Non-perturbative determination of the B meson decay constant in HQET

P. Fritzsch

Institut für Theoretische Physik
Westfälische-Wilhelms-Universität Münster

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19.06.2006

ALPHA Collaboration @

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search for a deeper understanding of all fundamental forces,
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in the case of QCD:

- well-defined QFT with Lagrangian

$$\mathcal{L}_{\mathrm{QCD}}[g_0, m_{\mathrm{f}}] = -\frac{1}{2g_0^2}\,\mathrm{Tr}\{F_{\mu\nu}F_{\mu\nu}\} + \sum_{\mathrm{f}=\mathrm{u},\mathrm{d},\ldots} \overline{\psi}_{\mathrm{f}}[\gamma_\mu(\partial_\mu + g_0 A_\mu) + m_{\mathrm{f}}]\psi_{\mathrm{f}}$$

- easy to write down, but much more difficult to 'solve' than QED
- more non-linearities due to structure of non-abelian gauge group $\rightsquigarrow$ confinement, asymtotic freedom
- spectrum extremly rich and exotic with various excitations over a wide energy range $\rightsquigarrow$ hadronic zoo

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only know, fully non-perturbative framework: Lattice QCD

Why B physics?
Relevant for what?

- the b-quark mass
- spectrum & lifetimes of b-hadrons
- determination of the CKM-parameters
  - "fundamental" parameters of nature
  - CP puzzle

weak eigenstates $\neq$ mass eigenstates $\Rightarrow$

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

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unitarity condition $V_{\rm CKM} V_{\rm CKM}^\dagger = \mathbb{1}$ in SM

$\leftrightsquigarrow$ 6 normalizations & 6 orthogonality relations like

$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0$$
$$V_{ud}^* V_{td} + V_{us}^* V_{ts} + V_{ub}^* V_{tb} = 0$$

Question: unitarity violation or not $\rightsquigarrow$ new physics? (NP)

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CKM-Matrix in *Wolfenstein parametrization* (1983)

$$V_{\text{CKM}} = \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - \mathrm{i}\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-\mathrm{i}\eta) & -A\lambda^2 & 1 \end{pmatrix} + \mathcal{O}(\lambda^4)$$

with CP-violating phase η and $\lambda = \sin\theta_C = 0.22$ (θ_C: Cabibbo angle)

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$$[(\bar{\rho} + \mathrm{i}\bar{\eta}) + (1-\bar{\rho}-\mathrm{i}\bar{\eta}) + (-1) + \mathcal{O}'(\lambda^2)]A\lambda^3 = 0$$

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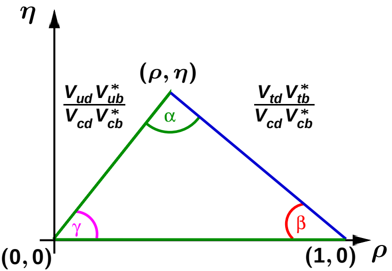

- side from Δm_s, $\Delta m_s/\Delta m_d$
- angle γ from $B \to h^+h^-$
- $\sin 2\beta$ from $J/\psi K_s$ decays

$$\begin{pmatrix} \bar{\rho} \\ \bar{\eta} \end{pmatrix} \equiv (1-\lambda^2/2)\begin{pmatrix} \rho \\ \eta \end{pmatrix}$$

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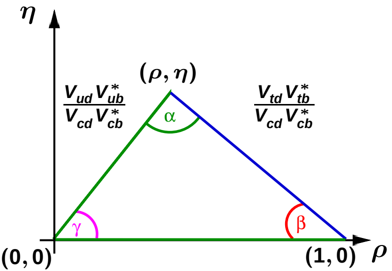

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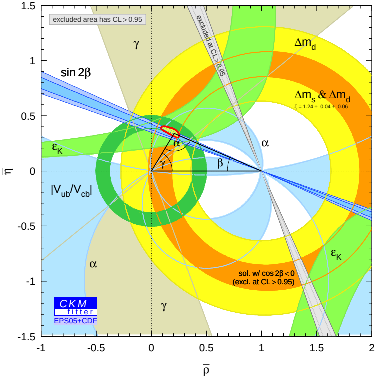
- $\sin 2\beta$ from $J/\psi K_s$ decays

$$\xi \equiv \frac{f_{B_s}\sqrt{B_{B_s}}}{f_{B_d}\sqrt{B_{B_d}}}$$

CKM Fitter Group

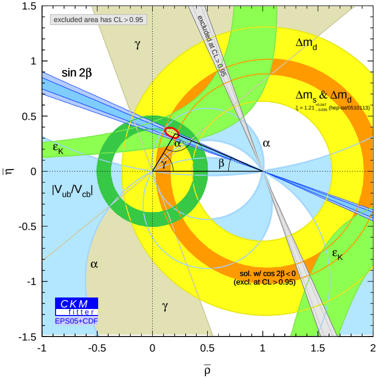
http://ckmfitter.in2p3.fr/

experimental dataset



$$\xi = 1.24 \pm 0.04 \pm 0.06$$

with lattice data [hep-lat/0510113]

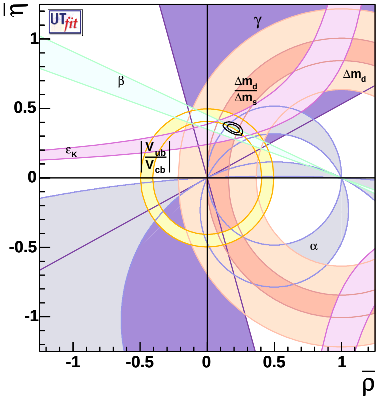


$$\xi = 1.21^{+0.047}_{-0.035}$$

UTfit collaboration

http://utfit.roma1.infn.it/

[hep-ph/0606167]



$$\bar{\rho} = 0.193 \pm 0.029$$

$$\bar{\eta} = 0.355 \pm 0.019$$

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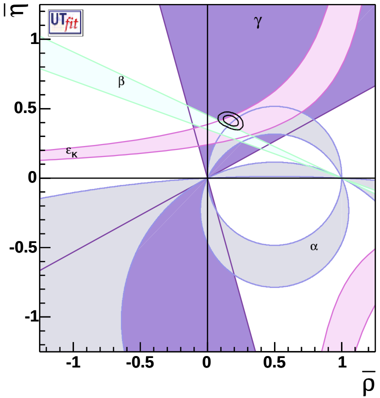

$$\overline{\rho} = 0.173 \pm 0.039$$

$$\overline{\eta} = 0.412 \pm 0.026$$

B factories

now and then

at SLAC since May 1999

at KEK since June 1999

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$\rightsquigarrow \mathcal{O}(10^8)$ $B\overline{B}$ pairs collected together so far

at CERN starting in autumn 2007

hope for a $e^+ - e^-$ “super-B factory” in a more distant future, with an increase of luminosity by up to two orders of magnitude

