial^p \bar{F}}{\partial \alpha^p} = \frac{(-\eta)^p p!}{2(2-\eta)^{p+1}} \int_0^\infty dt \frac{t^p}{\cosh^2 \left(\frac{t-\kappa}{2} \right) \left[1 + \left(\frac{\eta}{2-\eta} \alpha \right) t \right]^{p+1}}, \quad (2.220)$$

and by estimating

$$\left| \left[1 + \left(\frac{\eta}{2-\eta} \alpha \right) t \right]^{-(p+1)} \right| \leq 1 \quad (Re \alpha > 0), \quad (2.221)$$

as well as $\cosh^{-2} \left(\frac{t-\kappa}{2} \right) \leq 4e^{\kappa-t}$, one establishes a bound of type (1.14), similar to what we had for superrenormalizable amplitudes in (1.27):

$$\left| \frac{\partial^p \overline{F}}{\partial \alpha^p} \right| \leq \frac{2}{2-\eta} \left(\frac{k^2}{\Lambda^2} \right) \left(\frac{2-\eta}{\eta} \right)^{-p} (p!)^2. \quad (2.222)$$

Therefore we have Nevanlinna-Sokal conditions satisfied in the half-plane $Re \alpha > 0$, and \overline{F} is Borel summable with respect to α . (As a purely mathematical aside, we have shown the function \overline{F} of eq. (2.218) at $\eta = 1$ to be a *generating function* for the functions $2(-)^{p+1} p! Li_p(-e^\kappa)$, i. e. to generate them as coefficients when expanded with respect to α .)

The quasi-perturbative expansion (2.212), in this respect, is better behaved than the perturbative one, but we should not lose sight of the fact that there is now no easy way back from the former to the latter. The beautiful duality formula (2.215) became applicable only after we replaced the perturbation expansion in the strict sense by the rearranged series (2.210), which is no more the Taylor expansion at $\alpha = 0+$ of F . The rearrangement is infinite, in the sense that one quasi-perturbative term collects contributions from infinitely many perturbative terms and vice versa. As a result, in writing (2.212) we have "lost the perturbative limit": the formal limit of $\Lambda \rightarrow 0$ with α^p 's remaining finite, which previously could be obtained by simply dropping the last term in each of the coefficients c_p of (2.206), cannot now be recovered any more from one or a finite number of quasi-perturbative terms. In the *zeroth quasi-perturbative order*, the analog of the OPE object (2.56) which sums up the terms without powers of α but with all powers of Λ^2 , we have

$$\overline{F} \left(\frac{k^2}{\Lambda^2}; 0 \right) = \overline{c}_0 \left(\frac{k^2}{\Lambda^2} \right) = \frac{1}{1 - \left(\frac{\eta}{2} \right)} \frac{k^2}{k^2 + \Lambda^2}, \quad (2.223)$$

and although the limit $\Lambda \rightarrow 0$ here exists, it differs from the perturbative $F^{(0)pert} = 1$ by the factor $\left(1 - \frac{\eta}{2} \right)^{-1}$, which sums a geometric series coming from *all* perturbative terms. In the higher quasi-perturbative orders, the $\Lambda^2 \rightarrow 0$ limit does not even exist termwise; we could retrieve it only when undoing the rearrangement of eq. (2.210) by applying relation (2.208) backwards – i. e. by acknowledging that Λ^2 and α are, in fact, *not* unrelated. *In a calculation where we can determine only a finite number of quasi-perturbative terms, we therefore have no easy way of making contact with the perturbative series*, and of checking whether it comes out as a limiting case. One might take the view, why bother at all about a perturbative limit in this sense? After all, the very point of an RG-invariant scale is that it cannot be switched off in any way, and since the interesting regime we would like to explore for nonperturbative phenomena, in both

Quantum Chromodynamics and electroweak theory, is the regime $k^2 \lesssim \Lambda^2$, we could dismiss the $\Lambda \rightarrow 0$ limit as uninteresting. Still, the difficulty of staying in touch with the perturbative expansion remains a nuisance in some respects. Take, for example, the RG results on large-momentum behavior: they stipulate that the *perturbative* zeroth-order vertex functions should be approached, with logarithmic corrections obtainable through *perturbative* leading-log summations, and since these are exact limiting results one would like to preserve, one feels unhappy about losing the easy way back to their verification.

The loss is offset, to be sure, by considerable gain in the description of the small- k^2 regime. By eq. (2.203), we now have the correct behavior $F \rightarrow 0$ as $k^2 \rightarrow 0$ in each quasi-perturbative term. What is more, due to the analytic properties of the Li_p functions – for $p \geq 1$, they all have branch cuts at real argument larger than unity –, each term now also reflects the correct "spectral" singularities of F at Minkowskian values $k^2 \leq -\Lambda^2$, including the pole at $k^2 = -\Lambda^2$ carried by the $p = 0$ term (2.223) and the physical cut at $k^2 < -\Lambda^2$ in the $p \geq 1$ quasi-perturbative corrections. Thanks to the tight linkage between the Λ^2 and k^2 dependences, the quasi-perturbative resummation of $(\Lambda^2)^n$ powers in the resurgent symbol has automatically furnished a very satisfactory *analytic continuation in the k^2 plane*, which resummations of perturbation theory alone, even disregarding its false infrared singularities, could not provide. It would then seem that the quasi-perturbative series deserves a try in the "real" problem, and chap. 3 will sketch some steps in that direction.

3 Construction of Resurgent Amplitudes

3.1 The Approximating Sequence

We have seen that the operator-product expansion, in a strictly renormalizable and asymptotically free QFT, is of the form of a resurgent symbol in the coupling constant α . As such, it asserts that amplitudes in those QFT's are resurgent functions of α . We have also seen that there are good reasons for assuming that this symbol is summable, that is, the small- α expansion of a unique analytic function of α . In physics terms, there are good reasons for expecting the OPE to be complete, i. e. to leave no room for further terms that would escape the resurgent-symbol representation by having stronger-than-exponential suppression as $\alpha \rightarrow 0+$.

The implications of these two insights are quite strong. If we find methods of constructing systematically, or at least in a systematic approximation, the unique analytic coupling functions waiting behind the resurgent symbols, we will have accounted not just for one special type (as an easy misunderstanding might suggest) but for *the totality of the strongly nonanalytic coupling dependence*. It therefore seems worthwhile to look for and try out such methods. The method described in this, our last chapter, builds on the plausible hypothesis of summability of the resurgent symbols, and attempts to systematically reconstruct the strongly nonperturbative coupling dependence of correlation functions – the dependence manifesting itself in the resurgent symbol through powers of the RG-invariant scale Λ – from their equations of motion.

In developing such a method we will find ourselves compelled to deviate somewhat from standard strategy. Standard procedure relies on the function being a solution to some differential or differential-cum-functional equation. Its first step is then to obtain the function's resurgent-symbol expansion by purely formal manipulation, inserting the symbol into the governing equation and comparing terms. (If the equation is nonlinear, it becomes important at this point that the set of elementary resurgent symbols form an algebra under multiplication.) In mathematical theory, completion of this step is usually taken for granted; it belongs, so to speak, to a realm of pre-mathematical heuristics. The procedure then continues in the manner we illustrated in sect. 2.4 with our model amplitude: one "borelizes" the individual power series, separately for each n , and performs analytic continuation to obtain the Borel minors $b_n(s)$. If these are sufficiently nonsingular (integrable singularities, as pointed out by Sokal [SOK80], can be tolerated) and restricted in growth on the positive Borel axis, one may proceed to evaluating their Laplace transforms that appear in eq. (2.80). Finally, one confronts the

n series of that equation: if it has nonzero convergence radius, it may require another analytic-continuation step; if it is summably divergent, it will call for another BL resummation. If all goes well, one finally reaches the holy grail, a globally valid representation of the full resurgent function.

For physicists dealing with QFT, while this standard procedure should be in the back of their minds, there are cogent reasons for proceeding in different order. As we noted in studying our model, it is a consequence of the very definition of perturbation theory that individual b_n 's are in general afflicted with spurious Borel nonsummability, and thus force us to cope with spurious multivaluedness at all intermediate stages of the reconstruction, while we are convinced that the full function must ultimately be univalued at physical, positive couplings. We saw that the easiest way to avoid this is to keep perturbative and nonperturbative contributions tightly together by using a quasi-perturbative expansion, that is, by first performing the n summation in the symbol's nonperturbative direction at fixed p . You might choose to call this a "resummation of the OPE in the n direction", but you know by now why I am reluctant to use that term – it sounds as if the OPE, for realistic amplitudes, were something fully calculable *ab initio* that one could simply take and subject to some process of continuation. In truth, as we have emphasized, the OPE is not known *ab initio*, since it does not by itself determine its own strongly nonperturbative ingredients, the vacuum condensates. We will therefore be forced to recover our strongly nonperturbative contributions from exact equations of motion, which supply the dynamical principle the OPE is lacking. In the process, the vacuum condensates will lose some of their central role in describing the strongly nonperturbative effects, that role being taken over by *dimensionless* nonperturbative parameters more easily accessible through the equations of motion (and from which one may later recover the condensates if one so desires).

Of course, with realistic amplitudes there is again no way of performing the entire n sum at fixed p in one blow, as we did for our model in writing eqs. (2.206) or (2.214). But we may attempt to do it in terms of a suitable systematic sequence of approximations, amounting to resummations of the $(\Lambda^2)^n$ power series that grow progressively more comprehensive with increasing approximation order. These resummations, as again we learned in our model, will automatically provide a sequence of analytic continuations, increasingly accurate, to the momentum region of $Q^2 \lesssim \Lambda^2$.

As we stressed already at the end of sect. 1.2, it is not compulsory for us to use "Borelization" in doing this – all the more as we trust the n summation at fixed p to already have nonzero radius of convergence. Indeed we shall find other approximating sequences better adapted to the task. Enumerating successive approximants (we'll

discuss their detailed form in a moment) by an index $r = 0, 1, 2, \dots$, we denote

$$\Gamma^{[r,p]}(\Lambda^2) = r\text{-th approximant to } \sum_{n=0}^{\infty} c_{n,p} \left(\frac{\Lambda^2}{Q^2}\right)^n \quad (3.1)$$

at a fixed quasi-perturbative order p .

There is, to be sure, a subtle drawback to this necessity of employing approximants to the n sum. At finite level r , we can no more expect a quasi-perturbative expansion based on such approximants to achieve the complete cancellation of spurious poles in the combined Borel minor that we were entitled to expect from the full n summation. This is not much of a problem as long as we calculate amplitudes at an energy scale μ where the coupling $\alpha(\mu)$ is small enough to ensure good semi-convergence for the sequence of partial p sums,

$$\Gamma^{[r,p]} = \sum_{q=0}^p \Gamma^{[r,q]}(\Lambda^2) \alpha(\mu)^q \quad (r, p = 0, 1, 2, \dots), \quad (3.2)$$

as we do in most of present-day electroweak theory: then these sums may be taken directly as approximants to the amplitude, without ever confronting the Borel plane. But when $\alpha(\mu)$ is not small, as in the hadronic region of QCD, we will have to rely on a BL process for the p summation, and at low approximation levels r we will then be able to avoid only a small number, increasing with r , of the "nearest", i. e. next-to-the-origin, spurious Borel poles. Since nearest poles in the s plane dominate large- α behavior in the α -plane, and since in an asymptotically free theory the large- α regime is also the low-momentum regime, we may qualitatively expect our approximation to have its largest errors at low momenta and, on increasing r , to gradually work its way down from the large-momentum regime. Nevertheless, the crude analytic continuations it provides at low r to the low-momentum region will have finite error and be usable even down to $Q^2 = 0$, where the error of the original resurgent symbol, an expansion in terms of Λ^2/Q^2 , would become infinite.

If we then start our reconstruction scheme by inserting such a continuation-through-resummation, rather than the original OPE-type expansion, into the equations of motion, we reap several benefits. First, it turns out it is really only for $p = 0$ that we need to specify an explicit sequence of approximating functions: the *nonperturbatively modified, zeroth quasi-perturbative orders* $\Gamma^{[r,0]}$, whose exact counterparts we met in the form of eq. (2.56) for the OPE, and in the form of eq. (2.223) for our model amplitude. The higher quasi-perturbative corrections, $\Gamma^{[r,p]}$ with $p \geq 1$, will then be generated for us by the equations of motion themselves. Now choosing approximants for $\Gamma^{[r,0]}$ is a lot

easier than for general p , since they do not need to be saddled with typical complications that only the higher- p corrections are supposed to bring – in particular, with logarithmic corrections to large-momentum behavior, and with the associated dependence on μ (compare eq. (2.53) in our discussion of the OPE). Drawing on the body of experience we alluded to in connection with formulas (1.28) and (1.53), we therefore again choose *rational* approximants, but this time with respect to Λ and not to α – i.e. expressions of the form

$$\frac{\sum_{m=0}^{l'} f_m(\{k\}) (\Lambda^2)^m}{\sum_{n=0}^l g_n(\{k\}) (\Lambda^2)^n} \quad (3.3)$$

(for bosonic amplitudes). We seem to need *two* indices, the polynomial degrees l' and l , to label such an approximant, but here we reap our next benefit, since asymptotic freedom greatly simplifies the matter. Recall that the RG leads to exact asymptotic formulas for vertex functions when the entire set $\{k\}$ of momentum arguments are scaled up uniformly, and that these formulas simply feature the zeroth-order perturbative expression, $\Gamma^{(0) \text{pert}}$, multiplied by certain logarithmic corrections. But the logarithmic factors arise from (resummation of) perturbative corrections with $p \geq 1$, and in our context are therefore for the higher $\Gamma^{[r,p]}$'s to bring; they have no business with $\Gamma^{[r,0]}$. So of the latter we must demand *naive* asymptotic freedom:

$$\Gamma_N^{[r,0]}(\{k\}, \Lambda) \longrightarrow \Gamma_N^{(0) \text{pert}}(\{k\}) \quad (\text{all } k \gg \Lambda). \quad (3.4)$$

