A Little Study

On Analyticity Properties

Of the Ising Renormalization Group

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The Ising Model

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```

$s_x = \pm 1$

$$Z = \sum_s \exp[-\, H(s)]$$

$$H(s) = -\beta \sum_{<x,y>} s_x s_y$$

In the infinite volume limit, the free energy

$$f = \lim_{V \to \infty} \frac{1}{V} \ln Z$$

is *non-analytic* in β, $\tanh \beta$, $\exp(-4\,\beta)$, ...
at the critical point.

RG for the Ising Model

```
FINE LATTICE      DIVIDE IN BLOCKS    BLOCK LATTICE

+ + + + - -       + + | + + | - -     + + -
+ + + - - -       + + | + - | - -     + - +
+ + - + + +       ----|-----|----     + + +
+ - + - + +       + + | - + | + +
+ + + + + +       + - | + - | + +
- + - + + -       ----|-----|----
                  + + | + + | + +
                  - + | - + | + -     block size L=2
```

$P(\mu, s)$ = probability for block spin configuration μ, given the fine spin configuration s.

$P(\mu, s) \geq 0$

$\sum_\mu P(\mu, s) = 1$

Majority Rule:

$$P(\mu, s) = \prod_{x'} \frac{1}{2} \left[1 + \mu_{x'} \, \mathrm{sign} \left(\sum_{x \in x'} s_x \right) \right]$$

$$\mathrm{sign}(k) = \begin{cases} +1 & k > 0 \\ -1 & k < 0 \\ 0 & k = 0 \end{cases}$$

Effective Hamiltonian:

$$\exp\left[-H'(\mu)\right] = \sum_s P(\mu, s) \exp\left[-H(s)\right]$$

Partition Function remains unchanged:

$$\begin{aligned} \sum_\mu \exp[-H'(\mu)] &= \sum_\mu \sum_s P(\mu, s) \exp[-H(s)] \\ &= \sum_s \exp[-H(s)] \\ &= Z \end{aligned}$$

```
K1                           K1
^                            ^
| . 1                        |            . 1
|  . 2                       |
|    . 3                     |           . 2
|       .                    |         . 3
|            .               |      .
|                  * FP      |   .
|                            | .
|---------------------> K2   *-----------------> K2
```

$\xi < \infty$: Every step reduces correlation length by a factor L. Theory with small ξ can be evaluated with standard methods.

$\xi = \infty$: Flow converges towards a <u>fixed point</u>. Critical indices can be computed from the eigenvalues of the RGT linearized around the fixed point.

prize: Effective Hamiltonians with <u>∞-many couplings</u>

Some Numerical Results for 2D

$\beta_c = 0.4406868...$

```
couplings:  012
            345
            678

       1st step

01      0.353383 [187](054)
04      0.075220 [060](046)
02     -0.008110 [550](045)
05     -0.005486 [724](032)
08     -0.002491 [199](045)
0134   -0.015304 [154](056)
1345    0.006538 [092](028)
1357   -0.004699 [221](054)
0457    0.002874 [106](028)
0123    0.001419 [201](020)
0145    0.000875 [125](027)
0157   -0.001030 [120](019)
0148    0.000857 [047](020)
0247    0.000473 [107](027)
```

The Renormalization Group Philosophy

$$\begin{aligned} H(s) &= \sum_\alpha K_\alpha S_\alpha(s) \\ H'(\mu) &= \sum_\alpha K'_\alpha S_\alpha(\mu) \end{aligned}$$

RGT is regular map

$$K' = R(K) = K'(K)$$

- The new coupling constants K' should be analytic functions of the old ones, K, also in the critical regime.
- It should be allowed to deal with RG by studying the system in terms of small subsystems, e.g., by high temperature expansions.
- ...

1st RG-step, starting from $K_1 = 0.4406868$, PRB

block lattice size	K'_1	from
2	0.346450	KP
3	0.344247	KP
4	0.353871	KP
5	0.353361	Dudek et al.
32	0.35338(18)	KP

Parametrization of Ising Model Hamiltonians

$F(s) = \sum_X F_X \, s(X)$ sum over subsets of the lattice

$s(X) = \prod_{x \in X} s_x$ product of spins in set X

$F_X = 2^{-V} \sum_s F(s) \, s(X)$ inversion formula

```
1 2 3   . * *   * . .   . . .
4 5 6   . . .   * . .   . * *
7 8 9   . . .   . . .   . . .   ...   alpha= * *

        s2*s3   s1*s4   s5*s6
```

```
* * .   . * *
* * .   . * *                         alpha= * *
. . .   . . .   ...                          * *
```

$H(s) = \sum_\alpha K_\alpha \, S_\alpha(s)$ sum over classes of sets

$S_\alpha(s) = \sum_{X \in \alpha} s(X)$ interaction

2D Ising RG

```
x x | x x
x x | x x     majority       X  X
----|----    ----------->
x x | x x       rule         X  X
x x | x x

                   y                   d c
<xy> = x y    [xy] = x        {abcd} = a b
```

$$
\begin{aligned}
S_1(s) &= \sum\nolimits_{<xy>} s_x s_y \\
S_2(s) &= \sum\nolimits_{[xy]} s_x s_y \\
S_3(s) &= \sum\nolimits_{\{abcd\}} s_a s_b s_c s_d \\
\\
\ldots &= \ldots
\end{aligned}
$$