CP violation

The history so far

- 1964, first discovery of *indirect* CP violation in $K_L \to \pi^+\pi^-$ decays (branching ratio $\varepsilon_K \sim 10^{-3}$)
- CP-violating effects may also arise directly at the decay amplitude level $\rightsquigarrow$ *direct* CP violation; eventually established in 1999 through the NA48 (Cern) and KTeV (FNAL) collaborations

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- this decade, the main actor is the B-meson system, i.e. charged & neutral B mesons with the following valence-quark contents:

$$B^+ \sim u\bar{b}, \quad B_c^+ \sim c\bar{b}, \quad B_d^0 \sim d\bar{b}, \quad B_s^0 \sim s\bar{b}$$

  detectable by BaBar, Belle and at the Tevatron (CDF & D0 coll.s)

  - 2001, CP violation in $B_d \to J/\psi K_S$ decays by BaBar & Belle *1st observation outside the K system*; 'mixed-induced' CPv
  - 2004, direct CP violation detected in $B_d \to \pi^\mp K^\pm$ decays

see [Fleischer,hep-ph/0512253]

B Physics and the lattice

the two smallest CKM-matrix elements V_{ub}, V_{td} (mixing between 1st & 3rd generation) are the source of CP violation

b-quark decay inside the B meson always accompanied by a quark-gluon cloud

$\rightsquigarrow$ extraction of fund. parameters from experimental data difficult
$\rightsquigarrow$ lattice QCD is essential to calculate important B matrix elements

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Example: $B^0 - \overline{B}^0$ Mixing (2 neutral B mesons) with definitions of the

- mass difference (oscillation frequency) $(q = s, d)$

$$\Delta M_{B_q} = \frac{G_F^2 m_W^2}{6\pi^2} \eta_B S_0\left(\frac{m_t}{m_W}\right) M_{B_q} f_{B_q}^2 \widehat{B}_{B_q} |V_{tq} V_{tb}|^2$$

$f_{B_q}^2 \widehat{B}_{B_q}$: non-perturbative quantity to be computed on the lattice

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$f_{B_q}^2 \widehat{B}_{B_q}$: non-perturbative quantity to be computed on the lattice

- leptonic decay constant

$$\mathrm{i} f_{B_q} p_\mu = \langle 0 | A_\mu | B_q(p) \rangle$$

with a *heavy-light axial-vector current* $A_\mu = \bar{q}\gamma_5\gamma_\mu b$

B Physics and the lattice

- scale dependent B parameter B_{B_q}

$$\langle \bar{B}_q^0 | O^{\Delta B=2}(\mu) | B_q^0 \rangle = \frac{8}{3} B_{B_q}(\mu) f_{B_q}^2 M_{B_q}^2$$

with the $\Delta B = 2$ operator $O^{\Delta B=2} = \bar{q}\gamma_\mu(1-\gamma_5)b\bar{q}\gamma_\mu(1-\gamma_5)b$

B_d and B_s mesons differ in the valence light quark mass

see [hep-ph/0310329; hep-ph/0407221]

and [Duncan et al, Phys.Rev. D51 (1995); “Properties of B mesons in lattice QCD”]

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B_d and B_s mesons differ in the valence light quark mass
$\rightsquigarrow$ (as far as QCD is concerned) one can expect that the theoretical uncertainty largely cancels in the ratio

$$\frac{\Delta M_{B_s}}{\Delta M_{B_s}} = \frac{[G_F^2 m_W^2/6\pi^2]\eta_B M_{B_s} f_{B_s}^2 \hat{B}_{B_s} S_0\left(\frac{m_t}{m_W}\right)|V_{ts}V_{tb}|^2}{[G_F^2 m_W^2/6\pi^2]\eta_B M_{B_d} f_{B_d}^2 \hat{B}_{B_d} S_0\left(\frac{m_t}{m_W}\right)|V_{td}V_{tb}|^2}$$

$$\frac{\Delta M_{B_s}}{\Delta M_{B_d}} = \frac{|V_{ts}|^2}{|V_{td}|^2}\xi^2 \,, \quad \xi \equiv \frac{f_{B_s}\sqrt{M_{B_s}}}{f_{B_d}\sqrt{M_{B_d}}}$$

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Lattice QCD
Facts to remember

- discretisation of space and time by introduction of a minimal length scale a $\rightsquigarrow$ (ultra violet) lattice cutoff $a^{-1} \sim \Lambda_{\mathrm{UV}}$
- finite volume $L^3 \times L$ to fit lattice into computers memory
- Lattice action $S[U, \overline{\psi}, \psi] = S_G[U] + S_F[U, \overline{\psi}, \psi]$ with

$$\text{gauge part:} \quad S_G = \frac{1}{g_0^2} \sum_p \mathrm{Tr}\{\mathbb{1} - U(p)\}$$

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Functional integral representation of expectation values:

$$\mathcal{Z} = \int \mathcal{D}[U]\mathcal{D}[\overline{\psi},\psi]\mathrm{e}^{-S[U,\overline{\psi},\psi]} = \int \mathcal{D}[U]\prod_{\mathrm{f}} \det(\not{D} + m_{\mathrm{f}})\mathrm{e}^{-S_G[U]}$$

$$\langle O \rangle = \frac{1}{\mathcal{Z}} \int \prod_{x,\mu} \mathrm{d}U_\mu(x) \prod_{\mathrm{f}} \det(\not{D} + m_{\mathrm{f}})\mathrm{e}^{-S_G[U]}$$

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These days: from quenched case $\det(\cdots) \equiv 1$ to $N_{\mathrm{f}} = 2, 3, 4$

HQET – An asymtotic expansion of QCD

problems & physical picture

Problem: light quarks too light & b-quark to heavy

$$\lambda_\pi \sim 1/m_\pi \approx L \qquad\qquad \lambda_B \sim 1/m_b \approx a$$

