Analytical results for fundamental time-delayed feedback systems subjected to multiplicative noise

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We study the stochastic behavior of fundamental time-delayed feedback systems subjected to multiplicative noise. We derive exact results for the first and second moments and the autocorrelation function. For a particular class of systems we show how the variance depends on the amplitude of the multiplicative noise. Furthermore, we identify parameter regions of stationary solutions with finite and infinite variances. Finally, we suggest that delay-induced Lévy flights may occur in time-delayed feedback systems involving multiplicative noise.

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I. INTRODUCTION

Systems with time-delayed feedback can be found in the animate and inanimate world [1]. Especially, biological systems tend to involve feedback loops with retarded arguments [2–6] (see also the survey in Ref. [7]). These retarded arguments reflect finite transmission times related to the transport of matter, energy, and information through the systems. Since noise is an immanent property of delay systems in general and biological systems in particular, there is an increasing interest in the understanding and modeling of stochastic systems involving time-delayed feedback. In this context, a triangle of research activities has appeared in recent years with corners at the derivation of analytical results for the additive noise case [8], the description of stochastic delay systems by means of Fokker-Planck equations [9–13], and the investigation of delay systems involving parametric or multiplicative noise sources [14]; see Fig. 1. In particular, in line with general considerations on neural control mechanism [15], it has been argued that time-delayed feedback and multiplicative noise are indispensable ingredients for our understanding of the pupil light reflex [3,16], pointing movements [4], and balancing movements [6]. This triangle of research activities reveals that there is a need for the derivation of analytical results for stochastic delay systems subjected to multiplicative noise. In particular, such analytical results could be derived using the Fokker-Planck approach to stochastic delay systems. Therefore, we will derive in the present paper analytical expressions for the first and second moments and the autocorrelation functions of fundamental time-delayed feedback systems involving multiplicative noise and examine how these moments depend on the delay time and the amplitude of the multiplicative noise.

II. FUNDAMENTAL TIME-DELAYED FEEDBACK SYSTEMS WITH MULTIPLICATIVE NOISE

In Sec. II A we will briefly review how to derive evolution equations for probability densities $P(x,t)$, transition probability densities $P(x,t|x',t')$, and joint probability densities $P(x,t;x',t')$ of stochastic delay differential equations. In Sec. II B we will use these evolution equations to derive exact analytical results for time-delayed systems with multiplicative noise. As we will see below, we will primarily describe time in terms of a relative time variable z given by a mapping $t \to z(t)$ such that the aforementioned evolution equations are written in terms of the density functions $P(x,z)$, $P(x,z|x',z')$, and $P(x,z;x',z')$.

A. Fokker-Planck approach and the method of steps

Let us consider a univariate random variable $X(t)$ that is defined on the phase space Ω and satisfies the stochastic delay differential equation

$$\frac{d}{dt}X = h(X,X_\tau) + g(X,X_\tau)\Gamma(t), \tag{1}$$

interpreted according to the Ito calculus [11,14,17]. Here and in what follows, we use the notation $X_\tau(t)=X(t-\tau)$. The variable $\Gamma(t)$ denotes a Langevin force with $\langle\Gamma(t)\Gamma(t')\rangle=\delta(t-t')$ [18]. For $t\in[-\tau,0]$ the random variable X is given by $X(t)=\phi(t)$. Using

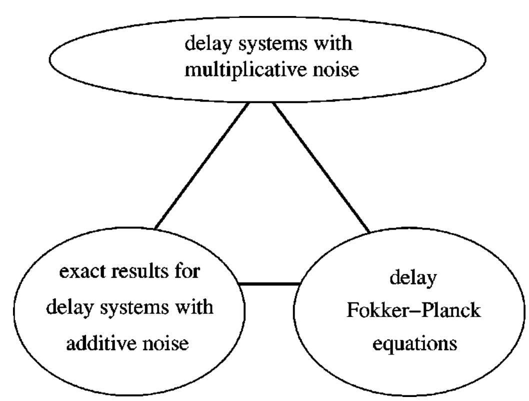

FIG. 1. Current research activities are calling for the derivation of exact results for delay systems with multiplicative noise by means of delay Fokker-Planck equations.

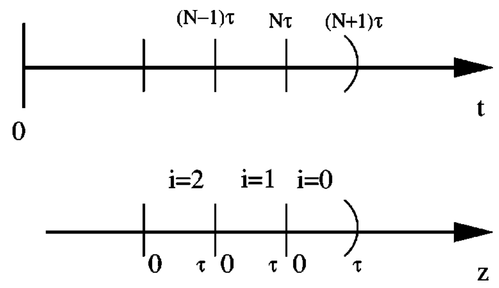

FIG. 2. Relationship between absolute time t and relative time z.

$$D(x,x') = \frac{1}{2}[g(x,x')]^2, \tag{2}$$

we get

$$\frac{d}{dt}X = h(X,X_\tau) + \sqrt{2D(X,X_\tau)}\,\Gamma(t). \tag{3}$$

Equation (1) can be treated by means of the method of steps [8,19]. To this end, we consider the time axis for events that fall into the interval $[0,(N+1)\tau)$ with $N \geqslant 0$ and describe time by means of a continuous counter $z \in [0,\tau)$ and an integer counter $i \in [0,N]$; see Fig. 2. More precisely, we decompose the interval $[0,(N+1)\tau)$ into $N+1$ intervals of length τ and label these intervals by means of the integer variable i such that $i=0$ corresponds to the interval $[N\tau,(N+1)\tau)$, $i=1$ corresponds to the interval $[(N-1)\tau,N\tau)$, and so on. In doing so, we look back in time when we increase i. We describe a time point t within an interval i by means of the continuous counter z such that if t falls into the ith interval given by $[(N-i)\tau,(N+1-i)\tau)$ we have $z=t-(N-i)\tau$. We will refer to z as relative time. Using this two-variable description of time, we can describe $X(t)$ in terms of a set of random variables $X_{i\tau}(z)$ defined on $[0,\tau)$. The variables $X_{i\tau}$ correspond to the original random variable $X(t)$ in one of the previously defined intervals i. In detail, for $t \in [N\tau,(N+1)\tau)$ we define the variable $X_0(t-N\tau)=X(t)$, for $t\in[(N-1)\tau,N\tau)$ we define the variable $X_\tau(t-(N-1)\tau)=X(t)$, and so on. In general, the variables $X_{i\tau}(z)$ are defined on the respective intervals i by $X_{i\tau}(z)=X(t)$ with $z=t-(N-i)\tau$.

In sum, we deal with an $(N+1)$-dimensional vector $\mathbf{X}(z) \in \Omega^{N+1}$ with components $X_{i\tau}(z)$ such as $\mathbf{X} = (X_0,X_\tau,X_{2\tau},\ldots,X_{N\tau})$. The joint probability density for $\mathbf{X}$ reads

$$P(\mathbf{x},z) = \langle \delta(\mathbf{x}-\mathbf{X}(z))\rangle \tag{4}$$

with $\mathbf{x}=x_0,\ldots,x_{N\tau}$. Note that for $t\in[N\tau,(N+1)\tau)$ and $z=t-N\tau$ we have

$$\begin{aligned} P(\mathbf{x},z) &= P(x_0,t;x_\tau,\,t-\tau;\,\ldots\,;x_{N\tau},t-N\tau)\\ &= \langle \delta(x_0-X(t)\cdots\delta(x_{N\tau}-X(t-N\tau))\rangle. \end{aligned} \tag{5}$$