In exploiting this condition, it is best to proceed in steps and deal first with the simplest case of a two-point vertex function ($N = 2$), or negative-inverse propagator, depending on a single scalar Euclidean k^2 ; vertices with $N \geq 3$ that require more momentum variables will be taken up later. Since Λ^2 is dimensionful, and we assume no Lagrangian masses present, the approximant (3.3) is then automatically rational with respect to k^2 too, with

$$f_m = \zeta_m (k^2)^{l' - m}, \quad g_n = \eta_n (k^2)^{l - n}, \quad (3.5)$$

and with real constants ζ_m and η_n since the complete approximant should be real at real Euclidean k^2 . (At this point, things would seem to become substantially more complicated when Lagrangian masses \hat{m} are present, since the ζ_m and η_n could then still depend on the dimensionless ratios \hat{m}^2 / k^2 – but let us continue to concentrate on the massless case.) Now the $\Gamma^{(0) \text{pert}}$ in condition (3.4), which are none other than the ordinary Feynman rules, are polynomial in k , and in particular for $N = 2$ and a massless scalar field,

$$-\Gamma_2^{(0) \text{pert}}(\{k\}) = k^2. \quad (3.6)$$

Thus condition (3.4) fixes the *relative* degrees of our approximant in advance at $l' = l + 1$. We do not face the prospect, as one usually does in working with rational approximants, of having to work through the entire Padé table at given total degree $l' + l$; we need to explore only the next-to-diagonal line $l' - l = 1$ of it. So in fact we need only one labeling index, and we will from now on adopt the *convention of using the denominator degree l as the classifying index r for our approximants to irreducible vertex functions*:

$$r = l = l' - 1 \quad (N = 2) \quad (3.7)$$

(For *propagators*, or connected two-point functions,

$$D \equiv G_2^{(conn)} = -(\Gamma_2)^{-1}, \quad (3.8)$$

and the index r is then also the *numerator* degree of rational propagator approximants.) Since without any loss of generality we may choose

$$\eta_0 = 1, \quad g_0(k) = (k^2)^r, \quad (3.9)$$

the level- r approximant is now characterized by a set of $2r + 2$ dimensionless, real parameters

$$\zeta_0^{[r]}, \zeta_1^{[r]}, \dots, \zeta_{r+1}^{[r]}; \quad \eta_1^{[r]}, \eta_2^{[r]}, \dots, \eta_r^{[r]} \quad (3.10)$$

and assumes the explicit form

$$-\Gamma_2^{[r,0]}(k^2) = \frac{N_2^{[r]}(k^2, \Lambda^2)}{\Delta_2^{[r]}(k^2, \Lambda^2)} \quad (3.11)$$

for a scalar two-point vertex, where

$$N_2^{[r]}(k^2, \Lambda^2) = \sum_{m=0}^{r+1} \zeta_m^{[r]} (k^2)^{r+1-m} (\Lambda^2)^m, \quad (3.12)$$

$$\Delta_2^{[r]}(k^2, \Lambda^2) = \sum_{n=0}^r \eta_n^{[r]} (k^2)^{r-n} (\Lambda^2)^n \quad (\eta_0^{[r]} = 1). \quad (3.13)$$

(Of course, if Γ_2 has Lorentz-tensor or Dirac-matrix structure, as do the fields in a gauge theory, such a formula applies separately to each of the Lorentz-scalar "invariant functions" in a decomposition of the vertex into basis tensors or basis matrices. For example, an $SU(N)$ gauge-field propagator $D_{ab}^{\mu\nu}(k)$, with Euclidean vector indices $\mu, \nu = 1 \dots 4$ and color indices $a, b = 1 \dots N^2 - 1$, would decompose into transverse and longitudinal pieces with respect to k ,

$$D_{ab}^{\mu\nu}(k) = \delta_{ab} [t^{\mu\nu}(k) D_T(k^2) + l^{\mu\nu}(k) D_L(k^2)], \quad (3.14)$$

with projectors $t^{\mu\nu}$ and $l^{\mu\nu}$ given by

$$t^{\mu\nu}(k) = \delta^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} = \delta^{\mu\nu} - l^{\mu\nu}(k), \quad (3.15)$$

and the functions $[D_T(k^2)]^{-1}$, $[D_L(k^2)]^{-1}$ would each have an approximant of the form (3.11). The ubiquitous indices $[r]$ are cumbersome but necessary at this stage to remind us that the coefficients of a rational approximant must be recalculated at every new order r .

You may ask at this point why we did not exploit condition (3.4) / (3.6) more fully – up to now, we only ensured

$$-\Gamma_2^{[r,0]}(k^2) \longrightarrow \zeta_0^{[r]} k^2 \quad (k^2 \gg \Lambda^2), \quad (3.16)$$

so why don't we impose $\zeta_0 = 1$, i. e. $f_0(k) = (k^2)^{r+1}$? Here I have to remind you of the experience with our model amplitude, and ask you to cast a glance at eq. (2.223) above. If we insist on establishing a quasi-perturbative expansion in the sense originally envisaged, and exemplified by eqs. (2.197) and (2.206), then indeed we should require $\zeta_0 = 1$. (For the inverse of the function $D_L(k^2)$ defined by eq. (3.14), the correct analog would be

$$[D_L(k^2)]^{-1} \longrightarrow \frac{1}{\xi} k^2 \quad (k^2 \gg \Lambda^2), \quad (3.17)$$

with ξ the gauge-fixing parameter, in the linear covariant gauge fixing usually employed.) But if we allow for the possibility that our approximation might turn out to be quasi-perturbative in the redefined sense of eq. (2.212) and (2.216), we must allow for a constant factor $\zeta_0 \neq 1$, analogous to the $(1 - \frac{\eta}{2})^{-1}$ of eq. (2.223), which amounts to a finite renormalization. As long as we have not begun to determine our approximant (3.11) from a dynamical equation, we cannot know whether that equation will reveal a tendency to prefer one or the other of the two expansions, or something in between. So caution would advise us to keep the option of $\zeta_0 \neq 1$ (or $\neq 1/\xi$) at this time.

It is trivial to recover from eq. (3.11) the expansion corresponding to what we met in the OPE context as eq. (2.56): by expanding (3.11) in powers of Λ^2/k^2 for $k^2 \gg \Lambda^2$, we have

$$\begin{aligned} -\Gamma_2^{[r,0]}(k^2) = k^2 \left\{ \zeta_0^{[r]} + \left(\zeta_1^{[r]} - \eta_1^{[r]} \zeta_0^{[r]} \right) \frac{\Lambda^2}{k^2} \right. \\ \left. + \left[\left(\zeta_2^{[r]} - \eta_1^{[r]} \zeta_1^{[r]} + \zeta_0^{[r]} \left(\eta_1^{[r]2} - \eta_2^{[r]} \right) \right] \left(\frac{\Lambda^2}{k^2} \right)^2 + \dots \right\}, \quad (3.18) \end{aligned}$$

so if we imposed $\zeta_0^{[r]} = 1$ we could identify, at level r ,

$$\begin{aligned} U_1^{(R)}(0, 0) &= \zeta_1^{[r]} - \eta_1^{[r]}, \\ U_2^{(R)}(0, 0) &= \zeta_2^{[r]} - \eta_1^{[r]} \left(\zeta_1^{[r]} - \eta_1^{[r]} \right) - \eta_2^{[r]}, \end{aligned} \quad (3.19)$$

etc. Indeed it would be nothing but standard Padé procedure to determine the ζ_m, η_n coefficients from the first $2r + 1$ of these equations if the $U_n(0, 0)$ were known quantities. They aren't, and so eqs. (3.19) are useful mainly as a kind of sum-rule check on the vacuum-condensate combinations $U_n^{(R)}(0, 0) (\Lambda_0^2)^n$ of eq. (2.57) that can in principle be computed once the ζ 's and η 's have been determined from dynamics.

More interesting, particularly with a view toward generalizations to higher vertex functions, is another simple rewriting of (3.11), namely the partial-fraction decomposition of its inverse (the propagator approximant) with respect to the variable k^2 . Decompose the numerator polynomial (3.12) into its root factors,

$$N_2^{[r]}(k^2, \Lambda^2) = \zeta_0 \prod_{l=1}^{r+1} \left(k^2 + \sigma_l^{[r]} \Lambda^2 \right) \quad (3.20)$$

then

$$D^{[r,0]}(k^2) = \sum_{l=1}^{r+1} \frac{\rho_l^{[r]}}{k^2 + \sigma_l^{[r]} \Lambda^2}, \quad (3.21)$$

with residues,

$$\rho_l^{[r]} = \frac{\sum_{n=0}^r \eta_n^{[r]} (-\sigma_l^{[r]})^{r-n}}{\zeta_0 \prod_{\substack{k=1 \\ (k \neq l)}}^{r+1} \left(\sigma_k^{[r]} - \sigma_l^{[r]} \right)} \quad (\eta_0^{[r]} = 1). \quad (3.22)$$

This may be viewed as arising from the Euclidean Källén-Lehmann representation,

$$D(k^2) = \int_0^\infty ds \frac{\rho_2(s)}{k^2 + s \Lambda^2}, \quad (3.23)$$

by a discrete approximation of the spectral function,

$$\rho_2(s) \longrightarrow \rho_2^{[r,0]}(s) = \sum_{l=1}^{r+1} \rho_l^{[r]} \delta \left(s - \sigma_l^{[r]} \right). \quad (3.24)$$

Since the spectral function is positive and has its support in the Minkowskian, we expect a sensible approximant to have all its ρ_l 's and σ_l 's non-negative; for the residues, comparison of the $k^2 \gg \Lambda^2$ limits of (3.21) and (3.11) gives the sum rule

$$\zeta_0 \sum_{l=1}^{r+1} \rho_l^{[r]} = 1. \quad (3.25)$$

The $2r + 2$ dimensionless quantities

$$\sigma_1^{[r]}, \sigma_2^{[r]}, \dots, \sigma_{r+1}^{[r]}; \quad \rho_1^{[r]}, \rho_2^{[r]}, \dots, \rho_{r+1}^{[r]} \quad (3.26)$$

represent a parametrization equivalent to (3.10); when $\zeta_0 = 1$, equation (3.25) turns into a relation among the ρ_l 's that reduces the number of independent parameters by one. For later convenience, we also note the corresponding decomposition of the vertex function (3.11) itself, which proceeds from the root-factor decomposition of the denominator polynomial (3.13),

$$\Delta_2^{[r]}(k^2, \Lambda^2) = \prod_{s=1}^r \left(k^2 + u_{2s}^{[r]} \Lambda^2 \right). \quad (3.27)$$

Since in this case the numerator polynomial is of higher degree, the decomposition now takes a slightly different form,

$$-\Gamma_2^{[r,0]} = \zeta_0^{[r]} \left[k^2 + u_1^{[r]} \Lambda^2 + \sum_{s=1}^r \frac{u_{2s+1}^{[r]} \Lambda^4}{k^2 + u_{2s}^{[r]} \Lambda^2} \right], \quad (3.28)$$

with the sum now representing a discrete approximation to the branch cut of the self-energy that mirrors the Källén-Lehmann propagator cut. By comparison with eq. (3.18), its parameters obey

$$u_1^{[r]} = \left(\frac{\zeta_1}{\zeta_0} - \eta_1 \right)^{[r]}, \quad (3.29)$$

$$\sum_{s=1}^r u_{2s+1}^{[r]} = \left(\frac{\zeta_2}{\zeta_0} - \eta_1 \frac{\zeta_1}{\zeta_0} + \eta_1^2 - \eta_2 \right)^{[r]},$$

etc.

If the field whose two-point correlation D represents had exactly one particle associated with it (as is usually assumed without discussion), then $\sqrt{\sigma_1} \Lambda$ and ρ_1 would approximate its mass and field-renormalization constant, and the higher ρ_l 's and σ_l 's

would provide a discrete approximation to its dressing cut. But in a nonperturbative context, where the zeroth-order perturbative spectrum may suffer not only small shifts but also qualitative changes, it is quite possible for a quantized field to give rise to more than one particle, or none at all, and in such cases we encounter a practical disadvantage of our rational approximation: at low degrees r , there is no simple way of deciding whether a given pole term in (3.21) approximates a genuine particle pole, or is merely part of the approximation to the dressing cut. The answer can only be found indirectly by observing the approximation as r increases: all experience with rational approximants indicates that poles approximating "true" poles then display reasonable convergence of their positions and residues, whereas the poles approximating a cut keep "hopping", and at the same time spread to ever higher values of s , since at each successive r the approximation invents a new set of positions and strengths to achieve an improved scan of the continuous part of the spectral function.

We should again pause, since some of you may have taken offense at this point with our apparent lightheartedness in talking about such treacherous objects as particle masses and field renormalizations. Haven't we all learned that these things are ultraviolet divergent, and can at best be traded for something experimental but not really computed? Here you should recall that at this stage we are dealing with a *zeroth quasi-perturbative order*, and at zeroth order in powers of α , renormalization constants are not yet divergent. The ultraviolet divergences will catch up with us soon enough once we start computing quasi-perturbative corrections, but at the zeroth order we are only into reconstruction of the quantity that in the OPE context appeared as eq. (2.56) – a quantity which we noted is RG-invariant and *finite*. At *this* level, things like the nonperturbative mass shift of eq. (2.104) do indeed have finite values. Even when it comes to computing quasi-perturbative corrections, you should still remember the pattern we noted in eq. (2.49) – it is only for the perturbative portion that we expect to have to cancel net ultraviolet divergences. For the nonperturbative remainder, what we are aiming at is a direct continuation-through-resummation of the n sum of eq. (2.49) with its ultraviolet-finite $V_{n,j}^{(R)}$ coefficients – we do not attempt to separately deal with vacuum condensates and Wilson coefficients, whose ultraviolet divergences then cancel by eq. (2.46). That the dynamical equations should provide a mechanism allowing for all these features is of course not trivial, but nevertheless true, as the next section will show.

