Fluctuation-dissipation theorems for nonlinear Fokker-Planck equations of the Desai-Zwanzig type and Vlasov-Fokker-Planck equations

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Abstract

For many-body systems described by nonlinear Fokker-Planck equations and Vlasov-Fokker-Planck equations we determine the linear system response to small external driving forces. Thus, we derive fluctuation-dissipation theorems that relate dissipative properties of the perturbed systems to the second-order fluctuations of the unperturbed systems.

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1 Introduction

Dynamical mean field models defined by nonlinear Fokker-Planck equations, in general, and Vlasov-Fokker-Planck equations, in particular, have found various applications in physics because they can describe both transient and stationary properties of many-body systems that involve mean field interactions. For

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example, nonlinear Fokker-Planck equations have been used to describe wetting transitions and surface growth [1–4], polymer dynamics [5–8], continuous and discontinuous phase transitions [9–21], reentrant phase transitions [22–27], and pattern formation [28–30]. Vlasov-Fokker-Planck equations are useful models to describe diffusion in plasmas [31–35] and have more recently been used to examine stochastic properties of particle bunches in electron storage rings [36–42].

Despite this plentitude of applications, only a few efforts have been made to elucidate the relationship between fluctuations and dissipation. For equilibrium systems this relationship is given by the famous Green-Kubo formula which relates transport coefficients of systems to their second-order statistics [43–45]. For nonequilibrium systems described by ordinary linear Fokker-Planck equation a fluctuation-dissipation theorem similar to the Green-Kubo formula has been derived by *Agarwal* [46]. This fluctuation-dissipation theorem describes the relationship between the linear response of a stochastic system to a small external driving force (reflecting dissipative properties of the system [45]) and the second-order statistics (fluctuations) of the system when the driving force is switched off. In this context it is important to note that a key step in the study by *Agarwal* is to exploit the formal description of transition probability densities defined by Fokker-Planck equations.

For many-body systems that involve interacting subsystems and are defined by nonlinear Fokker-Planck equations and Vlasov-Fokker-Planck equations basically two approaches have been made to generalize the Green-Kubo formula [47,48]. The first approach exploits the formal description of transition probability densities defined by Fokker-Planck equations. This approach, however, is centered around a quantity that vanishes in the thermodynamic limit of infinitely large systems for which nonlinear Fokker-Planck equations become relevant at all. The second approach starts off with the evolution of perturbations that do not vanish in the thermodynamic limit. However, this second approach is primarily concerned with the relationship between transport coefficients and second-order statistics and, consequently, the relationship between the results derived in this second approach and the fluctuation-dissipation theorem derived by *Agarwal* is mathematically involved. In sum, while both approaches have yield interesting insights into the problem, they are incomplete to certain extents.

In the present study, we will exploit simultaneously the advantages of both approaches. We will derive a fluctuation-dissipation formula that defines the relationship between the linear response of a many-body system to a small driving force and the second-order statistics of the system in the unperturbed state. In line with recent developments in the theory of nonlinear Fokker-Planck equations [42,49–56], our approach will involve explicitly the formal description of transition probability densities.

2 Fluctuation-dissipation theorems for dynamical mean field models

2.1 Nonlinear Fokker-Planck equations of the Desai-Zwanzig type

Let $X(t) \in \Omega = \mathbb{R}$ denote the random variable of a stochastic process with probability density $P_0(x,t;u) = \langle \delta(x - X(t)) \rangle$ and initial distribution $P_0(x,t_0;u) = u(x)$. We refer to Fokker-Planck equations with drift functions that involve the first moment $\langle X \rangle$ as nonlinear Fokker-Planck equations of the Desai-Zwanzig type. Accordingly, we assume that the evolution equation of $P_0(x,t;u)$ is given by

$$\frac{\partial}{\partial t} P_0(x,t;u) = -\frac{\partial}{\partial x} \left[h(x) - \kappa(x - \langle X \rangle_0) \right] P_0 + Q \frac{\partial^2}{\partial x^2} P_0 \qquad (1)$$

for $t \geq t_0$ and $\langle X \rangle_0 = \int_\Omega x P_0(x,t;u)\,\mathrm{d}x$. Note that Eq. (1) can alternatively be regarded as a Fokker-Planck equation that involves a free energy functional that depends on the variance K of the distribution function P_0 [57,58]. Therefore, we may also refer to Eq. (1) as the K-model. Now, let us assume that there is at least one stationary distribution for which the probability density current [59] vanishes such that

$$\Big[h(x) - \kappa(x - \langle X \rangle_{0,\mathrm{st}})\Big]\, P_{0,\mathrm{st}}(x) = Q \frac{\mathrm{d}}{\mathrm{d}x} P_{0,\mathrm{st}}(x) \ . \tag{2}$$