For the stochastic process described by Eq. (1) we can then derive the evolution equation of $P(\mathbf{x},z)$ in terms of the Fokker-Planck equation [10,11,13]

$$\frac{\partial}{\partial z}P(\mathbf{x},z) = \hat{F}(\mathbf{x},\boldsymbol{\nabla},z)P(\mathbf{x},z) \tag{6}$$

involving the operator

$$\begin{aligned} \hat{F}(\mathbf{x},\boldsymbol{\nabla},z) &= \sum_{i=0}^{N-1}\left[-\frac{\partial}{\partial x_{i\tau}}h(x_{i\tau},x_{(i+1)\tau}) + \frac{\partial^2}{\partial x_{i\tau}^2}D(x_{i\tau},x_{(i+1)\tau})\right]\\ &\quad -\frac{\partial}{\partial x_{N\tau}}h(x_{N\tau},\phi(z)) + \frac{\partial^2}{\partial x_{N\tau}^2}D(x_{N\tau},\phi(z)). \end{aligned} \tag{7}$$

The initial condition $P(\mathbf{x},0)$ for a particular integer N is given by the limiting distribution $\lim_{z\to\tau}P(\mathbf{x},z)$ for $N-1$. Therefore, we need to solve Eq. (7) in subsequent steps for $N=0,1,2$, and so on (method of steps). Comparing Eqs. (1) and (7), we see that we have transformed a non-Markovian process into a Markov diffusion process at the cost of introducing additional variables [10,11,13]. Note that such a procedure is frequently used in the theory of ordinary Langevin equations, for example, in order to treat univariate non-Markovian stochastic processes subjected to colored noise as multivariate Markov processes [18]. By analogy with Eq. (6), the Markov transition probability density $P(\mathbf{x},z|\mathbf{x}',z')$ defined by

$$P(\mathbf{x},z|\mathbf{x}',z') = \langle \delta(\mathbf{x}-\mathbf{X}(z))\rangle|_{\mathbf{x}'=\mathbf{X}(z')} \tag{8}$$

for $z,z' \in [0,\tau)$ and $z \geqslant z'$ satisfies

$$\frac{\partial}{\partial z}P(\mathbf{x},z|\mathbf{x}',z') = \hat{F}(\mathbf{x},\boldsymbol{\nabla},z)P(\mathbf{x},z|\mathbf{x}',z'). \tag{9}$$

Note that for $t,t' \in [N\tau,(N+1)\tau)$, $z=t-N\tau$, and $z'=t'-N\tau$ we have

$$\begin{aligned} P(\mathbf{x},z|\mathbf{x}',z') &= P(x_0,t;\,\ldots\,;x_{N\tau},t-N\tau|x_0',t';\,\ldots\,;x_{N\tau}',t'-N\tau)\\ &= \langle \delta(x_0-X(t))\cdots\delta(x_{N\tau}-X(t-N\tau))\rangle. \end{aligned} \tag{10}$$

Finally, the joint probability density $P(\mathbf{x},z|\mathbf{x}',z')$ with $z,z' \in [0,\tau)$ and $z\geqslant z'$ and

$$P(\mathbf{x},z;\mathbf{x}',z') = \langle \delta(\mathbf{x}-\mathbf{X}(z))\,\delta(\mathbf{x}'-\mathbf{X}(z'))\rangle \tag{11}$$

satisfies

$$\frac{\partial}{\partial z}P(\mathbf{x},z;\mathbf{x}',z') = \hat{F}(\mathbf{x},\boldsymbol{\nabla},z)P(\mathbf{x},z;\mathbf{x}',z'), \tag{12}$$

which can be verified by multiplying Eq. (10) with $P(\mathbf{x}',z')$. For $N\geqslant 1$ integrating Eq. (6) with respect to the variables $x_\tau,\ldots,x_{N\tau}$ yields

$$\begin{aligned} \frac{\partial}{\partial z}P(x,z) &= -\frac{\partial}{\partial x}\int_\Omega h(x,x_\tau)P(x,x_\tau,z)dx_\tau\\ &\quad + \frac{\partial^2}{\partial x^2}\int_\Omega D(x,x_\tau)P(x,x_\tau,z)dx_\tau \end{aligned} \tag{13}$$

with $P(x,z)=\int P(\mathbf{x},z)dx_\tau\cdots dx_{N\tau}$ and $P(x,x_\tau,z) = \int P(\mathbf{x},z)dx_{2\tau}\cdots dx_{N\tau}$. This evolution equation was previously derived in a study by Guillouzic *et al.* [9] when using Eq. (5) in order to transform the relation back to the absolute time frame described by t. Integrating with respect to the variables $x_\tau, \ldots, x_{N\tau}$ for $N \geqslant 1$ from Eq. (9) it follows that

$$\frac{\partial}{\partial z} P(x,z|x',z') = -\frac{\partial}{\partial x}\int_\Omega h(x,x_\tau)P(x,x_\tau,z|x',x_\tau,z')dx_\tau + \frac{\partial^2}{\partial x^2}\int_\Omega D(x,x_\tau)P(x,x_\tau,z|x',x'_\tau,z')dx_\tau. \quad (14)$$

Integrating with respect to $x_\tau, \ldots, x_{N\tau}$ and $x'_\tau, \ldots, x'_{N\tau}$ for $N \geqslant 1$ from Eq. (12) we obtain

$$\frac{\partial}{\partial z} P(x,z;x',z') = -\frac{\partial}{\partial x}\int_\Omega h(x,x_\tau)P(x,x_\tau,z;x',z')dx_\tau + \frac{\partial^2}{\partial x^2}\int_\Omega D(x,x_\tau)P(x,x_\tau,z;x',z')dx_\tau. \quad (15)$$

Note that Eqs. (6), (9), and (12) are closed evolution equations, whereas Eqs. (13)–(15) do not provide closed descriptions for the evolution of probability densities. Nevertheless, as we will see below, Eqs. (13)–(15) can be exploited to derive closed sets of equations to compute particular expectation values of probability densities.

B. Fundamental case

Now let us assume that the phase space Ω is given by the real line such that $X(t) \in \Omega = \mathbb{R}$. Accordingly, density functions are subjected to natural boundary conditions [18]. In what follows, we confine ourselves to the class of stochastic delay systems with multiplicative noise and linear drift terms. More precisely, we put $h(x,x_\tau) = -ax - bx_\tau$ with $a, b \geqslant 0$ and $a+b > 0$ and, consequently, Eq. (1) becomes

$$\frac{d}{dt}X = -aX - bX_\tau + \sqrt{2D(X,X_\tau)}\,\Gamma(t). \quad (16)$$

Likewise, Eqs. (13) and (15) now read

$$\frac{\partial}{\partial z}P(x,z) = \frac{\partial}{\partial x}\left[axP(x,z) + b\int_\Omega x_\tau P(x,x_\tau,z)dx_\tau\right] + \frac{\partial^2}{\partial x^2}\int_\Omega D(x,x_\tau)P(x,x_\tau,z)dx_\tau \quad (17)$$

and

$$\frac{\partial}{\partial z}P(x,z;x',z') = \frac{\partial}{\partial x}\left[axP(x,z;x',z') + b\int_\Omega x_\tau P(x,x_\tau,z;x',z')dx_\tau\right] + \frac{\partial^2}{\partial x^2}\int_\Omega D(x,x_\tau)P(x,x_\tau,z;x',z')dx_\tau, \quad (18)$$

respectively. In the stationary case, Eq. (17) reduces to

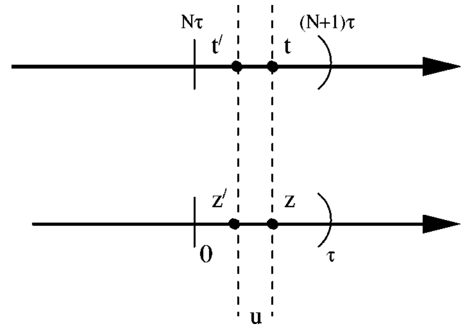

FIG. 3. Time differences u with respect to absolute and relative time.