Let us return to eqs. (3.21) and (3.28) where we exploited the tight linkage, inherent in the dimensionful character of Λ^2 , between the Λ^2 and k^2 dependences to reinterpret our original Λ^2 approximation as an approximation to the expected k^2 -plane analyticity

structure of our propagator or self-energy. It is this reinterpretation that provides the best guide for our next and slightly more difficult step: choosing zeroth-order approximants for higher-than-two-point vertex functions, consistent with the two-point form of eq. (3.11). Here again we start with the simplest and most important case of a single scalar three-point vertex function with Bose symmetry, depending on the three squared Euclidean momenta

$$k_1^2, \quad k_2^2, \quad k_3^2 \quad (3.30)$$

of its three external legs, and being otherwise dimensionless with unit perturbative limit,

$$\Gamma_3^{(0)pert}(\{k\}) = 1. \quad (3.31)$$

(We have tacitly adopted the widespread convention of defining the three-point vertex without a coupling factor \bar{g}_0 , so what actually enters into diagrams is $\bar{g}_0 \Gamma_3$ rather than Γ_3 alone.) This again is not precisely what occurs in a realistic gauge theory – a vertex for three $SU(N)$ gauge-vector-fields, for example, has mass dimension plus one rather than zero, and that dimension is carried by the Lorentz basis tensors in a decomposition of the full tensorial vertex. In the index notation adopted for eq. (3.14) above, the beginnings of such a decomposition would read like

$$\begin{aligned} & \left\{ \Gamma_{3V,abc}^{\mu\lambda\nu}(k_1, k_2, k_3) \right\}_{k_1+k_2+k_3=0} \\ &= f_{abc} \left\{ \delta^{\lambda\nu}(k_2 - k_3)^\mu F_0^{(f)}(k_2^2, k_3^2; k_1^2) \right. \\ & \quad + \delta^{\nu\mu}(k_3 - k_1)^\lambda F_0^{(f)}(k_3^2, k_1^2; k_2^2) \\ & \quad \left. + \delta^{\mu\lambda}(k_1 - k_2)^\nu F_0^{(f)}(k_1^2, k_2^2; k_3^2) + \dots \right\} \quad (3.32) \\ &+ d_{abc} \left\{ \delta^{\lambda\nu}(k_2 - k_3) F_0^{(d)}(k_2^2, k_3^2; k_1^2) \right. \\ & \quad + \delta^{\nu\mu}(k_3 - k_1)^\lambda F_0^{(d)}(k_3^2, k_1^2; k_2^2) \\ & \quad \left. + \delta^{\mu\lambda}(k_1 - k_2)^\nu F_0^{(d)}(k_1^2, k_2^2; k_3^2) + \dots \right\}, \end{aligned}$$

with f_{abc} and d_{abc} denoting respectively the antisymmetric and symmetric $SU(N)$ structure constants, which enter here in their capacity as Clebsch-Gordans for the two possible ways of coupling three adjoint-representation objects to a (global) gauge singlet.

What is dimensionless is then the $F_0^{(f)}$ and $F_0^{(d)}$ invariant functions, but they will of course display only the reduced Bose symmetry required to match the partial symmetry of their associated tensor structures, and thus be symmetric or antisymmetric only with respect to the first two of their arguments. The Feynman rule $\Gamma_{3V}^{(0)pert}$, as you will remember, has only the f_{abc} terms of eq. (3.32) with $F_0^{(f)}$'s replaced by unity, so we have the perturbative limits

$$\left(F_0^{(f)}\right)^{(0)pert} = 1, \quad \left(F_0^{(d)}\right)^{(0)pert} = 0, \quad (3.33)$$

as $\Lambda \rightarrow 0$.

The dots in eq. (3.32) in each case stand for a finite but lengthy list of other independent Lorentz-tensor structures (actually, no less than eleven of them), forming triplets or singlets under the group of permutations of the three vector-field legs, and each accompanied by its own invariant function. It will emerge only much further down the road (and be commented upon in our closing section) that in the zeroth quasi-perturbative order we are considering here, this cumbersome (2×14)-term tensorial structure fortunately simplifies – most of the Lorentz tensors not made explicit in eq. (3.32) are tensors involving three powers of momentum rather than one, and being therefore accompanied by invariant functions of mass dimension minus two rather than zero, and in the zeroth quasi-perturbative order these turn out to be excluded by considerations of renormalizability. Here, however, we should not let ourselves be sidetracked too much by these purely kinematical and enumerative aspects of tensor structure, so you will allow me to refer you rather summarily to the standard reference [BCU81] on these aspects. For the issues we wish to address, consideration of a purely schematic, scalar three-variable function characterized by eq. (3.31), or equivalently restriction to the $F_0^{(f)}$ function of eq. (3.32) with limit as in the first of eqs. (3.33), will be quite adequate.

For such a function, a rational approximant with respect to Λ^2 , of the form of eq. (3.3), must have $l' = l$ by virtue of condition (3.4), so again we need only one labelling index. But otherwise, eq. (3.3) alone is now much too unspecific to represent a computationally useful approximant. (All we can say at this stage about its f_n and g_n coefficients is that each must be of the form $(Q_n^2)^{l-n}$, where Q_n^2 is some dimension-two combination of the variables (3.30), times some dimensionless function of two independent dimensionless combinations.) It might be sufficient if the dynamical equations to be invoked were differential or integral equations *with respect to the coupling*, which would correspond to the situation called "equational resurgence" in mathematical theory [ECA81] [STS96]. But here the equations at our disposal will be integral equations with respect

to *momentum* variables, and for an approximant to be capable of exploiting these equations, it must then detail the momentum dependence in a manner comparable to eq. (3.11) for the one-variable case. In addition to the resurgent coupling structure, which we are resumming at the $p = 0$ level through functions (3.3), we need some additional input here. What can help us across this gap are considerations of analyticity. From the general properties of QFT amplitudes, we expect Γ_3 to be analytic (in the usual loose sense, i. e. except for a denumerable set of poles and cuts) in the complex plane of each of its variables (3.30) at fixed values of the other two, in a manner analogous to the single- k^2 analyticity embodied in the spectral representation (3.23). It therefore seems sensible, and not unduly restrictive, to postulate that our Γ_3 , *at least at the zeroth quasi-perturbative order*, fits into the form of a triple Cauchy-integral representation,

$$\Gamma_3^{(p=0)}(k_1^2, k_2^2, k_3^2; \Lambda^2) = \int_{\mathcal{C}_1} ds_1 \int_{\mathcal{C}_2} ds_2 \int_{\mathcal{C}_3} ds_3 \frac{\rho_3^{(0)}(s_1, s_2, s_3)}{(k_1^2 + s_1 \Lambda^2)(k_2^2 + s_2 \Lambda^2)(k_3^2 + s_3 \Lambda^2)}. \quad (3.34)$$

For a Bose-symmetric function, the spectral weight ρ_3 , which may include distribution-valued terms, should be symmetric in its three variables, and the contours \mathcal{C}_i in the three s_i planes should be identical to one \mathcal{C} . Also the full function must be real at real Euclidean momenta, so if \mathcal{C} comprises any sections away from the real s axis (which for off-shell variables like our k_i^2 cannot be excluded), these, as well as the associated portions of $\rho_3^{(0)}$, should only appear in complex-conjugate pairs. We will always understand the form (3.34) to allow for terms depending on less than three variables, by interpreting the compact notation $\rho_3^{(0)}$ as

$$\begin{aligned} \rho_3^{(0)} = & \xi_0 \prod_{i=1}^3 [k_i^2 \delta(s_i)] + \Lambda^2 \sum_{i < j} [k_i^2 \delta(s_i)][k_j^2 \delta(s_j)] \rho_{3,1}(s_k) \\ & + \Lambda^4 \sum_{k=1}^3 [k_k^2 \delta(s_k)] \rho_{3,2}(s_i, s_j) + \Lambda^6 \rho_{3,3}(s_1, s_2, s_3), \end{aligned} \quad (3.35)$$

with a positive constant ξ_0 , and with (i, j, k) cyclic in the second and third terms. This function then has large-momentum limit

$$\Gamma_3^{(p=0)} \rightarrow \xi_0 \quad (\text{all } k_i^2 \gg \Lambda^2), \quad (3.36)$$

and all our earlier comments on the $\zeta_0^{[r]}$ of eq. (3.16) are again applicable.

The main consequences of postulate (3.34), for our purposes, are that if we now generate an approximant to $\Gamma_3^{(0)}$ by adopting a discrete approximation analogous to

(3.24),

$$\begin{aligned}
\rho_3^{(0)}(s_1, s_2, s_3) &\longrightarrow \rho_3^{[r,0]}(s_1, s_2, s_3) = \sum_{n_1, n_2, n_3=0}^r \xi_{n_1 n_2 n_3}^{[r]} \\
&\times [\delta_{n_1 0} k_1^2 + (1 - \delta_{n_1 0}) \Lambda^2] [\delta_{n_2 0} k_2^2 + (1 - \delta_{n_2 0}) \Lambda^2] [\delta_{n_3 0} k_3^2 + (1 - \delta_{n_3 0}) \Lambda^2] \\
&\times \delta(s_1 - \tau_{n_1}^{[r]}) \delta(s_2 - \tau_{n_2}^{[r]}) \delta(s_3 - \tau_{n_3}^{[r]}) ,
\end{aligned} \tag{3.37}$$

then automatically that approximant will (i) be rational not only with respect to Λ^2 but also to the k_i^2 , and (ii) possess a *denominator factorizing* in the three k_i^2 which mirrors the product structure of the Cauchy-integral denominator. (To accommodate the structure (3.35) we add the convention that

$$\tau_0^{[r]} = 0 \quad (\text{all } r) \tag{3.38}$$

in eq. (3.37).) For full Bose symmetry we should have

$$\xi_{n_1 n_2 n_3}^{[r]} \quad \text{completely symmetric in } (n_1, n_2, n_3) . \tag{3.39}$$

Therefore if we put the entire approximant over a common denominator we arrive at the form

$$\Gamma_3^{[r,0]}(\{k_i^2\}; \Lambda^2) = \frac{N_3^{[r]}(\{k_i^2\}; \Lambda^2)}{\Delta_3^{[r]}(k_1^2; \Lambda^2) \Delta_3^{[r]}(k_2^2; \Lambda^2) \Delta_3^{[r]}(k_3^2; \Lambda^2)} \tag{3.40}$$

with a numerator polynomial of degree $3r$ in Λ^2 and of degree r in each of the squared momenta,

$$\begin{aligned}
N_3^{[r]}(\{k_i^2\}; \Lambda^2) &= \sum_{\lambda=0}^{3r} \sum_{m_1, m_2, m_3=0}^r \delta_{\lambda+m_1+m_2+m_3, 3r} x_{m_1 m_2 m_3}^{[r]} \\
&\times (k_1^2)^{m_1} (k_2^2)^{m_2} (k_3^2)^{m_3} (\Lambda^2)^\lambda ,
\end{aligned} \tag{3.41}$$

$$x_{m_1 m_2 m_3}^{[r]} \quad \text{completely symmetric in } (m_1, m_2, m_3) , \tag{3.42}$$

$$x_{rrr}^{[r]} = \xi_0^{[r]} \tag{3.43}$$

and with a denominator factorizing into three degree- r polynomials similar to (3.13),

$$\Delta_3^{[r]}(k^2; \Lambda^2) = \prod_{s=1}^r (k^2 + \tau_s^{[r]} \Lambda^2) . \tag{3.44}$$

In the notation of our original eq. (3.3) for the rational approximants, eq. (3.40) therefore has

$$l' = l = 3r \quad (N = 3), \quad (3.45)$$

rather than $l = r$ as in eq. (3.11). Indeed it will turn out that on this point analyticity has given us good guidance, since it will be just this combination – of $l = r$ for Γ_2 with $l = 3r$ for Γ_3 , and more matching l 's for the Γ_4 vertex – that will prove itself capable of self-consistency in the equations of motion.

When all $k_i^2 \gg \Lambda^2$, eq. (3.40) gives an expansion

$$\begin{aligned} & \Gamma_3^{[r,0]}(\{k_i^2\}; \Lambda^2) = \xi_0^{[r]} \\ & \times \left\{ 1 + \left[\frac{x_{r,r,r-1}^{[r]}}{\xi_0^{[r]}} - \left(\sum_{s=1}^r \tau_s^{[r]} \right) \right] \left(\frac{\Lambda^2}{k_1^2} + \frac{\Lambda^2}{k_2^2} + \frac{\Lambda^2}{k_3^2} \right) + \dots \right\} \end{aligned} \quad (3.46)$$

that may be regarded as a three-point counterpart of eq. (2.56) – a three-variable OPE, restricted here to terms of zeroth perturbation order with respect to α .

The foregoing discussion of $\Gamma_2^{[r,0]}$ and $\Gamma_3^{[r,0]}$ approximants has, I hope, provided sufficient illustration of the guidelines and technicalities that go into a construction of zeroth quasi-perturbative orders as a systematic sequence. You will therefore allow me to skip the corresponding construction for a four-boson vertex – the highest function for which such a construction will be found necessary. With no less than six independent kinematic invariants, and with a plethora of possible tensor structures, this function requires a much lengthier but not much more illuminating construction. Only much further along the way – that is, after having learned some lessons from trying to get these constructions self-consistent in the equations of motion – will it emerge that *for the zeroth quasi-perturbative order*, $\Gamma_4^{[r,0]}$, tensorial structures and functional dependences are strongly restricted, so that the final approximants are still manageable. For the moment, it may be more important to address your feelings that my remark about this being the highest construct necessary sounds completely obscure. The equations for correlation functions, in whatever variant, are always hierarchically coupled – so in order to establish a four-point amplitude, aren't we going to need a five-point one, and so on forever? In other words, mustn't we live in fear that unlike their perturbative zeroth-order counterparts – the Feynman rules $\Gamma_N^{(0)pert}$ – our new "rules" will endlessly *proliferate*? It seems nontrivial (at least to me) that such fears in fact should turn out to be unfounded.

