Supersymmetric Ward identities and their renormalization

Forschungsseminar
Quantenfeldtheorie

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agenda:

- Introduction: The Lagrangian of N=1-SUSY-Yang-Mills in the continuum
- Supersymmetric Invariance
- Ward identities
- the action on the lattice
- renormalization of the supercurrent

Spires database:

fin k supersymmetry $\rightsquigarrow$ 27354 matches

no experimental evidence

theoretical reasons:

- cancellation of divergencies
- hierachy problem
- GUT

1975 Haag, Lopuszanski, Sohnius

SUSY is the only possible extension of the known space-time symmetries of particle theories

$\rightsquigarrow$ supersymmetric gauge field theories

here: $N = 1$-SUSY (simplest case)

$\rightsquigarrow$ $N = 1$-Super-Yang-Mills theory

$$\mathcal{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \frac{1}{2} \bar{\lambda}^a \gamma_\mu (\mathcal{D}_\mu \lambda)^a$$

+gauge fixing+ghosts+auxiliary fields

where $a = 1, ..., N_c^2 - 1$ (here $N_c = 2$ or 3).

λ spinor field „Gluino“

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$

A_μ vector field „Gluon“

$\mathcal{D}_\mu = \partial_\mu + [A_\mu, \lambda]$ covariant derivative

We also write

$$\mathcal{L} = -\frac{1}{2g^2} \mathrm{Tr}\Big\{ F_{\mu\nu} F_{\mu\nu} \Big\} + \mathrm{Tr}\Big\{ \bar{\lambda} \gamma_\mu \mathcal{D}_\mu \lambda \Big\}$$

What are the differences compared to QCD ?

- λ is a Majorana spinor, i.e.

$$\bar{\lambda} = \lambda^t C$$

  C charge conjugation matrix

  $\rightsquigarrow$ only 2 complex degrees of freedom

  $\Rightarrow$ formally $N_f = \frac{1}{2}$ fermions

- λ transforms in the

  adjoint representation

  of the gauge group $SU(N_c)$

- under normal circumstances (i.e in QCD):

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \psi \to e^{i\theta^a T^a}\psi$$

T^a Pauli or Gell-Mann matrices
θ^a parameters of gauge transformation

- here: adjoint representation

representation space:

$N_c^2 - 1$ component vectors ϕ

$$\phi_b \to \left(e^{i\theta^a ad(T^a)}\right)_{bc}\phi_c$$

$ad(T^a)_{bc} = if_{abc}$ structure constants

alternatively:

$\phi = N_c \times N_c$-matrices with zero trace
$\rightsquigarrow N_c^2 - 1$ parameters

decomposition:

$$\phi = \phi^a T^a$$

with normalization

$$\mathrm{Tr}\Big\{T^a T^b\Big\} = \frac{1}{2}\delta_{ab}$$

then:

$$\phi \to e^{i\theta^a T^a} \phi e^{-i\theta^a T^a}$$

summary:

$$\lambda(x) \;\to\; \omega(x)\lambda(x)\omega(x)^{-1}$$

$$A_\mu(x) \;\to\; \omega(x)A_\mu(x)\omega(x)^{-1} - (\partial_\mu\omega(x))\omega(x)^{-1}$$

$\omega(x) \in$ gauge group

makes the theory gauge-invariant.

supersymmetry transformations

- relate fermions and bosons
- leave the action invariant
- depend on a parameter ε which is a spinor

$$\begin{aligned}
\delta A_\mu(x) &= -2g\bar{\lambda}(x)\gamma_\mu\varepsilon \\
\delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}F_{\rho\tau}(x)\varepsilon \\
\delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\varepsilon}\sigma_{\rho\tau}F_{\rho\tau}(x)
\end{aligned}$$

where $\sigma_{\rho\tau} = \frac{i}{2}[\gamma_\rho, \gamma_\tau]$.