$$axP_{\rm st}(x) + b\int_\Omega x_\tau P_{\rm st}(x,x_\tau)dx_\tau = -\frac{\partial}{\partial x}\int_\Omega D(x,x_\tau)P_{\rm st}(x,x_\tau)dx_\tau \quad (19)$$

and, using the relative time difference u defined by $u = z - z'$, Eq. (18) can be written as

$$\frac{\partial}{\partial u}P_{\rm st}(x,z'+u;x',z') = \frac{\partial}{\partial x}\left[axP_{\rm st}(x,z'+u;x',z') + b\int_\Omega x_\tau P_{\rm st}(x,x_\tau,z'+u;x',z')dx_\tau\right] + \frac{\partial^2}{\partial x^2}\int_\Omega D(x,x_\tau)P_{\rm st}(x,x_\tau,z'+u;x',z')dx_\tau. \quad (20)$$

From Eq. (19) it is clear that the mean value satisfies the relation $(a+b)\langle X\rangle_{\rm st} = 0$, that is, we have $\langle X\rangle_{\rm st} = 0$. Let us determine next the second moment $\langle X^2\rangle_{\rm st}$. To this end, we use the stationary autocorrelation function $C(u) = \langle X(t)X(t')\rangle_{\rm st}$ involving the time difference $u = t - t'$. Then, we get $C(0) = \langle X^2\rangle_{\rm st}$ and we see that C is a symmetric function: $C(u) = C(-u)$. In addition, it is reasonable to assume that $C(u)$ is a continuous function of u and, in particular, is continuous at $u=0$ and $u=\tau$:

$$\lim_{u\downarrow 0} C(u) = C(0), \quad \lim_{u\uparrow \tau} C(u) = C(\tau). \quad (21)$$

The autocorrelation function $C(u)$ can conveniently be determined if we discuss the problem within the relative time frame. Then, we have $C(u) = \langle X(z)X(z')\rangle_{\rm st}$ with $z, z' \in [0,\tau)$ and $u = z - z' \in [0,\tau)$; see Fig. 3.

Evaluating the equation for $P_{\rm st}(x)$, namely, multiplying Eq. (19) with x and integrating with respect to x, we obtain

$$aC(0) + b\,C(\tau) = \langle D\rangle_{\rm st}. \quad (22)$$

Evaluating the equation for $P_{\rm st}(x,z'+u;x',z')$, that is, using Eq. (20), the relation

$$\frac{dC(u)}{du} = -aC(u) - b\,C(u-\tau) \tag{23}$$

can be found for $u \in (0,\tau)$. Using the symmetry of $C(u)$, from Eq. (23) we obtain

$$\frac{d^2C(u)}{du^2} + (b^2 - a^2)C(u) = 0 \tag{24}$$

for $u \in (0,\tau)$. Using the continuity of $C(u)$, from Eq. (23) we obtain

$$\lim_{u \downarrow 0} \frac{dC(u)}{du} = -\langle D \rangle_{st}. \tag{25}$$

A detailed derivation of Eqs. (23)–(25) is given in the Appendix. Equations (22), (24), and (25) constitute the key relations for all subsequent calculations.

Introducing the parameter $\omega = \sqrt{|b^2 - a^2|}$ and considering the three cases $b > a$, $b = a$, and $b < a$ separately, Eq. (24) can be solved which gives us

$$b > a \geqslant 0{:}C(u) = C(0)\cos(\omega u) + e\,\sin(\omega u),$$

$$b = a > 0{:}C(u) = C(0) + e' u,$$

$$a > b \geqslant 0{:}C(u) = C(0)\cosh(\omega u) + e''\,\sinh(\omega u). \tag{26}$$

Next, we use Eq. (25) to determine e, e', e'' and thus obtain $e = e'' = -\langle D \rangle_{st}/\omega$ and $e' = -\langle D \rangle_{st}$; that is, we get

$$b > a \geqslant 0{:}C(u) = C(0)\cos(\omega u) - \frac{\langle D \rangle_{st}}{\omega}\sin(\omega u),$$

$$b = a > 0{:}C(u) = C(0) - \langle D \rangle_{st} u,$$

$$a > b \geqslant 0{:}C(u) = C(0)\cosh(\omega u) - \frac{\langle D \rangle_{st}}{\omega}\sinh(\omega u). \tag{27}$$

Let us evaluate these relations in the limiting case $u \to \tau$:

$$b > a \geqslant 0{:}C(\tau) = C(0)\cos(\omega \tau) - \frac{\langle D \rangle_{st}}{\omega}\sin(\omega \tau),$$

$$b = a > 0{:}C(\tau) = C(0) - \langle D \rangle_{st} \tau,$$

$$a > b \geqslant 0{:}C(\tau) = C(0)\cosh(\omega \tau) - \frac{\langle D \rangle_{st}}{\omega}\sinh(\omega \tau). \tag{28}$$

Consequently, if we deal with multiplicative noise sources that satisfy the relation

$$\langle D \rangle_{st} = f(C(0), C(\tau), \langle X \rangle_{st}), \tag{29}$$

then Eqs. (22), (28), and (29) with $\langle X \rangle_{st} = 0$ constitute a set of three equations for the three unknown variables $C(0)$, $C(\tau)$, and $\langle D \rangle_{st}$; that is, in this case we have a closed description at hand and can determine the unknown variables. We will demonstrate this point in the subsequent section for special choices of D.

In closing this section, let us derive some useful relations. First, by means of Eqs. (22) and (28) the variable $C(\tau)$ can be eliminated and Eq. (28) becomes

$$b > a \geqslant 0{:}C(0) = \langle D \rangle_{st} \frac{1 + b\,\omega^{-1}\sin(\omega\tau)}{a + b\,\cos(\omega\tau)},$$

$$b = a > 0{:}C(0) = \langle D \rangle_{st} \frac{1 + a\tau}{2a},$$

$$a > b \geqslant 0{:}C(0) = \langle D \rangle_{st} \frac{1 + b\,\omega^{-1}\sinh(\omega\tau)}{a + b\,\cosh(\omega\tau)} \tag{30}$$

for $a + b\,\cos(\omega\tau) \neq 0$. Using $C(u) = C(-u)$, Eq. (30) leads to

$$b > a \geqslant 0{:}C(u) = \langle D \rangle_{st} \left\{ \frac{1 + b\,\omega^{-1}\sin(\omega\tau)}{a + b\,\cos(\omega\tau)}\cos(\omega u) - \frac{\sin(\omega|u|)}{\omega} \right\},$$

$$b = a > 0{:}C(u) = \langle D \rangle_{st} \left\{ \frac{1 + a\tau}{2a} - |u| \right\},$$

$$a > b \geqslant 0{:}C(u) = \langle D \rangle_{st} \left\{ \frac{1 + b\,\omega^{-1}\sinh(\omega\tau)}{a + b\,\cosh(\omega\tau)}\cosh(\omega u) - \frac{\sinh(\omega|u|)}{\omega} \right\} \tag{31}$$

for $u \in [-\tau, \tau]$. In the special case $g = \sqrt{Q}$, we have $\langle D \rangle_{st} = Q/2$ and Eq. (31) recovers the result derived earlier for additive noise systems; see Ref. [8] and Eqs. (26) and (27) of Ref. [12].