Using the Fokker-Planck operator

$$\hat{F}_0(x, \partial/\partial x, M_1) = -\frac{\partial}{\partial x}\left[h(x) - \kappa(x - M_1)\right] + Q\frac{\partial^2}{\partial x^2} \ , \tag{3}$$

we can write Eqs. (1) and (2) as

$$\frac{\partial}{\partial t} P_0(x,t;u) = \hat{F}_0(x, \partial/\partial x, \langle X \rangle_0)\, P_0 \tag{4}$$

and $\hat{F}_{0,\mathrm{st}}(x, \partial/\partial x, \langle X \rangle_{0,\mathrm{st}}) P_{0,\mathrm{st}} = 0$. We confine ourselves to evolution equations (1) that correspond to strongly nonlinear Fokker-Planck equation such that Eq. (1) describes a nonlinear family of Markov diffusion processes [42]. In this case, every family member is labeled by the initial distribution u and has an individual transition probability density denoted by $P(x,t|x',t';u)$ and defined by

$$\frac{\partial}{\partial t} P_0(x,t|x',t';u) = \hat{F}_0\left(x, \partial/\partial x, \int_\Omega x P_0(x,t;u)\,\mathrm{d}x\right) P_0(x,t|x',t';u) \ . \tag{5}$$

Note that Eq. (5) is coupled with Eq. (4) by means of $\langle X \rangle_0 = \int_\Omega x P_0(x,t;u)\,\mathrm{d}x$. In the stationary case the Fokker-Planck operator (3) does not depend on t because M_1 is a constant which implies that in this case the formal solution of Eq. (5) can be written as [59]

$$P_{0,\mathrm{st}}(x,t|x',t';P_{0,\mathrm{st}}(x)) = \exp\{\hat{F}_0\Big(x, \partial/\partial x, \langle X \rangle_{0,\mathrm{st}}\Big)\,(t-t')\}\delta(x-x') \ . \tag{6}$$

In order to study perturbations of the stationary case, we consider a driving force $f(t)$ that acts on the system such that $P(x,t;u) = \langle \delta(x - X(t)) \rangle$ evolves like

$$\frac{\partial}{\partial t} P(x,t;u) = -\frac{\partial}{\partial x} \left[h(x) - \kappa(x - \langle X \rangle) + f(t) \right] P + Q \frac{\partial^2}{\partial x^2} P \tag{7}$$

for $t \geq t_0$, $P(x,t_0;u) = u(x)$ and $\langle X \rangle = \int_\Omega x P(x,t;u)\, \mathrm{d}x$. We assume that the amplitude of $f(t)$ is small and that the system is close to the stationary unperturbed state given by $P_{0,\mathrm{st}}(x)$. Consequently, we examine solutions of Eq. (7) that can be written as $P(x,t;u) = P_{0,\mathrm{st}}(x) + \epsilon(x,t)$ where ϵ is small and satisfies $\int_\Omega \epsilon(x,t)\, \mathrm{d}x = 0$. From Eq. (7) we then obtain the linear evolution equation for ϵ:

$$\frac{\partial}{\partial t} \epsilon(x,t) = \hat{F}_0\Big(x, \partial/\partial x, \langle X \rangle_{0,\mathrm{st}}\Big)\, \epsilon(x,t) - \left[f(t) + \kappa \langle X \rangle_\epsilon (t) \right] \frac{\mathrm{d}}{\mathrm{d}x} P_{0,\mathrm{st}}(x) \tag{8}$$

with $\langle X \rangle_\epsilon (t) = \int_\Omega x\, \epsilon(x,t)\, \mathrm{d}x$. A detailed calculation shows that

$$\hat{F}_0\Big(x, \partial/\partial x, \langle X \rangle_{0,\mathrm{st}}\Big) \left[x P_{0,\mathrm{st}} \right] = Q \frac{\mathrm{d}}{\mathrm{d}x} P_{0,\mathrm{st}}(x) \tag{9}$$

