Noncommutative geometrical formulation of the standard model of particle physics

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I. Traditional formulation of the standard model
II. Noncommutative geometry
III. Geometry of the standard model
IV. Fluctuations
V. Spectral action

Introduction

The action functional of the standard model unites several interactions between matter fields and force fields

- bosonic sector
  - $U(1)_{\rm hyper}$ Maxwell action (part of the photon)
  - $SU(2)$ Yang-Mills action, eventually describes the other part of the photon and W^+, W^-, Z
  - $SU(3)$ Yang-Mills action which describes gluons
  - action for doublet of complex scalar fields covariantly coupled $SU(2) \times U(1)_{\rm hyper}$ gauge fields
  - quartic self-interaction and negative mass square for the scalar fields (Higgs potential)

- fermionic sector: Dirac-Weyl actions for three families of . . .
  - left-handed leptons covariantly coupled to $SU(2) \times U(1)_{\mathrm{hyper}}$
  - right-handed leptons covariantly coupled to $U(1)_{\mathrm{hyper}}$
  - left-handed quarks covariantly coupled to $SU(3) \times SU(2) \times U(1)_{\mathrm{hyper}}$
  - two different types of right-handed quarks covariantly coupled to $SU(3) \times U(1)_{\mathrm{hyper}}$
- fermionic sector: Yukawa couplings between . . .
  - left-handed lepton, right-handed lepton and scalar doublet
  - left-handed quarks, one type of right-handed quarks and scalar doublet
  - left-handed quarks, the other type of right-handed quarks and the conjugated scalar doublet

- action: full $SU(3) \times SU(2) \times U(1)_{\mathrm{hyper}}$ symmetry

  vacuum: symmetry is only the subgroup $SU(3) \times U(1)_{\mathrm{em}}$

  $\longrightarrow$ spontaneous symmetry breaking

- standard model consists of many independent pieces, has 18 free parameters

... but it is extremely successful!

  – several predictions confirmed later

  – survived 35 years of experiments

  – precision tests confirmed the standard model and excluded more attractive models (GUTs)

- WHY IS THE STANDARD MODEL SO GOOD?

Geometric interpretation of the standard model

- Yang-Mills-Higgs models are natural geometric objects in differential geometry
  - Riemannian geometry for space and time
  - gauge theory located in fibre bundles over space and time
  - standard model is not distinguished
- standard model is essentially unique in noncommutative geometry
  - standard model arises from pure gravity (Riemannian geometry) for a noncommutative space, not from fibre bundles
- gauge groups restricted by noncommutative spin structure:
  - a single simple group cannot be realised
  - two factors: only $U(1) \times \{U(1), SO(2), U(2), SU(2)\}$
  - three factors: $U(1) \times \{U(1), SO(2), U(2), SU(2)\} \times U(n)$

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Continuum and discreteness

- Is light a wave or a particle?
  - quantum mechanics: it is both at the same time!
  - resolution of the continuous $\leftrightarrow$ discrete contradiction requires noncommuting operators
- Noncommutative geometry is the same spirit applied to differential geometry itself
  - geometry is encoded in operators on Hilbert space using algebraic and functional-analytic methods
  - has created new branches of mathematics
  - physics: no distinction between continuous and discrete spaces
  - promising framework for quantum theory of gravity

Basic idea behind noncommutative geometry

- Gelfand-Naimark theorem: Given a topological space X, take the algebra $C(X)$ of continuous functions on X, and forget X.
  Then the topological space X can be reconstructed from $C(X)$.

- $C(X)$ is a commutative C^*-algebra
  - essentially $\mathcal{B}(\mathcal{H})$ – bounded operators on Hilbert space

- idea: take noncommutative C^*-algebras
  - interesting example: von Neumann algebras (operator algebras in quantum mechanics, completely classified)
  - they describe measure theory, not differential geometry

- for differential geometry we need one additional operator which captures the metric information: the Dirac operator

What is algebraically a Dirac operator D? $\longrightarrow$ spectral triple

Connes: need algebra $\mathcal{A}$ represented on Hilbert space $\mathcal{H}$ and 7 axioms

