Multifractals in the Human Heartbeat

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outline:

- Introduction: Fractals and Dimensions
- The Multifractal Formalism
- Correlation in the Heart Rate Variability
- Multifractals in the Human Heart Beat

Fractal Dimensions

Compass dimension

measurement of the coast of Britain yields power law !

$$\log(u) \sim 0.3 \log\left(\frac{1}{s}\right) \qquad \Rightarrow u \sim \left(\frac{1}{s}\right)^{0.3}$$

Koch Curve also leads to power law !

compass setting: $s = \left(\frac{1}{3}\right)^k$
length: $u = \left(\frac{4}{3}\right)^k$

$\leadsto$ exponent:

$$\begin{aligned} \log_3 u &= d \cdot \log_3 \frac{1}{s} \\ \Rightarrow d &= \log_3 \frac{4}{3} = 0.2619 \end{aligned}$$

coast more convoluted than Koch curve

d: compass-dimension (divider dimension) measurement of length is an ill-posed problem

other ways to characterize complexity ?

Self-Similiarity Dimension

reduction factor r

number of pieces N

$$N = \frac{1}{r^{D_s}} \quad \Rightarrow D_s = -\frac{\log N}{\log r}$$

Koch-Curve: $r = \frac{1}{3} \qquad N = 4$

$$D_s = -\frac{\log 4}{\log 3} = 1.2619$$

In general

$$D_s = 1 + d$$

Box-Counting Dimension

$$D_{BC} = -\lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log \epsilon}$$

<u>Great Britain‘s coast line:</u>

$$D_{BC} = 1.31$$

<u>Cantor-Set:</u>

$\epsilon_n = \frac{1}{3}^n \Rightarrow N(\epsilon_n) = 2^n$

$\rightsquigarrow \qquad D_{BC} = \frac{\log 2}{\log 3} = 0.63$

<u>Koch curve:</u>

$$D_{BC} = 1.26$$

for self-similar structures:

$$D_{BC} = D_s$$

general Renyi-Dimensions

geometric set + measure μ

$$\mu_i = \mu(C_i)$$

for example:
chaotic system $\rightsquigarrow$ attractor

introduce measure on the attractor dependent on frequency of visits !

$$\mu_i = \lim_{T \to \infty} \frac{\#(C_i, x_0, T)}{T}$$

for C_i some cube.

define:

$$Z_q(\epsilon) := \sum_i \mu_i^q$$

i: cubes of size ϵ that cover the set

generalized dimension:

$$D_q = -\frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\log(Z_q(\epsilon))}{\log(\epsilon)}$$

for $q > 0$: cubes with large $\mu_i = \mu(C_i)$ have more weight !

<u>$q = 0$ (box-counting dimension)</u>

$$D_0 = -\lim_{\epsilon \to 0} \frac{\log(\#Cubes)}{\log(\epsilon)} = D_{BC}$$

<u>$q = 1$ (information dimension)</u>

$$D_1 = \lim_{q \to 1} D_q = \lim_{\epsilon \to 0} \frac{\sum_i \mu_i \log \mu_i}{\log \epsilon}$$

<u>$q = 2$ (correlation dimension)</u>

$$D_2 = -\frac{1}{1-q} \frac{\log \sum_i \mu_i^2}{\log \epsilon}$$

multifractals=measures on geometric supports with certain scaling properties

<u>example</u>: The Devil‘s Staircase

$$\mu_i = 2^{-n} \qquad l_i = 3^{-n}$$

$\rightsquigarrow$

$$\mu_i = l_i^{\alpha}, \quad \alpha = \frac{\log 2}{\log 3}$$

probability density:

$$\rho_i \sim \frac{\mu_i}{l_i} \sim l_i^{\alpha-1}$$

diverges for $l \to 0$ (since $0 < \alpha < 1$)