What does it mean „leave the action invariant“?

$$\delta\mathcal{L}(x) = \partial_\mu \overline{j_\mu}(x)\varepsilon$$

$\rightsquigarrow$ $\delta\mathcal{L}$=total divergence

$\rightsquigarrow$ $\delta S = 0$ since $S = \int d^4x\mathcal{L}(x)$

What do we need to prove the invariance ?

take $\mathcal{L}$, apply the transformations $\delta A_\mu, \delta\lambda, \delta\bar{\lambda}$ and make use of

- Leibniz-rule

$$\mathcal{D}_\mu(\bar{\lambda}F_{\mu\nu}) = (\mathcal{D}_\mu\bar{\lambda})F_{\mu\nu} + \bar{\lambda}\mathcal{D}_\mu F_{\mu\nu}$$

- Bianchi-identity

$$[D_\mu, F_{\rho\tau}] + [D_\rho, F_{\tau\mu}] + [D_\tau, F_{\mu\rho}] = 0$$

- Fierz-identity

$$\sum_\mu \sum_{a,b,c} f_{abc}(\bar{\lambda}^a\gamma_\mu\lambda^c)(\bar{\lambda}^b\gamma_\mu\varepsilon) = 0$$

$\rightsquigarrow$

$$\begin{aligned} \delta\mathcal{L}(x) &= \partial_\mu\overline{j_\mu}(x)\varepsilon \\ \overline{j_\mu}(x) &= \frac{i}{g}\mathrm{Tr}\Big\{\bar{\lambda}(x)F_{\rho\tau}(x)\Big\}\sigma_{\rho\tau}\gamma_\mu \end{aligned}$$

Noether theorem

symmetry $\Rightarrow$ conserved current

general case, $\varepsilon \neq \varepsilon(x)$

$$S[\phi] = \int d^4x \mathcal{L}(x)$$

$$\phi \rightarrow \phi + \delta\phi(x) \equiv \phi + \varepsilon\Delta\phi$$

Then

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\delta\phi) \\
&= \varepsilon\frac{\partial\mathcal{L}}{\partial\phi}\Delta\phi + \varepsilon\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\Delta\phi) \\
&= \varepsilon\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \varepsilon\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\Delta\phi \\
&= \partial_\mu j_\mu(x)\varepsilon
\end{aligned}$$

If the equations of motion hold

$$\partial_\mu j_\mu(x) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right)$$

$\Rightarrow$ Noether current

$$S_\mu(x) := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \Delta\phi(x) - j_\mu(x)$$

is conserved in the sense of

$$\partial_\mu S_\mu = 0$$

For $N = 1$-SUSY-Yang-Mills:

$$\overline{S}_\mu(x) = -2\frac{i}{g} \mathrm{Tr}\Big\{ \bar{\lambda}(x) F_{\rho\tau} \Big\} \sigma_{\rho\tau} \gamma_\mu$$

Ward identities

All this was on the classical level (no path integrals so far)

define:

$$\begin{aligned} Z &:= \int \mathcal{D}\phi e^{-S[\phi]} \\ S_{qu}[\phi, J] &:= \int d^4x \phi(x) J(x) \\ < \mathcal{O} > &:= \frac{1}{Z} \int \mathcal{D}\phi \mathcal{O}[\phi] e^{-S[\phi]} \end{aligned}$$

Then for a possibly local transformation

$$\begin{aligned} & \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]-S_{qu}[\phi]} \mathcal{O}[\phi] \\ = & \frac{1}{Z} \int \mathcal{D}\phi^{\varepsilon} e^{-S[\phi^{\varepsilon}]-S_{qu}[\phi^{\varepsilon}]} \mathcal{O}[\phi^{\varepsilon}] \\ = & \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi^{\varepsilon}]-S_{qu}[\phi^{\varepsilon}]} \mathcal{O}[\phi^{\varepsilon}] \end{aligned}$$

therefore we conclude

$$0 = \frac{\delta}{\delta\varepsilon}\Big|_{\varepsilon=0} \frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi^{\varepsilon}]-S_{qu}[\phi^{\varepsilon}]} \mathcal{O}[\phi^{\varepsilon}]$$

This yields

$$\left\langle \left(-\mathcal{O}\,\frac{\delta S}{\delta\varepsilon}\Big|_{\varepsilon=0} - \mathcal{O}\,\frac{\delta S_{qu}}{\delta\varepsilon}\Big|_{\varepsilon=0} + \frac{\delta\mathcal{O}}{\delta\varepsilon}\Big|_{\varepsilon=0} \right) e^{-S_{qu}} \right\rangle = 0$$

After putting the sources to zero this reads

$$\left\langle \mathcal{O}\partial_\mu S_\mu + \frac{\delta\mathcal{O}}{\delta\varepsilon}\Big|_{\varepsilon=0} \right\rangle = 0$$

Setting $\mathcal{O} = \mathbb{1}$ we obtain

$$< \partial_\mu S_\mu >= 0$$

classically conserved current is also conserved in the quantum theory !