$\rightsquigarrow$ propagating b on the lattice beyond today's computing resources
$\rightsquigarrow$ need for an effective theory of heavy quarks:
Heavy Quark Effective Theory [Eichten, 1988; Eichten & Hill, 1990]

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Physics: Momentum scales in heavy-light ($Q\bar{q}$) mesons

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Formal: $\mathcal{L}_{\rm HQET} = 1/m_b$-expansion of continuum QCD

- $\overline{\psi}_{\rm b}[\gamma_\mu D_\mu + m_{\rm b}]\psi_{\rm b} \to \mathcal{L}_{\rm stat} + \mathcal{L}^{(1)} + \ldots \qquad \mathcal{L}^{(1)} \sim O(1/m_b)$
- $\mathcal{L}_{\rm stat}(x) = \overline{\psi}_{\rm h}(x)[\gamma_0 D_0 + m_{\rm h}]\psi_{\rm h}(x)$

  $P_+\psi_{\rm h} = \psi_{\rm h} \quad \overline{\psi}_{\rm h}P_+ = \overline{\psi}_{\rm h}$ with $P_+ = (1\!\!1 + \gamma_0)/2 \quad \rightsquigarrow \quad$ 2 d.o.f.
- Accurate expansion for $m_{\rm h} \gg \Lambda_{\rm QCD}$

the axial vector-current $A_\mu(x) = \overline{\psi}(x)\gamma_\mu\gamma_5\psi(x)$

... between heavy and light quark

composite fields involving b-quarks, e.g. the time component of A_μ, also translate to the effective theory:

$$A_0(x) = \overline{\psi}_{\rm l}(x)\gamma_0\gamma_5\psi_b(x) \quad \xrightarrow{b\to h} \quad A_0^{\rm stat}(x) = \overline{\psi}_{\rm l}(x)\gamma_0\gamma_5\psi_{\rm h}(x)$$

What about renormalization?

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- *in HQET* ($A_\mu \to A_\mu^{\mathrm{stat}}$)
  there is no Ward identity $\rightsquigarrow$ *static-light axial current becomes explicit renormalization scale μ dependent*

$$(A_0^{\mathrm{stat}})_R(\mu) = Z_A^{\mathrm{stat}}(\mu)\overline{\psi}_l\gamma_0\gamma_5\overline{\psi}_h$$

Generic structure of the HQET-expansion . . .

. . . of QCD matrix elements

$$\boldsymbol{\Phi^{\rm QCD} \equiv f_B\sqrt{m_B} = Z_A\langle B|A_0|0\rangle = Z_A\Phi}$$

$\rightsquigarrow$ in HQET

$$\Phi^{\rm stat}(\mu) = Z_A(\mu)\langle B|A_0^{\rm stat}|0\rangle$$

focus on the μ & scheme independent renormalization group invariant (RGI) matrix element

$$\Phi_{\rm RGI}^{\rm stat} = \lim_{\mu\to\infty}\left[2b_0\bar{g}^2(\mu)\right]^{-\gamma_0/2b_0} \times \Phi^{\rm stat}(\mu)$$

with anomalous dim. $\gamma(\bar{g}) = (\mu/Z_{\rm A}^{\rm stat})(\partial Z_{\rm A}^{\rm stat}/\partial\mu) = -\gamma_0\bar{g}^2 + O(\bar{g}^4)$

$$\beta(\bar{g}) = \mu(\partial\bar{g}/\partial\mu) = -b_0\bar{g}^3 + O(\bar{g}^5)$$

Generic structure of the HQET-expansion …

… of QCD matrix elements

$$\Phi^{\rm QCD} \equiv f_B\sqrt{m_B} = Z_A\langle B|A_0|0\rangle = Z_A\Phi$$

$\rightsquigarrow$ in HQET

$$\Phi^{\rm stat}(\mu) = Z_A(\mu)\langle B|A_0^{\rm stat}|0\rangle$$

focus on the μ & scheme independent renormalization group invariant (RGI) matrix element

$$\Phi^{\rm stat}_{\rm RGI} = \lim_{\mu\to\infty}\left[2b_0\bar{g}^2(\mu)\right]^{-\gamma_0/2b_0} \times \Phi^{\rm stat}(\mu)$$

with anomalous dim. $\gamma(\bar{g}) = (\mu/Z_{\rm A}^{\rm stat})(\partial Z_{\rm A}^{\rm stat}/\partial\mu) = -\gamma_0\bar{g}^2 + O(\bar{g}^4)$ &

$$\Phi^{\rm QCD} = C_{\rm PS}(M_b/\Lambda_{\overline{\rm MS}}) \times \Phi^{\rm stat}_{\rm RGI} + O(1/M_b)$$

$$M_b = \lim_{\mu\to\infty}\left[2b_0\bar{g}^2(\mu)\right]^{-d_0/2b_0} \times \overline{m}_b(\mu)$$

$$\Lambda_{\overline{\rm MS}} = \lim_{\mu\to\infty}\mu\left[b_0\bar{g}^2_{\overline{\rm MS}}(\mu)\right]^{-b_1/2b_0^2} \times {\rm e}^{-1/2b_0\bar{g}^2_{\overline{\rm MS}}(\mu)}$$

with $\tau(\bar{g}) = (\mu/\overline{m})(\partial\overline{m}/\partial\mu) = -d_0\bar{g}^2 + O(\bar{g}^4)$ and

$$\beta(\bar{g}) = \mu(\partial\bar{g}/\partial\mu) = -b_0\bar{g}^3 + O(\bar{g}^5)$$

What is the meaning of $C_{\mathrm{PS}}(M_b/\Lambda_{\overline{\mathrm{MS}}})$

conversion to the *matching* scheme

Evaluation of the conversion factor for the axial current:

$$\Phi^{\mathrm{QCD}} = C_{\mathrm{PS}}(M_b/\Lambda_{\overline{\mathrm{MS}}}) \times \Phi^{\mathrm{stat}}_{\mathrm{RGI}} + O(1/M_b)$$

$$\stackrel{!}{=} C_{\mathrm{match}}(m_b/\mu) \times \Phi_{\overline{\mathrm{MS}}} + O(1/m_b)$$

