Time-dependent solutions for stochastic systems with delays: perturbation theory and applications to financial physics

T.D. Frank

Institute for Theoretical Physics, University of Münster, Wilhelm-Klemm-Str. 9, 48149 Münster, Germany

Abstract

First order approximations of time-dependent solutions are determined for stochastic systems perturbed by time-delayed feedback forces. To this end, the theory of delay Fokker-Planck equations is applied in combination with Bayes' theorem. Applications to a time-delayed Ornstein-Uhlenbeck process and the geometric Brownian walk of financial physics are discussed.

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1 Introduction

Low-dimensional analytical descriptions of complex high-dimensional systems help us to understand experimental findings and to predict new phenomena (e.g. [1]). In this context, complex high-dimensional systems with memory have frequently been described in terms of (low-dimensional) time-delayed evolution equations. Time-delayed evolution equations play important roles

[1] e-mail: tdfrank@uni-muenster.de

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in laser physics [2–8], chaos control [9], the theory of bistable systems [10,11], and engineering sciences [12–14]. Moreover, time-delayed evolution equations have often been used to model biological systems [15] such as motor control systems [16–24], neural network systems [25–31], and ecological systems [32]. Recently, applications of time-delayed evolution equations in financial physics have been discussed as well [33–36]. In particular, in life sciences time-delayed systems are subjected to fluctuating forces which can be accounted for as additive thermal noise sources [37] or multiplicative noise sources [17,22,38–43]. In contrast to the theory of deterministic time-delayed evolution equations (see e.g. [44,45]), the analysis of stochastic time-delayed evolution equations is still in its infancy. Some results for the stationary case can be found in [8,10,11,17,18,46,47,22,26,27,48–54]. However, only little is known about the non-stationary case. For example, in [55] evolution equations for moments of time-delayed stochastic systems have been studied. Our objective is to derive non-stationary solutions of stochastic evolution equations with delays by means of the delay Fokker-Planck approach proposed in [56] and further developed in [57–60]. To this end, we will generalize the perturbation theoretical approach that has been developed in [61–63] for the stationary case to the non-stationary case.

2 Time-dependent solutions derived by perturbation theory

2.1 Fundamentals

We will study systems that can be described by stochastic delay differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = h^{(0)}(X(t)) + h^{(1)}(X(t), X(t-\tau)) + g(X(t))\Gamma(t) \qquad (1)$$

where $X(t)$ describes a state variable, $\tau \geq 0$ is the time delay, $h^{(0)}$ and $h^{(1)}$ are a drift functions, and g denotes a (state-dependent) noise amplitude. In Eq. (1) the function $\Gamma(t)$ denotes to a Langevin force [64] normalized to unity like $\langle \Gamma(t)\Gamma(t')\rangle = \delta(t-t')$, where $\delta(\cdot)$ is the delta function and the brackets $\langle \cdot \rangle$ denote ensemble averaging. The noise term $g\Gamma$ will be treated in line with the Ito or Stratonovich calculus. Eq. (1) is subjected to an initial function $X(t) = \varphi(t)$ for $t \in [-\tau, 0]$. The function $h^{(1)}$ is a small quantity that can be regarded as a perturbating force, whereas the functions $h^{(0)}$ and g determine the unperturbed dynamics. Accordingly, we introduce a small parameter ϵ and assume that $h^{(0)}, g \propto O(\epsilon^0)$ and $h^{(1)} \propto O(\epsilon)$. From Eq. (1) the time-dependent probability density $P(x,t|\varphi) = \langle \delta(x - X(t))\rangle$ can be computed at least numerically. We will show how to determine first order approximations of $P(x,t|\varphi)$ analytically. In what follows, we will assume that we can compute $P(x,\tau|\varphi)$ from Eq. (1) with $X(t) = \varphi(t)$. The distribution $P(x,\tau|\varphi)$ serves then as the initial distribution for the perturbation theoretical approach that holds for $t \geq \tau$. For $t \geq \tau$ we approximate the exact time-dependent distribution $P(x,t|\varphi)$ by means of a first order approximation $P^{(1)}(x,t|\varphi)$ such that we have $P(x,t|\varphi) = P^{(1)}(x,t|\varphi) + O(\epsilon^2)$. The perturbation theoretical approach developed in [61] shows us that for $t \geq \tau$ the stochastic delay differential equation (1) is equivalent to the Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = h^{(0)}(X(t)) + h^{(1)}_{\mathrm{eff}}(X(t),t) + g(X(t))\Gamma(t) + O(\epsilon^2) \ . \qquad (2)$$

Accordingly, $P^{(1)}(x,t|\varphi)$ is defined by Eq. (2) when neglecting the $O(\epsilon^2)$-term. Eq. (2) involves the effective drift

$$h^{(1)}_{\mathrm{eff}}(x,t) = \int h^{(1)}(x,x_\tau) P^{(0)}(x_\tau, t-\tau|x,t)\,\mathrm{d}x_\tau \ , \qquad (3)$$

where $P^{(0)}(x_\tau, t-\tau|x,t)$ corresponds to the conditional probability density of the unperturbed stochastic process defined by $\mathrm{d}X/\mathrm{d}t = h^{(0)}(X(t)) + g(X(t))\Gamma(t)$. Note that we do not claim the equivalence of Eqs. (1) and (2) for higher order statistics. The equivalence of Eqs. (1) and (2) holds only for the first order statistics (we will return to this point in the conclusions).

