A Markov approach to nonlinear multivariate delay systems with noise

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Abstract

It is well known that non-Markov processes can be described by means of Markov processes by increasing the dimension of the non-Markov processes. In line with this approach, we derive a description of multivariate stochastic processes with delay in terms of Markov processes. In particular, a multivariate Langevin equation and the corresponding multivariate Fokker-Planck equation is obtained for multivariate systems with delay. In addition, an evolution equation of the Fokker-Planck type for systems with delay is derived that generalizes the evolution equation proposed by Guillouzic et al. to the multivariate case. Both Ito and Stratonovich calculus are considered.

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1. Introduction

In various disciplines ranging from population dynamics [1] to neurophysics [2, 3], biophysics [4–7], chaos synchronization [8–10] and laser physics [11–13], we deal with systems with delay. These delay systems typically correspond to reduced descriptions of systems without delay that involve transport processes and reaction chains. In delay systems the effects of transport processes and reactions chains are taken into account by the introduction of a delay. Delay systems are usually subjected to noise sources as well. In particular, in biological systems there are various noise sources [14–16]. In the inanimate world thermal noise is inevitable unless the systems operate at very low temperatures. In view of the relevance of systems with delay, on the one hand, and systems subjected to noise sources, on the other hand, there is a need for a stochastic theory of delay systems. Stochastic processes with delay are special cases of non-Markov processes which have been extensively studied in recent years [17–20]. For stochastic delay systems involving a single variable, exact stationary solutions [21–23], evolution equations for probability densities similar to Fokker-Planck equations [24, 25], and descriptions in terms of Markov processes [26] have been derived. For the multivariate case, however, similar studies have not yet been carried out.

In the present study, we derive a hierarchy of multivariate Langevin and Fokker-Planck equations from multivariate stochastic delay differential equations. We consider several boundary conditions and use both Ito and Stratonovich calculus. Finally, we generalize the evolution equation for delay systems proposed by by Guillouzic et al. [24, 25] to the multivariate case.

2. Multivariate stochastic delay systems

Let $(\xi_1, \ldots, \xi_N) \in \Omega = \Omega_1 \times \cdots \times \Omega_N$ denote a random vector with components ξ_i defined on $\Omega_i = [a_i, b_i]$ and $b_i > a_i$ or $\Omega_i = \mathbb{R}$. We consider a multivariate stochastic process defined by

$$\dot{\xi}_i = h_i[\{\xi_l(t)\}, \{\xi_l(t-\tau)\}] + g_i[\{\xi_l(t)\}, \{\xi_l(t-\tau)\}]\Gamma_i(t) \tag{1}$$

for $i = 1, \ldots, N$ and $t \geq 0$, where $\tau > 0$ denotes a delay, $h_i[\{\xi_l(t)\}, \{\xi_l(t -$

$\tau)\}] = h_i[\xi_1(t), \ldots, \xi_N(t), \xi_1(t-\tau), \ldots, \xi_N(t-\tau)]$ denote components of a drift force, and $g_i[\{\xi_l(t)\}, \{\xi_l(t-\tau)\}] = g_i[\xi_1(t), \ldots, \xi_N(t), \xi_1(t-\tau), \ldots, \xi_N(t-\tau)]$ denote components of a noise amplitude. $(\Gamma_1, \ldots, \Gamma_N)$ is a delta-correlated and Gaussian distributed Langevin force with $\langle \Gamma_i(t)\Gamma_k(t)\rangle = \delta_{i,k}\delta(t-t')$, where $\delta_{i,k}$ and $\delta(t-t')$ denote the Kronecker symbol and the delta-distribution, respectively [27]. Note that we do not sum over the index i in the expression $g_i\Gamma_i$. The multiplicative noise term $g_i\Gamma_i$ is interpreted according to the Ito and Stratonovich calculus. That is, using statistically independent Wiener processes $w_i(t) = \int^t \Gamma_i(t')\,\mathrm{d}t'$, Eq. (1) reads