3.2 Self-Consistency of the Generalized Feynman Rules

The best of the benefits from our n -resummed, quasi-perturbative reconstruction strategy are yet to be reaped. They emerge once we insert the matched set of zeroth-order approximants, as constructed in the last section, into the interaction (loop) terms of exact equations of motion. As announced, and as expected, these loop terms then start producing the quasi-radiative corrections $\Gamma^{[r,p]}$, $p \geq 1$, in what is basically the first step in an iteration process around a nonperturbatively improved starting point. But more importantly, we find four pleasant surprises. First, despite the strongly nonperturbative character of the $\Gamma^{[r,0]}$'s with their ubiquitous Λ -scale terms, the loops turn out to be readily *calculable* by the known methods of loop computation and regularization, simply due to their rational structure in momenta. Second, *in addition* to producing $p \geq 1$ corrections, the loop terms prove themselves capable of reproducing the nonperturbative parts of our zeroth-order input, thus conferring true *self-consistency* on the improved starting point. Third, while the $p \geq 1$ corrections continue to carry the ultraviolet loop divergences, the mechanism operative in the self-reproduction sees to it that the non-perturbative zeroth-order terms, the quantities $\Gamma^{[r,0]} - \Gamma^{(0)pert}$ at all levels r , establish themselves in a *finite* manner, as one would expect and demand from our consideration of the OPE. Fourth, the same mechanism relieves us from our nightmare of proliferation of Feynman rules, by *strictly* limiting the formation of zeroth-order proper vertices to a small finite set of these functions. It is these features together that seem to justify designation of this finite and self-consistent set of $\Gamma^{[r,0]}$'s as *generalized Feynman rules*.

The system of exact equations of motion we are going to exploit is the oldest known such system, the integral equations first established by Dyson [DYS49], who derived them by Feynman-graph summation, and by Schwinger [SCH51], whose functional derivation made it plain that they are in fact independent of any perturbative framework. They are referred to as the DS equations. Although they still were fundamental field-theory tools for such researchers as Schwinger, Bogolyubov, and Symanzik, they are rarely discussed in textbooks nowadays (two 1980's texts that give them more than passing mention are [ITZ80] and [RIV87]), but I hope you may have had at least a superficial encounter with them and will recognize a DS system (from its typical, hierarchical structure) when you see one. For the DS equations of non-abelian gauge theories, the standard original reference is [EIF74]; condensed statements of their derivation and main properties can also be found in the review articles [ROW94] and [AVS01], which concentrate on exact numerical solutions – another line of research which our focus on coupling dependence will not allow us to pursue in any detail.

To collect the minimum of formal relations we will need to refer to, we temporarily ignore the fact that we actually have to deal with non-abelian gauge-theories (these being the only ones for which our basic premise of asymptotic freedom is correct in the physical dimension $D = 4$) and write simplified equations for some generic field $\varphi(x)$, using notations as in eq. (1.25) above. The generating functional of Euclidean correlation functions,

$$Z_E[J] = \frac{1}{\mathcal{N}} \int \mathcal{D}[\varphi] e^{-S_E[\varphi] + (J, \varphi)} , \quad (3.47)$$

with normalization \mathcal{N} such that $Z_E[0] = 1$, is postulated to be invariant under simultaneous infinitesimal translations of all its integration variables,

$$\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \delta \varphi(x) \quad (\text{all } x) , \quad (3.48)$$

in the sense that $\mathcal{D}[\varphi'] = \mathcal{D}[\varphi]$. (For the mathematically minded, it is not unproblematic to use translation invariance for an infinite-dimensional measure, but since the procedure leads quickly and elegantly to equations of motion that can also be established independently by operator techniques, we will accept it here as a formal generating tool.) This gives the statement that

$$\frac{1}{\mathcal{N}} \int \mathcal{D}[\varphi] \frac{\delta}{\delta \varphi(x)} \left\{ e^{-S_E[\varphi] + (J, \varphi)} \right\} = 0 , \quad (3.49)$$

which may be written as

$$\left\langle -\frac{\delta S_E[\varphi]}{\delta \varphi(x)} + J(x) \right\rangle_J = 0 , \quad (3.50)$$

using the expectation-value notation

$$\langle A \rangle_J = \frac{1}{\mathcal{N}} \int \mathcal{D}[\varphi] A[\varphi] e^{-S_E[\varphi] + (J, \varphi)} \quad (3.51)$$

for functionals A of φ . Further rewriting along familiar lines puts this into the form

$$\left\{ \left(\frac{\delta S_E[\varphi]}{\delta \varphi(x)} \right)_{\varphi \rightarrow \frac{\delta}{\delta J}} - J(x) \right\} Z_E[J] = 0 , \quad (3.52)$$

a relation sometimes called the master DS equation. The equations for the 1-, 2-, 3-... point correlation functions then follow from this relation the same way that the functions themselves follow from (3.47): by functionally differentiating 0, 1, 2, ... times

with respect to J and putting $J = 0$. For example, after one differentiation with respect to $J(y)$ one arrives at

$$\left\langle \varphi(y) \frac{\delta S_E[\varphi]}{\delta \varphi(x)} \right\rangle_0 = \delta^D(y - x), \quad (3.53)$$

and if S_E , as usual, consists of a bilinear kinetic term

$$\varphi(x) K_x \varphi(x) \quad (3.54)$$

with symmetric, positive-definite differential operator K_x , plus tri- and quadrilinear interaction terms collected in a "potential" $V(\varphi(x))$, then (3.53) will be an integro-differential equation for the two-point function $G_2(y, x)$ of eq. (0.1) of the form

$$K_x G_2(y, x) = \delta^4(y - x) - \langle \varphi(y) V'(\varphi(x)) \rangle_0, \quad (3.55)$$

coupling that function to three- and four-point functions of the special forms

$$\langle \varphi(y) (\varphi(x))^2 \rangle_0, \quad \langle \varphi(y) (\varphi(x))^3 \rangle_0. \quad (3.56)$$

It is this characteristic coupling to the next higher functions that is referred to as hierarchical. Upon converting the equations from x space to momentum space, the identifications of two or three points in the functions (3.56) will turn into single or double internal-momentum integrations (loops). Finally, one transcribes the equations from full correlation functions G_N to connected, amputated, and 1 PI functions Γ_N to give them their most transparent form.

In Fig. 3.1 I have attempted to write diagrammatically the three lowest levels – the equations for two-, three- and four-point vertices – of what might be called a generic DS system. (These three equations, for our purpose, will actually turn out to bear a crucial intrinsic distinction, being the equations for the superficially ultraviolet-divergent functions). For each equation, Fig. 3.1 selects a set of typical and recurrent features: in the first line of each, the r. h. s. starts out with an "inhomogeneous" term representing a zeroth-order perturbative vertex $\Gamma_N^{(0)pert}$ (terms 2 A, 3 A, 4 A). Next on each r. h. s is a loop term containing no vertex function higher than the one on the l. h. s., so that with such an interaction term alone the equation would close upon itself (terms 2 B, 3 B, 4 B). Spoiling this closure is, on the second line of each equation, a loop term featuring the next higher vertex function, which brings in the hierarchical coupling (terms 2 C, 3 C, 4 C).

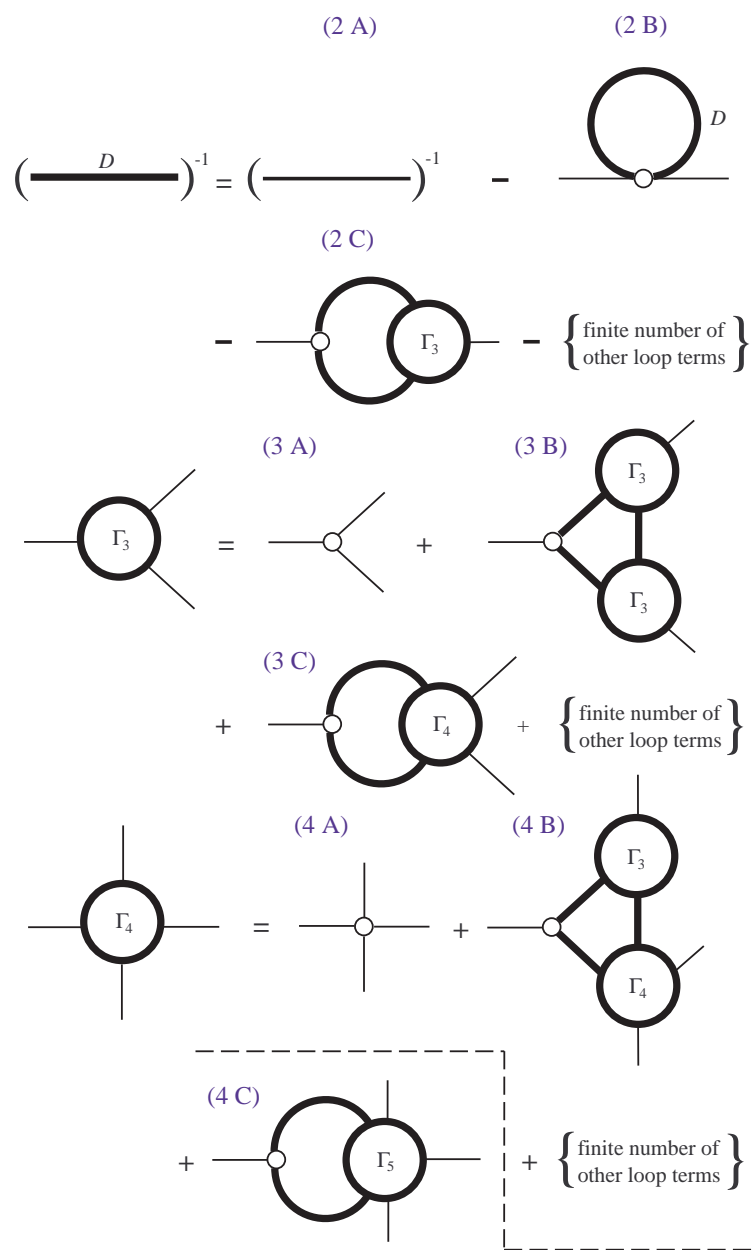


Fig. 3.1 Beginnings of a schematic Dyson-Schwinger hierarchy in diagrammatic form. Small circles represent bare vertices; heavy lines, exact propagators; large circles, exact proper vertices as specified. Dashed line in Γ_4 equation indicates limit beyond which loop terms do no more contribute to generation of zeroth quasi-perturbative order.

Also present in most cases, but not made explicit, are a finite number of other loop terms, which in the case of non-abelian gauge-boson vertices will include two-loop terms arising from the second type of function in (3.56). By suitably interpreting the dressed propagators (heavy lines) and dressed vertices (large circles) in their explicit terms, and suitably filling in the non-explicit ones, these three graphical equations may be viewed as encompassing the lower levels of practically every known DS hierarchy. To guard against misunderstanding, I must interject a remark here. Because of the hierarchical coupling, DS systems, like their non-relativistic counterparts in statistical mechanics, are usually solved by "decoupling" approximations whereby the next-higher vertices in some equation are expressed in terms of lower ones, thus closing the system. Although these decoupling approximations are efficient devices for generating partial resummations, and in my opinion do not deserve the low esteem in which they seem to be held by most contemporary particle theorists, it must be emphasized that what we are going to study in this section – the self-consistency of nonperturbatively modified zeroth orders in a quasi-perturbative resurgence reconstruction – has nothing to do with decoupling approximations. The one "decoupling" that will occur here will be both automatic and exact, and will arise from the basic property of renormalizable theories of having only a limited number of superficially divergent correlation functions. Decoupling approximations may be useful at later stages, particularly when dealing with the equations for superficially convergent higher amplitudes of which our Fig. 3.1 stops short, but for the purpose at hand, they play no role whatsoever.

The common form under which all the equations of Fig. 3.1 may be subsumed is

$$\Gamma_N = \Gamma_N^{(0)pert} + \bar{g}_0^2 \Phi_N[\Gamma] \quad (N = 2, 3, 4), \quad (3.57)$$

which makes explicit the fact that all interaction terms carry at least one power of the squared bare coupling, \bar{g}_0^2 . The DS interaction functional, Φ_N , consists of loop integrals over integrands involving all vertices Γ up to Γ_{N+1} (or Γ_{N+2} in the presence of quartic interactions), and is nonlinear, as is directly visible from the graphs of Fig. 3.1. Straightforward iteration of eqs. (3.57) around the zeroth perturbative orders is one way

of generating the perturbation series (0.4), but in such an iteration, in which at every stage one evaluates integrals over functions known from previous steps, one never fully faces the true challenge of (3.57), which is a *self-consistency* problem: find a set of Γ_N 's such that, when inserted into the Φ_N functionals, they are capable of reproducing themselves. Nonperturbative treatments, along whatever lines, must address that challenge: to determine, in mathematical terms, a fixed point of the complicated mapping defined by the r. h. s. of eqs. (3.57). In our specific context, not the least part of that challenge is to establish from eqs. (3.57) the nonperturbative parts of the zeroth-order input – the terms

$$\Gamma_N^{[r,0]} = \Gamma_N^{(0)pert} \quad (r = 0, 1, 2, \dots) \quad (3.58)$$

which carry powers Λ^n with $n \geq 1$ but no powers $(g^2)^p$ with $p \geq 1$ – in spite of the fact that the interaction terms always have at least one power of \bar{g}_0^2 in front. For this to be possible, the interaction functionals Φ_N must obviously be capable of producing terms with factors of $1/g^2$ from sufficiently nonperturbative input, and indeed that is what they will be seen to be doing below. (One might say that the basic "mistake" of perturbation theory is to miss precisely this possibility.)

