

Delay Fokker-Planck equations, perturbation theory, and data analysis for nonlinear stochastic systems with time delays

T. D. Frank

Institute for Theoretical Physics, University of Münster, Wilhelm-Klemm-Strasse 9, 48149 Münster, Germany

(Received 7 October 2004; published 21 March 2005)

We study nonlinear stochastic systems with time-delayed feedback using the concept of delay Fokker-Planck equations introduced by Guillouzic, L'Heureux, and Longtin. We derive an analytical expression for stationary distributions using first-order perturbation theory. We demonstrate how to determine drift functions and noise amplitudes of this kind of systems from experimental data. In addition, we show that the Fokker-Planck perspective for stochastic systems with time delays is consistent with the so-called extended phase-space approach to time-delayed systems.

DOI: 10.1103/PhysRevE.71.031106

PACS number(s): 05.40.-a, 02.30.Ks, 02.50.Ey

I. INTRODUCTION

There are various examples of stochastic systems with time-delayed feedback in the inanimate and animate world. The reason for this is that many complex stochastic systems exhibit self-regulating control mechanisms in terms of feedback loops. By means of such feedback loops, signals or other key quantities (such as information, matter, energy, and so on) are fed back from output to input interfaces. Usually, the transmission takes some time, which implies that the input signals are related to the output signals at earlier times. In order to study the implications of feedback delays, we may regard feedback loops as simple transport mechanisms that provide input variables in terms of time-delayed output variables. In this case, the systems at hand can be described in terms of stochastic systems with time-delayed feedback.

In life sciences, prominent examples of complex systems with time delays are population dynamics [1,2] (where delays are given by maturation times), the spread and the dynamics of infectious diseases [3] (where delays are related to the inactive infected phase), neural networks [4–12], and motor control systems (where delays can be found in terms of signal transmission times between receptors, muscles, and brain areas and in terms of receptor measurement times [13–15]). In the latter context, breathing [16], balancing [17], standing [18–21], pointing tasks [22,23], synchronized and coordinated movements [24–26], and the pupil light reflex [27] have been discussed. Further examples of biological systems involving time delays are reviewed in [28]. In the inanimate nature, some examples of complex systems with time delays are laser systems with optical feedback [29–43] centered around the Ikeda equation and the Lang-Kobayashi equation [44–47], VCSELs with time-delayed feedback control [48,49], hydrodynamic problems [50–52], chemical surface reactions [53], and feedback-regulated voltage-controlled oscillators [54–57].

In many cases, stochastic systems with time-delayed feedback can be described in terms of stochastic delay differential equations of the form

$$\frac{\partial}{\partial t}X(t) = h(X(t), X(t-\tau)) + g(X(t), X(t-\tau))\Gamma(t), \quad (1)$$

where $X(t)$ describes a state variable, $\tau > 0$ is the time delay, $h(x, y)$ is the drift function, $g(x, y)$ denotes a (state-dependent) noise amplitude, and $\Gamma(t)$ refers to a Langevin force [58] normalized to unity like $\langle \Gamma(t)\Gamma(t') \rangle = \delta(t-t')$ [here $\delta(\cdot)$ denotes the delta function and the brackets $\langle \cdot \rangle$ denote an ensemble average]. The advantage of Eq. (1) is that it appeals to our intuition. If we think of a stochastic system as a system that is affected by a deterministic force on the one hand, and a fluctuating force on the other hand, then Eq. (1) nicely corresponds to this picture, provided that we identify h with the deterministic force and the product $g\Gamma$ with the fluctuating force. The disadvantage of Eq. (1) is that it can hardly be solved analytically in a direct way. That is, the derivation of analytical expressions for mean values and variances, correlation functions, and probability distributions on the basis of Eq. (1) in a direct way is often mathematically involved. A solution to this problem can be easily found for stochastic systems without delay. For $h(x, y) = h(x)$ and $g(x, y) = g(x)$, Eq. (1) reduces to a Langevin equation [58]. In this case, analytical results can conveniently be derived in an indirect way, namely by solving the corresponding Fokker-Planck equation, see Fig. 1. Therefore, at issue is to treat stochastic systems with time delays in a similar way. That is, as shown in Fig. 1, the objective is to develop a Fokker-Planck perspective for systems described by Eq. (1) and to exploit this perspective in order to solve analytically the stochastic delay differential equation (1). The key step in this regard has been carried out in a study by Guillouzic *et al.* [59] in which a delay Fokker-Planck equation for Eq. (1) has been derived. Since then, several results have been obtained from this Fokker-Planck perspective [60–64] consistent with alternative studies in which the stochastic delay differential equation (1) has been solved directly for linear systems [65–67]. So far, however, nonlinear stochastic systems with time delays have not been treated analytically in general (for exceptions, see the small time delay approach [59] and the variable transformation approach [61,67]).

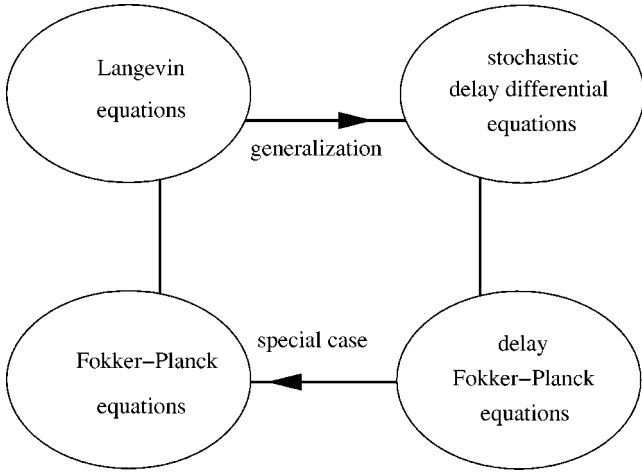


FIG. 1. Delay Fokker-Planck equations represent one of the four corners of a theory of stochastic systems with time delays.

In the present study, we will dwell on the development of a Fokker-Planck perspective for stochastic systems with time delays. In Sec. II, we will show that the delay Fokker-Planck equation derived by Guillouzic *et al.* is consistent with a technique that is nowadays frequently used: the extended phase-space approach [32,34,51,54,55,68]. In Sec. II A, we will exploit the Fokker-Planck approach to derive the first-order statistics (i.e., the stationary probability densities) for nonlinear stochastic systems with time delays. The focus will be on systems with feedback loops that only weakly affect the system dynamics. Such systems have previously been analyzed by means of master equations [48,69,70]. Examples will be given in Sec. II B. In Sec. II C, we will use the Fokker-Planck approach to study the second-order statistics of stochastic systems with time delays. In this context, we will briefly elucidate how to derive analytically autocorrelation functions for systems described by Eq. (1) and how to estimate the functions h and g involved in Eq. (1) on the basis of experimental observations.

II. DELAY FOKKER-PLANCK EQUATIONS AND EXTENDED PHASE-SPACE APPROACH

We consider a random variable $X(t) \in \Omega$ that is defined by Eq. (1) for $t > 0$ and subjected to particular boundary conditions (e.g., natural, periodic, or mixed boundary conditions). For $t \in [-\tau, 0]$, we assume that $X(t)$ is given by the initial function φ_0 : $X(-z) = \varphi_0(-z)$ with $z \in [0, \tau]$. In order to interpret the multiplicative noise term in Eq. (1), we first write Eq. (1) in the form of

$$X(t) = X(t^*) + \int_{t^*}^t h(X(s), X(s - \tau)) ds + \int_{t^*}^t g(X(s), X(s - \tau)) dW(s), \quad (2)$$

where $W(t)$ denotes a Wiener process [71]. Next, we note that expressions like $\int_{t^*}^t X^n(s) dW(s)$ with $n = 1, 2, \dots$ can be interpreted according to the Stratonovich or Ito calculus [71].

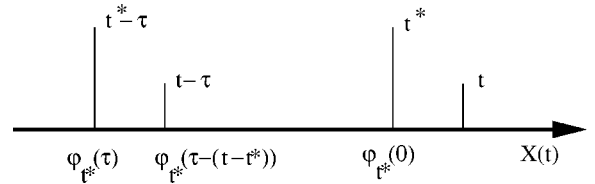


FIG. 2. Illustration of Eq. (5).

In contrast, expressions like $\int_{t^*}^t X^n(s - \tau) dW(s)$ with $\tau > 0$ can only be interpreted according to the Ito calculus [72]. As a result, the multiplicative noise integral $\int_{t^*}^t g(X(s), X(s - \tau)) dW(s)$ can be discretized with respect to time [71] in two different ways: we can either use the Stratonovich rule for the first argument of $g(x, y)$ and the Ito rule for the second argument, or we can use the Ito rule for both arguments. We will show below that these two options are also present in the delay Fokker-Planck equation corresponding to Eq. (1).

Now let us turn to the extended phase-space approach to delay differential equations. There are two closely related approaches. First, we can define a random path function $\varphi_t(z)$ of length τ at every time point t by

$$\varphi_t(z) = X(t - z) \quad (3)$$

for $z \in [0, \tau]$ [68]. In particular, we have

$$\varphi_t(0) = X(t), \quad \varphi_t(\tau) = X(t - \tau), \quad (4)$$

and reobtain the initial function $\varphi_0(-z) = X(-z)$ with $z \in [0, \tau]$. If we regard the random path function $\varphi_t(z)$ as a function of the arguments t and z , then it is defined on an extended domain of definition given by $[0, \infty) \times [0, \tau]$. That is, the stochastic process under consideration is embedded in a two-dimensional space. Second, a random path function $\varphi_{t_n}(z)$ can be defined at discrete time points $t_n = n\tau$ by $\varphi_{t_n}(z) = X(t)$ with $t = z + n\tau$, $z \in [0, \tau]$, $n = 0, 1, 2, \dots$. That is, we have $\varphi_{t_n}(z) = X(z + n\tau)$ [32,34,51]. Here the stochastic process is studied on the domain of definition $Z \times [0, \tau]$, where $Z = \{0, 1, 2, \dots\}$. In what follows, we will use the random path function $\varphi_t(z)$.

First, we assume that a particular random path of $X(t)$ is given on an interval $[t^* - \tau, t^*]$ and denote this path by $\varphi_{t^*}(z)$. Next we consider time points t that fall in the subsequent interval $[t^*, t^* + \tau]$. In this case, the time-delayed state variable $X(t - \tau)$ can be expressed by means of $\varphi_{t^*}(z)$ like

$$X(t - \tau) = \varphi_{t^*}(\tau) = \varphi_{t^*}(\tau - t + t^*), \quad (5)$$

see Fig. 2.