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$$\Rightarrow C_{\mathrm{PS}}(M_b/\Lambda_{\overline{\mathrm{MS}}}) = C_{\mathrm{match}}(1) \times \frac{\Phi_{\overline{\mathrm{MS}}}(\mu)}{\Phi_{\mathrm{RGI}}} \qquad \bar{g} = \bar{g}_{\overline{\mathrm{MS}}},\ \Lambda = \Lambda_{\overline{\mathrm{MS}}}$$

$$= \left[2b_0\bar{g}^2(m_b)\right]^{\gamma_0/2b_0} \exp\left\{-\int_0^{\bar{g}(m_b)} \mathrm{d}g \left[\frac{\gamma^{\mathrm{match}}(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

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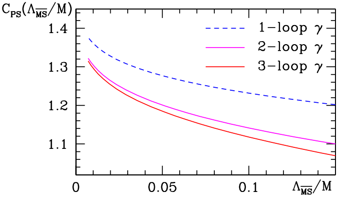


perturbatively under control
[Chetyrkin & Grozin, 2003]

- anom. dim. in the matching scheme:

$$\gamma^{\mathrm{match}}(g) = \gamma^{\overline{\mathrm{MS}}}(g) + \rho(\bar{g})$$

$\rho(\bar{g})$: contribution from C_{match}

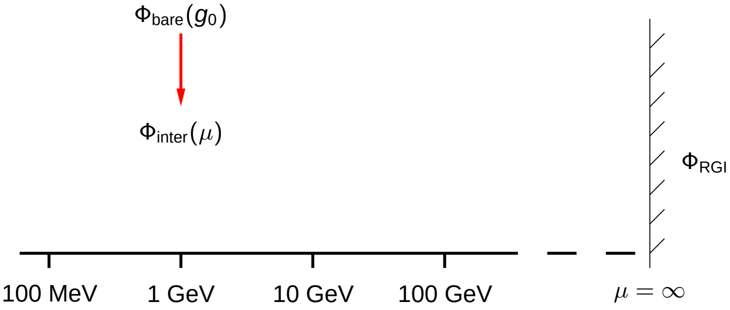
- Advantage of RGI-ration M/Λ: *can be fixed in lattice calculations without perturbative errors*

Realisation

overall computational strategy

- introduce an *intermediate finite-volume renormal. scheme*

$$\mathcal{O}_{\mathrm{inter}}(\mu) = Z(g_0, a\mu) \cdot \mathcal{O}_{\mathrm{bare}}(g_0)$$



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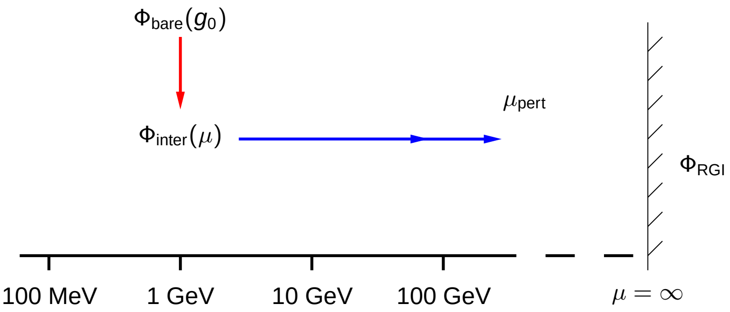

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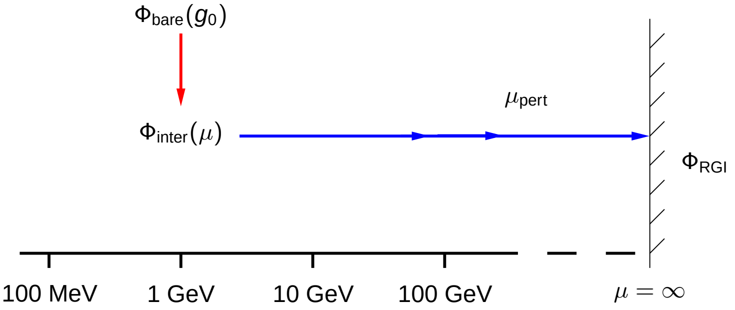

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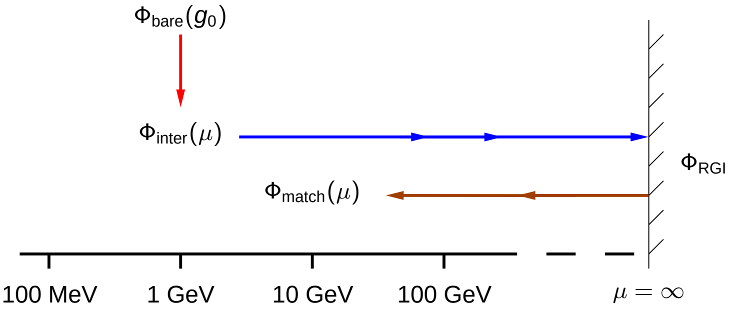
- Matching: convert into another scheme like $\overline{\text{MS}}$



Renormalization Group Invariant (RGI)

asymtotic $\mu \to \infty$

- at high energies (pert. scale μ_{pert}) use the perturbative evolution

$$\Phi_{\mathrm{RGI}} = \Phi_{\mathrm{inter}}(\mu_{\mathrm{pert}}) \left[2b_0\bar{g}^2(\mu_{\mathrm{pert}})\right]^{-\gamma_0/2b_0} \times \exp\left\{-\int_0^{\bar{g}(\mu_{\mathrm{pert}})} \mathrm{d}g \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

to connect Φ_{inter} at this scale with Φ_{RGI}

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  to connect Φ_{inter} at this scale with Φ_{RGI}

- the total renormalization is build out of

$$\Phi_{\text{match}}(\mu) = \frac{\Phi_{\text{match}}(\mu)}{\Phi_{\text{RGI}}} \times \frac{\Phi_{\text{RGI}}}{\Phi_{\text{inter}}(\mu_{\text{min}})} \times Z_{\text{inter}}(g_0, a\mu_{\text{min}}) \times \Phi_{\text{bare}}(g_0)$$