α: global scaling exponent, Hölder exponent

exponent of the singularity

A multifractal: The binomial measure

$$\mu_{0.\beta_1\beta_2\ldots\beta_k} = m_0^{n_0} m_1^{n_1}$$

local Hölder exponent

$$\alpha(x) = \lim_{\epsilon\to 0} \frac{\log \mu(B_\epsilon(x))}{\log \epsilon}$$

$\leadsto$

$$\mu_i \sim \epsilon^{\alpha_i}$$

binomial measure:

$$\alpha(0.\beta_1\beta_2\ldots\beta_k) = -\frac{1}{k}(n_0 \log_2 m_1 + n_1 \log_2 m_0)$$

$$\rho(x) \sim \epsilon^{\alpha(x)-1}$$

singularity whenever $\alpha(x) \neq 1$

$\alpha(x)$ local scaling exponent

define:

$$f(\alpha) \;:=\; D_{BC}\left(\{x|\alpha(x)=\alpha\}\right)$$

$$= \; -\lim_{\epsilon\to 0}\frac{\log(N(\epsilon))}{\log(\epsilon)}$$

$\leadsto$ number of cubes of size ϵ required to cover $\{x|\alpha(x)=\alpha\}$ is

$$N(\epsilon)\sim\epsilon^{-f(\alpha)}$$

therefore:

Cubes of size ϵ and $\alpha' \in [\alpha, \alpha + d\alpha]$

$$\sim \rho(\alpha)\epsilon^{-f(\alpha)}d\alpha$$

we had:

$$\begin{aligned} Z_q(\epsilon) &= \sum_i \mu_i^q \\ &= \int d\alpha' \rho(\alpha') \epsilon^{-f(\alpha')+q\alpha'} \\ &= -\ln(\epsilon) \int d\alpha' \rho(\alpha') e^{f(\alpha')-q\alpha'} \end{aligned}$$

main contribution: $f(\alpha') - q\alpha' \overset{!}{=} max$ then:

$$Z_q(\epsilon) \sim \epsilon^{f(\alpha')-q\alpha'}$$

because of

$$D_q = -\frac{1}{1-q} \lim_{\epsilon \to 0} \frac{\log(Z_q(\epsilon))}{\log(\epsilon)}$$

one finds

$$\tau(q) := -(q-1) D_q = f(\alpha(q)) - q\alpha(q)$$

and

$$\alpha(q) = -\tau'(q)$$

therefore Legendre-transform:

$$f(\alpha(q)) = \tau(q) - q\tau'(q)$$

$\tau(q)$=scaling exponent of the q^{th} moment:

$$Z_q(\epsilon) \sim \epsilon^{\tau(q)}$$

analogy to thermodynamics:

q	β
$-\log(\epsilon)$	V
$Z_q(\epsilon)$	$Z(\beta, V)$
$\tau(q) = -\frac{\log(Z_q(\epsilon))}{-\log(\epsilon)}$	$\beta f = -\frac{\log(Z(\beta,V))}{V}$
$\alpha(q) = \frac{\partial \tau(q)}{\partial q}$	$u = \frac{\partial(\beta f)}{\partial \beta}$
$f(\alpha) = \tau(q) - q\frac{\partial \tau(q)}{\partial q}$	$s = \beta\frac{\partial(\beta f)}{\partial \beta} - \beta f$

Correlation in the Human Heartbeat

The Human Heart: 3 Gigabeats in a life-time !

Heart Rate Variability (HRV): R-R-Intervals

record digitized electrocardiograms ($24h \sim 10^5$ beats)

$\rightsquigarrow$ time series $B(n)$ of interbeat intervals

The landscape of healthy subjects is more complex than the smoother pattern of persons with severe heart deseases !!

define: mean fluctuation function

$$f(n) = < |B(n' + n) - B(n')| >_{n'}$$

result:

$$f(n) \sim n^{\alpha}$$

- healthy heart: $\alpha \approx 0$
  $\rightsquigarrow$ increments anti-correlated