(see also [47]). As a result, Eq. (8) becomes

$$\begin{aligned} \frac{\partial}{\partial t} \epsilon(x,t) = {} & \hat{F}_0\Big(x, \partial/\partial x, \langle X \rangle_{0,\mathrm{st}}\Big)\, \epsilon(x,t) \\ & - \frac{1}{Q} \left[f(t) + \kappa \langle X \rangle_\epsilon (t) \right] \hat{F}_0\Big(x, \partial/\partial x, \langle X \rangle_{0,\mathrm{st}}\Big) \left[x P_{0,\mathrm{st}} \right] \,. \end{aligned} \tag{10}$$

In order to proceed further, recall that the inhomogeneous first order differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} a(t) = -\gamma a(t) + g(a(t), t) \tag{11}$$

has the implicit solution

$$a(t) = a(t_0)\exp\{-\gamma(t-t_0)\} + \int_{t_0}^{t} \exp\{-\gamma(t-t')\}g(a(t'),t')\,\mathrm{d}t' \ . \quad (12)$$

Using the notation $\hat{\hat{F}}_{0,\mathrm{st}} = \hat{F}_0(x, \partial/\partial x, \langle X\rangle_{0,\mathrm{st}})$, from Eq. (10) it follows that

$$\begin{aligned}\epsilon(x,t) = &\exp\left\{\hat{F}_{0,\mathrm{st}}\,(t-t_0)\right\}\epsilon(x,t_0)\\ &-\frac{1}{Q}\int_{t_0}^{t}\left[f(t') + \kappa\,\langle X\rangle_\epsilon\,(t')\right]\exp\left\{\hat{F}_{0,\mathrm{st}}(t-t')\right\}\hat{F}_{0,\mathrm{st}}\left[xP_{0,\mathrm{st}}\right]\,\mathrm{d}t' \quad (13)\end{aligned}$$

with $\epsilon(x,t_0) = u(x) - P_{0,\mathrm{st}}(x)$ (see also [48]). Since we have $\mathrm{d}\exp\{\hat{F}_{0,\mathrm{st}}t\}/\mathrm{d}t = \exp\{\hat{F}_{0,\mathrm{st}}t\}\hat{F}_{0,\mathrm{st}}$, we obtain

$$\begin{aligned}\epsilon(x,t) = &\exp\left\{\hat{F}_{0,\mathrm{st}}\,(t-t_0)\right\}\epsilon(x,t_0)\\ &-\frac{1}{Q}\int_{t_0}^{t}\left[f(t') + \kappa\,\langle X\rangle_\epsilon\,(t')\right]\frac{\partial}{\partial t}\exp\left\{\hat{F}_{0,\mathrm{st}}(t-t')\right\}\left[xP_{0,\mathrm{st}}\right]\,\mathrm{d}t' \ . \quad (14)\end{aligned}$$

Using $\epsilon(x,t_0) = \int_\Omega \delta(x-x_0)\epsilon(x_0,t_0)\,\mathrm{d}x_0$ and $xP_{0,\mathrm{st}}(x) = \int_\Omega \delta(x-x')x'P_{0,\mathrm{st}}(x')\,\mathrm{d}x'$, we get

$$\begin{aligned}\epsilon(x,t) = &\int_\Omega \exp\left\{\hat{F}_{0,\mathrm{st}}\,(t-t_0)\right\}\delta(x-x_0)\epsilon(x_0,t_0)\,\mathrm{d}x_0\\ -\frac{1}{Q}&\int_{t_0}^{t}\int_\Omega \left[f(t') + \kappa\,\langle X\rangle_\epsilon\,(t')\right]x'\frac{\partial}{\partial t}P_{0,\mathrm{st}}(x')\exp\left\{\hat{F}_{0,\mathrm{st}}(t-t')\right\}\delta(x-x')\,\mathrm{d}x'\,\mathrm{d}t'\end{aligned} \quad (15)$$

From Eq. (6) and $P_{0,\mathrm{st}}(x,t|x',t';P_{0,\mathrm{st}}(x))P_{0,\mathrm{st}}(x') = P_{0,\mathrm{st}}(x,t;x',t';P_{0,\mathrm{st}})$ it then follows that

$$\begin{aligned}\epsilon(x,t) = &\int_\Omega P_{0,\mathrm{st}}(x,t|x_0,t_0;P_{0,\mathrm{st}})\,\epsilon(x_0,t_0)\,\mathrm{d}x_0\\ &-\frac{1}{Q}\int_{t_0}^{t}\int_\Omega \left[f(t') + \kappa\,\langle X\rangle_\epsilon\,(t')\right]x'\frac{\partial}{\partial t}P_{0,\mathrm{st}}(x,t;x',t';P_{0,\mathrm{st}})\,\mathrm{d}x'\,\mathrm{d}t' \ . \quad (16)\end{aligned}$$