Besides a multitude of aspects of DS systems that will have to go unmentioned in the restricted time and space of these lectures (a few will be taken up in our closing section), there is one that is so rarely discussed in the literature, and so intricately connected to renormalizable ultraviolet-divergence structure, that I feel I should comment on it before plunging into the technicalities of self-consistency. In a theory with up to quartic terms in its action functional $S_E[\varphi]$, the "master DS equation" (3.52), of which our individual equations (3.57) are functional moments, is a functional-differential equation of the third order for the generating functional $Z_E[J]$. From the analogy with differential equations, one would therefore expect that the first three functional Taylor coefficients in the expansion

$$Z_E[J] = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^D x_1 \dots \int d^D x_N G_N(x_1 \dots x_N) J(x_1) \dots J(x_N) \quad (3.59)$$

will be left undetermined by the equation and will have to be specified as "initial conditions" (for the evolution of the functional away from $J = 0$). The $N = 0$ coefficient, the vacuum Norm $G_0 = Z_E[0] = \langle 0|0\rangle$, is fixed at unity by normalization, and the $N = 1$ coefficient, the single-field expectation $G_1 = \langle 0|\varphi(x)|0\rangle$, vanishes in theories without elementary scalar fields (or indeed remains an empirical parameter in Higgs-field theories, but then that's what we have been accustomed to). But the idea that the $N = 2$ coefficient, the propagator $G_2(x_1 - x_2)$, should have to be put in as an

initial datum sounds positively bewildering. I must refer you to the excellent discussion in sect. 1.4 of ref. [RIV87], which uses the so-called static ultra-local model of Caianiello and Scarpetta [CAI74], for insight into how the perturbation series seems to evade this problem – basically, it steers you tacitly into a particular solution in which the initial datum G_2 is assigned a special value, while displacing the freedom of choosing initial conditions into its undetermined "escaping" terms. (Again, analogs of this have been known in Écalle's theory of resurgent solutions to nonlinear differential equations, where the perturbation series is referred to as the "formal solution"; see the "epilogue" of [CNP93]). In any truly nonperturbative treatment, however, the initial-conditions problem is bound to reappear, and indeed the numerical studies of propagator equations described in [AVS01] have met with indeterminacies presumably related to this question. In our context, the problem will resurface in a very clear-cut and interesting way: while the standard DS system will produce self-consistency conditions determining the polynomials $N_2^{[r]}(k^2, \Lambda^2)$ of eq. (3.11) above, which at each level r encode the particle-pole positions of the propagator and the mesh points for the discretization of its continuum integral through their root-factor decompositions (3.20), conditions that would determine the coefficients of the polynomials $\Delta_2^{[r]}(k^2, \Lambda^2)$ of eqs. (3.13) and (3.27) – that is, the weights to be assigned to those poles and mesh points – will be found lacking. When reflecting on the origins of this indeterminacy, both in DS systems in general and in our more special problem of zeroth quasi-perturbative orders, we are led back to looking at the OPE in e. g. the form of eq. (2.54). We noted that the coefficients $U_n^{(R)}$ appearing there contain information about renormalized vacuum expectations of composite operators of mass dimension n – singular products of up to n elementary fields at the same space-time point. In the ordinary DS system, the loops do represent *some* of these composites – the quadratic and cubic composites of eq. (3.56) – but not more, while the OPE, when following it to higher n 's, obviously carries more information about higher composites. The natural conjecture then is that the indeterminacy revealed by the initial-conditions argument is related to insufficient composite-operator information in the ordinary DS equations. To supply the missing information in a way that is compatible with, and forms a natural extension of, the ordinary DS system, one may therefore think of using DS equations with increasing numbers of coalescing points in x space, which in momentum space would correspond to integrating out increasing numbers of the independent momenta. On the simplest level, for example, the $y \rightarrow x$ limit of eq. (3.53) would give

$$\left\langle \varphi(x) \frac{\delta S_E[\varphi]}{\delta \varphi(x)} \right\rangle_0 = 0 \quad (3.60)$$

(remembering that $\delta^D(0) = 0$ in dimensional regularization), which in momentum space

will turn into an equation for the integrated quantity

$$\int \frac{d^D k}{(2\pi)^D} \left[D^{(0) \text{pert}}(k) \right]^{-1} D(k) , \quad (3.61)$$

rather than for $D(k)$ itself. (This equation is a single-number condition, and at the order $r = 1$ of rational approximation, where only a single coefficient $u_2^{[1]} = \eta_1^{[1]}$ needs to be determined in the polynomial $\Delta_2^{[1]}$, it would already suffice for completing the "initial" conditions). On the next level, coalescence of two of the three spacetime points in a three-point equation would lead to an equation for the quantity

$$\int \frac{d^D q}{(2\pi)^D} \tilde{G}_3 \left(-p, \frac{1}{2}p + q, \frac{1}{2}p - q \right) , \quad (3.62)$$

which is still an equation for a function of p , and coalescing all three points would give the integral of this equation over p , which like the equation for (3.61) is a single-number condition. Equations of this kind have been termed "equation-of-motion condensates" in some OPE studies [AHL92], and have been used as checks on Wilson-coefficient calculations, but in our context they stand to assume a more fundamental role. In ordinary integral-equations theory with well-behaved functions and convergent integrals, once you have solved an equation, the integration of that equation over its domain of definition will not, of course, generate new information. In a theory with renormalizable ultraviolet divergences, the situation is different – composite-operator divergences require new renormalization steps beyond the usual field and parameter renormalizations, and the above-described *momentum-integrated DS equations* therefore do bring in new information. In these lectures we will take the view that they are an adequate means of completing the initial conditions.

When taking up self-consistency in earnest, it is again best to first concentrate on a partial task in which the basic mechanisms can be seen at work in an exemplary way. Let us therefore focus on the first, inverse-propagator equation of Fig. 3.1, and let us single out, among the interaction terms on its r. h. s., the one-DS-loop diagram (2 C), which is the simplest one to bring in the crucial hierarchical coupling. We take the D^{-1} on the l. h. s. to be a scalar function of k^2 , k being the external momentum flowing through the equation, with perturbative limit $(D^{(0) \text{pert}})^{-1}$ given by (3.6), and with zeroth quasi-perturbative order given, at level r of rational continuation, by (3.11) or (3.28). The equation reads, in the notation of (3.57),

$$D^{-1}(k^2) = k^2 - \bar{g}_0^2 \{ \Phi_{2C}[\Gamma; k^2] + [\text{other loop terms}] \} , \quad (3.63)$$

where, remembering our parenthetical remark after eq. (3.31)

$$\begin{aligned} \Phi_{2C} [\Gamma; k^2] &= \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \Gamma^{(0)pert} (q, -k, k - q) \\ &\times D(q^2) D((k - q)^2) \Gamma_3 (q - k, k, -q) . \end{aligned} \quad (3.64)$$

In order to get mass dimensions (and the associated degrees of divergence) right without having to delve again into the paraphernalia of gauge-theory tensor structure, we will patch up things a little and supply both Γ_3 and $\Gamma_3^{(0)pert}$ with an extra unit of momentum dimension in an *ad hoc* fashion, such that their product develops an extra contraction factor

$$[(k - q) - q]^\mu [-q - (q - k)]^\mu = (2q - k)^2 . \quad (3.65)$$

(Never mind this little dose of patchwork – it will do all the essential things right. If you feel uncomfortable with it, you may do an instructive piece of homework by setting up, say, the gluon-loop contribution to the gluon's transverse D_T^{-1} as defined by (3.14), with full tensor structure for $\Gamma_{3-gluon}$ as in eq. (3.32), working in Landau gauge to avoid mixing in D_L 's, and grind out the tensor contractions. You will find that our *ad hoc* factor does things qualitatively right for the ultraviolet-divergent terms, which turn out to be what matters here).

In a first iteration, we now substitute D and Γ_3 by their zeroth quasi-perturbative orders, using eq. (3.40) at level r of rational continuation, with perturbative limits given by (3.6) and (3.31). That is, we decide – in order to keep all inessential complications out of the following discussion of the self-consistency procedure – to only look for solutions with

$$\zeta_0^{[r]} = 1 , \quad \xi_0^{[r]} = 1 , \quad (3.66)$$

which represents a kind of boundary condition. Indeed we remember from sect. 2.4 that there may be at least two different meanings, related by finite rescaling, to the concept of quasi-perturbative zeroth-order, and we expect that the DS functionals in principle can produce both (and perhaps more), so a supplemental condition of some kind will be necessary. Eq. (3.63) assumes the form

$$\begin{aligned} &[D^{-1}(k^2)]^{[r,0]} + g^2 [D^{-1}(k^2)]^{[r,1]} + O(g^4) \\ &= k^2 - \bar{g}_0^2 \{ \Phi_{2C} [\Gamma^{[r,0]}; k^2] + [\text{other loop terms}]^{[r,0]} + O(g^2) \} , \end{aligned} \quad (3.67)$$

where

$$\begin{aligned} \Phi_{2C}[\Gamma^{[r,0]}; k^2] &= \left\{ \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} (2q - k)^2 \frac{\Delta_2^{[r]}(q^2, \Lambda^2) \Delta_2^{[r]}((k - q)^2, \Lambda^2)}{N_2^{[r]}(q^2, \Lambda^2) N_2^{[r]}((k - q)^2, \Lambda^2)} \right. \\ &\quad \left. \times \frac{N_3^{[r]}(q - k)^2, k^2, q^2; \Lambda^2}{\Delta_3^{[r]}((q - k)^2, \Lambda^2) \Delta_3^{[r]}(q^2, \Lambda^2)} \right\} \frac{1}{\Delta_3^{[r]}(k^2, \Lambda^2)}. \end{aligned} \quad (3.68)$$

At this point, to shorten the discussion, we anticipate a simplifying fact that will soon be obvious: the rational-approximant poles at $k^2 = -u_{2s}^{[r]} \Lambda^2$ ($s = 1 \dots r$), which according to eq. (3.11) we are postulating for the l. h. s. of eq. (3.67), can at zeroth quasi-perturbative order arise only from the last factor $1/\Delta_3^{[r]}$ of (3.68) but not from the bracketed integral. Therefore we immediately have a first set of self-consistency conditions, which in the notation of eq. (3.44) read

$$\tau_s^{[r]} = u_{2s}^{[r]} \quad (s = 1, \dots, r), \quad (3.69)$$

and which imply that quite simply

$$\Delta_3^{[r]}(k^2) = \Delta_2^{[r]}(k^2) \quad (\text{all } r). \quad (3.70)$$

We remark, without elaborating, that this situation will repeat itself when considering the three-point equation, to the effect that

$$\Delta_4^{[r]}(k^2) = \Delta_3^{[r]}(k^2) \quad (\text{all } r), \quad (3.71)$$

where $\Delta_4^{[r]}$ is the polynomial appearing in denominators of level- r approximants to the invariant functions of the 4-point vertex. In other words, the vertex-denominator (or propagator-numerator) polynomials $\Delta_2^{[r]}$, and the vertex-function branch cuts which they represent, turn out to be a common element in all zeroth-order, basic vertex functions. Depending on viewpoint, this element is "handed up" or "handed down" the ladder of DS equations by the hierarchical coupling, which is seen here playing a prominent role.

For our immediate task, this brings substantial simplification since Δ 's now cancel in the integrand of (3.68). For the remaining fraction, it is straightforward to write a partial-fraction decomposition, first with respect to the loop variables q^2 and $(k - q)^2$ and then also with respect to the external k^2 . The former step reduces (3.68) to a linear combination of the basis integrals

$$I_{m,n}^{[r]}(k^2, \Lambda^2) = \int \frac{d^D q}{(2\pi)^D} \frac{(2q - k)^2}{[(q - k)^2 + \sigma_m^{[r]} \Lambda^2][q^2 + \sigma_n^{[r]} \Lambda^2]} \quad (m, n = 1 \dots r + 1) \quad (3.72)$$

These may be evaluated in $D = 4 - 2\varepsilon$ by the standard techniques, the result being

$$I_{m,n}^{[r]}(k^2, \Lambda^2) = -\frac{1}{(4\pi)^2} \left[A_{m,n}^{[r]}(k^2, \Lambda^2; \varepsilon) \cdot k^2 + B_{m,n}^{[r]}(k^2, \Lambda^2; \varepsilon) \cdot \Lambda^2 \right]. \quad (3.73)$$

The functions $A_{m,n}^{[r]}$ and $B_{m,n}^{[r]}$ are given by the Feynman-parameter integrals

$$A_{m,n}^{[r]}(k^2, \Lambda^2; \varepsilon) = \int_0^1 dz \left[\Sigma_{m,n}^{[r]}(k^2, \Lambda^2; z) \right]^{-\varepsilon} \quad (3.74)$$

$$\times \left\{ [12z(1-z) - 1] \left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi \right] + 4z(1-z) + O(\varepsilon) \right\},$$

$$B_{m,n}^{[r]}(k^2, \Lambda^2; \varepsilon) = \int_0^1 dz \left[\Sigma_{m,n}^{[r]}(k^2, \Lambda^2; z) \right]^{-\varepsilon} \quad (3.75)$$

$$\times \left\{ 8[z\sigma_m^{[r]} + (1-z)\sigma_n^{[r]}] \left[\frac{1}{\varepsilon} - \gamma + \ln 4\pi + \frac{1}{2} \right] + O(\varepsilon) \right\},$$

and near $D = 4$ have weak, logarithmic dependences on k^2 and Λ^2 through the $(-\varepsilon)$ -th power of the quantity

$$\Sigma_{m,n}^{[r]}(k^2, \Lambda^2; z) = z(1-z)k^2 + [z\sigma_m^{[r]} + (1-z)\sigma_n^{[r]}] \Lambda^2. \quad (3.76)$$

(To maintain correct mass dimensions, we have stopped short of expanding the non-integer power). After collecting the debris of the partial-fraction decompositions, eq. (3.67) now assumes the form

$$\begin{aligned} & [D^{-1}(k^2)]^{[r,0]} + g^2 [D^{-1}(k^2)]^{[r,1]} + O(g^4) \\ &= k^2 + \left(\frac{\bar{g}_0}{4\pi} \right)^2 \left\{ r_0^{[r]}(k^2, \Lambda^2; \varepsilon) \cdot k^2 + r_1^{[r]}(k^2, \Lambda^2; \varepsilon) \cdot \Lambda^2 \right. \\ & \left. + \sum_{s=1}^r \frac{r_{2s+1}^{[r]}(k^2, \Lambda^2; \varepsilon) \Lambda^4}{k^2 + u_{2s} \Lambda^2} + [\text{other loop terms}]^{[r,0]} + O(g^2) \right\}, \end{aligned} \quad (3.77)$$

where the quantities $r_i^{[r]}$ have structure

$$r_i^{[r]} = \sum_{m,n=1}^{r+1} \left[a_{i,m,n}^{[r]} \cdot A_{m,n}^{[r]}(k^2, \Lambda^2; \varepsilon) + b_{i,m,n}^{[r]} \cdot B_{m,n}^{[r]}(k^2, \Lambda^2; \varepsilon) \right], \quad (3.78)$$

and therefore weak k^2 and Λ^2 dependence near $D = 4$. About the dimensionless coefficients $a_{i,m,n}^{[r]}$ and $b_{i,m,n}^{[r]}$, we need to know no more than that they depend linearly on the level- r , three-point vertex coefficients $x^{[r]}$, and in general nonlinearly on the set of parameters of the inverse propagator itself (the $u_i^{[r]}$ of eq. (3.28), or any equivalent set). With eq. (3.77), we are ready for posing the self-consistency question. It is, by the way, no later than at this stage that one realizes that condition (3.70) is a necessity.