Consequently, $X(t - \tau)$ can be eliminated in Eq. (2), which leads to the stochastic integral equation

$$X(t) = X(t^*) + \int_{t^*}^t \tilde{h}(X(s), s) ds + \int_{t^*}^t \tilde{g}(X(s), s) dW(s) \quad (6)$$

that involves the time-dependent coefficients \tilde{h} and \tilde{g} defined by

$$\tilde{h}(x, s) = h(x, \varphi_{t^*}(\tau - s + t^*)), \quad (7)$$

$$\tilde{g}(x, s) = g(x, \varphi_t^*(\tau - s + t^*)). \quad (8)$$

The stochastic integral equation (6) describes a Markov diffusion process with probability density $P(x, t) = \langle \delta(x - X(t)) \rangle$ and transition probability density $P(x, t | x', t')$ provided that the random path $\varphi_t^*(t)$ is given. Consequently, the evolution equations of $P(x, t | \varphi_t^*)$ and $P(x, t | x', t'; \varphi_t^*)$ are defined by

$$\frac{\partial}{\partial t} P(x, t | \varphi_t^*) = \hat{L}(x, \nabla_x, \varphi_t^*) P(x, t | \varphi_t^*), \quad (9)$$

$$\frac{\partial}{\partial t} P(x, t | x', t'; \varphi_t^*) = \hat{L}(x, \nabla_x, \varphi_t^*) P(x, t | x', t'; \varphi_t^*) \quad (10)$$

with

$$\hat{L}(x, \nabla_x, \varphi_t^*) = -\frac{\partial}{\partial x} \left[\tilde{h}(x, t) + \frac{\nu}{2} \tilde{g}(x, t) \frac{\partial \tilde{g}(x, t)}{\partial x} \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} \tilde{g}^2(x, t), \quad (11)$$

where $\nu=0$ holds for the Ito case and $\nu=1$ for the Stratonovich case. From Eqs. (7), (8), and (11), we read off that \hat{L} depends only on $\varphi_t^*(\tau - t + t^*) = X(t - \tau)$. The reason for this is that the stochastic feedback system (1) depends only on one particular previous time point $t_\tau = t - \tau$. Consequently, for all paths φ_t^* that involve the same state variable $X(t - \tau)$, the probability densities $P(x, t | \varphi_t^*)$ and $P(x, t | x', t'; \varphi_t^*)$ evolve in the same way at time t . In other words, the evolution of the probability densities at time t depends only on $X(t - \tau)$. Therefore, Eqs. (9)–(11) become

$$\frac{\partial}{\partial t} P(x, t | x_\tau, t_\tau) \Big|_{t_\tau = t - \tau} = \hat{L}(x, \nabla_x, x_\tau) P(x, t | x_\tau, t - \tau), \quad (12)$$

$$\frac{\partial}{\partial t} P(x, t | x', t'; x_\tau, t_\tau) \Big|_{t_\tau = t - \tau} = \hat{L}(x, \nabla_x, x_\tau) P(x, t | x', t'; x_\tau, t - \tau) \quad (13)$$

with

$$\hat{L}(x, \nabla_x, x_\tau) = -\frac{\partial}{\partial x} \left[h(x, x_\tau) + \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x} \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} g^2(x, x_\tau). \quad (14)$$

Note that [as indicated on the left-hand sides of Eqs. (12) and (13)] the partial derivatives with respect to t act only on the very first time arguments and do not act on the time variable involved in the constraint. Let us emphasize the meaning of Eqs. (12) and (13). Equation (12) describes how $P(x, t)$ changes at time t provided that the system is at time $t - \tau$ in the state x_τ . Accordingly, if we are dealing with a system that is at time $t - \tau$ in the state x_τ and evolves during the time interval $[t - \tau, t]$ such that it is distributed at time t like $P(x, t)$, then for a small time step Δt the distribution $P(x, t + \Delta t)$ is simply given by $P(x, t + \Delta t) = (1 + \Delta t \hat{L}) P(x, t) + O(\Delta t^2)$. Likewise, from Eq. (13) it follows that if we have a system at hand which is at time $t - \tau$ in the state x_τ which

is at time t' in the state x' , and evolves during the time interval $[t - \tau, t]$ such that it is distributed at time t like $P(x, t)$, then $P(x, t + \Delta t)$ is given by $P(x, t + \Delta t) = (1 + \Delta t \hat{L}) P(x, t) + O(\Delta t^2)$ again.

Multiplying Eq. (12) with $P(x_\tau, t - \tau) = \langle \delta(x_\tau - X(t - \tau)) \rangle$ and integrating the result with respect x_τ , we obtain the evolution equation

$$\frac{\partial}{\partial t} P(x, t) = \int_{\Omega} dx_\tau \hat{L}(x, \nabla_x, x_\tau) P(x, t; x_\tau, t - \tau), \quad (15)$$

which can alternatively be expressed as

$$\frac{\partial}{\partial t} P(x, t) = \hat{F}(x, \nabla_x, P) P(x, t) \quad (16)$$

for

$$\hat{F}(x, \nabla_x, P) = \int_{\Omega} dx_\tau \left\{ -\frac{\partial}{\partial x} \left[h(x, x_\tau) + \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x} \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} g^2(x, x_\tau) \right\} P(x_\tau, t - \tau | x, t). \quad (17)$$

We will refer to Eqs. (15) and (16) as delay Fokker-Planck equations. They have first been derived by Guillouzic *et al.* [59] using Ito's stochastic equation and can also be derived using a stepwise representation of stochastic processes with time delays [62]. Multiplying Eq. (13) with $P(x_\tau, t - \tau) = \langle \delta(x_\tau - X(t - \tau)) \rangle$ and integrating the result with respect to x_τ , we get the evolution equation

$$\frac{\partial}{\partial t} P(x, t | x', t') = \int_{\Omega} dx_\tau \hat{L}(x, \nabla_x, x_\tau) \times P(x, t; x_\tau, t - \tau | x', t'). \quad (18)$$

Multiplying Eq. (18) with $P(x', t') = \langle \delta(x' - X(t')) \rangle$, it follows that the joint probability density evolves like

$$\frac{\partial}{\partial t} P(x, t; x', t') = \int_{\Omega} dx_\tau \hat{L}(x, \nabla_x, x_\tau) P(x, t; x_\tau, t - \tau; x', t'). \quad (19)$$

This relation has been previously derived in [64] using the stepwise representation of stochastic processes with time delays mentioned earlier. In the following sections, we will use the delay Fokker-Planck equation (15) for $P(x, t)$ in order to determine the first-order statistics of $X(t)$. Likewise, we will use the delay Fokker-Planck equations (18) and (19) for $P(x, t | x', t')$ and $P(x, t; x', t')$ for the purpose of data analysis and in order to determine the second-order statistics of $X(t)$.

A. First-order statistics and perturbation theory

1. Perturbation theory for stationary probability densities

In what follows, we assume that the time-delayed feedback of a system interacts only weakly with the system dynamics such that the drift force h of the system can be de-

composed into $h(x, x_\tau) = h^{(0)}(x) + h^{(1)}(x, x_\tau)$, where $h^{(1)}$ can be regarded as a small perturbation. That is, we deal with perturbation theory and assume that the orders of magnitude are $h^{(0)} \propto O(0)$ and $h^{(1)} \propto O(1)$. For the sake of convenience, we put $g(x, x_\tau) = g^{(0)}(x) \propto O(0)$ (for a more general case, see Appendix A). Then, Eq. (1) becomes

$$\frac{\partial}{\partial t} X(t) = h^{(0)}(X(t)) + h^{(1)}(X(t), X(t - \tau)) + g(X(t))\Gamma(t). \quad (20)$$

Let us decompose the operator \hat{F} like $\hat{F} = \hat{F}^{(0)} + \hat{F}^{(1)}$ with

$$\hat{F}^{(0)} = -\frac{\partial}{\partial x} \left[h^{(0)}(x) + \frac{\nu}{2} g^{(0)}(x) \frac{\partial g^{(0)}(x)}{\partial x} \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [g^{(0)}(x)]^2, \quad (21)$$

$$\hat{F}^{(1)} = -\frac{\partial}{\partial x} \int_{\Omega} dx_\tau h^{(1)}(x, x_\tau) P(x_\tau, t - \tau | x, t) \quad (22)$$

such that $\hat{F}^{(0)} \propto O(0)$ and $\hat{F}^{(1)} \propto O(1)$. Furthermore, $P(x, t)$ and $P(x, t | x', t')$ may be decomposed like

$$P(x, t) = \underbrace{P^{(0)}(x, t) + \epsilon(x, t) + \chi(x, t)}_{P^{(1)}(x, t)} + O(2), \quad (23)$$

$$P(x, t | x', t') = P^{(0)}(x, t | x', t') + \epsilon(x, t | x', t') + O(2). \quad (24)$$

Here, the orders of magnitude are $P^{(0)}(x, t)$, $P^{(0)}(x, t | x', t') \propto O(0)$, and $\epsilon(x, t)$, $\epsilon(x, t | x', t') \propto O(1)$. The function χ deserves special attention. χ is a term of second-order [i.e., we have $\chi \propto O(2)$] but it does not contain all possible second-order terms. The explicit definition of χ will be given below. Substituting Eqs. (23) and (24) into Eq. (16) and collecting all terms of zeroth and first order gives us

$$\frac{\partial}{\partial t} P^{(0)}(x, t) = \hat{F}^{(0)} P^{(0)}(x, t), \quad (25)$$

$$\frac{\partial}{\partial t} \epsilon(x, t) = \hat{F}^{(0)} \epsilon(x, t) - \frac{\partial}{\partial x} \left[\int_{\Omega} dx_\tau h^{(1)}(x, x_\tau) P^{(0)}(x_\tau, t - \tau | x, t) P^{(0)}(x, t) \right]. \quad (26)$$

Furthermore, it is clear that the unperturbed transition probability density is defined by

$$\frac{\partial}{\partial t} P^{(0)}(x, t | x', t') = \hat{F}^{(0)} P^{(0)}(x, t | x', t'). \quad (27)$$

In addition, we require that χ satisfies the evolution equation

$$\frac{\partial}{\partial t} \chi(x, t) = \hat{F}^{(0)} \chi(x, t) - \frac{\partial}{\partial x} \int_{\Omega} dx_\tau h^{(1)}(x, x_\tau) P^{(0)}(x_\tau, t - \tau | x, t) \times [\epsilon(x, t) + \chi(x, t)]. \quad (28)$$

From Eq. (28) we see that χ is a term of second-order, al-

though χ does not account for all second-order contributions. Therefore, the expression $P^{(0)}(x, t) + \epsilon(x, t) + \chi(x, t)$ is only correct up to first-order terms, which is the reason why in Eq. (23) further terms of $O(2)$ occur on the right-hand side. We consider now the evolution of the probability density $P^{(1)}(x, t)$, which has been defined in Eq. (23) by $P^{(1)}(x, t) = P^{(0)}(x, t) + \epsilon(x, t) + \chi(x, t)$. From Eqs. (25), (26), and (28), it follows that

$$\frac{\partial}{\partial t} P^{(1)}(x, t) = [\hat{F}^{(0)} + \hat{F}^{(1,0)}] P^{(1)}(x, t) \quad (29)$$

with

$$\hat{F}^{(1,0)} = -\frac{\partial}{\partial x} \int_{\Omega} dx_\tau h^{(1)}(x, x_\tau) P^{(0)}(x_\tau, t - \tau | x, t) \quad (30)$$

and $P(x, t) = P^{(1)}(x, t) + O(2)$. This evolution equation can be solved in the stationary case. In the stationary case, $P^{(0)}(x, t | x', t')$ given by Eq. (27) corresponds to the transition probability density of a stationary Markov diffusion processes, which implies that $P^{(0)}(x, t | x', t')$ depends only on $|t - t'|$ [73]. In particular, we find the relationship

$$P_{\text{st}}^{(0)}(x_\tau, t - \tau | x, t) = P_{\text{st}}^{(0)}(x_\tau, t + \tau | x, t), \quad (31)$$

which should be regarded as one of the key ingredients of our perturbation theoretical approach to stochastic systems with time delays. From Eqs. (29) and (31), it follows that the stationary solution $P_{\text{st}}(x)$ is given by

$$0 = [\hat{F}^{(0)} + \hat{F}_{\text{st}}^{(1,0)}] P_{\text{st}}^{(1)}(x) \quad (32)$$

with

$$\hat{F}_{\text{st}}^{(1,0)} = -\frac{\partial}{\partial x} \int_{\Omega} dx_\tau h^{(1)}(x, x_\tau) P_{\text{st}}^{(0)}(x_\tau, t + \tau | x, t) \quad (33)$$

and $P_{\text{st}}(x) = P_{\text{st}}^{(1)}(x) + O(2)$. Introducing the stationary probability current $J = \text{const}$ and the delay-induced drift given by

$$\tilde{h}^{(1)}(x) = \int_{\Omega} dx_\tau h^{(1)}(x, x_\tau) P_{\text{st}}^{(0)}(x_\tau, t + \tau | x, t), \quad (34)$$