$$\xi_i(t+\epsilon) = \xi_i(t) + \int_t^{t+\epsilon} h_i[\{\xi_l(s)\}, \{\xi_l(s-\tau)\}]\mathrm{d}s + {}^I_S\!\int_t^{t+\epsilon} g_i[\{\xi_l(s)\}, \{\xi_l(s-\tau)\}]\,\mathrm{d}w_i(s) \tag{2}$$

for $i = 1, \ldots, N$. The integral ${}^I_S\!\int g_i[\{\xi_l(s)\}, \{\xi_l(s-\tau)\}]\mathrm{d}w(s)$ refers to the Ito and Stratonovich interpretation of stochastic integrals [27, 28]. These integrals are well-defined because the stochastic integral ${}^I_S\!\int g[\eta_1(s), \eta_2(s)]\mathrm{d}w(s)$ for the random variables η_1 and η_2 is defined irrespective of the vanishing or non-vanishing of correlations between η_1 and η_2. Eq. (1) is supplemented with the initial condition

$$\xi_i(t) = \phi_i(t),\ t \in [-\tau, 0]\ . \tag{3}$$

Boundary conditions will be formulated below in terms of constraints for probability currents. Using the method of steps [21, 26, 29], for $t \in [\tau M, (M+1)\tau]$ with $M \geq 0$ we introduce the time-dependent $N \times (M+1)$ matrices $\{\xi_i^k(t')\}_{i=1,k=0}^{N,M}$ and $\{\Gamma_i^k(t')\}_{i=1,k=0}^{N,M}$ defined by

$$\xi_i^k(t') = \xi_i(t),\ \Gamma_i^k(t') = \Gamma_i(t),\quad t' = t - k\tau,\ t' \in [0, \tau],\ k = 0, \ldots, M\ . \tag{4}$$

In turn, $\xi_i(t)$ and $\Gamma_i(t)$ can be expressed by means of these matrices like

$$\xi_i(t) = \xi_i^k(t'),\ \Gamma_i(t) = \Gamma_i^k(t'),\ t' = t - k\tau,\ k = \mathrm{int}\{\frac{t}{\tau}\}\ , \tag{5}$$

where $\mathrm{int}\{\cdot\}$ yields the integer value of $\{\cdot\}$. From Eq. (4) and the definition of Γ_i as independent Langevin forces, it follows that

$$\left\langle \Gamma_i^k(t)\Gamma_{i'}^{k'}(t')\right\rangle = \delta_{i,i'}\delta_{k,k'}\delta(t-t') \quad \text{for } t, t' \in [0, \tau]\ . \tag{6}$$

Note that the matrices $\{\xi_i^k\}$ and $\{\Gamma_i^k\}$ can alternatively be expressed as vectors $(\tilde{\xi}_1, \ldots, \tilde{\xi}_{N(M+1)})$ and $(\tilde{\Gamma}_1, \ldots, \tilde{\Gamma}_{N(M+1)})$ with $\tilde{\xi}_j = \xi_i^k$ and $\tilde{\Gamma}_j = \Gamma_i^k$ for $j = (i-1)*(M+1)+k$ and

$$\left\langle \tilde{\Gamma}_j(t)\tilde{\Gamma}_{j'}(t') \right\rangle = \delta_{j,j'}\delta(t-t') \quad \text{for } t, t' \in [0, \tau] \ . \tag{7}$$