This is, once more, a moment to pause – two observations are worth mentioning. First, while our integrand, with its Λ^2 terms all over the place, is manifestly a strongly nonperturbative object which has no power-series expansion in the coupling at all, nothing has kept us from computing the loop by the standard techniques. These techniques are sometimes, misleadingly, referred to as ”‘perturbative”, which sounds as if they were applicable only in pure perturbation-theory calculations, but in fact they depend, of course, only on the rational structure of integrands with respect to momenta.

Second, it is clear that from a purely technical standpoint, nothing would have stood in the way of our using directly the resurgent symbols – that is, the operator-product expansions – of D and of Γ^3 in the loop integral (3.64). Instead of (3.72) we would have encountered momentum integrals of type

$$\int \frac{d^D q}{(2\pi)^D} \frac{(q - \frac{1}{2}k)^2}{[(q - k)^2]^s [q^2]^t} \propto \left(\frac{1}{k^2}\right)^{s+t-3+\varepsilon}, \quad (3.79)$$

which are increasingly singular as $k^2 \rightarrow 0$, and we might have proceeded to matching the series involving these integrals to the series of eq. (2.56) at increasing orders n , thus determining in principle the resurgent-symbol coefficients $U_n^{(R)}(0, 0)$. But it would nevertheless have been foolish to do so. On the mathematical side, the coefficients of proportionality in (3.79) would be Feynman-parameter integrals increasingly divergent for s or $t > 2$, and would force us to throw in a new set of rather obscure regularizations that might (hopefully) go away in the symbols’s final resummation. Even if we were willing to adopt such an ugly formalism, after all the effort we would still be around with our resurgent symbol, a large- k^2 asymptotic representation, and would have made no progress at all toward knowing the function (D^{-1}) at intermediate and small k^2 – just as in Borel-Laplace reconstruction when we forget the crucial continuation step (1.51). On the physical side, we would have decomposed the well-defined physical effect described by our diagram Fig. 3.1 (2 C) – the self-dressing of a field quantum by virtual pairs of quanta of its own kind – in a totally unreasonable way: into a sum of increasingly infrared-singular terms with increasingly violent small- k^2 cancellations between them. Finally, this procedure would have broken down completely for the ”tadpole” diagram

of Fig. 3.1 (2B): by using the resurgent symbol for its single- D integrand, we would have decomposed it into a sum of scale-free integrals which vanish in dimensional regularization, whereas with any reasonable analytic continuation this term contributes a nonzero multiple of Λ^2 , in marked contrast to perturbation theory. Thus it is abundantly clear that (as mentioned already in the introduction) we cannot simply plug our resurgent symbols into the dynamical equations in order to determine their parameters – some amount of analytic continuation is necessary *from the outset* to deal with objects globally usable in momentum space, and it is such objects that we are trying to establish.

Returning to eq. (3.77), we note that although our task here is more intricate than the standard task of matching a rational approximant to an explicitly known function, the two problems seem to share the property that there is in principle a continuum of matching prescriptions for determining the approximant's coefficients – Padé I or II, as mentioned after our eq. (1.53) above, are just two among many (and Padé II in itself already represents a continuum of possibilities). But a moment's reflection will show that in the present problem there *is* a distinctly preferred matching strategy: we should clearly see to it that we get the *poles* (at the zeroes $k^2 = -u_{2s}^{[r]} \Lambda^2$ of the Δ_2 denominator) right, from the r. h. s. of (3.77), for the postulated zeroth-order $(D^{-1})^{[r,0]}$. If we failed to ensure this, the remainder in eq. (3.77), which is to give (the (2C)-type contribution to) the first-order-quasi-perturbative correction $g^2 (D^{-1})^{[r,0]}$, would have its typical one-loop analyticity properties contaminated by splinters from the zeroth-order poles, and thus would be plagued by approximation errors that become locally infinite in the k^2 plane. Having already ensured correct pole positions through conditions (3.69), we therefore need to match *residues* at those positions. Looking at eq. (3.28) (with $\zeta_0^{[r]} = 1$) above, we are led to imposing the self-consistency conditions

$$u_{2s+1}^{[r]} = \left(\frac{\bar{g}_0}{4\pi} \right)^2 r_{2s+1}^{[r]} (-u_{2s} \Lambda^2, \Lambda^2; \varepsilon) \quad (s = 1 \dots r). \quad (3.80)$$

For the $r_1 \Lambda^2$ term, there is no such obviously preferred matching prescription, so here we may freely choose a matching point $k^2 = -\omega_1 \Lambda^2$:

$$u_1^{[r]} = \left(\frac{\bar{g}_0}{(4\pi)} \right)^2 r_1^{[r]} (-\omega_1 \Lambda^2, \Lambda^2; \varepsilon) \quad (3.81)$$

(This freedom of choice is actually part of the freedom of performing finite renormalizations, since the u_1 term can be shifted around by a *local* Lagrangian counterterm). In all these conditions, the anxious question of how's-this-going-to-give-a-zeroth-order, with the explicit \bar{g}_0^2 factor in front, is still hanging over us.

But only apparently! With k^2 , in these conditions, now nailed to fixed multiples of Λ^2 , the $A_{m,n}$ and $B_{m,n}$ functions entering in eq. (3.78) now appear with values

$$A_{m,n}^{[r]}(-c\Lambda^2, \Lambda^2; \varepsilon) = (\Lambda^2)^{-\varepsilon} \frac{1}{\varepsilon} \{1 + [O(\varepsilon)](c)\}, \quad (3.82)$$

$$B_{m,n}^{[r]}(-c\Lambda^2, \Lambda^2; \varepsilon) = (\Lambda^2)^{-\varepsilon} \frac{1}{\varepsilon} \left\{4(\sigma_m + \sigma_n)^{[r]} + [O(\varepsilon)](c)\right\}, \quad (3.83)$$

so all the $r_i^{[r]}$ at matching points $-c\Lambda^2$ in conditions (3.80) and (3.81) are found to have factors of $(\Lambda^2)^{-\varepsilon} \frac{1}{\varepsilon}$ up front. Now take a closer look at the quantity

$$\left(\frac{\bar{g}_0}{4\pi}\right)^2 (\Lambda^2)^{-\varepsilon} \frac{1}{\varepsilon} = \left(\frac{g(\mu)}{4\pi}\right)^2 Z_\alpha \left(\frac{\Lambda^2}{\mu^2}\right)^{-\varepsilon} \frac{1}{\varepsilon}, \quad (3.84)$$

where relation (0.6) has been used. Proceed the way you are accustomed to, Laurent-expanding the last two factors around $\varepsilon = 0$. You have

$$\frac{1}{\varepsilon} \left(\frac{\Lambda^2}{\mu^2}\right)^{-\varepsilon} = -\ln\left(\frac{\Lambda^2}{\mu^2}\right) + \frac{1}{\varepsilon} + O(\varepsilon). \quad (3.85)$$

The logarithmic term survives the $\varepsilon \rightarrow 0$ limit in the same, familiar way as the well-known $\ln(k^2/\mu^2)$'s in a perturbative calculation, but it differs from them in one important respect: it leads them by one order in the quasi-perturbative $(g^2)^p$ classification. From eq. (2.25),

$$-\ln\left(\frac{\Lambda^2}{\mu^2}\right) = \left(\frac{4\pi}{g(\mu)}\right)^2 \frac{1}{\beta_0} [1 + O(g^2 \ln g^2)] \quad (3.86)$$

at the one-loop level. This is precisely the $1/g^2$ factor needed to "eat" the overall g^2 factor in (3.84) and make zeroth-order-in- g^2 terms possible. There's more: we now have

$$\left(\frac{\bar{g}_0}{4\pi}\right)^2 (\Lambda^2)^{-\varepsilon} \frac{1}{\varepsilon} = \frac{1}{\beta_0} \left[1 + \beta_0 \left(\frac{g}{4\pi}\right)^2 \frac{1}{\varepsilon} + \dots\right] Z_\alpha, \quad (3.87)$$

but Z_α , to the same order, is given by eq. (2.13), so that to the order at which we calculate here, the ultraviolet divergence too gets "eaten" in the special quantities that enter our matching conditions. From eq. (3.80), this in particular means that no nonlocal counterterms will be needed for the pole terms in eq. (3.77): as we expected from our discussion of the OPE, they establish themselves in a finite manner.

Our simple one-loop-level argument has established the values

$$A_{m,n}^{[r]}(-c\Lambda^2, \Lambda^2; 0) = \frac{1}{\beta_0}, \quad (3.88)$$

$$B_{m,n}^{[r]}(-c\Lambda^2, \Lambda^2; 0) = \frac{4}{\beta_0} \cdot (\sigma_m + \sigma_n)^{[r]} \quad (3.89)$$

at $D = 4$ only up to terms $O(g^2 \ln g^2)$, which still seem to be a potential nuisance in conditions (3.80) through (3.81). It is therefore useful to know that a more careful argument [STI96] establishes these values *exactly* to all orders. This argument avoids the use of the unrelated expansions (3.85) and (2.13) completely, and instead employs the exact integral representations for the two middle factors on the r. h. s. of eq. (3.84). For Z_α , we already noted t'Hooft's representation, eq. (2.12) above. For $(\Lambda_\varepsilon^2)^{-\varepsilon}$ – be careful here to consider the $\varepsilon > 0$ quantity formed with the full $\varepsilon > 0$ beta function (2.8) – it is a matter of a few lines of calculation to deduce from definition (2.20) the exact representation

$$\left(\frac{\Lambda_\varepsilon^2}{\mu^2}\right)^{-\varepsilon} = \frac{g_1^2}{g^2(\mu)} \exp \left\{ \int_{(g_1/4\pi)^2}^{(g/4\pi)^2} \frac{d\kappa'}{\kappa' + \varepsilon \chi(\kappa')} \right\}, \quad (3.90)$$

involving again the function $\chi(\kappa)$ defined by (2.11). Upon combining the two representations one sees that in the quantity (3.84), the dependence on $g^2(\mu)$, and therefore on μ , cancels exactly (remember g_1 is a trivially RG-invariant integration constant). Further analysis of the integral in

$$\left(\frac{\bar{g}_0}{4\pi}\right)^2 (\Lambda_\varepsilon^2)^{-\varepsilon} \frac{1}{\varepsilon} = \frac{g_1^2}{(4\pi)^2 \varepsilon} \exp \left\{ - \int_0^{(g_1/4\pi)^2} \frac{d\kappa'}{\kappa' + \varepsilon \chi(\kappa')} \right\} \quad (3.91)$$

by exactly the same technique as applied in eqs. (2.15) and (2.16) gives, finally,

$$\left(\frac{\bar{g}_0}{4\pi}\right)^2 (\Lambda_\varepsilon^2)^{-\varepsilon} \frac{1}{\varepsilon} = \frac{1}{\beta_0} [1 + O(\varepsilon, \varepsilon \ln \varepsilon)], \quad (3.92)$$

rigorously independent of $g^2(\mu)$. Despite the superficial similarity, this should not be confused with eq. (2.16); it says something conceptually quite different. While (2.16) is a representation for Z_α with a $g^2(\mu)$ dependence of the r. h. s. that is difficult to control as long as one does not know the exact RG beta function, the result (3.92) states the presence of an *exact RG invariant*, which, moreover, is *finite and scheme-independent* at $\varepsilon = 0$. Its point is the exact contragredience of the g^2 dependences in the quantities (2.12) and (3.90) *already at $\varepsilon \neq 0$* . Both a g^2 factor and an $\frac{1}{\varepsilon}$ divergence get "eaten" in this remarkable combination.

It is not necessary, in these seminars, for us to pursue the consequence of eqs. (3.88) and (3.89) through the $r_i^{[r]}$ functions to the self-consistency conditions (3.80)/(3.81) in every detail. It is sufficient to state that the latter now assume, at $\varepsilon = 0$, the general form

$$u_{2s+1}^{[r]} = \frac{1}{\beta_0} R_{2s+1} \left(\{u\}^{[r]}, \{x\}^{[r]} \right) \quad (s = 0 \dots r), \quad (3.93)$$

with functions R_{2s+1} depending linearly on the set $\{x\}^{[r]}$ of three-point vertex coefficients, and in general nonlinearly on the set $\{u\}^{[r]}$ of the inverse-propagator parameters u_i themselves. These conditions allow one, remarkably, to establish functions $(D^{-1})^{[r,0]}$ of zeroth quasi-perturbative order from DS loops with a \bar{g}_0^2 prefactor, and to do so in an ultraviolet-finite manner not requiring nonlocal counterterms.