Eq. (32) can equivalently be expressed as

$$J = \left[h^{(0)}(x) + \frac{\nu}{2} g^{(0)}(x) \frac{dg^{(0)}(x)}{dx} + \tilde{h}^{(1)}(x) \right] P_{\text{st}}^{(1)}(x) - \frac{1}{2} \frac{d}{dx} [g^{(0)}(x)]^2 P_{\text{st}}^{(1)}(x). \quad (35)$$

From this relation, the stationary probability density $P_{\text{st}}^{(1)}(x)$ of the stochastic process defined by Eq. (20) can conveniently be determined using the techniques developed for univariate ordinary Fokker-Planck equations [58,71]. In closing these considerations, we would like to note that Eq. (35) can be generalized to stochastic systems with noise amplitudes that depend on time-delayed state variables—as shown in Appendix A.

2. Moments and variance

Recall that $P_{st}^{(1)}(x)$ is the first-order approximation of the stationary probability density $P_{st}(x)$: $P_{st}(x) = P_{st}^{(1)}(x) + O(2)$. Therefore, all moments $\langle X^n \rangle_{st}^{(1)}$ computed from $P_{st}^{(1)}(x)$ are first-order approximations of the moments $\langle X^n \rangle_{st}$ of $P_{st}(x)$. Let $\sigma_{st}^{2(1)}$ denote the first-order approximation of the variance σ_{st}^2 of $P_{st}(x)$. Then, $\sigma_{st}^{2(1)}$ may be computed from $\sigma_{st}^{2(1)} = \langle X^2 \rangle_{st}^{(1)} - \langle X \rangle_{st}^{(1)2}$. Alternatively, we may use

$$\sigma_{st}^{2(1)} = \langle X^2 \rangle_{st}^{(1)} - [\langle X \rangle_{st}^{(1)}]^2. \quad (36)$$

Both expressions differ from σ_{st}^2 only by terms of second and higher order: $\sigma_{st}^2 = \sigma_{st}^{2(1)} + O(2)$.

3. Limiting cases $\tau \rightarrow 0$ and $\tau \rightarrow \infty$

For $\tau \rightarrow 0$, Eq. (35) reduces to

$$J = \left[h^{(0)}(x) + \frac{\nu}{2} g^{(0)}(x) \frac{\partial g^{(0)}(x)}{\partial x} + h^{(1)}(x, x) \right] P_{st}^{(1)}(x) - \frac{1}{2} \frac{\partial}{\partial x} [g^{(0)}(x)]^2 P_{st}^{(1)}(x). \quad (37)$$

This equation defines the exact stationary probability density of the stochastic process given by Eq. (20) for $\tau=0$. That is, we do not obtain a first-order approximation of the problem without delay but we recover the exact solution,

$$\lim_{\tau \rightarrow 0} P_{st}^{(1)}(x) = P_{st}(x). \quad (38)$$

The reason for this is that for $\tau=0$, Eqs. (25), (26), and (28) read

$$\frac{\partial}{\partial t} P^{(0)}(x, t) = \hat{F}^{(0)} P^{(0)}(x, t), \quad (39)$$

$$\frac{\partial}{\partial t} \epsilon(x, t) = \hat{F}^{(0)} \epsilon(x, t) - \frac{\partial}{\partial x} h^{(1)}(x, x) P^{(0)}(x, t), \quad (40)$$

$$\frac{\partial}{\partial t} \chi(x, t) = \hat{F}^{(0)} \chi(x, t) - \frac{\partial}{\partial x} h^{(1)}(x, x) [\epsilon(x, t) + \chi(x, t)]. \quad (41)$$

If we add this equation up, we see that $P^{(1)}(x, t) = P^{(0)}(x, t) + \epsilon(x, t) + \chi(x, t)$ satisfies exactly the Fokker-Planck equation of the nondelayed problem. It is also clear that if we neglect the function χ , then Eqs. (39) and (40) do not add up to the Fokker-Planck equation of the nondelayed problem. Therefore, the function $\chi(x, t)$ plays a crucial role here.

Let us now assume that for the unperturbed system, a stationary solution $P_{st}^{(0)}$ exists. Then for $\tau \rightarrow \infty$ we have $P_{st}^{(0)}(x_\tau, t + \tau | x, t) = P_{st}^{(0)}(x_\tau)$. Consequently, in this limiting case the delay-induced drift (34) reads

$$\tilde{h}_\infty^{(1)}(x) = \int_\Omega dx_\tau h^{(1)}(x, x_\tau) P_{st}^{(0)}(x_\tau) \quad (42)$$

and the stationary probability density $P_{st}^{(1)}(x)$ can be obtained from Eqs. (35) and (42). Since the noise-induced drift (42) is

independent of τ , the stationary probability density $P_{st}^{(1)}(x)$ becomes independent of τ in the limit $\tau \rightarrow \infty$.

4. Vanishing probability current

Finally, let us consider stochastic systems with time-delayed feedback for which $J=0$ holds in the stationary case (e.g., systems subjected to natural boundary conditions). Then, from Eq. (37) it follows that the stationary solution $P_{st}^{(1)}(x)$ is given by

$$P_{st}^{(1)}(x) = \frac{2}{Z g_0^2(x)} \times \exp \left\{ 2 \int^x dx' \frac{h^{(0)}(x') + \frac{\nu}{4} \frac{dg_0^2(x')}{dx'} + \tilde{h}^{(1)}(x')}{[g^{(0)}(x')]^2} \right\}, \quad (43)$$

where Z is a normalization constant. In particular, if Eq. (20) involves only additive noise, that is, if we have

$$\frac{d}{dt} X(t) = h^{(0)}(X(t)) + h^{(1)}(X(t), X(t - \tau)) + \sqrt{Q} \Gamma(t), \quad (44)$$

then we get the Boltzmann distribution

$$P_{st}^{(1)}(x) = \frac{1}{Z'} \exp \left\{ - \frac{2[V^{(0)}(x) + \tilde{V}^{(1)}(x)]}{Q} \right\} \quad (45)$$

that involves the potentials $V^{(0)}(x) = -\int^x dx' h^{(0)}(x')$ and $\tilde{V}^{(1)}(x) = -\int^x dx' \tilde{h}^{(1)}(x')$ and the normalization constant Z' .

B. Examples

1. Natural boundary conditions: Perturbed Ornstein-Uhlenbeck processes

An important class of systems can be described by delay differential equations of the form $dX(t)/dt = -aX(t) + h(X(t - \tau))$ with $a > 0$ [47]. Let us assume that (i) X is defined on the real line (i.e., we have $\Omega = \mathbb{R}$), (ii) the impact of the feedback loop given by h is weak by comparison with the linear force $-aX$, and (iii) in addition to $-aX$ and h , there is an additive fluctuating force. Then, we are dealing with a perturbed Ornstein-Uhlenbeck process defined by

$$\frac{d}{dt} X(t) = -aX(t) + h^{(1)}(X(t - \tau)) + \sqrt{Q} \Gamma(t). \quad (46)$$

The unperturbed transition probability density $P^{(0)}(x, t | x', t')$ corresponds to the transition probability density of an Ornstein-Uhlenbeck process with damping constant a and reads

$$P^{(0)}(x, t | x', t') = \sqrt{\frac{1}{2\pi K(t - t')}} \exp \left\{ - \frac{[x - x' e^{-a(t-t')}]^2}{2K(t - t')} \right\} \quad (47)$$

with $K(t-t')=Q[2a]^{-1}[1-\exp\{-2a(t-t')\}]$ [58,71]. Consequently, the delay-induced drift (34) reads

$$\tilde{h}^{(1)}(x) = \sqrt{\frac{1}{2\pi K(\tau)}} \int_{\Omega} dx_{\tau} h^{(1)}(x_{\tau}) \exp\left\{-\frac{[x_{\tau}-xe^{-a\tau}]^2}{2K(\tau)}\right\} \quad (48)$$

and the corresponding potential is given by

$$P_{\text{st}}^{(1)}(x) = \frac{1}{Z'} \exp\left\{-\frac{2[ax^2/2 - [2\pi K(\tau)]^{-1/2} \int dx' \int_{\Omega} dx_{\tau} h^{(1)}(x_{\tau}) \exp\left\{-\frac{[x_{\tau}-x'e^{-a\tau}]^2}{2K(\tau)}\right\}]}{Q}\right\} \quad (50)$$

and moments are given by

$$\langle X^n \rangle_{\text{st}}^{(1)} = \frac{1}{Z'} \int_{\Omega} dx x^n \exp\left\{-\frac{2[ax^2/2 - [2\pi K(\tau)]^{-1/2} \int dx' \int_{\Omega} dx_{\tau} h^{(1)}(x_{\tau}) \exp\left\{-\frac{[x_{\tau}-x'e^{-a\tau}]^2}{2K(\tau)}\right\}]}{Q}\right\}. \quad (51)$$

2. Comparison with exact results: Linear stochastic delay differential equation

Let us illustrate the perturbation theoretical approach for a system for which an exact analytical solution exists. To this end, we consider the linear model

$$\frac{d}{dt}X(t) = -aX(t) - bX(t-\tau) + \sqrt{Q}\Gamma(t) \quad (52)$$

with $X \in \Omega = \mathbb{R}$ and $a > b > 0$. In this case, the stationary probability density is given by

$$P_{\text{st}}(x) = \sqrt{\frac{1}{2\pi\sigma_{\text{st}}^2}} \exp\left\{-\frac{x^2}{2\sigma_{\text{st}}^2}\right\} \quad (53)$$

with

$$\sigma_{\text{st}}^2 = \frac{Q}{2} \left(\frac{1 + b\omega^{-1} \sinh(\omega\tau)}{a + b \cosh(\omega\tau)} \right) \quad (54)$$

with $\omega = \sqrt{a^2 - b^2}$ [63,65]. For $b \ll a$, perturbation theory applies. In this case, we have $h^{(0)}(x) = -ax$ and $h^{(1)}(x, x_{\tau}) = -bx_{\tau}$ and Eq. (48) gives us the delay-induced drift

$$\tilde{h}^{(1)} = -bx e^{-a\tau}. \quad (55)$$

The potentials satisfy $V^{(0)}(x) + \tilde{V}^{(1)}(x) = [a + be^{-a\tau}]x^2/2$ and from Eq. (50) we obtain

$$\tilde{V}^{(1)}(x) = -\sqrt{\frac{1}{2\pi K(\tau)}} \int dx' \int_{\Omega} dx_{\tau} h^{(1)}(x_{\tau}) \times \exp\left\{-\frac{[x_{\tau}-x'e^{-a\tau}]^2}{2K(\tau)}\right\}. \quad (49)$$