Let us consider now the case $u(x) = P_{0,\mathrm{st}}(x) \Rightarrow \epsilon(x,t_0) = 0$. Multiplying Eq. (16) with x and integrating with respect to x, one finds that

$$\langle X\rangle_\epsilon(t) = -\frac{1}{Q}\int_{t_0}^{t} [f(t') + \kappa\langle X\rangle_\epsilon(t')]\frac{\partial}{\partial t}\langle X(t)X(t')\rangle_{0,\mathrm{st}}\,\mathrm{d}t' \tag{17}$$

holds, where the autocorrelation $\langle X(t)X(t')\rangle_{0,\mathrm{st}}$ is defined by $\langle X(t)X(t')\rangle_{0,\mathrm{st}} = \int_\Omega\int_\Omega xx'P_{0,\mathrm{st}}(x,t;x',t';P_{0,\mathrm{st}})\,\mathrm{d}x\,\mathrm{d}x'$. Since in the stationary case the autocorrelation function depends only on time differences, we may introduce the function $C_0(z)$ defined by $\langle X(t)X(t')\rangle_{0,\mathrm{st}} = C_0(t-t')$. Likewise, one may write Eq. (17) like

$$\langle X\rangle_\epsilon(t) = -\frac{1}{Q}\int_{t_0}^{t} [f(t') + \kappa\langle X\rangle_\epsilon(t')]\frac{\partial}{\partial t}C_0(t-t')\,\mathrm{d}t'\ . \tag{18}$$

Finally, for $t_0 \to -\infty$ one can transform Eq. (18) into

$$\langle X\rangle_\epsilon(t) = -\frac{1}{Q}\int_{0}^{\infty} [f(t-z) + \kappa\langle X\rangle_\epsilon(t-z)]\frac{\mathrm{d}}{\mathrm{d}z}C_0(z)\,\mathrm{d}z\ . \tag{19}$$

Eqs. (18) and (19) relate the linear response $\langle X\rangle_\epsilon(t)$ of a dynamical mean field system to the driving force $f(t)$ and the stationary autocorrelation function $C_0(t-t')$. Introducing the Green's function

$$G(z) = -\frac{1}{Q}\frac{\mathrm{d}}{\mathrm{d}z}C_0(z) = -\frac{1}{Q}\frac{\mathrm{d}}{\mathrm{d}z}\langle X(z)X(0)\rangle_{0,\mathrm{st}}\ , \tag{20}$$

we can write Eq. (18) in a more concise way like

$$\langle X\rangle_\epsilon(t) = \int_{t_0}^{t} G(t-t')\,[f(t') + \kappa\langle X\rangle_\epsilon(t')]\,\mathrm{d}t'\ . \tag{21}$$

For $\kappa = 0$ Eqs. (18) and (19) reduce to the respective fluctuation-dissipation theorems by *Agarwal* for stochastic processes defined by linear Fokker-Planck equations [46]. Applying the Fourier transformation to Eq. (21), we can derive the susceptibility function of our system. To this end, we assume that $f(t \to -\infty) = 0$ holds and transform Eq. (19) by means of partial integration into

$$\begin{aligned} \langle X \rangle_\epsilon (t) = & \frac{\langle X^2 \rangle_{0,\mathrm{st}}}{Q} \left[f(t) + \kappa \langle X \rangle_\epsilon (t) \right] \\ & + \frac{1}{Q} \int_0^\infty C_0(z) \frac{\mathrm{d}}{\mathrm{d}z} \left[f(t-z) + \kappa \langle X \rangle_\epsilon (t-z) \right] \mathrm{d}z \ . \end{aligned} \tag{22}$$

Next, we use the Fourier transforms $\langle X \rangle_\epsilon (t) = \int_{-\infty}^{\infty} \exp\{i\omega t\} \langle X \rangle_\epsilon(\omega)\, \mathrm{d}\omega$, $f(t) = \int_{-\infty}^{\infty} \exp\{i\omega t\} f(\omega)\, \mathrm{d}\omega$, and $\tilde{C}_0(z) = \int_{-\infty}^{\infty} \exp\{i\omega z\} \tilde{C}_0(\omega)\, \mathrm{d}\omega$ with $\tilde{C}_0(z) = C_0(z)$ for $z \geq 0$ and $\tilde{C}_0(z) = 0$ for $z < 0$ which implies that $\tilde{C}_0(\omega) = [2\pi]^{-1} \int_0^\infty \exp\{-i\omega z\} \langle X(z)X(0) \rangle_{0,\mathrm{st}}\, \mathrm{d}z$. Then, Eq. (22) becomes

$$\langle X \rangle_\epsilon (\omega) = \frac{\langle X^2 \rangle_{0,\mathrm{st}} - 2\pi i \omega \tilde{C}_0(\omega)}{Q} \left[f(\omega) + \kappa \langle X \rangle_\epsilon (\omega) \right] \ . \tag{23}$$