Using ξ_i^k and $\Gamma_{i'}^{k'}$, we can transform Eq. (1) for $t \in [\tau M, (M+1)\tau]$ into

$$\frac{\mathrm{d}}{\mathrm{d}t'}\xi_i^k = h_i[\{\xi_l^k(t')\}, \{\xi_l^{k-1}(t')\}] + g_i[\{\xi_l^k(t')\}, \{\xi_l^{k-1}(t')\}]\Gamma_i^k(t'), \ t' \in [0, \tau] \tag{8}$$

with $i = 1, \ldots, N$, $k = 0, \ldots, M$, and $\xi_i^{-1}(t') = \phi_i(t' - \tau)$. Eq. (8) corresponds to a multivariate Langevin equation with multiplicative noise interpreted according to Ito or Stratonovich calculus. Note that drift and diffusion terms of the multivariate Langevin equation (8) depend explicitly on time due to the coefficients $\xi_i^{-1}(t') = \phi_i(t' - \tau)$. The initial conditions for Eq. (8) with $t \in [\tau M, (M+1)\tau]$ can be obtained from Eq. (8) with $t \in [\tau M', (M'+1)\tau]$ and $M' = M - 1$. That is, we deal with an iterative procedure. Let us dwell on this issue. We introduce the vectors $\mathbf{x}^k = (x_1^k, \ldots, x_N^k)$ for $k = 0, \ldots, M$ along with the joint probability density

$$P^M(\mathbf{x}^M, \ldots, \mathbf{x}^0, t') = \left\langle \prod_{i=1,k=0}^{N,M} \delta(x_i^k - \xi_i^k(t')) \right\rangle \ . \tag{9}$$

For $M = 0$ the variables $\xi_i^0(0) = \xi_i(0)$ are distributed according to $\phi_i(0)$, that is, we have the initial condition

$$P^{M=0}(\mathbf{x}^0, 0) = \prod_{i=1}^{N} \delta(x_i^0 - \phi_i(0)) \ . \tag{10}$$

Using Eq. (8) and the initial condition (10), we obtain $P^{M=0}(\mathbf{x}^0, t')$ with $t' \in [0, \tau]$ and, in particular, $P^{M=0}(\mathbf{x}^0, \tau)$. The distribution $P^{M=0}(\mathbf{x}^0, \tau)$ can then be used to construct the initial condition for $M = 1$. From Eq. (1) and the definition (4) of ξ_i^k it follows that

$$\begin{aligned} &\xi_i^1(0) \text{ distributed like } \xi_i^0(\tau) \\ &\Rightarrow P^{M=1}(\mathbf{x}^1, \mathbf{x}^0, 0) = P^{M=0}(\mathbf{x}^1, \tau) \prod_{i=1}^{N} \delta(x_i^0 - \phi_i(0)) \ . \end{aligned} \tag{11}$$

By analogy, the initial condition for $t \in [\tau M, (M+1)\tau]$ with $M \geq 1$ reads

$$\xi_i^k(0) \text{ distributed like } \xi_i^{k-1}(\tau),\ \ k = 0, \ldots, M$$
$$\Rightarrow P^M(\mathbf{x}^M, \ldots, \mathbf{x}^0, 0) = P^{M-1}(\mathbf{x}^M, \ldots, \mathbf{x}^1, \tau) \prod_{i=1}^{N} \delta(x_i^0 - \phi_i(0)) \ . \tag{12}$$

Eqs. (8) and (12) provide a description of the non-Markov process (1) related to a multivariate stochastic system with delay in terms of a hierarchy of multivariate Markov processes that can be solved iteratively (or step by step; whence the name: method of steps). The dimension of the multivariate Markov process (8) increases with time. For any $t \in [-\tau, T]$ with $T > 0$ we obtain $M = \mathrm{int}\{T/\tau\}$. That is, the dimension D equals $D = N(1 + \mathrm{int}\{T/\tau\})$. Consequently, in order to consider the stationary case $t \to \infty$ the multivariate Markov process becomes infinitely dimensional [26].