Since we have $\Omega = \mathbb{R}$ and natural boundary conditions hold, in the stationary case we have $J=0$, and Eq. (44) with $h^{(0)}(x) = -ax$ and $h^{(1)}(x, x_{\tau}) = h^{(1)}(x_{\tau})$ applies. Therefore, $P_{\text{st}}^{(1)}(x)$ is found as

$$P_{\text{st}}^{(1)}(x) = \sqrt{\frac{a + be^{-a\tau}}{\pi Q}} \exp\left\{-\frac{(a + be^{-a\tau})x^2}{Q}\right\}. \quad (56)$$

We see that for $\tau \rightarrow 0$, we obtain the exact stationary solution of the nondelayed problem (cf. Sec. II A 3). Let us demonstrate that for $\tau > 0$, Eq. (56) corresponds to the first-order approximation of the exact stationary solution (53) by comparing the moments of the Gaussian distributions (53) and (56). Since $\langle X \rangle_{\text{st}} = \langle X \rangle_{\text{st}}^{(1)} = 0$ holds, we need to consider only the variances. The variance $\sigma_{\text{st}}^{2(1)} = \langle X^2 \rangle_{\text{st}}^{(1)}$ computed with $P_{\text{st}}^{(1)}$ is given by

$$\sigma_{\text{st}}^{2(1)} = \frac{Q}{2(a + be^{-a\tau})}. \quad (57)$$

Expanding this expression with respect to b yields

$$\sigma_{\text{st}}^{2(1)} = \frac{Q}{2a} \left(1 - \frac{be^{-a\tau}}{a} \right) + O(b^2). \quad (58)$$

Likewise, expanding the variance (54) with respect to b leads to

$$\sigma_{\text{st}}^2 = \frac{Q}{2a} \left(1 - \frac{be^{-a\tau}}{a} \right) + O(b^2). \quad (59)$$

That is, the variances of both distributions differ only by terms of second and higher order. Since for Gaussian distributions with vanishing mean all higher even moments can be expressed in terms of powers of variances, we conclude that

all higher even moments of Eqs. (53) and (56) are equivalent up to terms $O(2)$, which implies that the distribution (56) indeed describes a first-order approximation of the exact Gaussian solution (53).

3. Shifts of fixed points: tanh model

In life sciences, sigmoidal conversion functions with saturation domains are often modeled by means of tanh and arctan functions. Therefore, we consider next a stochastic system subjected to a time-delayed feedback loop that involves such a sigmoidal conversion function. More explicitly, we study the stochastic process defined by

$$\frac{d}{dt}X(t) = -aX(t) - b \tanh[c(X(t-\tau) - \theta)] + \sqrt{Q}\Gamma(t) \quad (60)$$

with $X \in \Omega = \mathbb{R}$ and $a, c, \theta > 0$ and $b < 0$ (for similar models, see [19,22,25,74]).

In what follows, we will briefly show that the feedback mechanism for $\theta \neq 0$ can result in a shift of the fixed point of the unperturbed system. In this context, it is important to realize that for $a > 0$ and $b < 0$, the model describes the interplay of two contradictory forces: an attractive force $-aX$ and a repulsive force $-b \tanh(\cdot)$. Let us consider now the case $Q=0$. Let X_{st} denote the stationary solution for $Q=0$. For $\theta > 0$, the tanh function is shifted to the right and a system that is located close to the origin is driven to negative state values. Therefore, the fixed point X_{st} is shifted away from zero to negative values: $X_{st} < 0$. Let $u(t)$ denote a small deviation from the fixed point: $u(t) = X(t) - X_{st}$. Then, we obtain $du/dt = -au(t) - b'u(t-\tau)$ and one can show that for $|b|c < a$ and arbitrary $\theta > 0$, we have $|b'| < a$. This implies that the fixed point X_{st} is stable [1,68,75]. That is, the deterministic system is monostable. In what follows, we require that the inequality $|b|c < a$ holds. The case $|b|c > a$ will be considered in Sec. II B 4 below.

We assume now that $|b|$ is small such that the time-delayed feedback yields only a perturbation of the overall dynamics. For $Q > 0$, we can apply the perturbation theoretical approach developed in Sec. II A. We have $h^{(0)}(x) = -ax$ and $h^{(1)}(x, x_\tau) = -b \tanh[c(x_\tau - \theta)]$. From Eq. (48), it follows that

$$\begin{aligned} \tilde{h}^{(1)}(x) = & -b \sqrt{\frac{1}{2\pi K(\tau)}} \int_{\Omega} dx_\tau \tanh[c(x_\tau - \theta)] \\ & \times \exp\left\{-\frac{[x_\tau - xe^{-a\tau}]^2}{2K(\tau)}\right\} \end{aligned} \quad (61)$$

with $K(\tau) = Q[2a]^{-1}[1 - \exp\{-2a(\tau)\}]$. In particular, for $\tau=0$ we have $\tilde{h}_{1,0}^{(1)}(x) = -b \tanh[c(x - \theta)]$, whereas for $\tau \rightarrow \infty$ from Eq. (42) it follows that

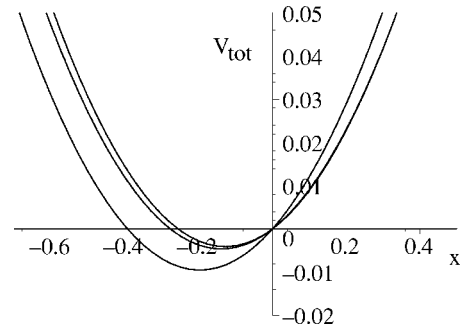


FIG. 3. Total potential $V_{tot}(x) = V^{(0)}(x) + \tilde{V}^{(1)}(x)$ of the tanh model (61) computed from Eq. (64) for $\tau=0$, $\tau=2$, and $\tau=\infty$ (from bottom to top). Other parameters are $a=0.5$, $b=-0.1$, $c=2.0$, $\theta=1.0$, and $Q=1.0$.

$$\tilde{h}_{\infty}^{(1)}(x) = -b \sqrt{\frac{a}{\pi Q}} \int_{\Omega} dx_\tau \tanh[b(x_\tau - \theta)] \exp\left\{-\frac{ax_\tau^2}{Q}\right\}. \quad (62)$$

Note that $\tilde{h}_{\infty}^{(1)}$ does not depend on x . Using some geometrical considerations, we see that for $\theta=0$ we have $\tilde{h}_{\infty}^{(1)}=0$, whereas for $\theta > 0$ we have $\tilde{h}_{\infty}^{(1)} < 0$. The potentials $V^{(0)}(x)$ and $\tilde{V}^{(1)}(x)$ are given by

$$\begin{aligned} V^{(0)}(x) + \tilde{V}^{(1)}(x) = & \frac{ax^2}{2} + b \sqrt{\frac{1}{2\pi K(\tau)}} \\ & \times \int_0^x dx' \int_{\Omega} dx_\tau \tanh[b(x_\tau - \theta)] \\ & \times \exp\left\{-\frac{[x_\tau - x'e^{-a\tau}]^2}{2K(\tau)}\right\}, \end{aligned} \quad (63)$$

where we have chosen the lower boundary of integration such that $\tilde{V}^{(1)}(0)=0$. For $\tau=0$, the potentials read

$$V^{(0)}(x) + \tilde{V}_0^{(1)}(x) = \frac{ax^2}{2} + \frac{b}{c} \ln \cosh[c(x - \theta)] - \frac{b}{c} \ln \cosh[c\theta], \quad (64)$$

whereas in the limit $\tau \rightarrow \infty$ the potentials are given by

$$V^{(0)}(x) + \tilde{V}_{\infty}^{(1)}(x) = \frac{ax^2}{2} - \tilde{h}_{\infty}^{(1)}x. \quad (65)$$

Examples for $V^{(0)}(x) + \tilde{V}^{(1)}(x)$ are depicted in Fig. 3. We see that for $\tau > 0$, the system is driven to the left-hand side just as in the case of $\tau=0$.

The stationary solution $P_{st}^{(1)}$ can be computed from Eq. (50) and reads

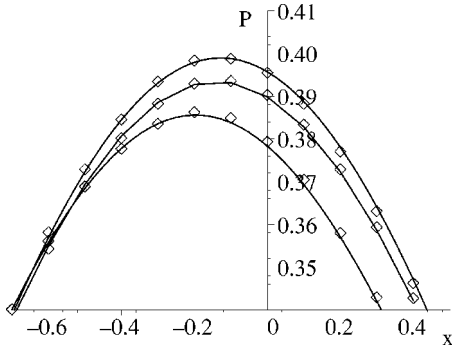


FIG. 4. Solid lines represent stationary probability densities $P_{\text{st}}^{(1)}(x)$ of the tanh model (61) computed from Eq. (67) for $\tau=0$, $\tau=2$, and $\tau=\infty$ (from bottom to top). Other parameters as in Fig. 3. Diamonds represent exact stationary distributions $P_{\text{st}}(x)$ of Eq. (61) obtained by solving Eq. (61) numerically using an Euler forward scheme [58] in combination with a Box-Muller algorithm for $\tau=0$, $\tau=2$, and $\tau=20$ (single time step $\Delta t=0.01$, number of realizations $N=10^8$).

$$P_{\text{st}}^{(1)}(x) = \frac{1}{Z'} \exp \left\{ -\frac{2}{Q} \left[\frac{ax^2}{2} + b \sqrt{\frac{1}{2\pi K(\tau)}} \int_0^x dx' \right. \right. \\ \left. \left. \times \int_{\Omega} dx_{\tau} \tanh[c(x_{\tau} - \theta)] \right. \right. \\ \left. \left. \times \exp \left(-\frac{[x_{\tau} - x' e^{-a\tau}]^2}{2K(\tau)} \right) \right] \right\}. \quad (66)$$

In the limiting cases $\tau \rightarrow 0$ and $\tau \rightarrow \infty$, we have

$$P_{\text{st}, \tau=0}^{(1)}(x) = \frac{1}{Z_0} \exp \left\{ -\frac{2}{Q} \left[\frac{ax^2}{2} + \frac{b}{c} \ln \cosh[c(x - \theta)] \right. \right. \\ \left. \left. - \frac{b}{c} \ln \cosh(c\theta) \right] \right\}, \quad (67)$$

$$P_{\text{st}, \tau \rightarrow \infty}^{(1)}(x) = \frac{1}{Z_{\infty}} \exp \left\{ -\frac{2}{Q} \left[\frac{ax^2}{2} - h_{1,\infty} x \right] \right\}, \quad (68)$$

where Z_0 and Z_{∞} are normalization constants. Note that $P_{\text{st}, \tau=0}^{(1)}$ is the exact stationary solution of the nondelayed problem (cf. Sec. II A 3). Examples for $P_{\text{st}, \tau=0}^{(1)}$, $P_{\text{st}}^{(1)}$, and $P_{\text{st}, \tau \rightarrow \infty}^{(1)}$ are given in Fig. 4. Solving the tanh model numerically, we see that for the selected parameters the probability densities $P_{\text{st}}^{(1)}(x)$ describe the system quiet well.