Second, using the relationship between $C(0)$ and $\langle D \rangle_{st}$ [see Eq. (30)], we can write Eq. (31) in terms of $C(0)$:

$$b > a \geqslant 0{:}C(u) = C(0) \left\{ \cos(\omega u) - \frac{a + b\,\cos(\omega\tau)}{1 + b\,\omega^{-1}\sin(\omega\tau)} \frac{\sin(\omega|u|)}{\omega} \right\},$$

$$b = a > 0{:}C(u) = C(0) \left\{ 1 - \frac{2a}{1 + a\tau}|u| \right\},$$

$$a > b \geqslant 0{:}C(u) = C(0) \left\{ \cosh(\omega u) - \frac{a + b\,\cosh(\omega\tau)}{1 + b\,\omega^{-1}\sinh(\omega\tau)} \frac{\sinh(\omega|u|)}{\omega} \right\} \tag{32}$$

for $u \in [-\tau, \tau]$. Let us briefly demonstrate the power of Eq. (32). Let us assume that Eq. (29) does not hold; that is, we assume that we cannot express $\langle D \rangle_{st}$ in terms of $C(0), C(\tau)$, and $\langle X \rangle_{st}$ [e.g., we deal with $D(x) = A + Bx^4$]. Then, we may determine numerically the variance $\langle X^2 \rangle_{st} = C(0)$ of the process at hand. If we put the estimate for $C(0)$ into Eq. (32), then Eq. (32) gives us the autocorrelation function $C(u)$.

Third, one may introduce the complex valued parameter $\tilde{\omega}=\sqrt{b^2-a^2}$ satisfying $\tilde{\omega}=\omega\neq 0$ for $b>a\geqslant 0$, $\tilde{\omega}=0$ for $b=a>0$, and $\tilde{\omega}=i\omega\neq 0$ for $a>b\geqslant 0$, where i is $i=\sqrt{-1}$. Using the relations $\cos(ix)=\cosh(x)$ and $\sin(ix)=i\sinh(x)$ and the limiting behavior $\sin(\epsilon x)/\epsilon=\sinh(\epsilon x)/\epsilon=x$ for $\epsilon\to 0$, all three cases $b>a$, $b=a$, and $b<a$ can be treated simultaneously. In particular, Eqs. (28), (31), and (32) become

$$C(\tau)=C(0)\cos(\tilde{\omega}\tau)-\frac{\langle D\rangle_{\mathrm{st}}}{\tilde{\omega}}\sin(\tilde{\omega}\tau), \tag{33}$$

$$C(u)=\langle D\rangle_{\mathrm{st}}\left\{\frac{1+b\ \tilde{\omega}^{-1}\sin(\tilde{\omega}\tau)}{a+b\ \cos(\tilde{\omega}\tau)}\cos(\omega u)-\frac{\sin(\tilde{\omega}|u|)}{\tilde{\omega}}\right\}, \tag{34}$$

$$C(u)=C(0)\left\{\cos(\tilde{\omega}u)-\frac{a+b\ \cos(\tilde{\omega}\tau)}{1+b\ \tilde{\omega}^{-1}\sin(\tilde{\omega}\tau)}\frac{\sin(\tilde{\omega}|u|)}{\tilde{\omega}}\right\} \tag{35}$$

for $a,b\geqslant 0$ and $a+b>0$, where the case $a=b\Rightarrow\tilde{\omega}=0$ is regarded as the limiting case $\tilde{\omega}\to 0$.

C. Examples

1. Systems with $g=\sqrt{A+Bx^2}$

We will first consider systems involving multiplicative noise sources that can be described by $g(x,x_\tau)=\sqrt{2D(x,x_\tau)}=\sqrt{A+Bx^2}$ with $A>0$ and $B\geqslant 0$. In fact, as we will see below, the results obtained for systems of this kind carry over to various different kinds of systems. The stochastic delay differential equation (16) now reads

$$\frac{d}{dt}X(t)=-aX(t)-bX_\tau(t)+\sqrt{A+BX^2(t)}\ \Gamma(t), \tag{36}$$

where $\Gamma(t)$ denotes the δ-correlated fluctuating force mentioned in Sec. II A.

Let us first consider $\tau=0$. Then, Eq. (36) reduces to the Ito-Langevin equation: $dX/dt=-(a+b)X(t)+\sqrt{A+BX^2(t)}\ \Gamma(t)$. For $B=0$ Eq. (36) becomes $dX/dt=-(a+b)X(t)+\sqrt{A}\ \Gamma(t)$ and describes the linear and instantaneous response of a system to the δ-correlated fluctuating force $\Gamma(t)$. Such a response system exhibits in the stationary case the Gaussian distribution of an Ornstein-Uhlenbeck process. For $B>0$ Eq. (36) reads $dX/dt=-(a+b)X(t)+\sqrt{A+BX^2}\ \Gamma(t)$. From the corresponding Fokker-Planck equation the stationary distribution can be obtained and is given by the power law distribution

$$P_{\mathrm{st}}(x)=\frac{1}{Z}\left[\frac{1}{A+Bx^2}\right]^{1+(a+b)/B} \tag{37}$$

with $Z=\int_\Omega[A+Bx^2]^{-1-(a+b)/B}\,dx$. We see that Z is finite and $P_{\mathrm{st}}(x)$ exists for all $a,b\geqslant 0$, $a+b>0$, $B>0$. The first moment vanishes: $\langle X\rangle_{\mathrm{st}}=0$. The second moment satisfies

$$\langle X^2\rangle_{\mathrm{st}}=A/[2(a+b)-B] \tag{38}$$

and is finite for $B<2(a+b)$ and infinite for $B\geqslant 2(a+b)$. Consequently, there is a parameter range in which we deal with stationary distributions that exhibit infinite variances (Lévy flights).

Next, let us consider $\tau>0$. By means of $D(x,x_\tau)=(A+Bx^2)/2$, Eq. (29) reads

$$\langle D\rangle_{\mathrm{st}}=\frac{A+BC(0)}{2}. \tag{39}$$

Exploiting Eqs. (22), (28), and (39) we get

$$b>a\geqslant 0{:}C(0)=\frac{A}{2}\frac{1+b\ \omega^{-1}\sin(\omega\tau)}{a-B/2+b[\cos(\omega\tau)-B(2\omega)^{-1}\sin(\omega\tau)]}, \tag{40a}$$

$$b=a>0{:}C(0)=\frac{A}{2}\frac{1+a\tau}{a-B/2+a(1-B\tau/2)}, \tag{40b}$$

$$a>b\geqslant 0{:}C(0)=\frac{A}{2}\frac{1+b\ \omega^{-1}\sinh(\omega\tau)}{a-B/2+b[\cosh(\omega\tau)-B(2\omega)^{-1}\sinh(\omega\tau)]}. \tag{40c}$$

It is clear that in the limit $b\downarrow a$ the first of the three relations converges to the second one. Likewise, in the limit $b\uparrow a$ the third relation converges to the second one. There are two helpful rearrangements of Eq. (40), which will be addressed next.