In the stationary case $h_{\text{eff}}^{(1)}$ becomes independent of time and can be determined by exploiting the relation $P^{(0)}(x_\tau, t-\tau|x,t) = P^{(0)}(x_\tau, t+\tau|x,t)$ [64,65] (for details see [61]). However, as can be seen from Eqs. (2) and (3), in the non-stationary case the effective drift term $h_{\text{eff}}^{(1)}$ will depend in general explicitly on time. In the non-stationary case $P^{(0)}(x_\tau, t-\tau|x,t)$ can be determined using the Bayes' theorem in form of

$$P^{(0)}(x_\tau, t-\tau|x,t) = \frac{P^{(0)}(x,t|x_\tau,t-\tau)P^{(0)}(x_\tau,t-\tau|\varphi)}{P^{(0)}(x,t|\varphi)} \ . \tag{4}$$

In terms of the initial distribution $P(x,\tau)$ we finally get

$$P^{(0)}(x_\tau, t-\tau|x,t) = \frac{P^{(0)}(x,t|x_\tau,t-\tau)\int P^{(0)}(x_\tau,t-\tau|x_0,\tau)P(x_0,\tau|\varphi)\mathrm{d}x_0}{\int P^{(0)}(x,t|x_0,\tau)P(x_0,\tau|\varphi)\mathrm{d}x_0} \ . \tag{5}$$

2.2 Perturbed Ornstein-Uhlenbeck process

As a fundamental example, we discuss a perturbed Ornstein-Uhlenbeck process defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}X = -aX(t) - bX(t-\tau) + \sqrt{Q}\Gamma(t) \tag{6}$$

with $\varphi(t) = 0$ and $a, b, Q > 0$, where b corresponds to a small parameter. The conditional probability density of the unperturbed problem (i.e. for $b = 0$) reads [64]

$$P^{(0)}(x,t|x',t') = \frac{1}{\sqrt{2\pi\sigma_{\text{OU}}^2(\Delta t)}} \exp\left\{-\frac{[x - x'm(\Delta t)]^2}{2\sigma_{\text{OU}}^2(\Delta t)}\right\} \tag{7}$$

with

$$m(\Delta t) = \exp\{-a\Delta t\} \ , \tag{8}$$
$$\sigma^2_{\mathrm{OU}}(\Delta t) = \frac{Q}{2a}\left(1 - \exp\{-2a\Delta t\}\right) \tag{9}$$

and $\Delta t = t - t'$. Exploiting some relationships for Gaussian conditional probability densities (e.g. [66], [67, Sec.3.9.2]) a detailed calculation gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}X = -\gamma(t)X(t) + \sqrt{Q}\Gamma(t) + O(b^2) \tag{10}$$

with

$$\gamma(t) = a + b\, m(\tau)\frac{\sigma^2_{\mathrm{OU}}(t-\tau)}{\sigma^2_{\mathrm{OU}}(t)} \ . \tag{11}$$

For $\tau \to 0$ we have $\gamma = a + b$ and the perturbation theoretical approach recovers the exact result [61]. Furthermore, from Eq. (11) it follows that the following special cases for the effective damping coefficient $\gamma(t)$ hold: $\gamma(\tau) = a$ and $\gamma(t \to \infty) = \gamma_{\mathrm{st}} = a + bm(\tau)$. More explicity, for the stationary case we have $\gamma_{\mathrm{st}} = a + b\exp\{-a\tau\}$ which recovers the result that has previously been obtain in [61].

Let us determine next the evolution of the variance $\sigma^2(t)$ of the model (6) in terms of its first order approximation $\sigma^{2(1)}(t)$: $\sigma^2(t) = \sigma^{2(1)}(t) + O(b^2)$. From Eqs. (10) it follows that

$$\sigma^{2(1)}(t) = \exp\{-2\int_\tau^t \gamma(t')\mathrm{d}t'\}\left[\sigma^2(\tau) + Q\int_\tau^t\left(\exp\{2\int_\tau^{t'}\gamma(t'')\mathrm{d}t''\}\right)\mathrm{d}t'\right] \tag{12}$$

for $t \geq \tau$. For $t \in [0,\tau]$ we have the exact result $\sigma^2(t) = \sigma^2_{\mathrm{OU}}(t)$. Let us simplify Eq. (12). For $t \gg \tau$ it is clear that $\sigma^2_{\mathrm{OU}}(t-\tau) \approx \sigma_{\mathrm{OU}}(t)$. Consequently, the Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X = -\gamma_{\mathrm{st}}X(t) + \sqrt{Q}\Gamma(t) + O(b^2) \tag{13}$$

does not only hold for the stationary case but also for the non-stationary case with $t \gg \tau$. From Eq. (13) we find that

$$\sigma^{2(1)}(t) = \frac{Q}{2\gamma_{\rm st}} - \left(\frac{Q}{2\gamma_{\rm st}} - \sigma^{2(1)}(t')\right) \exp\{-2\gamma_{\rm st}\Delta t\} \tag{14}$$

holds for $t, t' \gg \tau$ with $\Delta t = t - t'$.