1st RG step:

$$\begin{aligned} H &= -\beta\, S_1 \\ H' &= -K_1'\, S_1 - K_2' S_2 - K_3' S_3 - \ldots \end{aligned}$$

```
                                 ---------
                                 |mu1|mu2|
block spin configuration         |--------
                                 |mu3|mu4|
                                 ---------
```

$$\begin{aligned} K_1' &= 2^{-M^2} \textstyle\sum_\mu \mu_1\, \mu_2 & \ln Z(\mu) \\ K_2' &= 2^{-M^2} \textstyle\sum_\mu \mu_1\, \mu_4 & \ln Z(\mu) \\ K_3' &= 2^{-M^2} \textstyle\sum_\mu \mu_1\, \mu_2\, \mu_3\, \mu_4 & \ln Z(\mu) \end{aligned}$$

with

$$Z(\mu) = \sum_s P(\mu, s)\, \exp[\beta S_1(s)]$$

Appropriate way to compute effective couplings after 1st RG step:

Define

$$G(E,\mu) = \sum_s P(\mu,s)\,\delta[E - S_1(s)]$$

then

$$Z(\mu) = \sum_E G(E,\mu)\,\exp\,[\beta E]$$

Compute $G(E,\mu)$ for all admissable energies E and for all non-equivalent block spin configurations μ.

Number of non-equivalent block spin configurations μ (PRB) on an M by M block lattice:

M	2	3	4	5
#	4	13	479	86056

For lattices of moderate size, $G(E, \mu)$ can be computed exactly with transfer matrix methods.

$$\downarrow$$

$$K_X = 2^{-M^2} \sum_{\mu} \mu_X \ln[Z(\mu)]$$

$Z(\mu)$ is a polynomial in $u = \exp(4\beta)$.

Analyticity properties of K_X are determined by zeroes of $Z(\mu)$ in the complex u-plane.

High Temperature Expansions, naive

1st RG step

study expansion of K_i' in powers of $\beta = K_1$

coefficients of K_1':

order	2 by 2	3 by 3	4 by 4
1	0.28125000	0.28125000	0.28125000
2	0.75000000	0.67089844	0.75000000
3	0.75756836	0.79086304	0.95373535
4	0.85742188	1.47241116	1.27441406
5	0.22088814	0.72954550	-0.32214987
6	1.79092712	-2.43502268	-3.71962846
7	-5.83542468	-9.88575556	-5.28832809
8	0.33779963	-10.07229969	-5.86731428
9	-29.00932789	-5.05379215	-4.48936300
10	-133.70738020	81.01961518	32.53130737

Series has

- finite radius of convergence
- can be analytically continued to any $\beta > 0$

Cumulant Expansion

- keep exactly all interactions within blocks
- expand in block connecting interactions

```
0##0---0##0---0##0---
#  #   #  #   #  #
0##0---0##0---0##0---
|  |   |  |   |  |
0##0---0##0---0##0---
#  #   #  #   #  #
0##0---0##0---0##0---
|  |   |  |   |  |
0##0---0##0---0##0---
#  #   #  #   #  #
0##0---0##0---0##0---
```

1st RG step, started from $\beta = 0.440687$

coefficients:

order	K_1'	K_2'	K_3'
1	0.85117044	0.00000000	0.00000000
2	-0.03952204	0.48840288	-0.00000000
3	-0.24969285	-0.09438144	0.00000001
4	-0.02193745	-0.68109943	-0.58290436
5	-0.06170475	-0.21210601	-0.01608311
6	-0.17425370	0.55503437	1.10706175
7	0.44274740	0.17672024	0.09660566
8	0.27122868	-0.33289598	-1.27033469
9	-1.05061534	0.31500122	0.08576604
10	0.00089891	0.78637256	1.57653340

partial sums:

order	K'_1	K'_2	K'_3
0	0.00000000	0.00000000	0.00000000
1	0.37509957	0.00000000	0.00000000
3	0.34605461	0.08677271	0.00000000
5	0.34420165	0.05755933	-0.02225186
7	0.34435443	0.06219513	-0.01383131
9	0.34408166	0.06191906	-0.01558455
11	0.34437130	0.06208427	-0.01530046
13	0.34417820	0.06190167	-0.01534262
15	0.34427346	0.06199698	-0.01535127
17	0.34423802	0.06196858	-0.01533832
19	0.34424932	0.06197133	-0.01534444
21	0.34424516	0.06197309	-0.01534234
exact	0.34424654	0.06197233	-0.01534296

Conclusions & Questions

- There is some evidence that strong coupling expansion of 1st RG step can be analytically continued to all $\beta > 0$.

- Of course, analysis has to generalized to general step, i.e. #(couplings) >1 !

- Can one give more substantial evidence that no dangerous singularities appear in the infinite volume limit

- Can one do better by applying other methods?