4. A bistable stochastic system with time delay

Bistable stochastic systems involving time-delayed feedback loops that interact weakly with the system dynamics have previously been studied in terms of master equations that describe transitions between the two available stable states [48,69,70]. In contrast to these studies, we will apply the Fokker-Planck approach. To this end, we consider again the tanh model (61). For $\theta=0$, $\tau=0$, and $Q=0$, the tanh model has a stationary point at $X_{\text{st}}=0$ and the linearization at this point yields $du(t)/dt=(|b|c-a)u(t)$. We see that for

$|b|c > a$, the stationary point is unstable. In fact, for $|b|c > a$ the deterministic model describes a bistable system with two stable stationary points at $X_{\text{st}} \neq 0$. In order to focus on the essentials of a bistable stochastic system with time delay, we consider a special case of the tanh model: the limiting case $c \rightarrow \infty$. In this case, the tanh function becomes the sgn function. That is, we have $\lim_{c \rightarrow \infty} \tanh(cz) = \text{sgn}(z)$ with $\text{sgn}(z) = 1$ for $z > 0$, $\text{sgn}(z) = 0$ for $z = 0$, and $\text{sgn}(z) = -1$ for $z < 0$. The corresponding deterministic model reads $dX(t)/dt = -aX(t) - b \text{sgn}[X(t)]$ and exhibits an unstable fixed point at $X_{\text{st}}=0$ and two stable fixed points at $X_{\text{st}} = \pm |b|/a$. For $c \rightarrow \infty$, the stochastic tanh model with time delay given by Eq. (61) becomes

$$\frac{d}{dt} X(t) = -aX(t) - b \text{sgn}[X(t - \tau)] + \sqrt{Q}\Gamma(t) \quad (69)$$

with $X(t) \in \Omega = \mathbb{R}$ and $a > 0$, $b < 0$. In what follows, we will analyze this equation for small parameters $|b|$ such that the perturbation theoretical approach of Sec. II A can be applied. Accordingly, we have $h^{(0)}(x) = -ax$ and $h^{(1)}(x, x_{\tau}) = -b \text{sgn}[x_{\tau}]$ and Eq. (48) reads

$$\tilde{h}^{(1)}(x) = -b \sqrt{\frac{1}{2\pi K(\tau)}} \int_{\Omega} dx_{\tau} \text{sgn}(x_{\tau}) \\ \times \exp \left\{ -\frac{[x_{\tau} - x e^{-a\tau}]^2}{2K(\tau)} \right\} \quad (70)$$

with $K(\tau) = Q[2a]^{-1}[1 - \exp\{-2a(\tau)\}]$. Using the error function defined by $\text{erf}(x) = [\sqrt{2\pi}]^{-1} \int_{-\infty}^x \exp\{-z^2/2\} dz$, the noise-induced drift (70) can be written as

$$\tilde{h}^{(1)}(x) = -b \left[1 - 2 \text{erf} \left(-\frac{x e^{-a\tau}}{2K(\tau)} \right) \right]. \quad (71)$$

From Eq. (69), it is clear that for $\tau=0$ we have $\tilde{h}_0^{(1)}(x) = -b \text{sgn}(x)$. From Eq. (42), it follows that for $\tau \rightarrow \infty$ we have $\tilde{h}_{\infty}^{(1)}(x) = 0$. Note that these two limiting cases can also be deduced from Eq. (71) if we take into account that

$$\frac{e^{-a\tau}}{2K(\tau)} = \sqrt{\frac{2a}{Q[\exp\{2a\tau\} - 1]}} \quad (72)$$

holds and if we consider $\tau \rightarrow 0$ with $\lim_{A \rightarrow \infty} [1 - 2 \text{erf}(-Az)] = \text{sgn}(z)$ instead of $\tau=0$ on the one hand, and $\tau \rightarrow \infty$ with $\text{erf}(0) = 1/2$ on the other hand. The potentials $V^{(0)}(x) + \tilde{V}^{(1)}(x)$ satisfy

$$V^{(0)}(x) + \tilde{V}^{(1)}(x) = \frac{ax^2}{2} + b \left[x - 2 \int_0^x \text{erf} \left(-\frac{x' e^{-a\tau}}{2K(\tau)} \right) dx' \right]. \quad (73)$$

In the limiting cases $\tau \rightarrow 0$ and $\tau \rightarrow \infty$, we have

$$V^{(0)}(x) + \tilde{V}_0^{(1)}(x) = \frac{ax^2}{2} + b|x|, \quad (74)$$

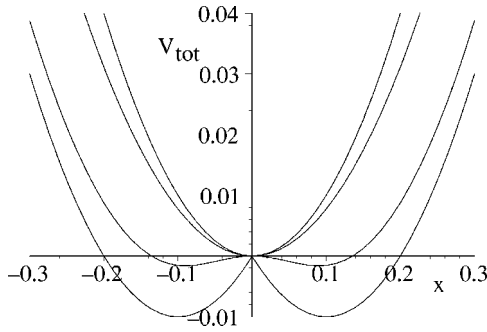


FIG. 5. Total potential $V_{\text{tot}}(x) = V^{(0)}(x) + \tilde{V}^{(1)}(x)$ of the double-well potential model (69) computed from Eq. (73) for $\tau=0$, $\tau=0.003$, $\tau=0.1$, and $\tau=\infty$ (from bottom to top). Other parameters are $a=2.0$, $b=-0.2$, and $Q=1.0$. Critical delay: $\tau^* \approx 0.006$ [see Eq. (77)]. For $\tau < \tau^*$ we are dealing with bistable potentials (see, e.g., $\tau=0.003$), whereas for $\tau > \tau^*$ we are dealing with monostable potentials (see, e.g., $\tau=0.1$).

$$V^{(0)}(x) + \tilde{V}_{\infty}^{(1)}(x) = \frac{ax^2}{2}. \quad (75)$$

Some examples of the potential $V_{\text{tot}}(x) = V^{(0)}(x) + \tilde{V}^{(1)}(x)$ are shown in Fig. 5.

For $\tau=0$ we have a double-well potential, whereas for $\tau \rightarrow \infty$ we get a monostable parabolic potential. Let us determine the critical value τ^* for which the double-well potential vanishes. The double-well potential vanishes if the second derivative of $V^{(0)}(x) + \tilde{V}^{(1)}(x)$ vanishes at $x=0$. Alternatively, we may say that the double-well potential vanishes if the first derivative of the force $h^{(0)}(x) + \tilde{h}^{(1)}(x)$ at $x=0$ vanishes. Using Eq. (70), we find that

$$\left. \frac{d}{dx} [h^{(0)}(x) + \tilde{h}^{(1)}(x)] \right|_{x=0} = -a - 2b \sqrt{\frac{a}{Q\pi[\exp\{2a\tau\} - 1]}}. \quad (76)$$

Then, τ^* can be computed by equating the left-hand side to zero. Thus, we get

$$\tau^* = \frac{1}{2a} \ln \left(1 + \frac{4b^2}{\pi a Q} \right). \quad (77)$$

The stationary distribution $P_{\text{st}}^{(1)}(x)$ can be obtained from the potentials $V^{(0)}(x)$ and $\tilde{V}^{(1)}(x)$ and reads

$$P_{\text{st}}^{(1)}(x) = \frac{1}{Z'} \exp \left\{ -\frac{2}{Q} \left[\frac{ax^2}{2} + b \left(x - 2 \int_0^x \text{erf} \left\{ -\frac{x' e^{-a\tau}}{2K(\tau)} \right\} dx' \right) \right] \right\}, \quad (78)$$

see Eqs. (45) and (50). In particular, we have

$$P_{\text{st}, \tau=0}^{(1)}(x) = \frac{1}{Z_0} \exp \left\{ -\frac{2}{Q} \left[\frac{ax^2}{2} + b|x| \right] \right\}, \quad (79)$$

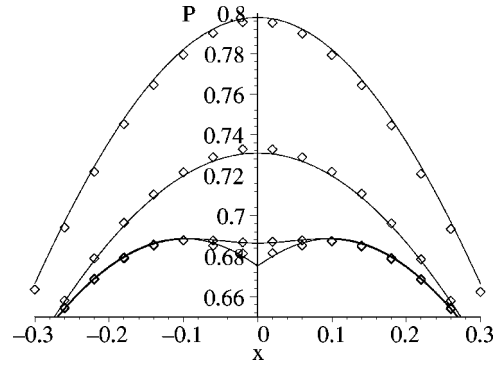


FIG. 6. Solid lines represent stationary probability densities $P_{\text{st}}^{(1)}(x)$ of the bistable model (69) computed from Eq. (78) for $\tau=0$, $\tau=0.003$, $\tau=0.1$, and $\tau=\infty$ (from bottom to top). Other parameters are as in Fig. 4. Diamonds represent exact stationary distributions $P_{\text{st}}^{(1)}(x)$ of Eq. (69) obtained by solving Eq. (69) numerically using an Euler forward scheme [58] in combination with a Box-Muller algorithm for $\tau=0$, $\tau=0.003$, $\tau=0.1$, and $\tau=5$ ($\Delta t=0.001$ for $\tau=0$, $\tau=0.003$; $\Delta t=0.01$ for $\tau=0.1$, $\tau=5$; $N=10^8$).

$$P_{\text{st}, \tau \rightarrow \infty}^{(1)}(x) = \frac{1}{\sqrt{\pi Q}} \exp \left\{ -\frac{ax^2}{Q} \right\} \quad (80)$$

in the limits $\tau=0$ and $\tau \rightarrow \infty$. Note that Eq. (79) is the exact stationary solution of Eq. (69) with $\tau=0$ (cf. Sec. II A 3). The stationary distribution $P_{\text{st}}^{(1)}(x)$ for several values of τ is depicted in Fig. 6. Numerical simulations of the stochastic delay differential equation (69) have been carried out as well. Figure 6 illustrates that for the selected parameters there is a good match between the numerical results and the first-order approximation $P_{\text{st}}^{(1)}(x)$ of the stationary distribution of Eq. (69).