Exploiting $g_i \Gamma_i^k = \sum_{l=1}^N g_{i,l} \Gamma_l^k$ with $g_{i,l} = g_i \delta_{i,l}$, we can compute the multivariate Fokker-Planck equation that corresponds to Eq. (8). Then, one obtains [27]

$$\frac{\partial}{\partial t'} P^M(\mathbf{x}^M, \ldots, \mathbf{x}^0, t') = -\sum_{i=1}^{N} \sum_{k=0}^{M} \frac{\partial}{\partial x_i^k} j_i^k(M) \ ,$$
$$j_i^k(M) = \left\{ h_i(\mathbf{x}^k, \mathbf{x}^{k-1}) + \nu \left[g_i(\mathbf{x}^k, \mathbf{x}^{k-1}) \frac{\partial}{\partial x_i^k} g_i(\mathbf{x}^k, \mathbf{x}^{k-1}) \right] \right\} P^M(\cdot, t')$$
$$-\frac{1}{2} \frac{\partial}{\partial x_i^k} \left[g_i(\mathbf{x}^k, \mathbf{x}^{k-1}) \right]^2 P^M(\cdot, t') \tag{13}$$

with $\mathbf{x}^{-1} = \mathbf{x}^{-1}(t) = (\phi_1(t-\tau), \ldots, \phi_N(t-\tau))$. For Ito and Stratonovich calculus we obtain $\nu = 0$ and $\nu = 1$, respectively. The variables j_i^k denote the components of the probability current matrix. In line with earlier studies [30, 26], from the ν-term in Eq. (13) we can read off that the variables involving a delay (i.e., $\xi_i(t-\tau)$) in the multiplicative noise term of Eq. (1) are not affected by the interpretation of Eq. (1) as Ito or Stratonovich delay equation. In particular, for $g_i[\{\xi_l(t)\}, \{\xi_l(t-\tau)\}] = g_i[\{\xi_l(t-\tau)\}]$ Ito and Stratonovich calculus yield the same Fokker-Planck equation (13). Given the joint probability density (9), we can compute the joint distribution

$$P(\mathbf{x}, t) = \left\langle \prod_{i=1}^{N} \delta(x_i - \xi_i(t)) \right\rangle \tag{14}$$

of the random vector $(\xi_1,\ldots,\xi_N)$ related to the original problem (1) from

$$P(\mathbf{x},t)=\prod_{l=0,l\neq k}^{M}\left[\int_\Omega \mathrm{d}[x^l]^N\right]P^M(\mathbf{x}^M,\ldots,\mathbf{x}^k=\mathbf{x},\ldots,\mathbf{x}^0,t-k\tau) \tag{15}$$

and $k=\mathrm{int}\{\frac{t}{\tau}\}$ for $t\in[-\tau,M\tau]$ with $\mathrm{d}[x^l]^N=\mathrm{d}x_1^l\cdots\mathrm{d}x_N^l$.

Our next objective is to derive boundary conditions for the Fokker-Planck equation (13). Recall that the variables ξ_i are defined on Ω_i and the vector $(\xi_1,\ldots,\xi_N)$ on $\Omega=\Omega_1\times\cdots\times\Omega_N$. Consequently, for a particular M the phase space ω_M of the variables $\xi_i^k\in\Omega_i$ with $i=1,\ldots,N$ and $k=0,\ldots,M$ is given by $\omega_M=\Omega^{M+1}$. Let $\partial\omega_M$ denote the boundary of ω_M. In the case of natural boundary conditions, $P^M(\cdot,t')$ satisfies $\forall i,k: P^M(\ldots,x_i^k\to\pm\infty,\ldots,t')\to 0$. We further assume that the drift vector $\mathbf{h}$ and the diffusion vector $\mathbf{g}$ have finite Taylor expansions and that $P^M(\cdot,t')$ falls off faster than any power law in the limit $x_i^k\to\pm\infty$ for every pair (i,k). Then, the coefficients j_i^k vanish for $x_l^m\to\pm\infty$ with arbitrary l and m. In other words, the probability current vanishes at the boundary of the phase space: $j_i^k(M)|_{\partial\omega_M}=0$. In the case of periodic boundary conditions, we confine ourselves to consider multivariate random variables with the same period. That is, we choose $\forall i:\Omega_i=[a,b]$ which implies a period p of $p=b-a$. Moreover, we require that $\mathbf{h}$ and $\mathbf{g}$ are p-periodic functions. Since ξ_i are p-periodic variables, ξ_i^k are p-periodic variables as well. Then, $P^M(\cdot,t')$ and $j_i^k(M)$ are p-periodic. Let $\partial\omega_M(1)$ and $\partial\omega_M(2)$ denote two points at the boundary of ω_M. Then, we get $j_i^k(M)|_{\partial\omega_M(1)}-j_i^k(M)|_{\partial\omega_M(2)}=0$. Finally, let us turn to reflective boundary conditions. Again, we consider random variables defined on the same domain: $\forall i:\Omega_i=[a,b]\to\forall i,k:\xi_i^k\in[a,b]$. Eq. (13) can be regarded as a continuity equation. In order to guarantee that the values of ξ_i^k are confined to the interval $[a,b]$, we require that the probability current vanishes at the boundary of the hypercube $\Omega=[a,b]^{N(M+1)}$ [31]: $j_i^k(M)|_{\partial\omega_M}=0$. Table I summerizes these considerations.