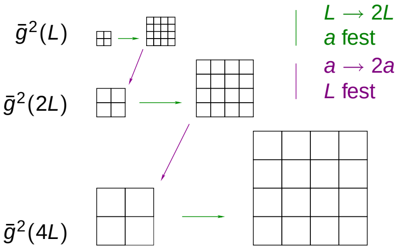
  with

$$\frac{\Phi_{\text{RGI}}}{\Phi_{\text{inter}}(\mu_{\text{min}})} = \frac{\Phi_{\text{RGI}}}{\Phi_{\text{inter}}(\mu_{\text{pert}})} \times \underbrace{\frac{\Phi_{\text{inter}}(\mu_{\text{pert}})}{\Phi_{\text{inter}}(\mu_{\text{min}})}}_{\text{factor of step scaling}}$$

recursive finite size scaling

climbing up the scales

1. choose a lattice with L/a points



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1. choose a lattice with L/a points
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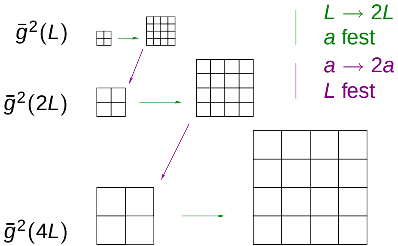

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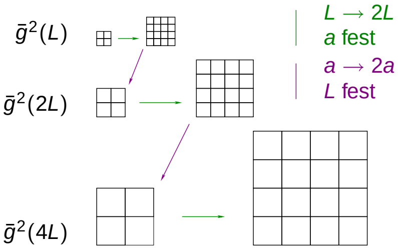

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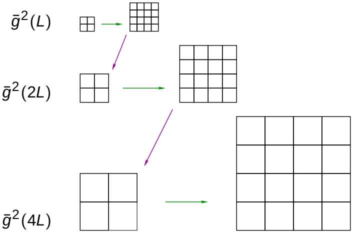
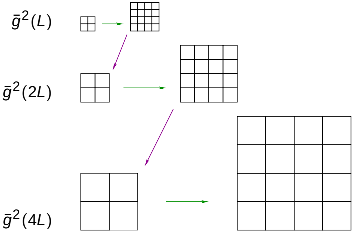
4. iterate 1 to 3 with several L/a and compute the continuum limit





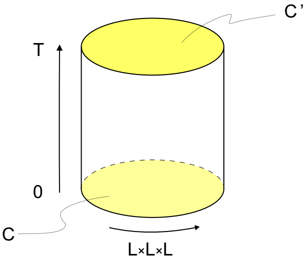
The Schrödinger functional

Definition

- defined on a $T \times L^3$ cylinder in Euclidian space with
  - periodic b.c. in space
  - Dirichlet b.c. in time
- partition function:

$$\mathcal{Z} \equiv \int_{T \times L^3} \mathcal{D}\left[U, \overline{\psi}, \psi\right] \mathrm{e}^{-S[U,\overline{\psi},\psi]}$$

- for convenience we set $T = L$



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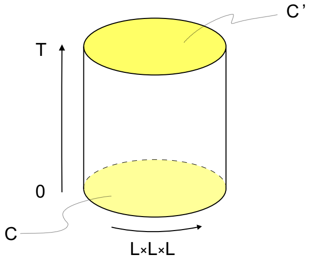

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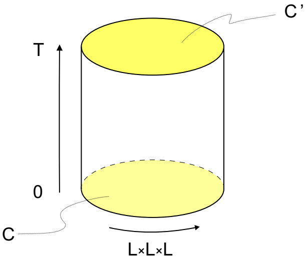

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Properties: explicit gauge invariance & mass independent
$\rightsquigarrow$ simple RGEs $\mu(\mathrm{d}\Phi_{\text{inter}}(\mu)\;/\mathrm{d}\mu\;) = \gamma\,(g) \cdot \Phi_{\text{inter}}(\mu)$

Lattice HQET setup
theoretical improvements

- starting point: discretization á la Eichten-Hill [1990]

$$S_{\rm h}^{\rm EH} = a^4 \sum_x \overline{\psi}_{\rm h}(x) \nabla_0^* \psi_{\rm h}(x)$$

$$\nabla_0 \psi_{\rm h}(x) = \frac{1}{a}\left[\psi_{\rm h}(x) - U^\dagger(x - a\hat{0}, 0)\psi_{\rm h}(x - a\hat{0})\right]$$

  with the usual gauge links U
- light quark in usual relativistic formulation

Problems in the past ...

(a) rapid grow of statistical errors

$$\frac{noise}{signal} \propto \exp\{x_0(E_{\rm stat} - m_\pi)\}$$

(b) new parameters in each order in the effective theory due to operator mixing $\rightsquigarrow$ continuum limit does not exist

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... now solved

(a) *alternative discretizations of HQET* called SOX, HYP1, HYP2 uses generalized gauge links $V \to W$ with equal symmetries [Della Morte et al, 2003/2005] $\rightsquigarrow$ better statistical precision

(b) Non-perturbative renormalization of HQET through a *non-perturbative matching to QCD in finite volume.* [J.H. & Sommer, 2004]

Correlation functions in the SF

The QCD transfer matrix formalism in the SF

the euclidean transfer matrix, defined by

$$\mathbb{T} = \exp\{-a\mathbb{H}\}\,, \quad \text{with QCD Hamiltonian } \mathbb{H}$$

allows to extract informations about *the energy spectrum from correlation functions*

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for Wilson fermions $\mathbb{T}$ can be constructed with all important properties (universality applies for $O(a)$ clover impr.) [Lüscher, 1977]

- **self-adjoint and bounded**
- **gauge invariant**
- **strictly positive (i.e. all eigenvalues larger than zero)**

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the action of $\mathbb{T}$ on a energy state is given by

$$\mathbb{T}|E_n^{(q)}\rangle = \exp\{E_n^{(q)}\}|E_n^{(q)}\rangle$$

with energy level $n \geq 0$ of states with q.n. $(q) = (J, P, C, \cdots)$

we denote the vacuum state as usual by $|0\rangle$

Correlation functions in the SF

The QCD transfer matrix formalism in the SF

in the SF we can define vacuum states at the boundaries by

$$|i,0\rangle \quad \text{for} \quad x_0 = 0$$
$$|f,0\rangle \quad \text{for} \quad x_0 = T$$

$\rightsquigarrow$ $|f,0\rangle = |i,0\rangle$ carries the quantum numbers of the vacuum

now we can apply some operator $\widehat{O}$ which creates a meson state

$$|i,M\rangle = \widehat{O}|i,0\rangle \quad \text{at} \quad x_0 = 0$$
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SF states are usual *no* eigenstates of $\mathbb{T}$