(a) Increase of variance with delay. For linear stochastic delay differential equations with additive noise one can show that the variance (i.e., $\langle X^2\rangle_{\mathrm{st}}$) increases with the delay τ [8,10]. Let us examine now the impact of the delay on $\langle X^2\rangle_{\mathrm{st}}$ for the multiplicative noise case. To this end, we write Eq. (40) as

$$b>a\geqslant 0{:}C(0)=\frac{A}{2}\frac{1+b\ \omega^{-1}\sin(\omega\tau)}{a+b\ \cos(\omega\tau)-B[1+b\ \omega^{-1}\sin(\omega\tau)]/2}, \tag{41a}$$

$$b=a>0{:}C(0)=\frac{A}{2}\frac{1+a\tau}{2a-B(1+a\tau)/2}, \tag{41b}$$

$$a>b\geqslant 0{:}C(0)=\frac{A}{2}\frac{1+b\ \omega^{-1}\sinh(\omega\tau)}{a+b\ \cosh(\omega\tau)-B[1+b\ \omega^{-1}\sinh(\omega\tau)]/2}. \tag{41c}$$

Note once again that for $B=0$ we obtain the result derived previously for the additive case; see Ref. [8] and Eq. (27) of Ref. [12]. Differentiating Eq. (41) with respect to τ gives us

$$b>a\geqslant 0{:}\frac{dC(0)}{d\tau}=\frac{2b^2}{A}[C(0)]^2\frac{1+\cos(\omega\tau+\varphi_1)}{[1+b\ \omega^{-1}\sin(\omega\tau)]^2}\geqslant 0,$$

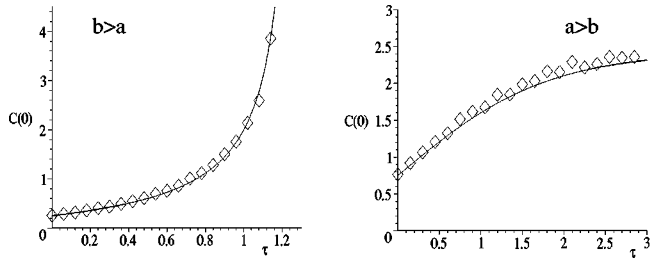

FIG. 4. Increase of the second moment $C(0)$ as a function of τ for $b>a$ (left panel) and $a>b$ (right panel). Solid lines: exact results given by of Eq. (41). Diamonds: $C(0)$ obtained by solving numerically Eq. (36) using an Euler forward algorithm. Parameters: $a=1$, $b=1.5$, $A=B=1$ (left); $a=1.5$, $b=1.0$, $A=3$, $B=1$ (right).

$$b=a>0: \frac{dC(0)}{d\tau} = \frac{4a^2}{A}[C(0)]^2 \frac{1}{[1+a\tau]^2} > 0,$$

$$a>b\geqslant 0: \frac{dC(0)}{d\tau} = \frac{2b^2}{A}[C(0)]^2 \frac{1+\cosh(\omega\tau+\varphi_2)}{[1+b\ \omega^{-1}\sinh(\omega\tau)]^2} > 0 \tag{42}$$

with $\tan\varphi_1=\omega/a$ and $\tanh\varphi_2=-\omega/a$. As a result, if $C(0)$ is finite then $C(0)$ increases monotonically as a function of τ. As we will see below, for $b>a\geqslant 0$ there is a boundary value τ^* for which $C(0)$ becomes infinite. Therefore, for $b>a\geqslant 0$ the variance $C(0)$ increases monotonically only on an interval $[0,\tau^*)$ as shown in Fig. 4 (left panel). Qualitatively the same behavior can be found for the cases $b=a>0$ with $B>0$ and $a>b\geqslant 0$ with $B>2\omega$. For $b=a>0$ with $B=0$ from Eq. (41b) it follows that $C(0)=A(1+a\tau)/(4a)$, which implies that $C(0)$ is finite for finite delays τ but runs to infinity when τ runs to infinity. For $a>b\geqslant 0$ and $B\leqslant 2\omega$ there does not exist a τ^* such that the numerator in Eq. (41c) vanishes (see below), which implies that for arbitrarily large but finite τ we have $C(0)<\infty$. Let us examine the asymptotic behavior of $C(0)$ in this case. For $B=2\omega$ from Eq. (40c) it follows that

$$C(0) = \frac{A}{2\omega}\exp\{\omega\tau\} \tag{43}$$

for large τ. Consequently, we have $C(0)\to\infty$ for $\tau\to\infty$. In contrast, for $B<2\omega$ from Eq. (40c) it follows that

$$\lim_{\tau\to\infty} C(0) = \frac{A}{2\omega - B}; \tag{44}$$

that is, we have $C(0)<\infty$ for $\tau\to\infty$, as illustrated in Fig. 4 (right panel).

(b) Domains of finite variance. The numerator in Eq. (40) may vanish for a particular boundary value τ^*. In this case, the second moment $C(0)$ becomes infinite. Let us determine τ^* and study the impact of B (i.e., the amplitude of the multiplicative noise) on τ^*. To this end, we transform Eq. (40) appropriately. For $b>a\geqslant 0$ Eq. (40) can be equivalently expressed as

$$C(0) = \frac{A}{2}\frac{1+b\ \omega^{-1}\sin(\omega\tau)}{\tilde{a}+\tilde{b}\cos(\omega\tau+\varphi)},$$

$$\tilde{a} = a - B/2,$$

$$\tilde{b} = b\sqrt{1+B^2/(4\omega^2)},$$

$$\tan\varphi = B/(2\omega). \tag{45}$$

Consequently, τ^* is found as

$$\tau^* = \frac{1}{\omega}\left[-\varphi + \arccos\left(-\frac{\tilde{a}}{\tilde{b}}\right)\right], \quad \varphi = \arctan\left(\frac{B}{2\omega}\right). \tag{46}$$

Using some geometrical considerations, we can read off from Eq. (46) that τ^* decreases as a function of B [23]. For $B=0$ the boundary value τ^* corresponds to the critical delay value τ_c for which the additive noise model becomes unstable [8,10,14] [see also Eq. (61) below]. For $B=2(a+b)$ from Eq. (40a) it follows that the numerator reads $b[\cos(\omega\tau)-1]-(a+b]\omega^{-1}\sin(\omega\tau)$ and vanishes for $\tau=\tau^*=0$. In sum, $\tau^*(B)$ decreases from $\tau^*=\tau_c$ at $B=0$ to $\tau^*=0$ at $B=2(a+b)$; see also Fig 5.

For $b=a>0$ Eq. (40) can be written as

$$C(0) = \frac{A}{2}\frac{1+a\tau}{2a - B(1+a\tau)/2}; \tag{47}$$

see Eq. (41b). The boundary value τ^* is given by

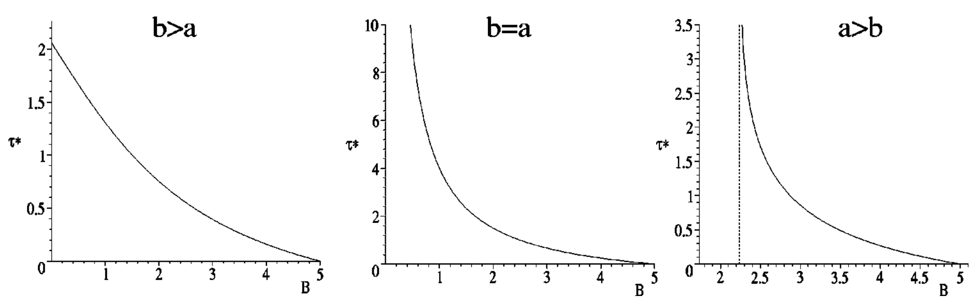

FIG. 5. The boundary value τ^* decreases when the amplitude of the multiplicative noise increases. Depicted are three cases: $b>a$ (left), $b=a$ (middle), and $a>b$ (right). τ^* was computed from Eqs. (46), (48), and (52). Parameters are chosen such that $2(a+b)=5$: $a=1$, $b=1.5$ (left); $b=a=1.25$ (middle); $a=1.5$, $b=1$ (right).