gauge theories on the lattice

$$U(c) = \mathcal{P} e^{-\int_c A_\mu(x) dx_\mu}$$

$$U(c_{x;\mu\nu}) = 1 - F_{\mu\nu}(x) dx_\mu dx_\nu$$

lattice

$$U_\mu(x) := \mathcal{P} e^{-\int_x^{x+a\hat{\mu}} dz A_\mu(z)}$$

$$U_{\mu\nu}(x) := U_\nu^\dagger(x) U_\mu^\dagger(x + a\hat{\nu}) U_\nu(x + a\hat{\mu}) U_\mu(x)$$

$$=$$

$$= 1 - a^2 F_{\mu\nu}(x) + \frac{1}{2} a^4 (F_{\mu\nu}(x))^2 + ...$$

Non-Abelian Stokes Formula:

$$
\begin{aligned}
& U_{\mu\nu}(x) \\
= \;& \mathcal{P}\,\mathrm{exp}\left\{-a^2\int_0^1 d\alpha\int_0^1 d\beta \mathcal{F}_{\mu\nu}(x+a\alpha\hat{\mu}+a\beta\hat{\nu})\right\} \\
\equiv \;& 1+\sum_{n=1}^{\infty}\prod_{i=1}^{n}\int_0^{\alpha_{i-1}} d\alpha_i\int_0^1 d\beta_i\Big\{ \\
& (-a^2)\mathcal{F}_{\mu\nu}(x+a\alpha_n\hat{\mu}+a\beta_n\hat{\nu})\cdots \\
& \cdots(-a^2)\mathcal{F}_{\mu\nu}(x+a\alpha_1\hat{\mu}+a\beta_1\hat{\nu})\Big\}
\end{aligned}
$$

in any case:

$$
\begin{aligned}
U_{\mu\nu}(x) &= 1-a^2F_{\mu\nu}(x)+\frac{1}{2}a^4(F_{\mu\nu}(x))^2+... \\
U_{\nu\mu}(x) &= 1+a^2F_{\mu\nu}(x)+\frac{1}{2}a^4(F_{\mu\nu}(x))^2+...
\end{aligned}
$$

$\rightsquigarrow$

$$
\begin{aligned}
S_g^{lat} &:= \frac{1}{2g^2}a^4\sum_x\sum_{1\le\mu\neq\nu\le 4} \\
&\qquad \frac{1}{a^4}\Big\{2N_c-\mathrm{Tr}\left(U_{\mu\nu}(x)+U_{\nu\mu}(x)\right)\Big\} \\
&\stackrel{a\to 0}{\longrightarrow} -\frac{1}{2g^2}\int d^4x\,\mathrm{Tr}\Big\{F_{\mu\nu}(x)F_{\mu\nu}(x)\Big\}
\end{aligned}
$$

fermions:

$$\begin{aligned} & \mathcal{D}_\mu^{lat,sym}\lambda(x) \\ = \; & \frac{1}{2a}\Big(U_\mu^\dagger(x)\lambda(x+a\hat{\mu})U_\mu(x) \\ & -U_\mu(x-a\hat{\mu})\lambda(x-a\hat{\mu})U_\mu^\dagger(x-a\hat{\mu})\Big) \end{aligned}$$

lattice action, first attempt

$$\begin{aligned} S_f^{lat} &= a^4\sum_x \mathsf{Tr}\Big\{\bar{\lambda}(x)\gamma_\mu\mathcal{D}_\mu^{lat,sym}\lambda(x)\Big\} \\ &\stackrel{a\to 0}{\longrightarrow} \int d^4x \mathsf{Tr}\Big\{\bar{\lambda}(x)\gamma_\mu\mathcal{D}_\mu\lambda(x)\Big\} \end{aligned}$$

problem:

fermion doubling
$\rightsquigarrow$ introduce Wilson mass $\sim \frac{r}{a}$

complete fermion action

$$\mathcal{L}_f(x) = \mathrm{Tr}\Bigg\{ \frac{1}{2a}\Big(\bar{\lambda}(x)(\gamma_\mu - r)U_\mu^\dagger(x)\lambda(x+a\hat{\mu})U_\mu(x) - \bar{\lambda}(x+a\hat{\mu})(\gamma_\mu + r)U_\mu(x)\lambda(x)U_\mu^\dagger(x)\Big) + \left(m_0 + \frac{4r}{a}\right)\bar{\lambda}(x)\lambda(x)\Bigg\}.$$