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SF states are usual *no* eigenstates of $\mathbb{T}$

they are a mixture of all states with the same quantum numbers q

$$|i,0\rangle = c_0|E_0^{(0)}\rangle + c_1|E_1^{(0)}\rangle + \ldots$$
$$|i,M\rangle = d_0|E_0^{(M)}\rangle + d_1|E_1^{(M)}\rangle + \ldots$$

Correlation functions in the SF

important correlation functions

the *partition function* $\mathcal{Z}$ can be written as a power of $\mathbb{T}$

$$\mathcal{Z} = \langle i, 0 | \mathbb{T}^{T/a} \mathbb{P} | i, 0 \rangle$$

with $\mathbb{P}$ projecting onto the gauge-invariant sector

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for *correlation functions* one obtains

$$f_X(x_0) = \frac{1}{\mathcal{Z}} \frac{L^3}{2} \langle i, 0 | \mathrm{e}^{-(T-x_0)\mathbb{H}} \mathbb{P} \mathbb{X} \mathrm{e}^{-x_0 \mathbb{H}} \mathbb{P} | i, M \rangle$$

$$f_1 = \frac{1}{\mathcal{Z}} \frac{1}{2} \langle i, M | \mathbb{T}^{T/a} \mathbb{P} | i, M \rangle$$

with $f_X = f_A, f_P$ and correponding operator $\mathbb{X} = A_0, P$

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spectral decomposition of correlator f_A:

$$f_A(x_0) = \frac{L^3}{2} \frac{\sum_{n,m} \exp[-(T - x_0) E_n^{(0)}] \exp[-x_0 E_m^{(M)}] c_n d_m \langle E_n^{(0)} | A_0 | E_m^{(M)} \rangle}{\sum_m c_m^2 \exp[-E_m^{(0)} T]}$$

Correlation functions in the SF

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- $x_0 \ll T/2$ sizeable contributions from excited meson states

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- $x_0 \gg T/2$ contributions from vacuum excitations

Correlation functions in the SF

important correlation functions

$$f_A(x_0) = \frac{L^3}{2} \frac{\sum_{n,m} \exp[-T(E_n^{(0)} - E_m^{(M)})/2] c_n d_m \langle E_n^{(0)} | A_0 | E_m^{(M)} \rangle}{\sum_m c_m^2 \exp[-E_m^{(0)} T]}$$

- $x_0 \ll T/2$ sizeable contributions from excited meson states
- $x_0 \gg T/2$ contributions from vacuum excitations
- $x_0 \approx T/2$ leading behaviour governed by lightest meson state $n = 0$:

$$f_A(x_0 \approx T/2) \propto \langle E_0^{(0)} | A_0 | E_0^{(M)} \rangle = F_{\mathrm{PS}} m_{\mathrm{PS}} / \sqrt{2 m_{\mathrm{PS}} L^3}$$

Correlation functions in the SF

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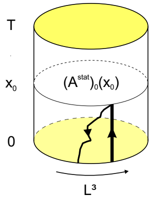
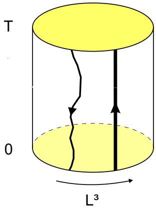
insert static-light axial current at $x_0 = T/2$



boundary-boundary correlator f_1 independent of x_0



Lattice Setup
special HQET observables

- renormalization condition for the static axial current, proposed in [Kurth, Sommer 2001]

$$X(0, L) = Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, L)\, X(g_0, L)$$

with ratio

$$X(g_0, L) = \frac{f_{\mathrm{A}}^{\mathrm{stat}}(L/2)}{\sqrt{f_1^{\mathrm{stat}}}}$$

$$f_{\mathrm{A}}^{\mathrm{stat}}(x_0) = -\frac{1}{2} \int \mathrm{d}^3\mathbf{y}\, \mathrm{d}^3\mathbf{z}\, \left\langle A_0^{\mathrm{stat}}(x)\, \overline{\zeta}_{\mathrm{h}}(\mathbf{y})\gamma_5\zeta_{\mathrm{l}}(\mathbf{z})\right\rangle$$

$$f_1^{\mathrm{stat}} = -\frac{1}{2L^6} \int \mathrm{d}^3\mathbf{u}\, \mathrm{d}^3\mathbf{v}\, \mathrm{d}^3\mathbf{y}\, \mathrm{d}^3\mathbf{z}\, \left\langle \overline{\zeta}_{\mathrm{l}}'(\mathbf{u})\gamma_5\zeta_{\mathrm{h}}'(\mathbf{v})\, \overline{\zeta}_{\mathrm{h}}(\mathbf{y})\gamma_5\zeta_{\mathrm{l}}(\mathbf{z})\right\rangle$$

Lattice Setup
special HQET observables

- renormalization condition for the static axial current, proposed in [Kurth, Sommer 2001]

$$X(0, L) = Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, L)\, X(g_0, L)$$

with ratio

$$X(g_0, L) = \frac{f_{\mathrm{A}}^{\mathrm{stat}}(L/2)}{\sqrt{f_1^{\mathrm{stat}}}}$$

$$f_{\mathrm{A}}^{\mathrm{stat}}(x_0) = -\frac{1}{2}\int \mathrm{d}^3\mathbf{y}\,\mathrm{d}^3\mathbf{z}\,\left\langle A_0^{\mathrm{stat}}(x)\,\bar{\zeta}_{\mathrm{h}}(\mathbf{y})\gamma_5\zeta_{\mathrm{l}}(\mathbf{z})\right\rangle$$

$$f_1^{\mathrm{stat}} = -\frac{1}{2L^6}\int \mathrm{d}^3\mathbf{u}\,\mathrm{d}^3\mathbf{v}\,\mathrm{d}^3\mathbf{y}\,\mathrm{d}^3\mathbf{z}\,\left\langle \bar{\zeta}_{\mathrm{l}}'(\mathbf{u})\gamma_5\zeta_{\mathrm{h}}'(\mathbf{v})\,\bar{\zeta}_{\mathrm{h}}(\mathbf{y})\gamma_5\zeta_{\mathrm{l}}(\mathbf{z})\right\rangle$$