Consequently, for $\kappa = 0$ we re-obtain the linear response equation $\langle X \rangle_\epsilon (\omega) = \chi_0(\omega) f(\omega)$ involving the susceptibility

$$\chi_0(\omega) = \frac{\langle X^2 \rangle_{0,\mathrm{st}} - 2\pi i \omega \int_0^\infty \exp\{-i\omega z\} \langle X(z)X(0) \rangle_{0,\mathrm{st}}\, \mathrm{d}z}{Q} \tag{24}$$

derived in [46]. In contrast, for $\kappa \neq 0$ the equation $\langle X \rangle_\epsilon (\omega)[1 - \kappa\chi_0(\omega)] = \chi_0(\omega) f(\omega)$ holds. That is, for $1 - \kappa\chi_0(\omega) \neq 0$ we get

$$\langle X \rangle_\epsilon (\omega) = \frac{\chi_0(\omega)}{1 - \kappa\chi_0(\omega)} f(\omega) \ . \tag{25}$$

Accordingly, the susceptibility of the many-body system with mean field interactions is given by $\chi_\kappa = \chi_0/[1 - \kappa\chi_0]$.

Let us illustrate the power of Eq. (19) for the nonlinear Fokker-Planck equation (1) with a linear drift force $h(x)$. Using $h(x) = -\gamma x$ with $\gamma > 0$, Eq. (1) reads

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}\left[\gamma x + \kappa(x - \langle X \rangle) - f(t)\right] P + Q\frac{\partial^2}{\partial x^2}P \ . \tag{26}$$

As suggested by *Shimizu* and *Yamada* for $f = 0$ this model may describe the contraction of muscles due to the collective pulling strokes of so-called muscular cross-bridges [60,61]. In the following, we use $\kappa > 0$. Then, one can show that stationary distributions exist and correspond to Gaussian distributions with $\langle X \rangle_{0,\mathrm{st}} = 0$ [52]. It is clear from Eq. (26) that the first moment evolves like

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X \rangle (t) = -\gamma \langle X \rangle + f(t) \ . \tag{27}$$

From Eq. (27) and $\langle X \rangle_{0,\mathrm{st}} = 0$ it follows that the deviation $\langle X \rangle_\epsilon (t) = \int_\Omega x\epsilon(x,t)\,\mathrm{d}x$ evolves in the same way. That is, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X \rangle_\epsilon (t) = -\gamma \langle X \rangle_\epsilon (t) + f(t) \ . \tag{28}$$

Furthermore, for $f = 0$ the autocorrelation function in the stationary case is given by [52]

$$C_0(t-t') = \langle X(t)X(t') \rangle_{0,\mathrm{st}} = \frac{Q}{\gamma+\kappa}e^{-(\gamma+\kappa)(t-t')} \ . \tag{29}$$

Let us demonstrate that the fluctuation-dissipation theorem (18) indeed establish a link between Eqs. (28) and (29). Differentiating Eq. (29) with respect to t and substituting the result into Eq. (18) give us

$$\langle X \rangle_\epsilon (t) = \int_{t_0}^{t} \left[f(t') + \kappa \langle X \rangle_\epsilon (t')\right] e^{-(\gamma+\kappa)(t-t')}\,\mathrm{d}t' \ . \tag{30}$$

Differentiating Eq. (30) with respect to t leads to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle X\rangle_\epsilon(t) &= -(\gamma+\kappa)\langle X\rangle_\epsilon(t) + f(t) + \kappa\langle X\rangle_\epsilon(t)\\ &= -\gamma\langle X\rangle_\epsilon(t) + f(t)\ .\end{aligned} \tag{31}$$

That is, for the Shimizu-Yamada model one can directly prove that Eqs. (28) and (29) are related to each other by means of the fluctuation-dissipation theorem (18).