5. Periodic boundary conditions: Sine loop with time delays

Let us turn now to stochastic systems with periodic state variables that involve time-delayed self-regulating feedback mechanisms. Prominent examples are laser systems with optical feedback described by the Ikeda equation and self-regulated voltage-controlled oscillators (see the Introduction for references). Let $X \in \Omega = [-T/2, T/2]$ denote a periodic variable with period $T > 0$. Then, the sine loop with time-delayed feedback [54,56] and additive noise is described by

$$\frac{d}{dt} X(t) = -\epsilon \sin \left[\frac{m\pi}{T} X(t-\tau) \right] + \sqrt{Q} \Gamma(t) \quad (81)$$

with $m=1, 2, \dots$. We consider the case in which ϵ is small such that the time-delayed feedback results only in a perturbation of the diffusion process $dX/dt = \sqrt{Q} \Gamma(t)$ subjected to periodic boundary conditions. In this case, we find that $h^{(0)}(x) = 0$ and $h^{(1)}(x, x_\tau) = -\epsilon \sin[m\pi x_\tau / T]$. Furthermore, $P^{(0)}(x, t | x', t')$ is the transition probability density of a Wiener process defined by

$$\frac{\partial}{\partial t} P^{(0)}(x, t | x', t') = \frac{Q}{2} \frac{\partial^2}{\partial x^2} P^{(0)}(x, t | x', t') \quad (82)$$

satisfying periodic boundary conditions. We find [71]

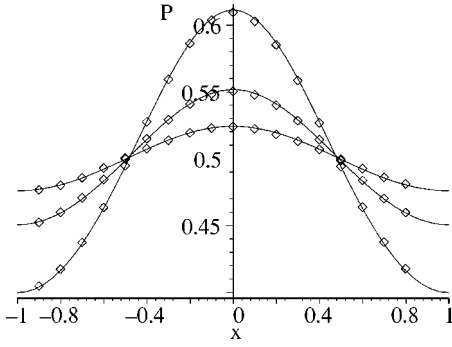


FIG. 7. Solid lines represent stationary probability densities $P_{\text{st}}^{(1)}(x)$ of the sine loop model (81) computed from Eq. (85) for $\tau=0$, $\tau=0.5$, and $\tau=1.0$ (from top to bottom). Other parameters are $\epsilon=0.1$, $m=1$, $T=1$, and $Q=0.3$. Diamonds represent exact stationary distributions $P_{\text{st}}(x)$ of Eq. (81) obtained by solving Eq. (81) numerically using an Euler forward scheme [58] in combination with a Box-Muller algorithm for the respective τ values ($\Delta t=0.01$, $N=10^8$).

$$P^{(0)}(x, t|x', t') = 1 + \sum_{n=1}^{\infty} \exp\left\{-\frac{Qn^2\pi^2(t-t')}{2T^2}\right\} \times \left[\cos\left(\frac{n\pi}{T}x\right)\cos\left(\frac{n\pi}{T}x'\right) + \sin\left(\frac{n\pi}{T}x\right)\sin\left(\frac{n\pi}{T}x'\right) \right]. \quad (83)$$

Consequently, the delay-induced drift (34) is given by

$$\tilde{h}^{(1)}(x) = -\epsilon e^{-Qm^2\pi^2\tau[2T^2]} \sin\left(\frac{m\pi}{T}x\right). \quad (84)$$

Since we are dealing with periodic boundary conditions and $h^{(1)}(x_\tau)$ is a T -periodic function, the stationary probability current J vanishes in Eq. (35) and Eq. (45) can be used in order to get

$$P_{\text{st}}^{(1)}(x) = \frac{1}{Z'} \exp\left\{\frac{2\epsilon T e^{-Qm^2\pi^2\tau[2T^2]}}{Qm\pi} \cos\left(\frac{m\pi}{T}x\right)\right\}. \quad (85)$$

Let us consider the limits $\tau \rightarrow 0$ and $\tau \rightarrow \infty$. For $\tau=0$ from Eq. (81) it follows that $\tilde{h}^{(1)}(x) = -\epsilon \sin(m\pi/Tx)$. For $\tau \rightarrow \infty$ from Eq. (42) it follows that $\tilde{h}^{(1)}(x) = 0$. Note that these limiting cases can also be computed from Eq. (84). Accordingly, for $\tau=0$ we have

$$P_{\text{st}, \tau=0}^{(1)}(x) = \frac{1}{Z_0} \exp\left\{\frac{2\epsilon T}{Qm\pi} \cos\left(\frac{m\pi}{T}x\right)\right\}, \quad (86)$$

which corresponds to the exact stationary solution of Eq. (81) with $\tau=0$ (cf. Sec. II A 3). For $\tau \rightarrow \infty$, we obtain the uniform distribution $P_{\text{st}}^{(1)} = 1/T$. The probability density $P_{\text{st}}^{(1)}(x)$ is shown in Fig. 7 for several values of τ .

C. Second-order statistics, correlation functions, and data analysis

1. Correlation functions and second-order statistics

Equation (19) can be used to determine correlation functions. To this end, we consider the stationary case in which the joint probability density $P(x, t; x', t')$ depends only on the time difference $u = t - t'$ for an arbitrary reference time t' . If we put $t' = 0$, Eq. (19) becomes

$$\frac{\partial}{\partial u} P_{\text{st}}(x, u; x', 0) = \int_{\Omega} dx_\tau \hat{L}(x, \nabla_x, x_\tau) P_{\text{st}}(x, u; x_\tau, u - \tau; x', 0). \quad (87)$$

The evolution of the correlation function $C(u) = \langle A(X(u))B(X(0)) \rangle_{\text{st}}$ involving two functions A and B can be determined by multiplying Eq. (87) with $A(x)$ and $B(x')$. Integrating with respect to x and x' and using partial integrations, we obtain

$$\frac{d}{du} C(u) = \left\langle \frac{dA(X(u))}{dx} \left[h(X(u), X(u-\tau)) + \frac{\nu}{4} \frac{dg^2(X(u), X(u-\tau))}{dx} \right] B(X(0)) \right\rangle + \left\langle \frac{d^2A(X(u))}{dx^2} g^2(X(u), X(u-\tau)) B(X(0)) \right\rangle. \quad (88)$$

In general, this evolution equation does not provide us with a closed description for $C(u)$. In special cases, however, the right-hand side of Eq. (88) can be expressed in terms of $C(u)$ like [63,64]

$$\frac{d}{du} C(u) = f[C(u), C(u-\tau)]. \quad (89)$$

In these cases, the correlation function $C(u)$ can be computed from Eq. (89) and appropriately defined boundary conditions for $C(0)$ and $dC(0)/du$. For details, see [63,64].

2. Data analysis

Using the Fokker-Planck perspective for stochastic systems with time delays, the drift function $h(x, y)$ and the noise amplitude $g(x, y)$ can be estimated from experimental data. Although this technique has already been discussed in previous studies [63,76], we would like to dwell on this issue in this section. In particular, we will show how the data-analysis technique proposed in [63,76] is related to the extended phase-space approach leading to the delay Fokker-Planck equation (18). In addition, we will elucidate the physical meaning of the expressions used in the data-analysis technique and we will address the implementation of the data-analysis technique.

To begin with, let us determine the evolution of $\langle X(t) \rangle$ under the condition that $X(t') = x'$ and $X(t-\tau) = x_\tau$. Multiplying Eq. (18) with x and integrating with respect to x , we get

$$\begin{aligned} \left. \frac{d}{dt} \langle X(t) \rangle \right|_{X(t')=x', X(t-\tau)=x_\tau} &= \int_{\Omega} dx \left[h(x, x_\tau) \right. \\ &+ \left. \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x} \right] \\ &\times P(x, t | x', t'; x_\tau, t - \tau) \end{aligned} \quad (90)$$

(assuming that surface terms arising due to partial integration vanish). In the limiting case $t \rightarrow t'$ for which $P(x, t | x', t'; x_\tau, t - \tau) = \delta(x - x')$ holds, Eq. (90) reduces to

$$\left. \frac{d}{dt} \langle X(t) \rangle \right|_{X(t)=x, X(t-\tau)=x_\tau} = h(x, x_\tau) + \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x}. \quad (91)$$

Likewise, multiplying Eq. (18) with x^2 , one can show that the relation

$$\begin{aligned} \left. \frac{d}{dt} \langle X^2(t) \rangle \right|_{X(t)=x, X(t-\tau)=x_\tau} &= 2x \left[h(x, x_\tau) + \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x} \right] \\ &+ g^2(x, x_\tau) \end{aligned} \quad (92)$$

holds. Combining Eqs. (91) and (92) leads to

$$\left. \frac{d}{dt} [\langle X^2(t) \rangle - \langle X(t) \rangle^2] \right|_{X(t)=x, X(t-\tau)=x_\tau} = g^2(x, x_\tau), \quad (93)$$

see also [77]. Equations (91) and (93) are open to an interesting interpretation. Accordingly, we see that the first moment $M_1(t)$ is related to the drift function h , whereas the variance $\sigma^2(t)$ is related to the noise amplitude g . More precisely, the conditional changes of the first moment $M_1(t)$ and the variance $\sigma^2(t)$ determine drift and diffusion,

$$\left. \frac{d}{dt} M_1 \right|_{X(t)=x, X(t-\tau)=x_\tau} = h(x, x_\tau) + \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x}, \quad (94)$$

$$\left. \frac{d}{dt} \sigma^2 \right|_{X(t)=x, X(t-\tau)=x_\tau} = g^2(x, x_\tau), \quad (95)$$

where it is understood that first the time derivatives are computed and then the conditional averages are carried out.

In order to apply these relations to experimental data and implement them on a computer, we need to account for the properties of experimental data sets. Real time series are recorded with a sampling frequency f . Therefore, we express the differential quotient d/dt as $dA(t)/dt = [A(t+\Delta t) - A(t)]/\Delta t$, where $\Delta t = 1/f$ should be small. Furthermore, since experimental data sets contain only a finite number of data, we can determine the drift function and noise amplitude only with a finite resolution Δx . As a result, Eqs. (91) and (93) become

$$\begin{aligned} \frac{1}{\Delta t} \langle X(t + \Delta t) - X(t) \rangle \Big|_{X(t) \in [x, x + \Delta x], X(t-\tau) \in [x_\tau, x_\tau + \Delta x]} &\approx h(x, x_\tau) \\ &+ \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x}, \end{aligned} \quad (96)$$

$$\frac{1}{\Delta t} \langle [X(t + \Delta t) - X(t)]^2 \rangle \Big|_{X(t) \in [x, x + \Delta x], X(t-\tau) \in [x_\tau, x_\tau + \Delta x]} \approx g^2(x, x_\tau). \quad (97)$$

As stated in the Introduction, the average corresponds to an ensemble average involving the realizations $X^i(t)$ of the state variable $X(t)$. Given a data set with N realizations, we finally get an implementation of Eqs. (91) and (93) in terms of

$$\begin{aligned} \frac{1}{\Delta t} \frac{\sum_{i \in I(x, x_\tau)} [X^i(t + \Delta t) - X^i(t)]}{\sum_{i \in I(x, x_\tau)} 1} &\approx h(x, x_\tau) \\ &+ \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x}, \end{aligned} \quad (98)$$

$$\frac{1}{2\Delta t} \frac{\sum_{i \in I(x, x_\tau)} [X^i(t + \Delta t) - X^i(t)]^2}{\sum_{i \in I(x, x_\tau)} 1} \approx \frac{g^2(x, x_\tau)}{2}, \quad (99)$$

where $I(x, x_\tau)$ is the set of indices that belong to the realizations X^i satisfying the constraint $X^i(t) \in [x, x + \Delta x], X^i(t - \tau) \in [x_\tau, x_\tau + \Delta x]$. If the system is ergodic and stationary, we may replace the ensemble average by means of a time average. In this case, we may use a single stationary trajectory composed of the data points $X_i = X(t_i)$ with $t_i = i\Delta t + t_0$. Then, as opposed to Eqs. (98) and (99), we obtain an implementation of Eqs. (91) and (93) in terms of

$$\frac{1}{\Delta t} \frac{\sum_{i \in I(x, x_\tau)} [X_{i+1} - X_i]}{\sum_{i \in I(x, x_\tau)} 1} \approx h(x, x_\tau) + \frac{\nu}{2} g(x, x_\tau) \frac{\partial g(x, x_\tau)}{\partial x}, \quad (100)$$

$$\frac{1}{2\Delta t} \frac{\sum_{i \in I(x, x_\tau)} [X_{i+1} - X_i]^2}{\sum_{i \in I(x, x_\tau)} 1} \approx \frac{g^2(x, x_\tau)}{2}, \quad (101)$$

where $I(x, x_\tau)$ is the set of indices that belong to the data X_i satisfying the constraint $X_i \in [x, x + \Delta x], X_{i-m} \in [x_\tau, x_\tau + \Delta x]$ and the delay τ is given by $\tau = m\Delta t$. For examples and further details, see [63,76]. Note also that the data-analysis method based on Eqs. (91)–(101) generalizes the data-analysis method for Markov diffusion processes that was introduced a while ago [78,79] and since then seems to attract more and more researchers (see, e.g., [80–85] and references therein).