Let us illustrate our results by an example. Figure 1 shows the variance of the perturbed Ornstein-Uhlenbeck model for fixed parameters a, b, and Q and three different time delays $\tau_0 = 0 < \tau_1 < \tau_2$. The solid lines describe first order approximations obtained from the simplified variance equation (14) for $t \geq \tau$ and $t' = \tau$ (and from $\sigma^2_{\rm OU}(t)$ for $t \leq \tau$). Numerical results are shown as diamonds and correspond nicely to the analytical results. We also plotted the numerical results versus analytical results given by the more accurate variance equation (12). However, for the fixed parameters used in our example we did not obtain a visible difference. The reason for this is that the difference between the variances given by Eqs. (12) and (14) is very small. Figure 2 shows that there is a significant difference between the variance functions defined by Eqs. (12) and (14) but this difference is for the parameters used in our example of the order of 1/1000 and is not visible to the naked eye on a scale of the order of 1/10 as given in Fig. 1.

Insert Figures 1 and 2 about here

2.3 Perturbed geometric Brownian walk

The geometric Brownian walk describes an exponential growth process $\mathrm{d}X/\mathrm{d}t = AX$ involving a growth factor A that is composed of a deterministic part and a fluctuating part: $\mathrm{d}X/\mathrm{d}t = X(a + \sqrt{Q}\Gamma(t))$ with $a, Q > 0$ [68]. Assuming that the deterministic growth part is subjected to a memory, we will consider a time-delayed version of the geometric Brownian walk defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}X = X(t)\left[a + \sqrt{Q}\Gamma(t)\right] + bX(t-\tau) \tag{15}$$

for $X \geq 0$ and $a, b, Q > 0$, where the multiplicative noise term is treated by the Stratonovich calculus. For the sake of simplicity, we choose as initial condition a jump function given by $X = 0$ for $t \in [-\tau, 0)$ and $X = x_0 > 0$ at $t = 0$. It is clear that the geometric Brownian walk is a non-stationary process. In order to apply our perturbation theoretical approach, we assume that b is small. The unperturbed process is given by the aforementioned original process $\mathrm{d}X/\mathrm{d}t = X(t)\left[a + \sqrt{Q}\Gamma(t)\right]$. Using $Y = \ln(X/x_0) - at$ with $Y(0) = 0$, we get the Wiener process $\mathrm{d}Y/\mathrm{d}t = \sqrt{Q}\Gamma(t)$ and its conditional probability density

$$P^{(0)}(y, t|y', t') = \frac{1}{\sqrt{2\pi\sigma_{\mathrm{W}}^2(\Delta t)}} \exp\left\{-\frac{[y-y']^2}{2\sigma_{\mathrm{W}}^2(\Delta t)}\right\} \tag{16}$$

with $\sigma_{\mathrm{W}}^2(\Delta t) = Q\Delta t$. In order to compute the effective drift term (3), we need to evaluate the b-term of Eq. (15). That is, we need to solve the integral

$$\begin{aligned} I(x,t) &= b\int x_\tau P^{(0)}(x_\tau, t-\tau|x, t)\,\mathrm{d}x_\tau \\ &= bx_0 \exp\{a(t-\tau)\}\int \exp\{y_\tau\}P^{(0)}(y_\tau, t-\tau|y, t)\,\mathrm{d}y_\tau \ . \end{aligned} \tag{17}$$

A detailed calculation shows that

$$\begin{aligned} I(x,t) &= bx_0 \exp\left\{a(t-\tau) + \frac{y}{f(t)} + \frac{\sigma_{\mathrm{W}}^2(\tau)}{2f(t)}\right\} \\ &= bx_0 \exp\left\{\left(\frac{Q}{2f(t)} - a\right)\tau\right\} e^{at}\left[\frac{x}{x_0}e^{-at}\right]^{1/f(t)} \end{aligned} \tag{18}$$

with $f(t) = \sigma_{\mathrm{W}}(t)/\sigma_{\mathrm{W}}(t-\tau)$. Consequently, the first order statistics of the time-delayed geometric Brownian walk can be described in terms of the Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X = X(t)\left[\gamma(X,t) + \sqrt{Q}\Gamma(t)\right] + O(b^2) \tag{19}$$

which involves the growth factor $\gamma(x,t)$:

$$\gamma(x,t) = a + b\exp\left\{\left(\frac{Q}{2f(t)} - a\right)\tau\right\}\left[\frac{x}{x_0}e^{-at}\right]^{-1+1/f(t)} \quad . \tag{20}$$