$$\tau^* = \frac{4a}{B} - 1. \qquad (48)$$

It is clear that τ^* decreases as a function of B. In the limit $B \to 0$ we have $\tau^* \to \infty$; that is, there does not exist a finite boundary value τ^*. For $B = 2(a+b) = 4a$ we have $\tau^* = 0$; see also Fig. 5. For $a > b \geq 0$ we can write down Eq. (40) for three different cases such as

$$B < 2\omega: C(0) = \frac{A}{2} \frac{1 + b\ \omega^{-1}\sinh(\omega\tau)}{\tilde{a} + \tilde{b}\cosh(\omega\tau + \varphi)},$$

$$\tilde{a} = a - B/2,$$

$$\tilde{b} = b\sqrt{1 - B^2/(4\omega^2)},$$

$$\tanh\varphi = -B/(2\omega); \qquad (49)$$

$$B = 2\omega: C(0) = \frac{A}{2} \frac{1 + b\ \omega^{-1}\sinh(\omega\tau)}{\tilde{a} + b\exp\{-\omega\tau\}}; \qquad (50)$$

$$B > 2\omega: C(0) = \frac{A}{2} \frac{1 + b\ \omega^{-1}\sinh(\omega\tau)}{\tilde{a} - \tilde{b}\sinh(\omega\tau + \varphi)},$$

$$\tilde{a} = a - B/2,$$

$$\tilde{b} = b\sqrt{B^2/(4\omega^2) - 1},$$

$$\tanh\varphi = -2\omega/B. \qquad (51)$$

For $B \leq 2\omega$ we have $\tilde{a} = a - B/2 \geq a - \omega$, which implies $\tilde{a} \geq a - \sqrt{a^2 - b^2} > 0$. Therefore, for $B \leq 2\omega$ we have $\tilde{a} + \tilde{b}\cosh(\omega\tau + \varphi) > 0$ and $\tilde{a} + b\exp\{-\omega\tau\} > 0$; that is, the numerators in both cases are positive and $C(0)$ is finite for arbitrarily large but finite τ. For $B > 2\omega$ we get

$$\tau^* = \frac{1}{\omega}\left[-\varphi + \mathrm{arcsinh}\left(\frac{\tilde{a}}{\tilde{b}}\right)\right], \quad \varphi = \mathrm{arctanh}\left(-\frac{2\omega}{B}\right). \qquad (52)$$

The boundary value τ^* decreases as a function of B [24]. For $B \downarrow 2\omega$ we have $\tau^* \to \infty$. For $B = 2(a+b)$ from Eq. (40c) it follows that the numerator reads $b[\cosh(\omega\tau) - 1) - (a + b]\omega^{-1}\sinh(\omega\tau)$ and vanishes for $\tau = \tau^* = 0$. In sum, we have $\tau^* = \infty$ at $B = 2\omega$. For $B > 2\omega$ the boundary value τ^* decreases and vanishes at $B = 2(a+b)$; see Fig. 5. Finally, note that for $b = 0$ there does not exist an interval $(2\omega, 2(a+b))$ [because for $b = 0$ we have $2\omega = 2(a+b)$], whereas for $b > 0$ one can show that the inequality $2\omega < 2(a+b)$ holds (under the constraint $a > b \geq 0$) which means that for $b > 0$ it is indeed important to take the parameter range $B \in [2\omega, 2(a+b)]$ into account.

2. Further systems

The results obtained in the preceding section also apply to several stochastic delay differential equations different from Eq. (36). Let us mention a few of them.

For $g(x, x_\tau) = \sqrt{2D} = \tilde{A} + \tilde{B}x$ Eq. (16) reads

$$\frac{d}{dt}X(t) = -aX(t) - bX_\tau(t) + [\tilde{A} + \tilde{B}X(t)]\Gamma(t). \qquad (53)$$

We have $D(x, x_\tau) = [\tilde{A}^2 + 2\tilde{A}\tilde{B}x + \tilde{B}^2x^2]/2$ which implies that Eq. (39) holds for $A = \tilde{A}^2$ and $B = \tilde{B}^2$ because of $\langle X\rangle_{\rm st} = 0$. Consequently, Eqs. (36) and (53) yield the same variables $C(0)$, $C(\tau)$, and $\langle D\rangle_{\rm st}$ if we put $A = \tilde{A}^2$ and $B = \tilde{B}^2$.

For $g(x, x_\tau) = \sqrt{2D} = \sqrt{A + Bx_\tau^2}$ Eq. (16) can be written as

$$\frac{d}{dt}X(t) = -aX(t) - bX_\tau(t) + \sqrt{A + BX_\tau^2(t)}\ \Gamma(t) \qquad (54)$$

and we have $D(x, x_\tau) = [A + Bx_\tau^2]/2$. Consequently, Eq. (39) holds and the variables $C(0)$, $C(\tau)$, and $\langle D\rangle_{\rm st}$ of Eqs. (36) and (54) assume the same values.

If we put $g(x, x_\tau) = \sqrt{2D} = \tilde{A} + \tilde{B}x_\tau$ then Eq. (16) is given by

$$\frac{d}{dt}X(t) = -aX(t) - bX_\tau(t) + [\tilde{A} + \tilde{B}X_\tau(t)]\Gamma(t) \qquad (55)$$

and we have $D(x, x_\tau) = [\tilde{A}^2 + 2\tilde{A}\tilde{B}x_\tau + \tilde{B}^2x_\tau^2]/2$. Since we have $\langle X\rangle_{\rm st} = 0$, we obtain Eq. (39) for $A = \tilde{A}^2$ and $B = \tilde{B}^2$ and the variables $C(0)$, $C(\tau)$, and $\langle D\rangle_{\rm st}$ of the processes given by Eqs. (36) and (55) have the same expectation values $C(0)$, $C(\tau)$, and $\langle D\rangle_{\rm st}$.

We may also consider stochastic delay differential equations involving the Stratonovich calculus. The idea is to map these equations to equivalent stochastic delay differential equations with Ito calculus. For example, let us consider a stochastic process defined by

$$\frac{d}{dt}X(t) = -aX(t) - b'X_\tau(t) + \underbrace{\sqrt{A + BX^2(t)}\ \Gamma(t)}_{\text{Stratonovich}}. \qquad (56)$$

As shown in Refs. [11,14], the corresponding evolution equation involving Ito calculus reads

$$\frac{d}{dt}X(t) = -aX(t) - \left(b' - \frac{B}{2}\right)X_\tau(t) + \sqrt{A + BX^2(t)}\ \Gamma(t). \qquad (57)$$

Therefore, if we put $b = b' - B/2$ then Eqs. (36) and (56) exhibit the same values for $C(0)$, $C(\tau)$, and $\langle D\rangle_{\rm st}$. In other words, the parameter b' in combination with the amplitude B of the multiplicative noise source gives rise to an effective parameter b. Such shifts $b' \to b$ in the parameter space are discussed in detail in a study by Mackey and Nechaeva [14] and in Ref. [11].

D. Considerations on stationary and nonstationary solutions

Having obtained exact results for the boundary value τ^* the following question arises: how does the systems behave for delay times larger than τ^*. Basically there are two options. The systems either exhibit stationary distributions with infinite variances or they become nonstationary.