2.3 Vlasov-Fokker-Planck equations

We consider now a many-body system with a two-dimensional μ-space: $(p, q) \in \Omega_p \times \Omega_q$. We assume that the evolution of an isolated subsystem is described by a Hamilton function $H_0(p, q)$. We assume further that the non-isolated subsystem is subjected to a mean field force that is produced by all other subsystems and is defined by the gradient of the mean field Hamiltonian

$$H_{\mathrm{MF}}(q, P) = \int_{\Omega_q} V_{\mathrm{MF}}(q - q')P(q')\mathrm{d}q'\ . \tag{32}$$

Finally, we assume that due to the contact with the environment the dynamics of a subsystem is subjected to a damping force and a fluctuating force. Thus, we arrive at the self-consistent Langevin equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}q(t) &= \frac{\partial}{\partial p}H_0(p,q)\\ \frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -\frac{\partial}{\partial q}H_0(p,q) - \frac{\partial}{\partial q}H_{\mathrm{MF}}(q,P) - \gamma p + \sqrt{Q}\,\Gamma(t)\ ,\end{aligned} \tag{33}$$

where $\Gamma(t)$ denotes a Gaussian distributed fluctuating force with $\langle\Gamma(t)\rangle=0$ and $\langle\Gamma(t)\Gamma(t')\rangle = 2\delta(t-t')$ [59]. Here, $Q > 0$ measures the strength of the fluctuating force. For sake of conveniency, we consider a Hamiltonian H_0 that includes a parabolic potential and consider rescaled variables p and q such that H_0 reads

$$H_0(p,q) = \frac{p^2}{2} + \frac{q^2}{2} \ . \tag{34}$$

The self-consistent Langevin equation (33) is related to the Vlasov-Fokker-Planck equation for $P_0(p,q,t) = \langle \delta(p-p(t))\delta(q-q(t)) \rangle$ that reads

$$\frac{\partial}{\partial t} P_0(p,q,t) = \hat{F}_0(p,q,\nabla,P_0)P_0 \tag{35}$$

with $\nabla = (\partial/\partial p, \partial\partial q)$ and

$$\hat{F}_0(p,q,\nabla,W) = -p\frac{\partial}{\partial q} + \left(q + \frac{\partial}{\partial q}\int_{\Omega_q} V_{\rm MF}(q-q')W(q')\mathrm{d}q' \right)\frac{\partial}{\partial p} + \gamma\frac{\partial}{\partial p}p + Q\frac{\partial^2}{\partial p^2} \ . \tag{36}$$

In fact, if Eq. (35) corresponds to a strongly nonlinear Fokker-Planck equation then one can show that Eqs. (33) and (35) are equivalent [52]. In this case, the transition probability density $P(p,q,t|p',q',t')$ is defined by

$$\frac{\partial}{\partial t} P_0(p,q,t|p',q',t') = \hat{F}_0(p,q,\nabla,P_0(q,t))P_0(p,q,t|p',q',t') \tag{37}$$

and is coupled to Eq. (35) due to the occurrence of $P_0(q,t) = \int_{\Omega_p} P_0(p,q,t)\,\mathrm{d}p$ in the Fokker-Planck operator. In the stationary case Eq. (35) is solved by $P_{0,\rm st}(p,q,) = P_{0,\rm st}(p)P_{0,\rm st}(q)$ with

$$P_{0,\rm st}(p) = \sqrt{\frac{\gamma}{2\pi Q}}\, e^{-\frac{\gamma p^2}{2Q}} \ ,$$

$$P_{0,\rm st}(q) = \frac{1}{Z}\, e^{-\frac{\gamma}{Q}\left[\frac{q^2}{2} + \int_{\Omega_q} V_{\rm MF}(q-q')P_{0,\rm st}(q')\mathrm{d}q'\right]} \ , \tag{38}$$

where Z is normalization constant and $P_{0,\rm st}(q)$ is only implicitly defined. Likewise, in the stationary case Eq. (37) has the formal solution

$$P_{0,\mathrm{st}}(p,q,t|p',q',t') = \exp\{\hat{F}_0(p,q,\nabla,P_{0,\mathrm{st}}(q))\,(t-t')\}\delta(p-p')\delta(q-q')\ . \tag{39}$$

Let us study now the impact of a perturbation in terms of a small driving force $f(t)$. Then, Eq. (33) becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}q(t) &= \frac{\partial}{\partial p}H_0(p,q) \\
\frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -\frac{\partial}{\partial q}H_0(p,q) - \frac{\partial}{\partial q}H_{\mathrm{MF}}(q,P) + f(t) - \gamma p + \sqrt{Q}\,\Gamma(t)
\end{aligned} \tag{40}$$