For $\tau = 0$ we obtain $\gamma = a + b$, that is, the perturbation theoretical approach becomes exact [61]. In the special case $t \to \tau$ we have $f(\tau) \to \infty$, which implies that $\gamma(x,\tau) = a + bx_0/x$. Eq. (19) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}X = X(t)\left[a + \sqrt{Q}\Gamma(t)\right] + bx_0 \ , \tag{21}$$

which is consistent with the initial condition $\varphi(t)$ defined earlier. In the special case $t \to \infty$, we have $f(t \to \infty) = 1$, which implies that the growth factor $\gamma(x,t)$ becomes a constant γ_{st} defined by

$$\gamma_{\mathrm{st}} = a + b\exp\left\{\left(\frac{Q}{2} - a\right)\tau\right\} \ . \tag{22}$$

As a result, the Langevin equation (19) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}X = X(t)\left[\gamma_{\mathrm{st}} + \sqrt{Q}\Gamma(t)\right] + O(b^2) \ . \tag{23}$$

For $t \gg \tau$ we also have $f \approx 1$. Consequently, Eq. (23) also holds for $t \gg \tau$.

Let us illustrate the impact of the time delay τ on the growth process described by the time-delayed model (15). Since b is small, the dynamics of the model is dominated by the exponential growth with growth factor a. We eliminate this impact by looking at the logarithmic scale defined by the variable $Y(t) = \ln(X(t)/x_0) - at$. From Eqs. (15) and (23) is is clear that the first order approximation $\langle Y(t)\rangle^{(1)}$ of $Y(t)$ evolves like

$$\langle Y(t)\rangle^{(1)} = \begin{cases} 0 & \text{for} \quad t \in [0,\tau] \\ \langle Y(t')\rangle^{(1)} + (\gamma_{\mathrm{st}} - a)\Delta t & \text{for} \quad t, t' \gg \tau \end{cases} \tag{24}$$

with $\Delta t = t - t'$. For $t > \tau$ but $t \not\gg \tau$ a simple analytical result cannot be derived because we are dealing with a highly nonlinear and time-dependent Langevin equation, see Eqs. (19) and (20). Nevertheless, we see from Eq. (22) that for $Q > 2a$ the stationary growth factor $\gamma_{\rm st}$ increases as a function of the time delay, whereas for $Q < 2a$ it decreases. Figure 3 illustrates the case $Q > 2a$ for several time delays: $\tau_0 = 0 < \tau_1 < \tau_2$. Analytical results obtained from Eq. (24) for $t' = \tau$ are shown versus numerical results computed from Eq. (15). We see that there is a good match of both results although the time point t' does not satisfy the constraint $t' \gg \tau$. We speculate that the situation resembles the one observed for the perturbed Ornstein-Uhlenbeck process. There, we found that there was a difference between the more accurate Langevin equation with time-dependent coefficients and the simplified Langevin equation with time-independent coefficients but this difference was negligible on our scale of interest (see Fig. 2).

Insert Figure 3 about here

2.4 Geometric Brownian walk in financial physics

The geometric Brownian walk has, in particular, relevance for financial physics [68–70]. It can describe the stochastic evolution of a stock price $S(t)$. However, several authors have pointed out that stock price models should account for memory effects because trading in general is also influenced by events that happen in the past. Therefore, the geometric Brownian walk model has been generalized and various kinds of time-delayed stochastic differential equations have been proposed [33–36]. In line with these proposals, we may regard Eq. (15) as a time-delayed evolution equation for a stock price $S(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t} S = S(t) \left[\mu_1 + \sigma \Gamma(t) \right] + \mu_2 S(t - \tau) \ , \tag{25}$$

where $\mu_1, \mu_2 > 0$ are drift parameters and σ is the so-called volatility. From Eqs. (19) and (23) it follows that for small parameters μ_2, the first order statistics can be described by the Langevin equations

$$\frac{\mathrm{d}}{\mathrm{d}t}S = S(t)\left[\gamma(S,t) + \sigma\Gamma(t)\right] + O(\mu_2^2) \tag{26}$$

for $t \geq \tau$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}S = S(t)\left[\gamma_{\mathrm{st}} + \sigma\Gamma(t)\right] + O(\mu_2^2) \tag{27}$$

for $t \gg \tau$. Here, $\gamma(S,t)$ and γ_{st} are given by Eqs. (20) and (22) with $a = \mu_1$, $b = \mu_2$, and $Q = \sigma^2$. From Eq. (26) we conclude that for $t \propto O(\tau)$ the time-delayed feedback may result in a price dynamics which is qualitatively different from the dynamics given by an ordinary geometric Brownian walk. In contrast, for $t \gg \tau$ we are dealing with a non-delayed geometric Brownian walk model involving an effective drift parameter γ_{st}. In what follows, let us discuss some implications of these findings.