As we have anticipated, apart from equations such as (3.70) and (3.71) that establish a common set of denominator coefficients $u_{2s}^{[r]}$ for all vertex functions, we have not found conditions that would really determine these coefficients. We have already sketched how one may go about filling this gap by invoking higher degrees of compositeness; setting up equations for the quantities (3.61) and (3.62) one obtains the r additional conditions needed. Apart from the technical complication that such equations now have at least two-loop terms on their right-hand sides, and involve the trickery of extracting two-loop divergences, their treatment requires no truly new ideas; in particular, relation (3.92) again plays a central role. We therefore skip details of these "integrated" conditions entirely. For the same reasons, we do not enter into details of the next higher vertex equations, which involve application of the same technology to more variables and lengthier loop integrals. For example, the middle (three-point) equation of Fig. 3.1, when analyzed for terms of zeroth quasi-perturbative order, will produce conditions for the three-point parameters $\{x\}^{[r]}$ of the general form

$$x_{m_1 m_2 m_3}^{[r]} = \frac{1}{\beta_0} S_{m_1 m_2 m_3} \left(\{u\}^{[r]}, \{x\}^{[r]}, \{t\}^{[r]} \right), \quad (3.94)$$

with right-hand sides now involving the set $\{t\}^{[r]}$ of four-point numerator coefficients. The structure of these conditions, which after multiplication with enough denominators become a system of nonlinear algebraic equations, mirrors quite closely the structure of the underlying DS system with its nonlinearities and hierarchical couplings.

As for the remainder terms in eq. (3.77) that we expect to yield the first quasi-

perturbative correction, we are now left with

$$\begin{aligned}
g^2 [D^{-1}(k^2)]^{[r,1]} &= \left(\frac{\bar{g}_0}{4\pi}\right)^2 \left\{ r_0^{[r]}(k^2, \Lambda^2; \varepsilon) k^2 + \right. \\
&\left. \left[r_1^{[r]}(k^2, \Lambda^2; \varepsilon) - r_1^{[r]}(-\omega_1 \Lambda^2, \Lambda^2; \varepsilon) \right] \Lambda^2 \right. \\
&+ \sum_{s=1}^r \frac{r_{2s+1}^{[r]}(k^2, \Lambda^2; \varepsilon) - r_{2s+1}^{[r]}(-u_{2s} \Lambda^2, \Lambda^2; \varepsilon)}{k^2 + u_{2s} \Lambda^2} \Lambda^4 \\
&\left. + [\text{contributions from other loop terms}] \right\}
\end{aligned} \tag{3.95}$$

The unsubtracted term $r_0^{[r]} k^2$ carries the $\Lambda^2 \rightarrow 0$ "perturbative limit" as it would come out from the usual perturbative calculation of the loop, *including* the perturbative $\frac{1}{\varepsilon}$ ultraviolet divergence. The other terms, which the self-consistency process has left subtracted, are both ultraviolet finite and (for the terms with Λ^4) regular at $k^2 = -u_{2s} \Lambda^2$, since by eqs. (3.78), (3.74) and (3.75) they are linear combinations of the quantities

$$\begin{aligned}
&\left[\Sigma_{m,n}^{[r]}(k^2, \Lambda^2; z) \right]^{-\varepsilon} - \left[\Sigma_{m,n}^{[r]}(-c \Lambda^2, \Lambda^2; z) \right]^{-\varepsilon} = \\
&-\varepsilon (k^2 + c \Lambda^2) z(1-z) \int_0^1 dw \left\{ wz(1-z) k^2 \right. \\
&\left. + \left[z \sigma_m^{[r]} + (1-z) \sigma_n^{[r]} - (1-w) cz(1-z) \right] \Lambda^2 \right\}^{-(1+\varepsilon)}.
\end{aligned} \tag{3.96}$$

With respect to renormalization, we are therefore led into precisely the situation we expected from our OPE discussion around eq. (2.49): we have to follow the "two-track" procedure of supplying an explicit counterterm, proportional to $g^2 \frac{1}{\varepsilon}$, for the perturbative portion, while the terms with powers of Λ^2 establish themselves in an ultraviolet-finite manner. As for the \bar{g}_0^2 in expression (3.95), we note that its rigorously g^2 -independent combination (3.92) with the quantity $(\Lambda^2)^{-\varepsilon} \frac{1}{\varepsilon}$ no more occurs, and since we are working within a quasi-perturbative scheme, we must therefore revert to the usual pattern of treating it as $\mu^{2\varepsilon} g^2(\mu) + O(g^4)$, the $\mu^{2\varepsilon}$ serving as usual to render the $(\Sigma^{[r]})^{-\varepsilon}$ terms in the quantities (3.74) and (3.75) dimensionless.

It is clear that our condition (3.66) of maintaining strict perturbative limits bars us from "oversubtracting" our result (3.77), in the sense of subtracting also the $r_0^{[r]}$ term at some point $k^2 = -\omega_0 \Lambda^2$ and invoking the mechanism of eq. (3.92). Such a procedure would inevitably lead us away from the perturbative limit $\xi_0 = 1$ and into a different version of the quasiperturbative expansion, analogous to the one we noted for our model amplitude in eqs. (2.212) and (2.214). We are not going to explore this route here, interesting as it may be, since it would also force us to reorganize the entire perturbative-renormalization procedure, and therefore would lead us too far away from our stated focus.

Several features of this self-consistency mechanism are worth commenting on. First, the final self-consistency conditions at $D = 4$ (i. e. $\varepsilon = 0$) are independent of the chosen subtraction points $-u_{2s} \Lambda^2$ and $-\omega_1 \Lambda^2$, since the latter only enter into the terms $O(\varepsilon)$ in eqs. (3.82) and (3.83). The subtraction points do, however, enter into the first-order, quasi-perturbative correction $[D^{-1}]^{[r,1]}$ of eq. (3.95), and are essential for keeping it uncontaminated by the zeroth-order rational-approximation poles.

Second, the conditions at $D = 4$, while derived here in a dimensional-regularization scheme, make no reference any more to a special renormalization scheme, since β_0 is scheme-independent. Now different schemes, as noted in eq. (2.23) above, have Λ scales differing by constant numerical factors. This leads us to suspect that upon scaling Λ by a factor, the conditions will remain invariant when scaling all the nonperturbative x and u coefficients inversely to the Λ^2 powers with which they are associated. This *scaling property* indeed turns out to be present in the system, and has the effect that the conditions in fact determine only the ratios of all coefficients to one of them. The undetermined coefficient, with its associated Λ^2 power, effectively defines a modified scale $\tilde{\Lambda}$, which can be fixed only by calculating and matching some dimensionful observable.

Third, it is manifest that the entire mechanism is intimately *tied to the presence of the ultraviolet loop divergences*: the crucial term (3.86), and the ensuing terms (3.88) and (3.89), while themselves finite, have survived the removal-of-regulator limit $\varepsilon \rightarrow 0$ only due to their association with a $\frac{1}{\varepsilon}$ divergence. We have all known such an effect from our experience with perturbative logarithms, but here it assumes a new and much more prominent role. The self-reproduction of zeroth quasi-perturbative orders is therefore not only a genuine quantum effect (as it comes from loops) but also a genuine quantum-field effect (as it comes from divergent loops).

This has the immediate consequence that, fourth, the formation of zeroth quasi-perturbative orders becomes rigorously restricted to the small finite number of superficially divergent vertices – by and large, the same functions that already possess a zeroth

perturbative order. You need no big calculation to see this. In a superficially convergent amplitude, the one-loop contributions have factors of \bar{g}_0^2 but are, by definition, genuinely convergent, so they cannot trigger the mechanism of eq. (3.92) and remain quasi-perturbative corrections of the first order. The two-loop contributions, with factors of \bar{g}_0^4 , may exhibit ultraviolet divergences, but only in the form of one-loop subdivergences producing a single $\frac{1}{\epsilon}$ factor, and generally in the higher l -loop terms the number of $\frac{1}{\epsilon}$ factors produced lags behind the number of overall \bar{g}_0^2 factors by at least one unit. So the mechanism of eq. (3.92) can "eat" at most $l - 1$ of the l factors of \bar{g}_0^2 , and all contributions remain $p \geq 1$ quasi-perturbative corrections.

The same consideration applies to terms such as (4 C) of Fig. 3.1 that couple the highest superficially divergent to the lowest superficially convergent vertices. Since there is no $\Gamma_5^{(0)pert}$ in a renormalizable action, the lowest contributions to term (4 C) have Γ_5 replaced by its convergent one-loop terms and two factors of \bar{g}_0^2 in front, of which at most one can be eaten. Thus *as far as self-reproduction of zeroth orders* is concerned, the hierarchical coupling automatically stops acting after the four-boson vertex equation. Once more we have a fine line (depicted in the third equation of Fig. 3.1) marking a fundamental divide, this time within the strictly renormalizable theory itself: between the equations for superficially divergent and for superficially convergent functions.

The self-consistency conditions, fifth, do nevertheless retain an infinite-expansion aspect of a different kind, at least in the somewhat conservative treatment we have afforded them here. We have been classifying the contributions to the right-hand sides of the integral equations in the time-honored way by their number of loops, or independent momentum integrations, that arise upon iterating the equations around the newly established, nonperturbative starting points $\Gamma^{[r,0]}$. It is clear that this will translate into an analogous loop classification of contributions to the right-hand sides of the self-consistency conditions. In a superficially divergent amplitude, there is no lagging of the number of \bar{g}_0^2 prefactors; an l -loop diagram here produces terms of type

$$(\bar{g}_0^2)^l \cdot \left\{ 1, \frac{1}{\epsilon}, \frac{1}{\epsilon^2}, \dots, \frac{1}{\epsilon^l} \right\} \quad (3.97)$$

and through the mechanism of eq. (3.92) therefore gives contributions to the quasi-perturbative series of the forms

$$(g^2)^l, \quad \frac{1}{\beta_0} (g^2)^{l-1}, \quad \frac{1}{\beta_0^2} (g^2)^{l-2}, \quad \dots, \frac{1}{\beta_0^l}, \quad (3.98)$$

all the way down to the zeroth order: a given loop number l produces not only terms of order $(g^2)^l$ but also corrects all previous orders $(g^2)^p$ with $p < l$. That the tight coupling

between l and p orders characteristic of a *perturbative* single-coupling theory should get loosened is presumably something that happens in any truly nonperturbative treatment. (It complicates diagrammatic argument, in that a quasi-perturbative correction $\Gamma^{[r,p]}$ with $p \geq 1$ can no more simply be written as a set of p -loop diagrams, but must include finite counterterms to account for the pieces that go into orders $p' < p$). But here it has the specific consequence that the self-consistency condition for a generic coefficient $x^{[r]}$ of a zeroth-order nonperturbative approximant in principle gets corrections from all higher loop orders that form a series in powers of $1/\beta_0$:

$$x^{[r]} = \frac{1}{\beta_0} X_{(l=1)}^{[r]} + \frac{1}{\beta_0^2} X_{(l=2)}^{[r]} + \dots \quad (3.99)$$

This type of series will not create new summability problems of its own, since from (3.97) you see that it arises from the "most strongly divergent terms", terms proportional to $(1/\varepsilon)^l$, in all loop orders l , and these terms, like the "leading logarithms" to which they are closely related, generally do not exhibit factorial divergence but form a series with nonzero convergence radius, behaving roughly like a geometric series. But I have not yet had enough experience with this formalism to have a ready suggestion for its practical resummation, which at this time remains a (somewhat technical but not unimportant) research problem. In pure $SU(N)$ Yang-Mills theory, where $\beta_0 = \frac{11}{3}N$, eq. (3.99) clearly is of the nature of an $\frac{1}{N}$ expansion (though only formally, since $X_l^{[r]}$ may well contain contributions comparable to $(\beta_0)^l$ of its own), and since general experience with $\frac{1}{N}$ expansions indicates they tend to work much better than anyone has a right to expect, it seems not unreasonable to start out by drawing conclusions from the lowest orders of (3.99) only.

Sixth and last for these comments, while we have stopped short of technical detail for three-point and four-point vertex equations (on that subject I must refer you to refs. [DRI99]), it is already clear from the inverse-propagator equation alone, in e. g. the form (3.77), that the mechanism allows us, in the well-defined approximation of a zeroth quasi-perturbative order, to watch the spectacle of the *spontaneous generation of mass* for a field that was massless on the Lagrangian level – the process we referred to in eq. (2.104). If the field decides to form a single massive particle, its squared mass at level r will be given by the first root of the propagator denominator (3.20),

$$(m^2)^{[r]} = \sigma_1^{[r]} \Lambda^2, \quad (3.100)$$

with $\sigma_1^{[r]}$ calculable from the self-consistent u_i coefficients. Because of the scaling property, solution of the $[r, 0]$ self-consistency problem alone does not, of course, make this

an absolute prediction, since at this stage you do not yet know which Λ scale you are working with. But ratios of particle masses do in principle become predictable. It is interesting to note that in some sense this is already true on the $r = 0$ level (which is not to be confused with the perturbative limit). The $r = 0$ approximation is so crude that it can be accused of internal inconsistency – as you see from (3.41), it keeps the three-point, and in fact also the four-point, vertex functions at their perturbative forms $\Gamma_{3,4}^{(0) \text{pert}}$, at least under our "boundary condition" (3.66) above, but it does allow the inverse propagator a nonperturbative mass term proportional to Λ^2 ,

$$(D^{-1})^{[0,0]} = k^2 + u_1^{[0]} \Lambda^2, \quad (3.101)$$

so that $\sigma_1^{[0]} = u_1^{[0]}$. With such a crudely restricted scheme, you are not surprised when you work out the self-consistency condition and find the form $u_1^{[0]} = \text{const.} \cdot u_1^{[0]}$ which allows only the trivial solution $u_1^{[0]} = 0$, since for realistic values of β_0 the constant refuses to be unity. But it is interesting that for an asymptotically free theory with $\beta_0 > 0$, the constant is positive, so that even at this oversimplified level the system sends a muted signal of its tendency to produce a spontaneous mass.