(a) Sufficient conditions for nonstationary solutions. Using Eq. (5), we transform Eq. (17) back into the absolute time frame which gives us

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left[axP(x,t) + b\int_\Omega x_\tau P(x,t;x_\tau,t-\tau)dx_\tau\right] + \frac{\partial^2}{\partial x^2}\int_\Omega D(x,x_\tau)P(x,t;x_\tau,t-\tau)dx_\tau \qquad (58)$$

for $t \geqslant \tau$, whereas for $t \in [0,\tau)$ from Eq. (36) it follows that

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}[ax + b\phi(t-\tau)]P(x,t) + \frac{\partial^2}{\partial x^2}D(x,\phi(t-\tau))P(x,t). \qquad (59)$$

Multiplying Eqs. (58) and (59) with x and integrating with respect to x, we obtain an evolution equation for $M_1(t) = \langle X(t)\rangle$ that reads

$$\frac{d}{dt}M_1(t) = -aM_1(t) - bM_1(t-\tau) \qquad (60)$$

for $t \geqslant 0$ with $M_1(t)=\phi(t)$ for $t \in [-\tau,0)$ and exhibits the stationary value $M_{1,\mathrm{st}}=0$. Let us define the critical delay τ_c by

$$\tau_c = \tau^*(B=0, b>a) = \frac{1}{\omega}\arccos\left(-\frac{a}{b}\right). \qquad (61)$$

It can be shown that $M_{1,\mathrm{st}}=0$ is stable for (i) $b>a\geqslant 0$ with $\tau<\tau_c$, (ii) $b=a>0$ with $\tau \geqslant 0$, and (iii) $a>b\geqslant 0$ with $\tau\geqslant 0$. $M_{1,\mathrm{st}}=0$ is unstable for $b>a\geqslant 0$ with $\tau>\tau_c$. In particular, there is a Hopf bifurcation at $\tau=\tau_c$ which links the stationary solution with nonstationary oscillatory solutions [20,21]. Consequently, a sufficient condition for the existence of nonstationary solutions is $b>a\geqslant 0$ with $\tau>\tau_c$. In line with a study by Mackey and Nechaeva [14], we conclude that irrespective of the explicit structure of the multiplicative noise source for $b>a\geqslant 0$ and $\tau>\tau_c$ stationary solutions do not exist or they exist but they are unstable. Figure 6 illustrates an oscillatory nonstationary solution of Eq. (36).

(b) Numerical evidence for domains of stable stationary solutions with infinite variance. We now make the hypotheses that for some of the multiplicative noise systems described by Eq. (16) the stability of the first moment determines completely the asymptotic behavior of their solutions. Systems of this kind may exhibit solutions with finite as well as infinite variances. Let us further assume that these systems exhibit a simple boundary value τ^* such that $\langle X^2\rangle_{\mathrm{st}}<\infty$ for $\tau<\tau^*$ and $\langle X^2\rangle_{\mathrm{st}}=\infty$ for $\tau\geqslant\tau^*$ and $\tau^*\leqslant\tau_c$. According to our hypotheses, we deal with stationary solutions exhibiting finite variance for $\tau<\tau^*$, with stationary solutions exhibiting

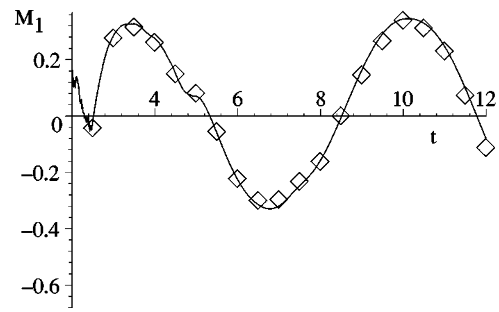

FIG. 6. Oscillatory nonstationary behavior for $\tau>\tau_c$. After a transient period determined by the initial function $\phi(t)$, the first moment starts to oscillate. Solid line: numerical solution of Eq. (60). Diamonds: first moment obtained by solving numerically Eq. (36). Parameters: $a=1$, $b=1.5$, $\tau=2.5$, $A=B=1$ (here we have $\tau_c \approx 2.05$).

infinite variances for $\tau\in[\tau^*,\tau_c)$, and with nonstationary solution for $\tau>\tau_c$. In particular, if Eq. (36) belongs to this class of systems we obtain the classification scheme shown in Table I. In fact, we have found numerical evidence that our hypothesis is correct and that at least for Eq. (36) the asymptotic behavior is determined by the asymptotic behavior of the first moment.

In detail, we have solved Eq. (36) numerically for the two cases $b>a$ with $\tau\in[\tau^*,\tau_c)$ and $a>b$ with $\tau>\tau^*$ given an initial function $\phi(t)$ composed of Gaussian distributed random numbers. We have plotted the distributions $P(x,t) = \langle\delta(x-X(t))\rangle$ for both cases for several times t and, in doing so, have found numerical evidence that transient distributions converge to stationary ones for $t\to\infty$. Figure 7 shows the distributions $P(x,t)$ for three different times t^* (dashed lines), t_1 (solid lines), and t_2 (diamonds) with $t^*\ll t_1\ll t_2$. We have chosen t^* such that we obtained transient distributions that differ from the respective stationary ones. In contrast, we have chosen t_1 such that we obtained stationary distributions. In order to illustrate that the distributions $P(x,t_1)$ describe stationary distributions, we have plotted the distributions $P(x,t)$ for another time $t=t_2$ with $t_2\gg t_1$. Indeed, there is a good match of the distributions taken at times t_1 and t_2. In sum, we have good reason to believe that in the two aforementioned cases (i.e., for $b>a$ with $\tau\in[\tau^*,\tau_c)$ and $a>b$ with $\tau>\tau^*$) the distributions $P(x,t)$ converge to stationary distributions for $t\to\infty$.

III. CONCLUSIONS

We have discussed stochastic processes that occur in time-delayed feedback systems subjected to multiplicative

TABLE I. Parameter regions of different types of asymptotic behaviors.

Asymptotic behavior	$b>a\geqslant 0$	$b=a>0$	$a>b\geqslant 0$
Nonstationary	$\tau>\tau_c$		
Stationary with $\langle X^2\rangle_{\mathrm{st}}=\infty$	$\tau\in[\tau^*,\tau_c)$	$\tau\geqslant\tau^*$	$\tau\geqslant\tau^*$
Stationary with $\langle X^2\rangle_{\mathrm{st}}<\infty$	$\tau<\tau^*$	$\tau<\tau^*$	$\tau<\tau^*$

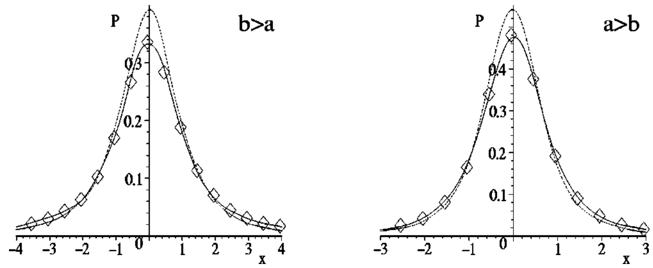

FIG. 7. Transient (dashed lines) and stationary distributions (solid lines and diamonds) for $\tau > \tau^*$ and $b > a$ (left panel) and $a > b$ (right panel). Left panel parameters: $t^* = 1000$, $t_1 = 10\,000$, $t_2 = 30\,000$, $a = 1$, $b = 1.5$, $A = B = 1$, $\tau = 1.5$ (here we have $\tau^* \approx 1.30$ and $\tau_c \approx 2.05$). Right panel parameters: $t^* = 300$, $t_1 = 10\,000$, $t_2 = 30\,000$, $a = 1.5$, $b = 1.0$, $A = 2$, $B = 2.5$, $\tau = 3$ (here we have $\tau^* \approx 1.72$ and $B > 2\omega \approx 2.23$). See text for details.

noise. We have derived exact results for the first and second moments and the autocorrelation functions of these processes. To this end, we have exploited a Fokker-Planck approach that was advocated by Guillouzic *et al.* [9]. The results thus derived generalize previous findings by Küchler and Mensch [8] that are devoted to delay systems with additive noise.