1. dimension: asymptotics of spectrum of D (hear shape of the drum)
2. first order differential operator: $[[D, f], \bar{g}] = 0$ for all $f, g \in \mathcal{A}$
   commutative case: $D = \mathrm{i}\gamma^\mu \partial_\mu \quad \Rightarrow \quad [D, f] = \mathrm{i}\gamma_\mu \frac{\partial f}{\partial x^\mu}$
3. smoothness of the algebra: f and $[D, f]$ belong to the domain of $|D|^n$ in $\mathcal{H}$, for all $f \in \mathcal{A}$
4. orientability: there is a selfadjoint chirality χ $(\equiv \gamma^5)$ on $\mathcal{H}$, with $(\chi)^2 = 1$ and $D\chi + \chi D = 0$, which plays rôle of the volume form
5. finiteness: smooth part of $\mathcal{H}$ has the form $p\mathcal{A}^n$ for a projector p
6. Poincaré duality: a topological condition
7. real structure: charge conjugation $\mathcal{C}$ sending $f \mapsto \bar{f} = \mathcal{C} f \mathcal{C}^{-1}$ with $\mathcal{C}^2 = \pm 1, \mathcal{C}\chi = \pm\chi\mathcal{C}, \mathcal{C}D = \pm D\mathcal{C}$ and $[f, \bar{g}] = 0$

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The spectral triple of the standard model (Euclidean)

- start with Hilbert space (for fermions)
  to represent $\gamma^5, \mathcal{C}$ we need 4 parts: left/right, particles/antiparticles

- Dirac operator
  - traditionally only $\mathrm{i}\gamma^\mu\partial_\mu$ in Dirac equation $(\mathrm{i}\gamma^\mu\partial_\mu + m)\psi = 0$
  - in NCG: mass m is part of the Dirac operator!

$$D = \begin{pmatrix} \mathrm{i}\,\partial\!\!\!/ & \gamma^5\mathcal{M} & 0 & 0 \\ \gamma^5\mathcal{M}^* & \mathrm{i}\,\partial\!\!\!/ & 0 & 0 \\ 0 & 0 & \mathrm{i}\,\partial\!\!\!/ & \gamma^5\overline{\mathcal{M}} \\ 0 & 0 & \gamma^5\mathcal{M}^T & \mathrm{i}\,\partial\!\!\!/ \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \begin{pmatrix} \mathcal{H}_L \\ \mathcal{H}_R \\ \mathcal{H}_L^c \\ \mathcal{H}_R^c \end{pmatrix}$$

  - $\mathrm{i}\,\partial\!\!\!/ = \mathrm{i}\gamma^a e_a^\mu(\partial_\mu + \frac{1}{8}\omega_\mu^{bc}[\gamma_b, \gamma_c])$ – Dirac op. for spin connection
  - $\mathcal{M}$ – fermionic mass matrix

- chirality and charge conjugation

$$\chi = \begin{pmatrix} -\gamma^5 & 0 & 0 & 0 \\ 0 & \gamma^5 & 0 & 0 \\ 0 & 0 & -\gamma^5 & 0 \\ 0 & 0 & 0 & \gamma^5 \end{pmatrix} \qquad \mathcal{C} = \begin{pmatrix} 0 & 0 & \gamma^0\gamma^2 & 0 \\ 0 & 0 & 0 & \gamma^0\gamma^2 \\ \gamma^0\gamma^2 & 0 & 0 & 0 \\ 0 & \gamma^0\gamma^2 & 0 & 0 \end{pmatrix} \circ \text{c.c.}$$

- algebra $C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})) \ni (a, b, c)$

action on $\mathcal{H}$: $\quad \rho(a, b, c) = \begin{pmatrix} \rho_L(b) & 0 & 0 & 0 \\ 0 & \rho_R(a) & 0 & 0 \\ 0 & 0 & \rho_L^c(a, c) & 0 \\ 0 & 0 & 0 & \rho_R^c(a, c) \end{pmatrix}$

Details

- left-handed and right-handed fermions

$$\begin{pmatrix} \boldsymbol{u}_L \\ \boldsymbol{d}_L \\ \nu_L \\ e_L \end{pmatrix} \in \mathcal{H}_L \qquad \begin{pmatrix} \boldsymbol{u}_R \\ \boldsymbol{d}_R \\ e_R \end{pmatrix} \in \mathcal{H}_R$$