Insert Table I about here.

Finally, we would like to derive an evolution equation for the probability density $P(\mathbf{x},t)$ given by Eq. (14). To this end, we compute $P(\mathbf{x},t)$ from Eq. (15) for $k=M$ and differentiate with respect to t which yields

$$\frac{\partial}{\partial t}P(\mathbf{x},t) = \prod_{l=0}^{M-1}\left[\int_\Omega \mathrm{d}[x^l]^N\right]\frac{\partial}{\partial t'}P^M(\mathbf{x},\mathbf{x}^{M-1},\ldots,\mathbf{x}^0,t') \tag{16}$$

for $t' = t - M\tau$. Substituting the Fokker-Planck equation (13) into Eq. (16), we obtain

$$\frac{\partial}{\partial t}P(\mathbf{x},t) = -\underbrace{\prod_{l=0}^{M-1}\left[\int_\Omega \mathrm{d}[x^l]^N\right]\sum_{i=1}^{N}\sum_{k=0}^{M-1}\frac{\partial}{\partial x_i^k}j_i^k(M)}_{Y}$$
$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\int_\Omega \mathrm{d}[x^{M-1}]^N\left\{h_i(\mathbf{x},\mathbf{x}^{M-1}) + \nu\left[g_i(\mathbf{x},\mathbf{x}^{M-1})\frac{\partial}{\partial x_i}g_i(\mathbf{x},\mathbf{x}^{M-1})\right]\right\}\tilde{P}^M(\mathbf{x},\mathbf{x}^{M-1},t')$$
$$+\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\int_\Omega \mathrm{d}[x^{M-1}]^N\left[g_i(\mathbf{x},\mathbf{x}^{M-1})\right]^2\tilde{P}^M(\mathbf{x},\mathbf{x}^{M-1},t') \tag{17}$$

with

$$\tilde{P}^M(\mathbf{x},\mathbf{x}^{M-1},t') = \prod_{l=0}^{M-2}\left[\int_\Omega \mathrm{d}[x^l]^N\right]P^M(\mathbf{x},\mathbf{x}^{M-1},\ldots,\mathbf{x}^0,t')$$
$$= P(\mathbf{x},t;\mathbf{x},t-\tau)\ . \tag{18}$$

The expression Y vanishes due to the boundary conditions, cf. Table I. Substituting Eq. (18) into Eq. (17), we obtain for all $t \geq \tau$ the evolution equation

$$\frac{\partial}{\partial t}P(\mathbf{x},t)$$
$$= \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\int_\Omega \mathrm{d}y^N\left\{h_i(\mathbf{x},\mathbf{y}) + \nu\left[g_i(\mathbf{x},\mathbf{y})\frac{\partial}{\partial x_i}g_i(\mathbf{x},\mathbf{y})\right]\right\}P(\mathbf{x},t;\mathbf{y},t-\tau)$$
$$+\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\int_\Omega \mathrm{d}y^N\left[g_i(\mathbf{x},\mathbf{y})\right]^2 P(\mathbf{x},t;\mathbf{y},t-\tau) \tag{19}$$