In closing these considerations, let us dwell on the implementation of the data-analysis approach. First, above we have written conditional averages in terms of averages for random variables that fall into particular small intervals (or boxes). Basically, this means that we have expressed probability densities in terms of box estimators (or, mathemati-

cally speaking, in terms of indicator functions) such that a probability density $P(x)$ is written like $P(x) \approx \sum_i \chi_i(x, x_i, \Delta x) p_i(x_i, \Delta x)$, where p_i is the probability to find a random variable in the interval $[x_i, x_i + \Delta x]$ and χ_i equals 1 for $x \in [x_i, x_i + \Delta x]$ and is zero otherwise. It is clear that this is only one possible choice of an estimator function. One may use different estimators depending on the observed (or expected) statistics of the system under consideration. Second, one should realize that the number of available data points determines the accuracy of the result obtained by the data analysis method because in order to determine the drift function and noise amplitude at a particular state with coordinates (x, y) , only data points close to this point can be evaluated (e.g., see the box-estimator technique). Therefore, if there are only few data points available, we may confine ourselves to determine only expansion coefficients of drift functions h and noise amplitudes g . Accordingly, one may expand h and g into a set of functions $\phi_i(x, y)$ with $\int_{\Omega^2} \phi_i(x, y) \phi_k^{-1}(x, y) dx dy = \delta_{ik}$. The expansion coefficients of the functions ϕ_i can then be determined by multiplying the conditional averages occurring in the data analysis technique (see Sec. II C 2) by ϕ_i^{-1} and integrating the results with respect to x and y . In doing so, all available data points are evaluated, not only those close to a particular state (x, y) . For further details and examples, see [86].

III. CONCLUSIONS

We have shown that delay Fokker-Planck equations proposed by Guillouzic *et al.* can be derived by means of a method that is frequently used in the theory of delay differential equations: the extended phase-space approach. Not only do these delay Fokker-Planck equations describe the evolution of transient probability densities $P(x, t)$, but they also describe the evolution of transition and joint probability densities $P(x, t | x', t')$ and $P(x, t; x', t')$. Although delay Fokker-Planck equations are not closed and consequently cannot be solved by integration, they are very helpful tools for the analysis of stochastic systems with time delays. For example, as shown in Sec. II A, from the evolution equation for $P(x, t)$ we have derived the stationary distributions of stochastic systems that are perturbed by the impacts of time-delayed feedback loops. This result is of particular importance because in this case we can treat analytically systems that involve nonlinear drift force. Furthermore, as shown in Sec. II C, from the evolution equation for $P(x, t; x', t')$ correlation functions can be derived, whereas the evolution equation for $P(x, t | x', t')$ can be exploited in order to estimate drift functions and noise amplitudes from experimental data.

Let us briefly address some benefits and limitations of the results derived in this study. First, the determination of stationary distributions of weakly perturbed stochastic systems with time delays involves the transition probability densities of the unperturbed (i.e., nondelayed) systems. Closed analytical expressions of such transition probability densities of Markov diffusion processes can only be found in some special cases, such as the Ornstein-Uhlenbeck process (see Sec.

II B 1) and the Wiener process (see Sec. II B 5). However, for small time differences, transition probability densities of Markov diffusion processes can be written in terms of path integral solutions [58,87], which implies that for small time delays the required transition probability densities are available. In general, one may evaluate the eigenfunction expansion of transition probability densities of Markov diffusion processes [58,71] [for an example, see Eq. (83)]. In doing so, analytical expressions for stationary distributions that involve infinite series of functions can be obtained. Second, the perturbation theoretical approach yields correct results when the time-delayed feedback results in the perturbation of a reference state given by a stationary Markov diffusion process. As we have illustrated explicitly in the example section (Sec. II B), this assumption holds for various systems. This assumption, however, usually fails when we consider systems with time delays beyond their bifurcation threshold (e.g., beyond a Hopf bifurcation threshold).

APPENDIX A: PERTURBATION THEORETICAL APPROACH TO SYSTEMS WITH TIME-DELAYED MULTIPLICATIVE NOISE

Let us generalize some of the results obtained in Sec. II A. To this end, we introduce the diffusion coefficient $D_2 = g^2/2$. Then, Eq. (1) reads

$$\frac{d}{dt} X(t) = h(X(t), X(t - \tau)) + \sqrt{2D_2(X(t), X(t - \tau))} \Gamma(t). \quad (\text{A1})$$

The total drift term occurring in the operator \hat{L} [see Eq. (14)] now reads

$$D_1(x, x_\tau) = h(x, x_\tau) + \frac{\nu}{2} \frac{\partial}{\partial x} D_2(x, x_\tau). \quad (\text{A2})$$

Next, we decompose h and D_2 like

$$h(x, x_\tau) = h^{(0)}(x) + h^{(1)}(x, x_\tau), \quad (\text{A3})$$

$$D_2(x, x_\tau) = D_2^{(0)}(x) + D_2^{(1)}(x, x_\tau), \quad (\text{A4})$$

which implies that Eq. (A1) becomes

$$\begin{aligned} \frac{d}{dt} X(t) = & h^{(0)}(X(t)) + h^{(1)}(X(t), X(t - \tau)) \\ & + \sqrt{2[D_2^{(0)}(X(t)) + D_2^{(1)}(X(t), X(t - \tau))]} \Gamma(t). \end{aligned} \quad (\text{A5})$$

In line with the assumption that time-delayed feedback only perturbs the system dynamics, we put $h^{(0)}, D_2^{(0)} \propto O(0)$ and $h^{(1)}, D_2^{(1)} \propto O(1)$. In this case, we find $D_1 = D_1^{(0)} + D_1^{(1)}$ with $D_1^{(0)} \propto O(0)$ and $D_1^{(1)} \propto O(1)$ and

$$D_1^{(0)}(x) = h^{(0)}(x) + \frac{\nu}{2} \frac{\partial}{\partial x} D_2^{(0)}(x), \quad (\text{A6})$$

$$D_1^{(1)}(x, x_\tau) = h^{(1)}(x, x_\tau) + \frac{\nu}{2} \frac{\partial}{\partial x} D_2^{(1)}(x, x_\tau). \quad (\text{A7})$$

Just as in Sec. II A, we write the operator \hat{F} [see Eq. (17)] like $\hat{F} = \hat{F}^{(0)} + \hat{F}^{(1)}$. Then Eqs. (21) and (22) are generalized to

$$\hat{F}^{(0)} = -\frac{\partial}{\partial x} D_1^{(0)}(x) + \frac{\partial^2}{\partial x^2} D_2^{(0)}(x), \quad (\text{A8})$$

$$\begin{aligned} \hat{F}^{(1)} = & -\frac{\partial}{\partial x} \int_{\Omega} dx_\tau D_1^{(1)}(x, x_\tau) P(x_\tau, t - \tau | x, t) \\ & + \frac{\partial^2}{\partial x^2} \int_{\Omega} dx_\tau D_2^{(1)}(x, x_\tau) P(x_\tau, t - \tau | x, t). \end{aligned} \quad (\text{A9})$$

It is clear that Eqs. (25)–(29) now hold when we replace Eq. (30) by

$$\begin{aligned} \hat{F}^{(1,0)} = & -\frac{\partial}{\partial x} \int_{\Omega} dx_\tau D_1^{(1)}(x, x_\tau) P^{(0)}(x_\tau, t - \tau | x, t) \\ & + \frac{\partial^2}{\partial x^2} \int_{\Omega} dx_\tau D_2^{(1)}(x, x_\tau) P^{(0)}(x_\tau, t - \tau | x, t). \end{aligned} \quad (\text{A10})$$

Accordingly, the stationary distribution $P_{\text{st}}^{(1)}(x)$ is defined by

$0 = [\hat{F}^{(0)} + \hat{F}_{\text{st}}^{(1,0)}] P_{\text{st}}^{(1)}(x)$ [see Eq. (32)] with $\hat{F}^{(0)}$ defined by Eq. (A8) and

$$\begin{aligned} \hat{F}_{\text{st}}^{(1,0)} = & -\frac{\partial}{\partial x} \int_{\Omega} dx_\tau D_1^{(1)}(x, x_\tau) P_{\text{st}}^{(0)}(x_\tau, t + \tau | x, t) \\ & + \frac{\partial^2}{\partial x^2} \int_{\Omega} dx_\tau D_2^{(1)}(x, x_\tau) P_{\text{st}}^{(0)}(x_\tau, t + \tau | x, t) \end{aligned} \quad (\text{A11})$$

instead of Eq. (33). Accordingly, Eq. (35) can be generalized and reads

$$J = [D_1^{(0)}(x) + \tilde{D}_1^{(1)}(x)] P_{\text{st}}^{(1)}(x) - \frac{d}{dx} [D_2^{(0)}(x) + \tilde{D}_2^{(1)}(x)] P_{\text{st}}^{(1)}(x) \quad (\text{A12})$$

with

$$\tilde{D}_1^{(1)}(x) = \int_{\Omega} dx_\tau D_1^{(1)}(x, x_\tau) P_{\text{st}}^{(0)}(x_\tau, t + \tau | x, t), \quad (\text{A13})$$

$$\tilde{D}_2^{(1)}(x) = \int_{\Omega} dx_\tau D_2^{(1)}(x, x_\tau) P_{\text{st}}^{(0)}(x_\tau, t + \tau | x, t). \quad (\text{A14})$$

As pointed out in Sec. II A, Eq. (A12) can be solved with respect to $P_{\text{st}}^{(1)}(x)$ using standard techniques available for conventional univariate Fokker-Planck equations [58,71].