2.4.1 Expected return

The expected (logarithmic) return R of a stock over a period T is given by $R = \langle \ln[S(T)/S(0)] \rangle$ [70]. For stocks that satisfy the dynamics (25) the expected return R is a function of the delay τ. For $t, t' \gg \tau$ and a unit period $T = \Delta t = 1$ from Eqs. (22) and (24) with $a = \mu_1$, $b = \mu_2$, and $Q = \sigma^2$, it follows that

$$R = \gamma_{\mathrm{st}} = \mu_1 + \mu_2 \exp\left\{\left(\frac{\sigma^2}{2} - \mu_1\right)\tau\right\} \tag{28}$$

up to order of $O(\mu_2^2)$. Comparing $R(\tau)$ with the expected return of a stock satisfying a non-delayed geometric Brownian walk (Eq. (24) for $\tau = 0$) with $R_0 = \mu_1 + \mu_2$, we can draw the following conclusions: the expected return is smaller than R_0 and decreases with τ for stocks with low volatility ($\sigma < \sqrt{2\mu_1}$) and is larger than R_0 and increases with τ for highly volatile stocks ($\sigma > \sqrt{2\mu_1}$), see Fig. 4 panel (a).

Insert Figure 4 about here

2.4.2 Value at risk

The value at risk (VaR) is a central measure in risk management. Roughly speaking, the VaR measure can be expressed as an amount of money and describes a threshold. Accordingly, a VaR threshold tells us that a company will loose due to market fluctuations an amount of money that is larger than the VaR threshold only with a particular probability p over a particular time horizon T. That is, if $p = 0.01$, VaR equals \$1m and $T = 10$ days, then there is only a change of 1 percent that in ten days the company will loose one million dollar or more than that. It is recommended that companies hold three times of their VaR values as capital such that they have enough capital to cover unlikely events [71]. In order to determine the VaR measure of a company, we need, among other things, to determine the VaR measure of its stocks.

Let $S(0)$ denote the stock price today and $S(T)$ the stock price at time T, where $S(t)$ satisfies Eq. (25). Using the first order approximation of the price dynamics given by Eq. (27), a detailed calculation [71] shows that the stock price $S(T)$ will be larger than or equal to S_c given by

$$S_c = S(0) \exp\left(\left[\mu_1 + \mu_2 \exp\left\{\left(\frac{\sigma^2}{2} - \mu_1\right)\tau\right\}\right] T - \sigma\sqrt{T} N_{0,1}^{-1}(c)\right) \tag{29}$$

with a probability of c. Here, $N_{0,1}^{-1}$ is the inverse function of the cumulative standard normal distribution $N_{0,1}(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} \exp(-z^2/2)\,\mathrm{d}z$. That is, over the horizon T the probability is c that the loss is less than $S_c - S(0)$. The difference $S_c - S(0)$ is the c-VaR measure: VaR $= S_c - S(0)$. For example, suppose we find $T = 1$ week, $S(0) =$ \$1m, $S_c =$ \$0.93m with $c = 95\%$, then there is a 95% chance that less than \$0.07m will be lost and a 5% percent chance that more than \$0.07m will be lost over a time horizon of one week. The 95% value at risk equals −\$0.07m. From Eq. (29) it is clear that for stocks with low volatility ($\sigma < \sqrt{2\mu_1}$) the VaR measure decreases as a function of the time delay τ (becomes more negative because S_c decreases), whereas for highly volatile stocks ($\sigma > \sqrt{2\mu_1}$) VaR increases (becomes less negative) with τ, see Fig. 4 panel (b). In the low volatility case, for $\tau \to \infty$ VaR converges to a finite limiting value which corresponds to the VaR measure of a stock satisfying Eq. (25) with $\mu_2 = 0$. In the high volatility case, the deterministic part becomes more and more dominant when τ increases such that all fluctuation-induced risks eventually vanish. That is, VaR increases and becomes zero for a critical delay τ^* given by $[\mu_1 + \mu_2 \exp\{(\sigma^2/2 - \mu_1)\tau^*\}]T = \sigma\sqrt{T}N_{0,1}^{-1}(c)$. These observations are consistent with the relationship between expected return and delay discussed in the previous section. For example, in the previous section we found that in the case of low volatility the expected return is smaller than the expected return of a stock with non-delayed price dynamics. On the basis of this finding, we would anticipate that S_c is lower and the amount of VaR is larger for a stock with $\tau > 0$ than for a stock with $\tau = 0$. As we have seen above, this is indeed the case.

2.4.3 Risk premium (Sharpe ratio)

The expected return of a stock usually exceeds the return r of a risk-free asset (e.g. the return of a cash-bond that investors get when they put money on a bank account). The reason for this is that stock holders get a premium for the risk that they take by holding a stock. The risk premium is often simply defined by $R - r$. A key measure in financial physics is the risk premium per unit volatility (Sharpe ratio) defined by $s = (R - r)/\sigma$ [71]. If a stock or a collection of stocks called portfolio can be described by our geometric Brownian walk model with delay, the risk premium per unit volatility (first order approximation) is given by

$$s = \frac{\gamma_{\rm st} - r}{\sigma} = \frac{\mu_1 + \mu_2 \exp\{(\sigma^2/2 - \mu_1)\,\tau\} - r}{\sigma}\,. \tag{30}$$