SUSY-transformation on the lattice

$$\begin{aligned}
\delta U_\mu(x) &= -agU_\mu(x)\bar{\varepsilon}\gamma_\mu\lambda(x) \\
&\quad -ag\bar{\varepsilon}\gamma_\mu\lambda(x+a\hat{\mu})U_\mu(x) \\
\delta U_\mu^\dagger(x) &= +ag\bar{\varepsilon}\gamma_\mu\lambda(x)U_\mu^\dagger(x) \\
&\quad +agU_\mu^\dagger(x)\bar{\varepsilon}\gamma_\mu\lambda(x+a\hat{\mu}).
\end{aligned}$$

$$\begin{aligned}
\delta\lambda(x) &= -\frac{i}{g}\sigma_{\rho\tau}\mathcal{G}_{\rho,\tau}(x)\varepsilon \\
\delta\bar{\lambda}(x) &= \frac{i}{g}\bar{\varepsilon}\sigma_{\rho\tau}\mathcal{G}_{\rho,\tau}(x)
\end{aligned}$$

two SUSY-transformations
$\rightsquigarrow$ infinitesimal translation in space-time

$\Rightarrow$ lattice discretization breaks SUSY

$$\delta_{lat}\mathcal{L}^{lat}(x) = \Delta_\mu \overline{j_\mu}^{lat}(x)\varepsilon + \mathcal{X}(x)$$

where

$$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2$$

- $\mathcal{X}_1$:
- due to discretization itself
- Bianchi and Leibniz fail on the lattice
- irrelevant at tree level: $\mathcal{X}_1 = \mathcal{O}(a^2)$

- $\mathcal{X}_2$:
- due to Wilson-fermions
- $\mathcal{X}_2 = r\mathcal{O}(a)$

symmetry breaking also has consequences for the WI

continuum

$$\left\langle \mathcal{O}\partial_\mu S_\mu - \mathcal{O}D_S + \frac{\delta\mathcal{O}}{\delta\varepsilon}\Big|_{\varepsilon=0} \right\rangle = 0$$

lattice

$$\left\langle \mathcal{O}\Delta_\mu S_\mu^{lat} - \mathcal{O}D_S + \frac{\delta\mathcal{O}}{\delta\varepsilon}\Big|_{\varepsilon=0} - \mathcal{O}X_S \right\rangle = 0$$

X_S depends on

- the choice of the lattice SUSY-transformations
- the choice of the current

investigate the WIs within perturbation theory

general expecation values $< ... >$ not necessarily finite

$\rightsquigarrow$ renormalization

- of the fields
- of the couplings
- of the masses

- of possible composite operators $\rightsquigarrow Z_S, Z_{D_S}, ...$

Question:
How to define finite operators that respect the WIs ?

theorem:
composite operators mix with operators of lower or equal dimension that have the same quantum numbers and symmetries

X_S mixes with

- $\Delta_\mu S_\mu$
- $\Delta_\mu T_\mu = \Delta_\mu \mathrm{Tr}\{\mathcal{G}_{\mu\nu}\gamma_\nu\lambda\}$
- $D_S = 2m_0 \mathrm{Tr}\{\sigma_{\rho\tau}\mathcal{G}_{\rho\tau}\lambda\}$

$$\overline{X_S} \;=\; X_S + (Z_S - 1)\Delta_\mu S_\mu + 2\bar{m} D_S + Z_T \Delta_\mu T_\mu$$

bare WI

$$\Delta_\mu S_\mu = 2m_0 D_S + X_S$$

renormalized WI

$$\Delta_\mu S_\mu = 2m_0 D_S - (Z_S - 1)\Delta_\mu S_\mu - 2\bar{m} D_S - Z_T \Delta_\mu T_\mu$$

renormalization condition $< \bar{X}_S > \to 0 \rightsquigarrow$

$$\underbrace{Z_S \Delta_\mu S_\mu + Z_T \Delta_\mu T_\mu}_{\Delta_\mu \widehat{S}_\mu} = 2(m_0 - \bar{m}) D_S$$

renormalized current:

$$\widehat{S}_\mu = Z_S S_\mu + Z_T T_\mu$$

How to determine Z_S, Z_T and $\bar{m}$?

1) non-perturbatively

$$\frac{Z_S}{Z_T} < \Delta_\mu S_\mu \mathcal{O}_i > + < \Delta_\mu T_\mu \mathcal{O}_i >$$

$$= 2\frac{m_0 - \bar{m}}{Z_T} < D_S \mathcal{O}_i >$$

2 different operators $\mathcal{O}_i$, $i = 1, 2$

$\Rightarrow$ solve for $\frac{Z_S}{Z_T}$ and $\frac{m_0 - \bar{m}}{Z_T}$

2) by means of lattice perturbation theory