By means of the corresponding Vlasov-Fokker-Planck equation for the probability density $P(p,q,t) = \langle \delta(p-p(t))\delta(q-q(t))\rangle$, we can derive an evolution equation for the perturbations $\epsilon(p,q,t) = P(p,q,t) - P_{0,\mathrm{st}}(p,q)$ from the unperturbed stationary state. This evolution equation reads

$$\begin{aligned}
\frac{\partial}{\partial t}\epsilon(p,q,t) =& \hat{F}_0(p,q,\nabla,P_{0,\mathrm{st}}(q))\,\epsilon(p,q,t) \\
&- \left(f(t) - \frac{\partial}{\partial q}\int_{\Omega_q} V_{\mathrm{MF}}(q-q')\epsilon(q',t)\mathrm{d}q' \right) \frac{\partial}{\partial p}P_{0,\mathrm{st}}(p,q)
\end{aligned} \tag{41}$$

with $\epsilon(q,t) = \int_{\Omega_p} \epsilon(p,q,t)\,\mathrm{d}p$. In order to analyze Eq. (41), we proceed as in Sec. 2.1. To this end, we first decompose $\hat{F}_0$ like $\hat{F}_0 = -p\partial/\partial q + R(p,q,\partial/\partial p, W)$. Then, one can show that the relation

$$\hat{F}_0(p,q,\nabla,P_{0,\mathrm{st}}(q))\,[qP_{0,\mathrm{st}}] = -pP_{0,\mathrm{st}} \tag{42}$$

holds. Using $\partial P_{0,\mathrm{st}}/\partial p = -\gamma p P_{0,\mathrm{st}}/Q$, we eliminate in Eq. (41) the expression $\partial P_{0,\mathrm{st}}/\partial p$ and thus obtain

$$\begin{aligned}
&\frac{\partial}{\partial t}\epsilon(p,q,t) = \hat{F}_0(p,q,\nabla,P_{0,\mathrm{st}}(q))\,\epsilon(p,q,t) \\
&-\frac{\gamma}{Q}\left(f(t) - \frac{\partial}{\partial q}\int_{\Omega_q} V_{\mathrm{MF}}(q-q')\epsilon(q',t)\mathrm{d}q' \right) \hat{F}_0(p,q,\nabla,P_{0,\mathrm{st}}(q))\,[qP_{0,\mathrm{st}}]\ .
\end{aligned} \tag{43}$$

Next, we use the implicit solution of Eq. (43), write $qP_{0,\mathrm{st}}(p,q) = \int_{\Omega_q}\int_{\Omega_p}\delta(p-p')\delta(q-q')q'P_{0,\mathrm{st}}(p',q')\,\mathrm{d}p'\,\mathrm{d}q'$, and exploit Eq. (39). In doing so, we can transform Eq. (43) into

$$\begin{aligned}\epsilon(p,q,t) = &\int_{\Omega_q}\int_{\Omega_p} P_{0,\mathrm{st}}(p,q,t|p_0,q_0,t_0)\,\epsilon(p_0,q_0,t_0)\,\mathrm{d}p_0\,\mathrm{d}q_0\\ -&\frac{\gamma}{Q}\int_{t_0}^{t}\left(f(t') - \frac{\partial}{\partial q}\int_{\Omega_q} V_{\mathrm{MF}}(q-q')\epsilon(q',t')\mathrm{d}q'\right) q'\frac{\partial}{\partial t}P_{0,\mathrm{st}}(p,q,t;p',q',t')\,\mathrm{d}p'\,\mathrm{d}q'\,\mathrm{d}t'\end{aligned} \tag{44}$$

(see also Sec. 2.1).

Let us consider the special case $\epsilon(p_0,q_0,t_0)=0$. If we put $w(z) = -\mathrm{d}V(z)/\mathrm{d}z$ and integrate with respect to p, then from Eq. (44) we get

$$\epsilon(q,t) = -\frac{\gamma}{Q}\int_{t_0}^{t}\left(f(t') + \int_{\Omega_q} w(q-q')\,\epsilon(q',t')\mathrm{d}q'\right) q'\frac{\partial}{\partial t}P_{0,\mathrm{st}}(q,t;q',t')\,\mathrm{d}q\,\mathrm{d}t' \tag{45}$$