$\boldsymbol{u}, \boldsymbol{d}$ of the form $\boldsymbol{q} = \begin{pmatrix} q^r \\ q^b \\ q^g \end{pmatrix}$

each $q^{\{r,b,g\}}, \nu, e$ is of the form Dirac spinor $\otimes \mathbb{C}^3$ (3 families)

- mass matrix $\mathcal{M} : \mathcal{H}_R \to \mathcal{H}_L$

$$\mathcal{M} = \begin{pmatrix} I_3 \otimes M_u & 0 & 0 \\ 0 & I_3 \otimes C_{KM} M_d & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_e \end{pmatrix}$$

M_u, M_d, M_e real diagonal 3×3 matrices

- $a \in C^\infty(M)$, $b = \left(\begin{smallmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{smallmatrix}\right)$, $\alpha, \beta \in C^\infty(M)$, $c \in C^\infty(M) \otimes M_3(\mathbb{C})$

$$\rho_L(b) = \begin{pmatrix} \alpha I_3 & -\bar\beta I_3 & 0 & 0 \\ \beta I_3 & \bar\alpha I_3 & 0 & 0 \\ 0 & 0 & \alpha & -\bar\beta \\ 0 & 0 & \beta & \bar\alpha \end{pmatrix} \otimes I_3 \qquad \rho_R(a) = \begin{pmatrix} a I_3 & 0 & 0 \\ 0 & \bar a I_3 & 0 \\ 0 & 0 & \bar a \end{pmatrix} \otimes I_3$$

$$\rho_L^c(a,c) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & \bar a & 0 \\ 0 & 0 & 0 & \bar a \end{pmatrix} \otimes I_3 \qquad \rho_R^c(a,c) = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \bar a \end{pmatrix} \otimes I_3$$

- $[\rho_L, \rho_L^c] = 0$ and $[\rho_R, \rho_R^c] = 0$ required by axioms
particles see hyper-weak, antiparticles see hyper-strong

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The gauge group

- $\mathcal{U}(\mathcal{A}) = \{\text{unitaries } \in \mathcal{A}\} = C^\infty(M) \otimes (U(1) \times SU(2) \times U(3))$
  one $U(1)$ is too much $\quad \longrightarrow$ spin lift and central extension

- adjoint action of $\mathcal{U}(\mathcal{A})$ on $\mathcal{H}$: $\psi \mapsto \psi^u = u\mathcal{C}u\mathcal{C}^{-1}\psi$
  with Dirac operator: $(D\psi)^u = \underbrace{u\mathcal{C}u\mathcal{C}^{-1} D (u\mathcal{C}u\mathcal{C}^{-1})^{-1}}_{D^u} \underbrace{u\mathcal{C}u\mathcal{C}^{-1}\psi}_{\psi^u}$

$$\begin{aligned} \Rightarrow D^u &= u\mathcal{C}u\mathcal{C}^{-1} D \mathcal{C}u^*\mathcal{C}^{-1}u^* \\ &= u\mathcal{C}u\mathcal{C}^{-1}[D, \mathcal{C}u^*\mathcal{C}^{-1}]u^* + u\mathcal{C}u\mathcal{C}^{-1}\mathcal{C}u^*\mathcal{C}^{-1}Du^* \\ &= \mathcal{C}u\mathcal{C}^{-1}[D, \mathcal{C}u^*\mathcal{C}^{-1}] + uDu^* \\ &= D + u[D, u^*] + \mathcal{C}u[D, u^*]\mathcal{C}^{-1} = D + A^u + \mathcal{C}A^u\mathcal{C}^{-1} \end{aligned}$$

  $A^u = u[D, u^*]$ is pseudo-force due to gauge transformation

- equivalence principle: true forces and pseudo-forces are locally indistinguishable
  $\Rightarrow$ true forces are described by $A = \sum_i f_i[D, g_i]$ for $f_i, g_i \in \mathcal{A}$