with $P(\mathbf{x},t;\mathbf{y},t-\tau) = \left\langle\prod_{i=1}^{N}\delta(x_i-\xi_i(t))\,\delta(y_i-\xi_i(t-\tau))\right\rangle$. In addition, for $t\in[0,\tau]$ we directly obtain from Eq. (1) the evolution equation

$$\frac{\partial}{\partial t}P(\mathbf{x},t) = \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left\{h_i(\mathbf{x},\mathbf{z}(t)) + \nu\left[g_i(\mathbf{x},\mathbf{z}(t))\frac{\partial}{\partial x_i}g_i(\mathbf{x},\mathbf{z}(t))\right]\right\}P(\mathbf{x},t)$$
$$+\frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2}\left[g_i(\mathbf{x},\mathbf{z}(t))\right]^2 P(\mathbf{x},t) \tag{20}$$

with $\mathbf{z}(t) = (\phi_1(t-\tau),\ldots,\phi_N(t-\tau))$. Eqs. (19) and (20) are linked by the interface condition at $t=\tau$ described by Eq. (11) which reads here

$$P(\mathbf{x},\tau;\mathbf{y},0) = P(\mathbf{x},\tau)\prod_{i=0}^{N}\delta(y_i-\phi_i(0))\ . \tag{21}$$

3. Conclusions

Markov diffusion processes can be described in terms of Langevin and Fokker-Planck equations. In the linear case (Ornstein-Uhlenbeck processes), stationary solutions can be computed from Langevin equations and Fokker-Planck equations. In the nonlinear case, stationary solutions are usually derived from Fokker-Planck equations. Since the stochastic processes with delay discussed in this study include Markov diffusion processes as special cases, most probably we need to employ a Fokker-Planck description in order to derive stationary solutions of nonlinear stochastic delay systems. Such a Fokker-Planck descriptions is provided by the the hierarchy of evolution equations (13) and the interface condition (12). The Fokker-Planck equation (13) can be solved iteratively and, therefore, corresponds to a closed description of the stochastic process (1). The dimension of the Fokker-Planck equation (13), however, is not time-invariant and increases with time. In contrast, Eq. (19) can be regarded as a time-invariant evolution equation for the probability density of a stochastic delay system. Eq. (19) generalizes the evolution equation proposed in Ref. [24, 25] to the multivariate case. The evolution equation (19), however, does not provide a closed description for the stochastic process (1) because it involves the joint-probability density $P(\mathbf{x}, t; \mathbf{x}, t-\tau)$ but does not determine the evolution of $P(\mathbf{x}, t; \mathbf{x}, t-\tau)$. Nevertheless, in the one-dimensional case, Eq. (19) has been applied to determine stationary solutions of stochastic delay differential equations [23, 24]. Therefore, the hierarchy of closed evolution equations (13) as well as the time-invariant but not closed evolution equation (19) should be regarded as complementary tools to examine stochastic processes with delays as defined by Eq. (1).

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TABLE I: Boundary conditions

natural b.c.	periodic b.c.	reflective b.c.
$\omega_M = \mathbb{R}^{N(M+1)}$	$\omega_M = [a,b]^{N(M+1)}$	$\omega_M = [a,b]^{N(M+1)}$
$P^M(..,x_i^k \to \pm\infty,..) = 0$	$P^M(..,x_i^k = a,..) = P^M(..,x_i^k = b,..)$	$P^M(..,x_i^k \notin \omega_M,..) = 0$
$j_i^k(M)\vert_{\partial\omega_M} = 0$	$j_i^k(M)\vert_{\partial\omega_M(1)} - j_i^k(M)\vert_{\partial\omega_M(2)} = 0$	$j_i^k(M)\vert_{\partial\omega_M} = 0$