- multiplicative renormal. $\zeta_{\mathrm{R}} = Z_\zeta \zeta, \ldots$ and $(A_{\mathrm{R}}^{\mathrm{stat}})_0 = Z_{\mathrm{A}}^{\mathrm{stat}} A_0^{\mathrm{stat}}$ leads to

$$\frac{(f_{\mathrm{A}}^{\mathrm{stat}})_{\mathrm{R}}}{((f_1^{\mathrm{stat}})_{\mathrm{R}})^{1/2}} = \frac{Z_{\zeta_{\mathrm{l}}} Z_{\zeta_{\mathrm{h}}} Z_{\mathrm{A}}^{\mathrm{stat}} f_{\mathrm{A}}^{\mathrm{stat}}}{Z_{\zeta_{\mathrm{l}}} Z_{\zeta_{\mathrm{h}}} \sqrt{f_1^{\mathrm{stat}}}} = Z_{\mathrm{A}}^{\mathrm{stat}} \frac{f_{\mathrm{A}}^{\mathrm{stat}}}{\sqrt{f_1^{\mathrm{stat}}}}$$

and X scales like $X_{\mathrm{R}} = Z_{\mathrm{A}}^{\mathrm{stat}} X$

Lattice Setup
Lattice Step Scaling Function

- use $O(a)$ improved ratio

$$X_{\mathrm{I}}(g_0, L) = \frac{f_{\mathrm{A}}^{\mathrm{stat}}(L/2) + a c_{\mathrm{A}}^{\mathrm{stat}} f_{\delta\mathrm{A}}^{\mathrm{stat}}(L/2)}{\sqrt{f_1^{\mathrm{stat}}}}$$

$c_{\mathrm{A}}^{\mathrm{stat}}$: improvement coefficient (pert. known)
$f_{\delta\mathrm{A}}^{\mathrm{stat}}$: O(a) correction

- definition of the step scaling function

$$\Sigma_{\mathrm{A}}^{\mathrm{stat}}(u, a/L) = \frac{Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, 2L/a)}{Z_{\mathrm{A}}^{\mathrm{stat}}(g_0, L/a)}\,, \quad \text{with} \quad u = \bar{g}^2(L) \quad \text{and} \quad m_{\mathrm{q}} = 0$$

- so continuum limit exists and can be taken in each step i.e. for different coupling values $\{u\}$

$$\sigma_{\mathrm{A}}^{\mathrm{stat}}(u) \equiv \lim_{a \to 0} \Sigma_{\mathrm{A}}^{\mathrm{stat}}(u, a/L)\Big|_{\bar{g}^2 = u, m_{\mathrm{q}} = 0}$$

climbing up the scales

full step scaling factor

$$\frac{\Phi(\mu_{\rm pert})}{\Phi(\mu_{\rm min})} = \frac{\Phi(\mu_{\rm pert})}{\Phi(\mu_{\rm pert}/2)}\frac{\Phi(\mu_{\rm pert}/2)}{\Phi(\mu_{\rm pert}/4)} \times \ldots = [\sigma_{\rm A}^{\rm stat}(u_n)]^{-1} \cdots [\sigma_{\rm A}^{\rm stat}(u_0)]^{-1}$$

with $u_k = \bar{g}^2(L_k)$ and $\mu_k = 1/L_k = 2^k/L_{\rm max}$

climbing up the scales

full step scaling factor

$$\frac{\Phi(\mu_{\rm pert})}{\Phi(\mu_{\rm min})} = \frac{\Phi(\mu_{\rm pert})}{\Phi(\mu_{\rm pert}/2)}\,\frac{\Phi(\mu_{\rm pert}/2)}{\Phi(\mu_{\rm pert}/4)} \times \ldots = [\sigma_{\rm A}^{\rm stat}(u_n)]^{-1} \cdots [\sigma_{\rm A}^{\rm stat}(u_0)]^{-1}$$

with $u_k = \bar{g}^2(L_k)$ and $\mu_k = 1/L_k = 2^k/L_{\rm max}$

$$L_{\rm max} = \mathcal{O}\left[\tfrac{1}{2}\,{\rm fm}\right]: \quad {\rm HS} \longrightarrow {\rm SF}(\mu = 1/L_{\rm max})$$

$$\downarrow \quad \sigma_{\rm A}^{\rm stat}(u_0)$$

$${\rm SF}(\mu = 2/L_{\rm max})$$

$$\downarrow \quad \sigma_{\rm A}^{\rm stat}(u_1)$$

•
•
•

$$\downarrow \quad \sigma_{\rm A}^{\rm stat}(u_n)$$

$${\rm SF}(\mu = 2^n/L_{\rm max})$$

$$\textbf{PT}\downarrow$$

$$\overline{\rm MS}\text{-scheme} \overset{\textbf{PT}}{\longleftarrow} \Lambda_{\rm QCD}, M, \Phi_{\rm RGI}$$

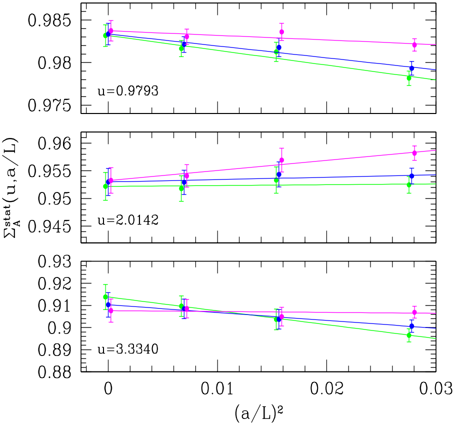
Lattice Results

fit to continuum limit (CL)

	Hyp1		
L/a	$Z_A^{stat}(g_0, L/a)$	$Z_A^{stat}(g_0, 2L/a)$	$\Sigma_A^{stat}(u, a/L)$
6	0.9363(5)	0.9169(6)	0.9793(8)
8	0.9295(5)	0.9126(9)	0.9818(11)
12	0.9231(3)	0.9066(7)	0.9821(9)
...			
6	0.8332(12)	0.7504(20)	0.9007(28)
8	0.8184(13)	0.7396(34)	0.9037(44)
12	0.8078(13)	0.7339(33)	0.9085(44)

- ✓ well-behaved error, estimated by jackknife analysis within whole data set
- ✓ $O(a)$ improvement verified $\Rightarrow$ fitting in $x = (a/L)^2$ possible

lattice step scaling function: $\sigma_{\mathrm{A}}^{\mathrm{stat}}(u) \equiv \lim_{a\to 0} \Sigma_{\mathrm{A}}^{\mathrm{stat}}(u, a/L)\Big|_{\bar{g}^2=u, m=0}$