In particular, we have shown that for all systems with linear drift terms the stationary autocorrelation function $C(u)$ evolves qualitatively in the same way irrespective of the explicit structure of the multiplicative noise source; see Eq. (32). We have examined in more detail stochastic delay differential equations that may involve different noise sources but exhibit in the stationary case the same averaged noise amplitude $\langle D \rangle_{\rm st}$ given by $\langle D \rangle_{\rm st} = [A + B\ C(0)]/2$ [here $C(0)$ corresponds to the variance in the stationary case]. We have found that for stochastic processes described by this kind of evolution equations the variance increases monotonically with the time delay. We have determined a boundary value τ^* for the time delay for which variances become infinite. In line with a study by Mackey and Nechaeva [14], we have derived sufficient conditions for the instability of stationary solutions. We would like to point out that in their study stochastic delay different equations are evaluated, whereas in the present manuscript Fokker-Planck equations have been studied.

We have closed our considerations with the hypothesis that there are stochastic delay systems with multiplicative noise that exhibit stable stationary distribution with infinite variances. This hypothesis seems to be plausible because for vanishing delays it is well-known that such systems exist. Moreover, we have presented numerical evidence to support our hypothesis. A striking implication of our hypothesis is that when time delay is increased a system subjected to multiplicative noise may leave a parameter regime in which it exhibits a stationary distribution with finite variance and enters a parameter regime in which it exhibits a stationary distribution with infinite variance. In this sense, there might be delay-induced Lévy flights. Increasing the time delay even further, some of these multiplicative noise systems will leave the parameter regime of stable stationary Lévy flights and enter a parameter regime in which they exhibit nonstationary solutions such as oscillatory solutions.

It seems to be rewarding to study the issue of stable stationary distributions with infinite variances in more detail. We feel, however, that to this end numerical studies may be employed which are beyond the scope of the present study. Future studies may also be devoted to discuss the impact of colored noise sources as opposed to the δ-correlated noise sources that have been considered in the present manuscript. In particular, one may consider colored noise sources described by Ornstein-Uhlenbeck processes. Such systems can often be treated analytically by increasing the dimensionality of the problem. Accordingly, the one-dimensional system subjected to a colored noise source is described by means of a two-dimensional system involving a δ-correlated noise source [18]. In our context, this would mean that we need to study multivariate stochastic delay differential equations. In the mathematical literature for a particular two-dimensional stochastic delay differential equation the second moment of one of the two system variables has been derived as a function of the systems parameters [22]. In addition, the Fokker-Planck approach to stochastic delay differential equations has been extended to the multivariate case [13]. In view of these works, we are inclined to say that it should be possible to extend the results of the present manuscript to stochastic delay systems involving colored noise sources.

APPENDIX: DERIVATION OF EQS. (23)–(25)

Multiplying Eq. (20) with x and x' and integrating with respect to x and x', we obtain

$$\begin{aligned}\frac{d}{du}C(u) = &- a\int_\Omega \int_\Omega xx' P_{\rm st}(x,z'+u;x',z')dxdx' \\ &- b\int_\Omega \int_\Omega \int_\Omega x' x_\tau P_{\rm st}(x,x_\tau,z' \\ &+ u;x',z')\ dxdx'dx_\tau \\ &- \int_\Omega \int_\Omega \int_\Omega x' \frac{\partial}{\partial x} D(x,x_\tau) P_{\rm st}(x,x_\tau,z' \\ &+ u;x',z')\ dxdx'dx_\tau \end{aligned} \tag{A1}$$

for $u > 0$. We need to distinguish between the a-term, b-term, and D-term. The a-term reads

$$a\text{-term} = -a\ C(u). \tag{A2}$$

The b-term reads

$$\begin{aligned} b\text{-term} &= -b\langle X_\tau(z'+u)X(z')\rangle - b\langle X(z'+u-\tau)X(z')\rangle \\ &= -b\ C(u-\tau). \end{aligned} \tag{A3}$$

The D-term involves an integral that can be evaluated by means of the Gaussian integral relation:

$$D\text{-term} = -\int_V \frac{\partial}{\partial x} f(x,x',x_\tau)\, dx\, dx'\, dx_\tau$$
$$= -\int_{-s}^{s}\int_{-s}^{s} [f(s,x',x_\tau) - f(-s,x',x_\tau)]\, dx'\, dx_\tau, \tag{A4}$$

where V denotes the phase space

$$V = \{x \in [-s,s], x' \in [-s,s], x_\tau \in [-s,s]\}$$

in the limit $s \to \infty$. From Eq. (A1) we read off that $f(x,x',x_\tau) = x'D(x,x_\tau)P_{\text{st}}(x,x_\tau,z'+u;x',z')$. Taking natural boundary conditions into account, we assume that P_{st} decays to zero for $|x| \to \infty$, $|x_\tau| \to \infty$, $|x'| \to \infty$ such that $x'D(x,x_\tau)P_{\text{st}} \to 0$ at the surface of V. Consequently, the D-term vanishes and Eq. (A1) becomes Eq. (23). We proceed now just as in the additive case [12]. Exploiting the symmetry $C(u) = C(-u)$, from Eq. (23) it follows that

$$\frac{dC(u)}{du} = -aC(u) - b\, C(\tau - u). \tag{A5}$$

Next, we differentiate Eq. (A5) with respect to u to obtain

$$\frac{d^2C(u)}{du^2} = -a\frac{dC(u)}{du} - b\frac{dC(\tau-u)}{du}. \tag{A6}$$

From Eq. (23) it follows that

$$\frac{dC(\tau-u)}{du} = aC(\tau-u) + b\, C(u). \tag{A7}$$

Substituting Eqs. (A5) and (A7) into Eq. (A6) gives us Eq. (24). Since the autocorrelation function $C(u)$ is assumed to be continuous in $u \in [0,\tau]$ and differentiable in $u \in (0,\tau)$, we can compute the limit $u \downarrow 0$ of Eq. (23):

$$\lim_{u\downarrow 0}\frac{dC(u)}{du} = -aC(0) - b\, C(\tau) \tag{A8}$$

[by exploiting Eq. (21)]. Substituting Eq. (22) into Eq. (A8), we obtain Eq. (25). Note that in general $C(u)$ is not differentiable at $u=0$. Exploiting the symmetry of $C(u)$, from Eq. (25) it follows that dC/du jumps from $+\langle D\rangle_{\text{st}}$ to $-\langle D\rangle_{\text{st}}$ at $u=0$. In particular, in the additive case given by $D(x,x_\tau) = Q/2$ the expression dC/du jumps from $+Q/2$ to $-Q/2$ at $u=0$ [12].

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[23] Proof: if $B(2)>B(1)$ then $\tilde{a}(2)<\tilde{a}(1)$ and $\tilde{b}(2)>\tilde{b}(1)$ which implies that $(-\tilde{a}/\tilde{b})(2)>(-\tilde{a}/\tilde{b})(1)$. Since $\cos(x)$ decreases monotonically in $[0,\pi]$, we have $\arccos(-\tilde{a}/\tilde{b})(2)<\arccos(-\tilde{a}/\tilde{b})(1)$. Furthermore, we have $\varphi(2)>\varphi(1)$ for $B(2)>B(1)$. In sum, we have $\tau^*(2)<\tau^*(1)$ for $B(2)>B(1)$.

[24] Proof: if $B(2)>B(1)$ then $\tilde{a}(2)<\tilde{a}(1)$ and $\tilde{b}(2)>\tilde{b}(1)$ which implies $(\tilde{a}/\tilde{b})(2)<(\tilde{a}/\tilde{b})(1)$. Since $\sinh(x)$ is a monotonically increasing function, we have $\operatorname{arcsinh}(\tilde{a}/\tilde{b})(2)<\operatorname{arcsinh}(\tilde{a}/\tilde{b})(1)$. Furthermore, we have $\varphi<0$ and $\varphi(2)>\varphi(1)$ for $B(2)>B(1)$. In sum, we have $\tau^*(2)<\tau^*(1)$ for $B(2)>B(1)$.