3.3 Things done and not done

At the end of these seminars, let us take a stroll along the periphery of the landscape that we tried to begin to explore, casting glances at the few things that have been done or attempted in the matter of construction of coupling-resurgent amplitudes, and at the larger number of things that have not been done but could be worthwhile taking up if you find these questions interesting.

The method described in the last two sections for constructing resurgent vertex functions from quasi-perturbative series, based on a set of generalized Feynman rules, has not at this time been tested to a degree that one could call comprehensive, but there has been a reasonably complete application to massless QCD at $r = 1$, the lowest non-trivial level of rational continuation, for which I must refer you to the articles [DRI99]. This study employs Landau gauge fixing and plausibly motivated restrictions on the oppressively complicated tensor structure of the four-gluon vertex, and concentrates on the case of at most two flavors of massless quarks. It finds that the two superficially divergent vertices for Faddeev-Popov ghosts, *at the zeroth quasi-perturbative order*, remain perturbative in Landau gauge fixing, a conclusion I would like to see corroborated by independent studies. For the zeroth-order self-consistency problem of the remaining five superficially divergent functions (three for the gluon and two for the fermion sector),

it finds acceptable solutions with all-real approximants, which after throwing out a few on physical grounds are even remarkably unique. Unlike the pure Yang-Mills case (no quarks) which is also considered, the $r = 1$ transverse-gluon propagator obtained is not entirely satisfactory, since its zero at $k^2 = -u_2^{[1]} \Lambda^2$ slips a bit into the Euclidean region, thus violating reflection positivity on a small Euclidean interval – an effect coming from a strong coupling to virtual quark-antiquark pairs, which the approximation obviously exaggerates. You will tolerate my not entering into further discussion of the propagators obtained in those articles – they are interesting, but their interpretation would drag us into issues of confinement far removed from the focus of these talks. Here I am quoting refs. [DRI99] mainly as a demonstration of technical feasibility.

One problem for which refs. [DRI99] do offer a reasonably complete solution, even for arbitrary r , looks at first like a purely technical one. But it turns out to have a bearing on the fundamental issue of unitarity (total-probability conservation), and therefore let me briefly stop at it on our stroll. When we found we had to impose eq. (3.70) as a self-consistency condition, you may have wondered in passing what would have happened in the diagram of Fig. 3.1 (2 C), or its analytic expression (3.68) at level r , without that condition. Then we would have had on each of the two internal lines, in addition to the r -level propagator poles of eq. (3.21) that provide a discretized Källén-Lehmann description of the propagation of the virtual particle, r additional poles from an $1/\Delta_3$ factor that would plainly contradict the postulated propagator approximant. Condition (3.70) saved us from that embarrassment, but only because in that diagram both internal lines end in a bare vertex on the left-hand side. But now take a look at the second of the DS equations of Fig. 3.1, the three-point vertex equation, and its loop term (3 B). In that loop, the right-hand part involving the vertical internal line represents a dressed one-particle exchange of the form

$$\Gamma_3(\dots, \dots, Q) D(Q^2) \Gamma_3(Q, \dots, \dots), \quad (3.102)$$

with Q the momentum of the vertical line. Since that line now extends between two *dressed* vertices, only one of the two $1/\Delta_3$'s from these vertices gets cancelled by the Δ_2 numerator of $D(Q^2)$ by virtue of eq. (3.70), and now the embarrassment of the "superfluous poles" seems to be really on us: such poles will, by the diagram-cutting rules for unitarity relations, produce contributions to these relations that do not correspond to any accessible external states, so unitarity would be endangered. It is a minor surprise (at least to me) to see that the formalism solves that problem automatically by its own devices: one finds that the four-point vertex Γ_4 in diagram (3 C) unavoidably develops poles in its crossed channels that are the exact opposites of the "superfluous" poles in (3.102). (That conclusion, remarkably, follows from residue-taking operations

in the three-point and two-point equations alone, that is, from *lower* equations in the hierarchy). Note that since the denominators in these "compensating poles" are not propagators of any of the elementary fields of the theory, their presence does not contradict the one-particle-irreducible nature of Γ_4 . Their effect in the present context is to turn the vertical internal line described by (3.102) into what refs. [DRI99] denote by a *dotted internal line*, that is, a partial-fraction decomposition of expression (3.102) in which only the "true" propagator poles have been kept. Further analysis quickly reveals that these dotted lines really are almost everywhere, being present on every internal line between two dressed vertices. They are an important ingredient for keeping the formalism in line with the unitarity requirement. One amusing consequence of their presence that has not even begun to be exploited is that they provide a natural discretization of Bethe-Salpeter bound-state equations that refines itself with increasing r .

What is entirely lacking at present – and that brings us to the part of the countryside where the things not done are lying around – is tests of the formalism at levels $r > 1$ of rational continuation. (They are necessary not only for numerical refinement but also because rational approximants have a reputation for sometimes behaving erratically at low orders). The reasons are, to a large extent, purely technical and have little to do with physics: the nonlinear algebraic systems of self-consistency equations turn out to be a surprisingly tough assignment. Refs. [DRI99] were still able to get away with ad hoc methods of solution, exploiting accidental simplifying features of their Landau-gauge, $r = 1$ system, but higher r 's call for more systematic methods. From present experience, it seems fair to say that no presently known algorithm or available routine is capable of locating the rapidly growing number of solutions of realistic $r \geq 2$ systems *completely* in tolerable amounts of computing time, and most methods even have a hard time coping with $r = 1$ systems. So here the formalism also creates a new challenge for applied mathematics. The provisional solution currently available is to rely on minimization methods that do not guarantee to find all solutions, to let them search a reasonable portion of the space of starting values as finely as one can afford, and to rely on the experience that a fully real and physically acceptable solution, as a rule, stands out quite conspicuously and essentially uniquely.

The correlation functions of QCD, by the way, may not represent the optimal proving ground for first tests of the method described here – or, for that matter, of any method you may invent for coupling-resurgent reconstruction. The fact that the elementary QCD fields have no asymptotically detectable particles associated with them tends to intermix and obscure all questions of nonperturbative methodology, in which we are interested here, with questions of the physical interpretation of various features

of the functions obtained – particularly of the propagators. A much cleaner environment conceptually is the $SU(2)$ gauge system of electroweak theory, with coupling to fermions but without strong interactions. Here, quasi-perturbative corrections are almost certainly small at energies around the relevant Λ scale, and resurgent amplitudes of the zeroth quasi-perturbative order should allow one in principle to explore with some amount of systematics the question of precisely how much can and cannot be done without the help of a Higgs field, particularly with respect to dynamical mass generation. All elementary fields here have asymptotically detectable particles, so the requirements on and interpretation of propagators are clean and simple.

The same environment may be well suited for studying the question of when-and-how-and-how-much the coupling-resurgent amplitudes break symmetries, particularly the local gauge symmetries. Here I may remind you of the fact that quantum local symmetries place much more stringent invariance requirements on the functional measure of the generating functional (3.47) than the simple translation invariance we used for deriving the DS equation (3.52). The DS equations, consequently, are more fundamental than the Slavnov-Taylor identities that express local gauge invariance on the level of quantum correlation functions; they reserve the right of producing symmetry-breaking solutions. This possibility becomes particularly acute in connection with nonperturbative solutions containing Λ terms – since the self-consistency of the latter, as we saw in sect. 3.2, is a genuine quantum effect arising from ultraviolet-divergent loops, they have in principle the power to break classical symmetries, much like anomalies in that respect. On the other hand the perturbative limits of amplitudes, denoted $\Gamma_N^{(0)pert}$ in eq. (0.4), do preserve the quantum-symmetry identities, and that is all that is both necessary and sufficient for maintaining the most important practical consequence of gauge invariance – the perturbative renormalizability [WEI96]. I could therefore imagine (why not speculate a little on a casual stroll like this) that a picture of "subasymptotic symmetry breaking" might emerge in general, where perturbative $\Gamma^{(0)pert}$'s as large-momentum limits, with their proven Slavnov-Taylor compliance, uphold renormalizability, while the Λ terms coming into view one by one on moving down the momentum scale increasingly deviate from Slavnov-Taylor. In QCD, things are probably different – the simple fact that gluons obviously do not get saddled with b - or even t -quark masses (through virtual quark-antiquark pairs) would seem to require much stricter adherence to Slavnov-Taylor, and the fact that the solutions of refs. [DRI99] are not doing very well in that respect is another indicator that the approximation level there is still too crude.

A purely methodological but definitely challenging cluster of things-not-done centers around the question of whether the DS equations, as employed here, really offer the

best framework for the reconstruction of resurgent amplitudes. Clever rearrangements of the pure-and-simple DS system, as it emerges from eq. (3.52) and is sketched in Fig. 3.1, have long been known. Already in his first paper on the subject [DYS49], Dyson used what may be called a Bethe-Salpeter resummation of the QED three-point vertex equation, which allows one to replace the bare three-point vertices at the left ends of diagrams (3 B) and (3 C) in Fig. 3.1 by dressed ones, while replacing the right-hand portions of both diagrams by a Bethe-Salpeter kernel K_4 one- and two-particle irreducible in the horizontal channel. Such a rearrangement presumably shifts a substantial amount of physics from higher into lower loop orders, and may make a crucial difference when studying problems of symmetry breaking. It also provides partial, though not complete, relief from a perennial nuisance in most DS equations with $N \geq 3$ – the lack of manifest Bose or Fermi symmetry due to the unsymmetric distinction of the leftmost leg that corresponds to the primary functional differentiation in eq. (3.49). In a non-abelian gauge theory, however, one pays for the progress with equally substantial new complication: the information on the four-point amplitude, already by far the most complicated of the superficially divergent ones, now enters the system in two different ways, as Γ_4 and as K_4 , whose relation to each other must be stated as an additional dynamical equation. This roughly doubles the effort required for dealing with the superficially divergent subsystem. Still more sophisticated rearrangements, originating in work by Symanzik [SYM61] and carried out for non-abelian gauge theory by Baker and Lee [BAL77], allow the elimination of bare vertices in favor of dressed ones to be continued down into the propagator equations (thus creating a system with the remarkable property that its perturbative iteration does not produce diagrams with overlapping ultraviolet divergences). But it once more brings substantial additional complication in that internal lines now carry momentum derivatives of propagators in addition to the propagators themselves. In the search for improvements along these lines, just about everything seems to come at a heavy price. None of these improvements touches the one feature of all DS-type dynamical equations that the mathematician studying resurgence would find most disappointing: that they make no direct reference, either differential or integral or otherwise, to coupling dependence, which enters them only in a somewhat indirect, parametric way. Equations like the t’Hooft-Weinberg RG equations, which directly react to coupling dependence through their trademark $\beta(g) \frac{\partial}{\partial g}$ terms, would be much more after the heart of the resurgence theorist, but while formulating important restrictions on correlation functions, they are not statements of their full dynamics. Equations combining the coupling sensitivity of the t’Hooft-Weinberg with the full dynamical content of the DS equations could perhaps be found among various versions of the exact RG differential equations of Polchinski [POL84]; one example that I happen to know of is ref. [WIE97].

Equations of this kind could be well suited for studying nonperturbative structure of the resurgent type. Once more the subject has not even begun to be explored.

Still strolling on, we dimly perceive from a distance a set of problems that lie at the borderline to mathematics, and may well be the deepest and most long-term ones. I emphasized at the outset that I would be appealing only to a few simple notions from what may be called the surface of resurgence theory, not to its deeper constructs and methods. But to bring these into play in the elucidation of nonperturbative coupling structure may be fascinating in the short and unavoidable in the long run. To mention one example, the ultraviolet-renormalon poles on the left-hand, real Borel axis that we found to persist in our resurgent model amplitude are presumably a common feature of most if not all correlations in asymptotically free theories. In trying to characterize quantitatively the features of coupling dependence controlled by these singularities, and in bringing to the fore the common traits of different correlations arising from them, Écalle's alien-derivative calculus, with its beautiful constructive tool of resurgence monomials, may have a nontrivial contribution to make. Another conjecture brings us, once more, to QCD. There, the quasi-perturbative corrections are not small enough at the scale Λ to be treated by "pragmatic" semiconvergence (as one would not hesitate to do in electroweak theory). Accordingly, the ultimate methods of solution may well be those that abandon completely the n -versus- p -direction dichotomy still prevailing in our quasi-perturbative scheme, and perform partial resummations that cut across the entire resurgent symbol. The prominent role, already noted, that resurgent symbols with support on the non-negative integers play in Écalle's treatment of differential equations may have heuristic significance here: perhaps the resummations needed may be differential approximants in the coupling, resurgent solutions to a well-conceived sequence of differential equations, characterizable by holomorphic invariants of presently unknown physical meaning ...

Our little stroll has led us on to flights of imagination. For here and now, in my opinion, the greater value of resurgence theory lies not in specific techniques, intriguing as they may be. It lies in providing a framework for thinking, and in connecting disparate pieces of physical knowledge. The study of QFT, to the extent that it has used continuum methods, has been dominated conceptually by the perturbation expansion to such an extent that "summing the series" was all but synonymous with establishing the full solution for generations of physicists. Resurgent functions provide a much wider yet precise framework, wide enough presumably to cover the strongly nonperturbative coupling dependence of asymptotically free theories. They quite literally open up a second

dimension. From this, I would expect, will arise their longer-term impact.

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