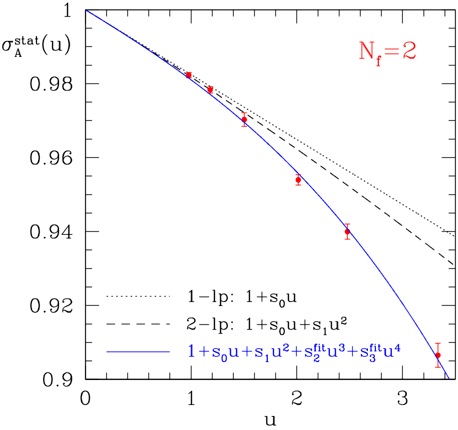
(a) fit for each discretization $\Sigma_{\mathrm{A,i}}^{\mathrm{stat}}(u, x) = \sigma_{\mathrm{A,i}}^{\mathrm{stat}}(u) + b_i \cdot x$

(b) fit to universal CL $\Sigma_{\mathrm{A}}^{\mathrm{stat}}(u, x) = \sigma_{\mathrm{A}}^{\mathrm{stat}}(u) + c_i \cdot x$

Continuum Results

continuum step scaling function

u	$\sigma^{\rm stat}_{\rm A,HYP1}$	$\sigma^{\rm stat}_{\rm A}$
0.9793	0.9834(13)	0.9834(12)
1.1814	0.9791(16)	0.9792(16)
1.5031	0.9712(25)	0.9710(25)
2.0142	0.9530(24)	0.9529(24)
2.4792	0.9428(35)	0.9428(35)
3.3340	0.9103(55)	0.9104(54)



fitting step scaling function: $\sigma^{\rm stat}_{\rm A}(u) = 1 + s_0 u + s_1 u^2 + s_2 u^3 + \ldots$

Continuum Results

scale evolution of the renormalized matrix element

non-perturbative vs. perturbative evaluation of

$$\Phi(\mu)/\Phi_{\mathrm{RGI}} = \left[2b_0\bar{g}^2(\mu)\right]^{\gamma_0/2b_0} \exp\left\{\int_0^{\bar{g}(\mu)} \mathrm{d}g \left[\frac{\gamma(g)}{\beta(g)} - \frac{\gamma_0}{b_0 g}\right]\right\}$$

- 3-loop β-function

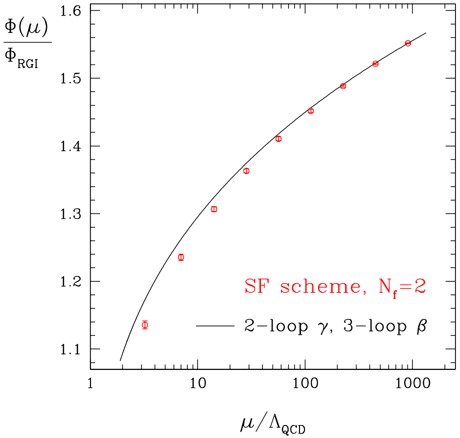
  $$\beta(\bar{g}) = -\bar{g}^3 \cdot (b_0 + b_1\bar{g}^2 + b_2\bar{g}^4)$$

  with universal b_0, b_1
- 2-loop γ-function

  $$\gamma(\bar{g}) = -\bar{g}^2 \cdot (\gamma_0 + \gamma_1\bar{g}^2)$$

  with universal γ_0

rel. deviation at hadronic scale: 2.7%



Results

$$\Phi_{\text{match}}(\mu) = \frac{\Phi_{\text{match}}(\mu)}{\Phi_{\text{RGI}}} \times \frac{\Phi_{\text{RGI}}}{\Phi_{\text{inter}}(\mu_{\text{min}})} \times Z_{\text{inter}}(g_0, a\mu_{\text{min}}) \times \Phi_{\text{bare}}(g_0)$$

✓ Universal result referring to the continuum limit

$$\frac{\Phi_{\text{inter}}(\mu)}{\Phi_{\text{RGI}}} = 1.143(16)$$

or without coarsest lattice $L/a = 6$ and fit to constant

$$\frac{\Phi_{\text{inter}}(\mu)}{\Phi_{\text{RGI}}} = 1.136(10)$$

at $\mu = 1/L_{\max}$ or rather $\bar{g}^2(L_{\max}) = 4.61$

✓ determination of the Z-factor at the low-energy reference scale in the intermediate (SF) scheme (done rencently)

► conversion into the $\overline{\text{MS}}$-scheme (matching scheme)

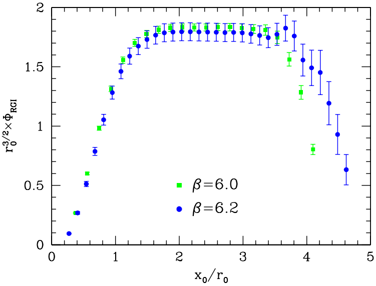
Outlook

Further improvements by new methods ...

Wave functions (still in use)

$$\omega(r) \sim r^n \exp(-r/r_H)$$

at the boundaries of the SF-cylinder to suppress excited B-meson state contributions to correlators [Duncan, 1992]



Outlook

Further improvements by new methods ...

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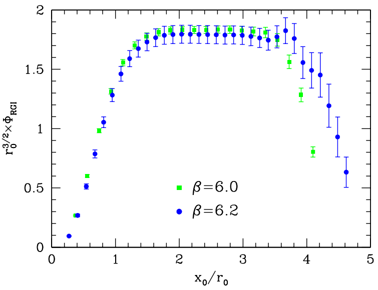
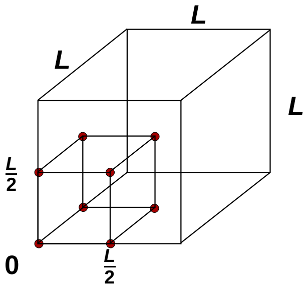


lower momenta strategy (in use) to reduce computational effort in some $1/m$ correlators ($\propto L^6$) by skipping higher momenta $k_{\rm min} \leq k \leq k_{\rm max}$

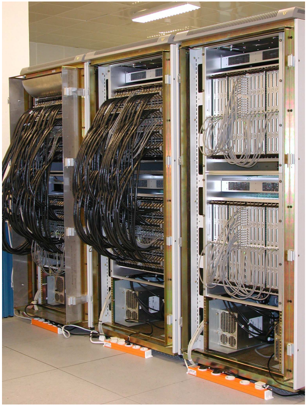
"**8 sources** are better than one" [Billoire et al, 1985]



Outlook

... and new computers

APEmille



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