Just as the expected return R and the volatility σ, the Sharpe ratio s is an important measure to compare the performance of two portfolios. However, it is known that two portfolios that differ in the model parameters μ_1, μ_2, and σ of Eq. (25) for $\tau = 0$ but have the same underlying fluctuating force exhibit under particular market circumstances the same Sharpe ratio [71]. This result carries over to portfolios exhibiting a time-delayed price dynamics with a delay $\tau > 0$ as far as the effective price dynamics given by Eq. (27) is concerned. Let $\mu_1^{(i)}, \mu_2^{(i)}, \sigma^{(i)}$, and $\tau^{(i)}$ denote the model parameters of two perfectly correlated portfolios $i = 1, 2$. Then, we conclude that

$$s = \frac{\mu_1^{(i)} + \mu_2^{(i)} \exp\left\{\left([\sigma^{(i)}]^2/2 - \mu_1^{(i)}\right)\tau^{(i)}\right\} - r}{\sigma^{(i)}} \tag{31}$$

will correspond to the same constant for $i = 1, 2$ at least up to terms of $O(\mu_2^2)$ given the aformentioned particular market conditions are satisfied.

2.4.4 Ito calculus

So far, the multiplicative noise term in Eq. (25) has been interpreted according to the Stratonovich calculus [64]. If Ito calculus is used, we need to replace μ_1 by $\mu_1 + \sigma^2/2$ in Eqs. (28), (29), and (30) [40,56]. Consequently, the expected return reads

$$R = \mu_1 + \frac{\sigma^2}{2} + \mu_2 \exp\left(-\mu_1 \tau\right) \tag{32}$$

and the value at risk measure is given by $\mathrm{VaR} = S_c - S(0)$ with

$$S_c = S(0) \exp\left(\left[\mu_1 + \frac{\sigma^2}{2} + \mu_2 \exp\left(-\mu_1 \tau\right)\right] T - \sigma\sqrt{T} N_{0,1}^{-1}(c)\right) . \tag{33}$$

Comparing Eqs. (32) and (33) with Eqs. (28) and (29), we realize that in the Ito case there is no need to distinguish between stocks with low and high volatility. In the Ito case we see that R, S_c, and VaR decrease with the time delay τ irrespective of the ratio between μ_1 and σ. Finally, the risk premium per unit volatility s (first order approximation) is given by

$$s = \frac{\mu_1 + \sigma^2/2 + \mu_2 \exp\left(-\mu_1 \tau\right) - r}{\sigma} \tag{34}$$

and will correspond under particular market conditions to a constant for all perfectly correlated portfolios that may differ in the parameters $\mu_1, \mu_2, \sigma, \tau$.

2.4.5 Black-Scholes formula

The Black-Scholes formula of option prices [72,73] is based on the assumption that stock prices evolve according to Eq. (25) with $\tau = 0$. Our analysis reveals that the Black-Scholes formula for the time-delayed stock price model (25) with $\tau > 0$ is still valid in the sense that the Black-Scholes formula yields a first order approximation if μ_2 is a small parameter and $t \gg \tau$. In contrast, on a time scale $t \propto O(\tau)$ the Black-Scholes formula may fail to predict accurate option prices because on that time scale the time-delayed geometric Brownian walk can behave qualitatively different from a ordinary geometric Brownian walk. Numerical simulations as performed in Sec. 2.3 can clarify whether or not the time-independent approximative Langevin equation (27) also holds on the critical time scale $t \propto O(\tau)$ (see e.g. Fig. 3). Note that these qualitative considerations hold both for the Stratonovich and the Ito version of Eq. (25).

3 Conclusions

We studied a perturbation theoretical approach to derive time-dependent solutions for stochastic systems with delays. To this end, we used a particular delay Fokker-Planck equation that can be transformed into an approximative ordinary Fokker-Planck equation and its corresponding Langevin equation. These approximative descriptions involve the conditional probability densities

$P(x, t_\tau - \tau | x, t)$ of the relevant unperturbed stochastic Markov processes. The conditional probability densities $P(x, t_\tau - \tau | x, t)$ were determined by means of Bayes' theorem. In doing so, first order approximations of time-dependent solutions of stochastic systems with delays were obtained from approximative Langevin equations. We applied our approach to an Ornstein-Uhlenbeck process and a geometric Brownian walk and perturbed the two kinds of processes by means of linear time-delayed feedback forces. In the latter context we discussed stock price dynamics and implications for financial physics.

Both examples have in common that they reveal three time domains, see Table 1. For $t \leq \tau$ we found that the stochastic delay differential equations correspond to Langevin equations with external driving forces defined by initial functions. For $t \geq \tau$ but $t \not\gg \tau$ we found that the stochastic delay differential equations correspond to approximative Langevin equations with time-dependent coefficients. In this context, we may think again of the impact of external driving forces. These external driving forces, however, are not necessarily related to initial functions because in our first example (the perturbed Ornstein-Uhlenbeck process) the initial function was constant. Finally, for $t \gg \tau$ we found that the stochastic delay differential equations correspond to approximative Langevin equations with time-independent coefficients. In view of these findings, the question arises whether or not this scheme is specific to the two models discussed in the previous sections or applies to all kinds of stochastic delay differential equations. Future studies may be devoted to answer this question.