[1] N. S. Goel, S. C. Maitra, and E. W. Montroll, *Rev. Mod. Phys.* **43**, 231 (1971).
 [2] J. M. Cushing, *Integrodifferential Equations and Delay Models in Population Dynamics* (Springer, Berlin, 1977).
 [3] R. V. Culshaw and S. Ruan, *Math. Biosci.* **165**, 27 (2000).
 [4] H. Sompolinsky, D. Golomb, and D. Kleinfeld, *Phys. Rev. A* **43**, 6990 (1991).
 [5] E. Niebur, H. G. Schuster, and D. M. Kammen, *Phys. Rev. Lett.* **67**, 2753 (1991).
 [6] S. Kim, S. H. Park, and H. Pyo, *Phys. Rev. Lett.* **82**, 1620 (1999).
 [7] M. K. S. Yeung and S. H. Strogatz, *Phys. Rev. Lett.* **82**, 648 (1999).
 [8] H. Haken, *Brain Dynamics* (Springer, Berlin, 2002).
 [9] D. Goldobin, M. Rosenblum, and A. Pikovsky, *Phys. Rev. E* **67**, 061119 (2003).
 [10] A. Hutt, M. Bestehorn, and T. Wennekers, *Network Comput. Neural Syst.* **14**, 351 (2003).
 [11] V. K. Jirsa and M. Ding, *Phys. Rev. Lett.* **93**, 070602 (2004).
 [12] M. G. Rosenblum and A. S. Pikovsky, *Phys. Rev. Lett.* **92**, 114102 (2004).
 [13] D. H. Fender, in *The Control of Eye Movements*, edited by P. Bach-Y-Rita and C. C. Collins (Academic Press, New York, 1971), pp. 539–543.
 [14] H. Shibasaki, in *Evoked Potentials in Clinical Testing*, edited by A. M. Halliday (Churchill Livingstone, Edinburgh, 1982), pp. 471–482.
 [15] C. Gerloff, C. Toro, N. Uenishi, L. G. Cohen, L. Leocani, and M. Hallett, *Electroencephalogr. Clin. Neurophysiol.* **102**, 106 (1997).
 [16] M. C. Mackey and L. Glass, *Science* **197**, 287 (1977).
 [17] J. L. Cabrera and J. G. Milton, *Phys. Rev. Lett.* **89**, 158702 (2002).
 [18] C. W. Eurich and J. G. Milton, *Phys. Rev. E* **54**, 6681 (1996).
 [19] T. Ohira and Y. Sato, *Phys. Rev. Lett.* **82**, 2811 (1999).
 [20] T. Ohira and T. Yamane, *Phys. Rev. E* **61**, 1247 (2000).
 [21] R. J. Peterka, *Biol. Cybern.* **82**, 335 (2000).
 [22] K. Vasilakov and A. Beuter, *J. Theor. Biol.* **165**, 389 (1993).
 [23] A. Beuter, H. Haverkamp, L. Glass, and L. Carriere, *Int. J. Neurosci.* **83**, 281 (1995).
 [24] P. Tass, J. Kurths, M. G. Rosenblum, G. Guasti, and H. Hefter, *Phys. Rev. E* **54**, R2224 (1996).
 [25] R. Engbert, C. Scheffczyk, R. T. Krampe, M. Rosenblum, J. Kurths, and R. Kliegl, *Phys. Rev. E* **56**, 5823 (1997).
 [26] Y. Chen, M. Ding, and J. A. S. Kelso, *Phys. Rev. Lett.* **79**, 4501 (1997).
 [27] A. Longtin, J. G. Milton, J. E. Bos, and M. C. Mackey, *Phys. Rev. A* **41**, 6992 (1990).
 [28] G. A. Bocharov and F. A. Rihan, *J. Comput. Appl. Math.* **125**, 183 (2000).
 [29] F. A. Hopf, D. L. Kaplan, H. M. Gibbs, and R. L. Shoemaker, *Phys. Rev. A* **25**, 2172 (1982).
 [30] H. M. Gibbs, F. A. Hopf, D. L. Kaplan, and R. L. Shoemaker, *Phys. Rev. Lett.* **46**, 474 (1981).
 [31] D. Lenstra, *Opt. Commun.* **81**, 209 (1991).
 [32] F. T. Arecchi, G. Giacomelli, and A. Lapucci, *Phys. Rev. A* **45**,

- R4225 (1992).
- [33] I. Fischer, O. Hess, W. Elsässer, and E. Göbel, *Phys. Rev. Lett.* **73**, 2188 (1994).
- [34] G. Giacomelli, R. Meucci, A. Politi, and F. T. Arecchi, *Phys. Rev. Lett.* **73**, 1099 (1994).
- [35] B. Krauskopf, G. R. Gray, and D. Lenstra, *Phys. Rev. E* **58**, 7190 (1998).
- [36] M. Yousefi and D. Lenstra, *IEEE J. Quantum Electron.* **35**, 970 (1999).
- [37] A. P. Fischer, O. K. Andersen, M. Yousefi, S. Stolte, and D. Lenstra, *IEEE J. Quantum Electron.* **36**, 375 (2000).
- [38] M. Bestehorn, E. V. Grigorieva, H. Haken, and S. A. Kaschenko, *Physica D* **145**, 110 (2000).
- [39] G. Huyet, P. A. Porta, S. P. Hegarty, J. G. McInerney, and F. Holland, *Opt. Commun.* **180**, 339 (2000).
- [40] S. Wiczorek, B. Krauskopf, and D. Lenstra, *Opt. Commun.* **183**, 215 (2000).
- [41] A. N. Pisarchik, R. Meucci, and F. T. Arecchi, *Phys. Rev. E* **62**, 8823 (2000).
- [42] V. S. Udaltsov, J. P. Goedgebuer, L. Larger, and W. T. Rhodes, *Opt. Commun.* **195**, 187 (2001).
- [43] C. Masoller, *Phys. Rev. Lett.* **88**, 034102 (2002).
- [44] R. Lang and K. Kobayashi, *IEEE J. Quantum Electron.* **16**, 347 (1980).
- [45] K. Ikeda, H. Daido, and O. Akimoto, *Phys. Rev. Lett.* **45**, 709 (1980).
- [46] K. Ikeda and O. Akimoto, *Phys. Rev. Lett.* **48**, 617 (1982).
- [47] T. Erneux, L. Lager, M. W. Lee, and J. Goedgebuer, *Physica D* **194**, 49 (2004).
- [48] C. Masoller, *Phys. Rev. Lett.* **90**, 020601 (2003).
- [49] J. Houlihan, D. Goulding, T. Busch, C. Masoller, and G. Huyet, *Phys. Rev. Lett.* **92**, 050601 (2004).
- [50] E. Villermaux, *Phys. Rev. Lett.* **75**, 4618 (1995).
- [51] S. Boccaletti, D. Maza, H. Mancini, R. Genesio, and F. T. Arecchi, *Phys. Rev. Lett.* **79**, 5246 (1997).
- [52] D. Maza, H. Mancini, S. Boccaletti, R. Genesio, and F. T. Arecchi, *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **8**, 1843 (1998).
- [53] N. Khrustova, G. Vesper, A. Mikhailov, and R. Imbühl, *Phys. Rev. Lett.* **75**, 3564 (1995).
- [54] W. Wischert, A. Wunderlin, A. Pelster, M. Olivier, and J. Gros-lambert, *Phys. Rev. E* **49**, 203 (1994).
- [55] C. Simmendinger, A. Wunderlin, and A. Pelster, *Phys. Rev. E* **59**, 5344 (1999).
- [56] M. Schanz and A. Pelster, *Phys. Rev. E* **67**, 056205 (2003).
- [57] L. Larger, J. Goedgebuer, and T. Erneux, *Phys. Rev. E* **69**, 036210 (2004).
- [58] H. Risken, *The Fokker-Planck Equation—Methods of Solution and Applications* (Springer, Berlin, 1989).
- [59] S. Guillouzic, I. L'Heureux, and A. Longtin, *Phys. Rev. E* **59**, 3970 (1999).
- [60] S. Guillouzic, I. L'Heureux, and A. Longtin, *Phys. Rev. E* **61**, 4906 (2000).
- [61] T. D. Frank and P. J. Beek, *Phys. Rev. E* **64**, 021917 (2001).
- [62] T. D. Frank, *Phys. Rev. E* **66**, 011914 (2002).
- [63] T. D. Frank, P. J. Beek, and R. Friedrich, *Phys. Rev. E* **68**, 021912 (2003).
- [64] T. D. Frank, *Phys. Rev. E* **69**, 061104 (2004).
- [65] U. Küchler and B. Mensch, *Stoch. Stoch. Rep.* **40**, 23 (1992).
- [66] M. C. Mackey and I. G. Nechaeva, *Phys. Rev. E* **52**, 3366 (1995).
- [67] A. A. Budini and M. O. Caceres, *Phys. Rev. E* **70**, 046104 (2004).
- [68] J. Hale, *Theory of Functional Differential Equations* (Springer, Berlin, 1977).
- [69] L. S. Tsimring and A. Pikovsky, *Phys. Rev. Lett.* **87**, 250602 (2001).
- [70] D. Huber and L. S. Tsimring, *Phys. Rev. Lett.* **91**, 260601 (2003).
- [71] C. W. Gardiner, *Handbook of Stochastic Methods*, 2nd ed. (Springer, Berlin, 1997).
- [72] S. A. Mohammed, in *Stochastic Analysis and Related Topics VI*, edited by I. Decreusefond, J. Gjerde, B. Oksendal, and A. S. Üstünel (Birkhäuser, Boston, 1998), pp. 1–77.
- [73] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1963), Vol. 1.
- [74] H. Lu, *Phys. Lett. A* **298**, 109 (2002).
- [75] R. D. Driver, *Ordinary and Delay Differential Equations—Applied Mathematical Sciences* (Springer, New York, 1977), Vol. 20.
- [76] T. D. Frank, P. J. Beek, and R. Friedrich, *Phys. Lett. A* **328**, 219 (2004).
- [77] T. D. Frank and R. Friedrich, *Physica A* **347**, 65 (2005).
- [78] R. Friedrich and J. Peinke, *Phys. Rev. Lett.* **78**, 863 (1997).
- [79] R. Friedrich and J. Peinke, *Physica D* **102**, 147 (1997).
- [80] R. Friedrich, J. Peinke, and C. Renner, *Phys. Rev. Lett.* **84**, 5224 (2000).
- [81] C. Renner, J. Peinke, R. Friedrich, O. Chanal, and B. Chabaud, *Phys. Rev. Lett.* **89**, 124502 (2002).
- [82] G. R. Jafari, S. M. Fazeli, F. Ghasemi, S. M. V. Allaei, M. R. Tabar, A. I. Zad, and G. Kavei, *Phys. Rev. Lett.* **91**, 226101 (2003).
- [83] J. Gradisek, R. Friedrich, E. Govekar, and I. Grabec, *Mechanica* **38**, 33 (2003).
- [84] H. U. Bödeker, M. C. Röttger, A. Liehr, T. D. Frank, R. Friedrich, and H. G. Purwins, *Phys. Rev. E* **67**, 056220 (2003).
- [85] M. Waechter, A. Kouzmitchev, and J. Peinke, *Phys. Rev. E* **70**, 055103 (2004).
- [86] T. D. Frank, R. Friedrich, and P. J. Beek, *Stoch. Dyn.* (to be published).
- [87] H. Haken, *Synergetics: Introduction and Advanced Topics* (Springer, Berlin, 2004).