- diseased heart: $\alpha \approx 0.5$
  $\rightsquigarrow$ uncorrelated random walk

different statistical properties ? No !

define: increments

$$I(n) := B(n+1) - B(n)$$

histograms described by Levy stable distribution:

$$P(I, \psi, \gamma) = \frac{1}{\pi} \int_0^{\infty} e^{-\gamma q^{\psi}} \cos(qI) dq$$

where $\psi = 1.7$ and $\gamma > 0$

$\rightsquigarrow$ correlations related to ordering of the increments

$\to$ underlying dynamics of the heart

power spectrum of $I(n)$

expectation:

$$S_I(f) \sim f^{\beta}$$

where $\beta = 1 - 2\alpha$.

- $\beta = 0$: no correlation, white noise
- $-1 < \beta < 0$: $I(n)$ correlated
- $0 < \beta < 1$: $I(n)$ anti-correlated

- long-range-anti-correlations: organizing principle, system is driven away from a single stable state

- lack of a characteristic scale prevents system from mode-locking, no dominant frequency mode

Multifractality in the Human Heartbeat

The wavelet transform

signal $s(t)$

define:

$$T_\psi(t_0, a) := \frac{1}{a}\int_{-\infty}^{\infty} s(t)\psi\left(\frac{t-t_0}{a}\right)dt$$

mathematical microscope,

allows to study $s(t)$ at scale a

If

$$s(t_0 + t) = \text{polynomial}(t) + \text{const} \cdot |t|^{h(t_0)}$$

then

$$T_\psi(t_0, a) \sim a^{h(t_0)} \quad \text{for } a \to 0$$

example: The Devil‘s Staircase

for a given signal $s(t)$ define partition function as:

$$Z_q(a) = \sum_{t_i(a)} |T_\psi(t_i(a), a)|^q \sim a^{\tau(q)}$$

sum only over the modulus maxima !

- monofractal self-affine time series: one global Hurst exponent H

  $\rightsquigarrow$ $\tau(q) = qH - 1$ and $D(h) =$ const

- multifractal signal: local Hurst exponents $h(t)$
  $\rightsquigarrow$ $\tau(q)$ nonlinear,

  $D(h) = -\left(\tau(q) - q\frac{d\tau(q)}{dq}\right)$

  its Legendre-Transform

- for healthy and deseased subjects $Z_q(a)$ scales as a power-law

- for healthy subjects $\tau(q)$ is nonlinear

- for healthy subjects $D(h)$ shows a broad range of local Hurst exponents

- no difference between night and day

- $0 < h < 0.3 \leadsto$ anti-correlation

conclusion:

the healthy heartbeat is a multifractal signal for subjects with pathological condition:

- $\tau(q)$ close to linear
- $D(h) \neq 0$ only in a narrow range of h
- $h \approx 0.22 \rightsquigarrow$ closer to 0.5: less correlated

conclusion:

the deseased heartbeat is a monofractal signal

power spectrum analysis only provides one scaling exponent !

multifractal analysis reveals complex nature of healthy heart dynamics !

tests with surrogate time series

- shuffle the interbeat intervals of a healthy subject
  $\rightsquigarrow$ distribution preserved, but long-range correlation destroyed

  $\rightsquigarrow$ random walk
  single Hurst exponent $H = 0.5$

- randomize Fourier phases of healthy time series
  $\rightsquigarrow$ monofractal behaviour, $H \approx 0.22$, linear correlations preserved

  $\rightsquigarrow$ phase correlations in the healthy case

- randomize Fourier phases of heart failure time series
  $\rightsquigarrow$ minor changes in the fractal spectrum

  $\rightsquigarrow$ only weak phase-correlations in diseased case

comparison with other methods

multifractal analysis provides three values:

- σ of interbeat intervals
- $\tau(q = 3)$ from scaling of $Z_3(a)$
- degree of multifractality $h_{max} - h_{min}$

$\leadsto$ analysis incorporating the multifractal method may add diagnostic power to ordinary time-series analysis.