Insert Table 1 about here

Our perturbation theoretical approach has been developed for stochastic systems exhibiting noise amplitudes g that do not depend on the time-delayed variable, see Eq. (1). In general, the noise amplitude g in Eq. (1) may depend on the time-delayed variable. The results presented in Sec. 2.1 can be generalized to this case. The this end, the considerations in Appendix A of Ref.[61]

can be exploited. In doing so, the diffusion coefficient $D(x, x_\tau) = g^2(x, x_\tau)/2$ will become a time-dependent effective diffusion coefficient $D_{\text{eff}}(x, t)$. Again, the key step is to exploit Bayes' theorem in order to determine the conditional probability density $P(x, t_\tau - \tau|x, t)$ from which $D_{\text{eff}}(x, t)$ can then be computed.

The perturbation theoretical approach presented in Sec. 2 and in related previous studies [61–63] is centered around the first order statistics. The approximative Langevin equation (2) should not be used to compute second order statistical measures. The reason for this is that in contrast to alternative perturbation theoretical approaches [57], our approach is based on the delay Fokker-Planck equation of time-delayed systems and yields an effective ordinary Fokker-Planck equation for the single time point probability density $P^{(1)}(x, t|\varphi)$ given for a particular initial function φ. The probability density $P^{(1)}(x, t|\varphi)$ describes first order statistical properties but does not provide information about second order statistical quantities. However, the analysis of second order statistical quantities such as the eigenvalues λ of Fokker-Planck operators and autocorrelation functions $\langle X(t)X(t')\rangle$ has crucially improved our understanding of complex systems subjected to fluctuations. For example, in the context of the noise-induced transitions the eigenvalue spectrum has been derived for the Hongler model [41]. Therefore, future studies may generalize the results derived earlier for the stationary case and derived in the present study for the non-stationary case to the stationary and non-stationary case with respect to second order statistical quantities, see Table 2.

Insert Table 2 about here

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Table 1

Interpretation of stochastic delay differential equations

Time	Interpretation
$[0,\tau]$	exact Langevin equations with driving forces
$t \propto O(\tau)$	non-autonomous approximative Langevin equations
$t \gg \tau$	autonomous approximative Langevin equations

Table 2

State of the art and possible future perspectives of perturbation theoretical approaches to time-delayed stochastic systems

	Stationary	Non-stationary	
1st order statistics	$P_{\rm st}^{(1)}(x)$	$P^{(1)}(x,t)$	state of the art
2nd order statistics	$\lambda^{(1)}$	$\langle X(t)X(t')\rangle^{(1)}$	future studies

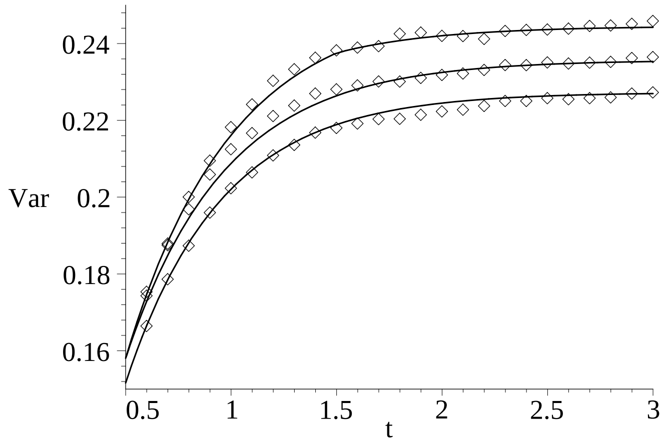

Fig. 1. Variance of the perturbed Ornstein-Uhlenbeck process (6) as a function of time for several time delays. Diamonds represent exact results as obtained by solving Eq. (6) numerically. Solid lines represent first order approximations computed from Eq. (14) for $t' = \tau$ and $t \geq \tau$. Time delays: $\tau_0 = 0$, $\tau_1 = 0.5$, $\tau_2 = 1.5$ (from bottom to top). Other parameters: $a = 1.0$, $b = 0.1$, $Q = 0.5$. For the numerical simulation an Euler forward scheme was used (single time step $\delta t = 0.001$; number of realizations $N = 10^5$; Gaussian random numbers generated by means of Box-Muller algorithm; for details see [53]).

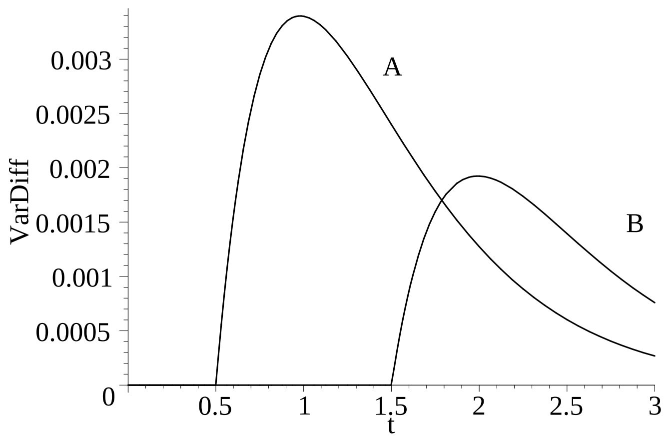

Fig. 2. The variance difference 'VarDiff' defined by the difference of functions (12) and (14) for τ_1 (label A) and τ_2 (label B) and the parameters used in Fig. 1: $\mathrm{VarDiff} = \sigma_{18}^{2(1)} - \sigma_{20}^{2(1)}$.