Spin lift and central extension

- remember that we look for pure gravity on nc space!
  - $\mathcal{U}(\mathcal{A}) \sim$ Lorentz group, but on $\mathcal{H}$ there acts the spin group
  - spin group $\mathcal{S}(G)$ is universal cover of rotation group G, e.g. $G = SO(4), \mathcal{S}(SO(4)) = Spin(4)$
  - spin lift $L : SO(4) \ni \exp(\omega) \mapsto \exp(\frac{1}{8}\omega^{ab}[\gamma_a, \gamma_b]) \in Spin(4)$
  - projection $\pi : Spin(4) \ni u \mapsto i_u \in SO(4)$ $\qquad \pi \circ L = \mathrm{id}$
    i_u acts on $x \in \mathbb{R}^4$ by $i_u(x) = \boldsymbol{\gamma}^{-1}(u\boldsymbol{\gamma}(x)u^{-1}), \quad \boldsymbol{\gamma}(x) = \gamma_\mu x^\mu$

- $Spin(\mathcal{H}) = \{ u \in \mathcal{B}(\mathcal{H}),\ uu^* = u^*u = \mathrm{id},\ u\mathcal{C} = \mathcal{C}u,\ u\chi = \chi u,$
  $i_u(f) = \rho^{-1}(u\rho(f)u^{-1}) \in \mathcal{A}$ for all $f \in \mathcal{A}\}$

  $\mathcal{U} = \pi(Spin(\mathcal{H})) = C^\infty(M) \otimes (SU(2) \times SU(3))$ $\qquad$ no $U(1)$!

- restore $U(1)$ by central extension $\mathcal{U}^Z(\mathcal{A}) = \mathcal{U}(\mathcal{A}) \cap Z(\mathcal{A})$
  $i_{vu} = i_u$ for $v \in L(\mathcal{U}^Z(\mathcal{A}))$, double-valuedness gives correct $U(1)$

- fluctuated Dirac operator $D_A = D + A + \mathcal{C} A \mathcal{C}^{-1}$

$$A = A^* = \begin{pmatrix} A_L & H & 0 & 0 \\ H^* & A_R & 0 & 0 \\ 0 & 0 & A_L^c & 0 \\ 0 & 0 & 0 & A_R^c \end{pmatrix}$$

$$A_L = \textstyle\sum_i \rho_L(b_i)[\mathrm{i}\,\partial\!\!\!/, \rho_L(b_i')] \qquad = \mathrm{i}\gamma^\mu \rho_L(W_\mu)$$

$$A_R = \textstyle\sum_i \rho_R(a_i)[\mathrm{i}\,\partial\!\!\!/, \rho_R(a_i')] \qquad = \mathrm{i}\gamma^\mu \rho_R(A_\mu)$$

$$A_L^c = \textstyle\sum_i \rho_L^c(a_i, c_i)[\mathrm{i}\,\partial\!\!\!/, \rho_L^c(a_i', c_i')] \qquad = \mathrm{i}\gamma^\mu \rho_L^c(A_\mu, G_\mu)$$

$$A_R^c = \textstyle\sum_i \rho_R^c(a_i, c_i)[\mathrm{i}\,\partial\!\!\!/, \rho_R^c(a_i', c_i')] \qquad = \mathrm{i}\gamma^\mu \rho_R^c(A_\mu, G_\mu)$$

$$H = \textstyle\sum_i \rho_L(b_i)(\rho_L(b_i')\mathcal{M} - \mathcal{M}\rho_R(a_i'))$$

$$= \begin{pmatrix} \phi_1 I_3 \otimes M_u & -\overline{\phi_2} I_3 \otimes C_{KM} M_d & 0 \\ \phi_2 I_3 \otimes M_u & \overline{\phi_1} I_3 \otimes C_{KM} M_d & 0 \\ 0 & 0 & -\overline{\phi_2} \otimes M_e \\ 0 & 0 & \overline{\phi_1} \otimes M_e \end{pmatrix}$$

Fermionic action

$$S_F = \langle \psi, D_A \psi \rangle \qquad \text{for} \qquad \psi \in \mathcal{H}$$