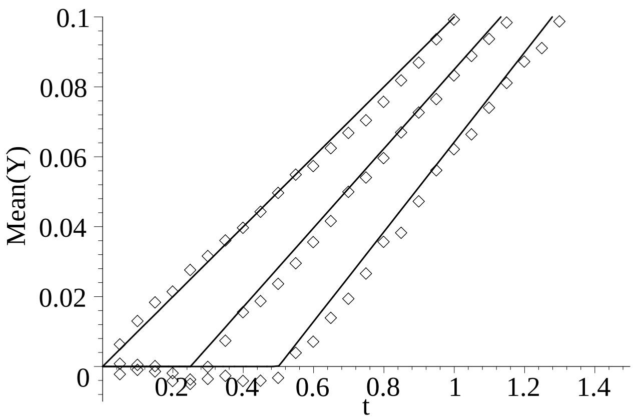

Fig. 3. Mean value of the random variable $Y(t)$ defined by the perturbed geometric Brownian walk (15) as a function of time. Diamonds represent exact results as obtained by solving Eq. (15) numerically. Solid lines represent first order approximations computed from Eq. (24) with $t' = \tau$. Time delays: $\tau_0 = 0$, $\tau_1 = 0.25$, $\tau_2 = 1.50$ (from top to bottom). Other parameters: $a = 1.0$, $b = 0.1$, $Q = 3.0$ ($\delta t = 0.001$; $N = 5 \times 10^4$; Box-Muller algorithm).

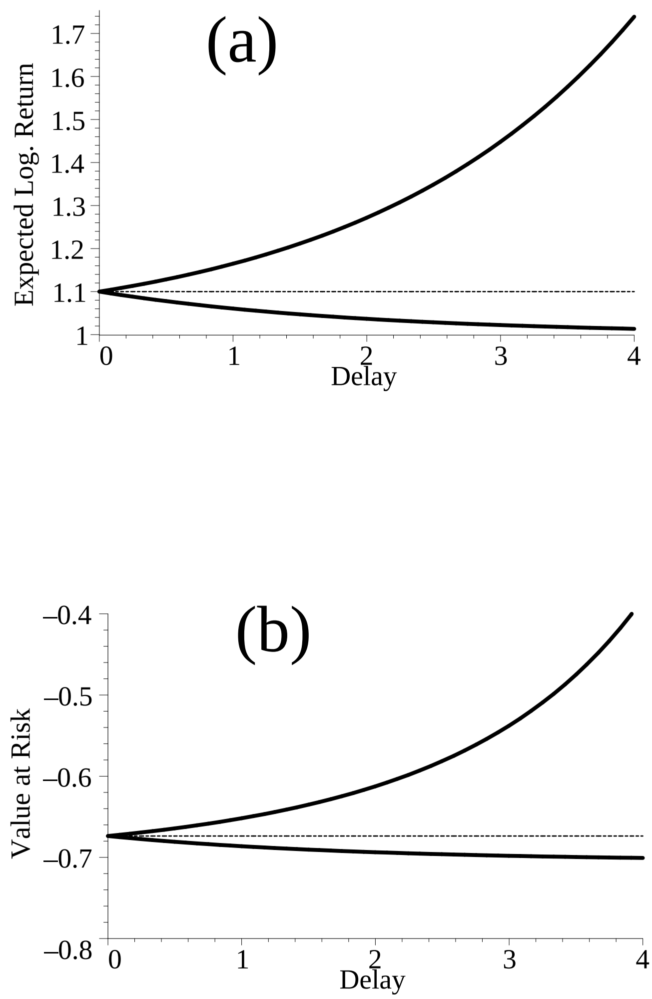

Fig. 4. Panel (a): expected logarithmic return R as a function of the time delay τ computed from Eq. (28) for a stock with high (upper thick line) and low (lower thick line) volatility. The horizontal thin line represents the reference value for $\tau = 0$. Parameters: $\mu_1 = 1.0$, $\mu_1 = 0.1$, $\sigma^2 = 3.0$ (high volatility case), $\sigma^2 = \mu_1$ (low volatility case). Panel (b): 90% value at risk for a horizon of one time unit as a function of τ drawn from $\mathrm{VaR} = S_c - S(0)$ and Eq. (29) for a stock with high (upper thick line) and low (lower thick line) volatility. The horizontal thin line represents the reference value for $\tau = 0$. Parameters: $S(0) = 1$, $T = 1$, $c = 0.9$. Other parameters as in panel (a).