reproduces Euclidean fermionic action of the standard model

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Spectral action principle

- Philosophy [Connes]:
  The bosonic action depends only on the spectrum of D_A.
- most general form: $S[D_A] = \mathrm{tr}\left(f\left(\frac{D_A^2}{\Lambda^2}\right)\right)$ (Λ – a scale)
  $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) \to 0$ sufficiently fast for $x \to \infty$
- Laplace transformation $\mathrm{tr}\left(f\left(\frac{D_A^2}{\Lambda^2}\right)\right) = \int_0^\infty dt\ \mathrm{tr}\left(\mathrm{e}^{-t\frac{D_A^2}{\Lambda^2}}\right)\tilde{f}(t)$
  alternatively: one-loop effective action for quantum fermions coupled to classical gauge fields is proportional to Yang-Mills action and computable by heat kernel
- heat kernel expansion:
  $$\mathrm{e}^{-tD^2} = \sum_{0\le k\le \frac{n}{2}} t^{k-\frac{n}{2}} \int d^n x \sqrt{\det g}\, \underbrace{a_{2k}(D^2)}_{\text{Seeley coefficients}} + \mathcal{O}(t)$$

- in $n = 4$ dimensions:

$$\mathrm{tr}\Big(f\Big(\frac{D_A^2}{\Lambda^2}\Big)\Big) = \frac{1}{16\pi^2}\int d^4x\,\sqrt{\det g}$$

$$\times\big(\Lambda^4 f_0 a_0(D^2) + \Lambda^2 f_2 a_0(D^2) + f_4 a_4(D^2) + \mathcal{O}(\Lambda^{-2})\big)$$

$$f_0 = \int_0^\infty dt\, t\, f(t)\ , \quad f_2 = \int_0^\infty dt\, f(t)\ , \quad f_4 = f(0)$$

- Seeley coefficients

$$a_0(D^2) = \mathrm{tr}(1)$$

$$a_2(D^2) = \tfrac{1}{6} R\, \mathrm{tr}(1) - \mathrm{tr}(E)$$

$$a_4(D^2) = \tfrac{1}{360}(5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\mathrm{tr}(1)$$

$$+\tfrac{1}{12}\mathrm{tr}(\Omega_{\mu\nu}\Omega^{\mu\nu}) - \tfrac{1}{6}R\,\mathrm{tr}(E) + \tfrac{1}{2}\mathrm{tr}(E^2)$$

for $D_A^2 = \Delta + E$,

$$\Delta = -g^{\mu\nu}(\nabla_\mu\nabla_\nu - \Gamma^\rho_{\mu\nu}\nabla_\rho)\ , \quad \Omega_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$$

Result

of elementary but very long calculation and reparametrisation:

$$
\begin{aligned}
S[D_A] = \int d^4x \, \sqrt{\det g} \\
& \times \Big(\tfrac{2\Lambda_c}{16\pi G} - \tfrac{1}{16\pi G} R + a(5R^2 - 8R_{\mu\nu}R^{\mu\nu} - 7R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \\
& + \tfrac{1}{4g_2^1} F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2g_2^2} \mathrm{tr}(W^*_{\mu\nu}W^{\mu\nu}) + \tfrac{1}{2g_3^2} \mathrm{tr}(G^*_{\mu\nu}G^{\mu\nu}) \\
& + \tfrac{1}{2}(D_\mu\phi)^*(D^\mu\phi) + \lambda|\phi|^4 - \tfrac{\mu^2}{2}|\phi|^2 + \tfrac{1}{12}R|\phi|^2 \Big)
\end{aligned}
$$

where

$$\Lambda_c = \frac{6f_0}{f_2}\Lambda^2 \qquad G = \frac{\pi}{2f_2}\Lambda^{-2} \qquad a = \frac{f_4}{960\pi^2}$$

$$g_2^2 = g_3^2 = \tfrac{5}{3}g_1^2 = \frac{\pi^2}{f_4} \qquad \lambda = \frac{\pi^2}{3f_4} \quad \mu^2 = \frac{2f_2}{f_4}\Lambda^2$$

- suggests Λ as grand unification scale
- one-loop renormalisation group flow to $\Lambda_{SM} = m_Z$ leads to
  $188\,\mathrm{GeV} \leq m_H \leq 201\,\mathrm{GeV}$

Summary of the achievements

- unification of standard model with gravity at the level of classical field theories
- all 7 relative signs of the various terms come out correctly for Euclidean space
- Higgs field is gauge field in discrete direction, Higgs potential is Yang-Mills Lagrangian for discrete field strength
- understanding why symmetry group of the standard model cannot be simple
- strong interactions must couple vectorially, all colours must have the same mass, whereas weak coupling must be chiral
- predicts vanishing neutrino masses (from Poincaré duality)