Next, using the Fourier transforms $\epsilon(q,t) = \int_{\Omega_k} e^{ikq}\epsilon(k,t)\,\mathrm{d}k$ and $\epsilon(k,t) = [2\pi]^{-1}\int_{\Omega_q} e^{-ikq}\epsilon(q,t)\,\mathrm{d}q$, Eq. (45) can be transformed into

$$\begin{aligned}\epsilon(k,t) = -\frac{\gamma}{2\pi Q}\int_{t_0}^{t}\Bigg[&f(t')\frac{\partial}{\partial t}\left\langle e^{-ikq(t)}q(t')\right\rangle_{0,\mathrm{st}}\\ &+2\pi\int_{\Omega_k} w(k')\epsilon(k',t')\frac{\partial}{\partial t}\left\langle e^{-i(k-k')q(t)}q(t')\right\rangle_{0,\mathrm{st}}\mathrm{d}k'\Bigg]\,\mathrm{d}t'\,.\end{aligned} \tag{46}$$

Introducing the Green's function

$$G(k,z) = -\frac{\gamma}{2\pi Q}\frac{\mathrm{d}}{\mathrm{d}z}\left\langle e^{-ikq(z)}q(0)\right\rangle_{0,\mathrm{st}}\,, \tag{47}$$

we can express Eq. (46) as

$$\epsilon(k,t) = \int_{t_0}^{t} \int_{\Omega_k} G(k-k', t-t') \left[\delta(k') f(t') + 2\pi\, w(k')\, \epsilon(k',t')\right] \mathrm{d}k' \, \mathrm{d}t' \ . \quad (48)$$

Just as in the case of the conventional fluctuation-dissipation theorem of the linear response theory [45], Eqs. (46) and (48) relate the linear response $\epsilon(k,t)$ of a system to the driving force $f(t)$ to the second-order statistics of the unperturbed system. This second-order statistics is given in terms of the correlation functions $\left\langle e^{-ikq(t)} q(t') \right\rangle_{0,\mathrm{st}}$.

2.4 Example: Kuramoto-Shinomoto-Sakaguchi model with inertia

From Eq. (48) we read off that in general the evolution of the Fourier mode $\epsilon(k,t)$ depends on the evolution of all other Fourier modes $\epsilon(k',t)$. However, if V_{MF} has a discrete Fourier spectrum and, accordingly, $w(k)$ is finite only for particular k-values $k_1, k_2, \ldots$ then we can derive for the modes $\epsilon(k_i,t)$ a closed set of evolution equations. The modes $\epsilon(k,t)$ with $k \notin \{k_1, k_2, \ldots\}$ will then depend only on these fundamental modes $\epsilon(k_i,t)$. Let us illustrate this issue for the Kuramoto-Shinomoto-Sakaguchi model with inertia term.

A model that describes the collective behavior of phase oscillators was proposed by *Kuramoto*, *Shinomoto*, and *Sakaguchi* [62–64] and many others (for reviews see [65,66]). Accordingly, a single oscillator i is described by a phase $\phi_i(t) \in \Omega_\phi = [0, 2\pi]$ and the evolution of all oscillators is given by $\mathrm{d}\phi_i(t)/\mathrm{d}t = \Omega_i - \kappa N^{-1} \sum_k \sin[\phi_i(t) - \phi_k(t)] + \sqrt{Q}\,\Gamma(t)$ for $i = 1, \ldots, N$ in the limit $N \to \infty$. This limiting case implies that we can replace $N^{-1} \sum_k \sin[\phi_i(t) - \phi_k(t)$ by $\int_{\Omega_\phi} \sin[\phi_i(t) - \phi) P(\phi, t))$. Here, Ω_i is the natural oscillation frequency of the ith oscillator. Let us consider the case $\Omega_i = \Omega$. Then, the parameter Ω can be eliminated by studying the system in a moving frame. That is, the model can be simplified by means of a variable transformation yielding $\phi \to \phi - \Omega t$. In line with a study by *Acebron* et al. [67], we may study for $\Omega_i = \Omega$ the impacts of inertia terms using the model

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &= p \ ,\\ \frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -\kappa \int\limits_{\Omega_\phi} \sin[\phi(t)-\phi] P(\phi,t)\,\mathrm{d}\phi - \gamma p + \sqrt{Q}\,\Gamma(t)\end{aligned} \tag{49}$$

with $p \in \Omega_p = \mathbb{R}$. This evolution equation corresponds to the Vlasov-Fokker-Planck equation (33) for $H_0 = p^2/2$, $V_{\mathrm{MF}}(z) = -\kappa\cos(z) \Rightarrow w(z) = -\kappa\sin(z)$ and $q = \phi$. Taking an external driving force into account, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\phi(t) &= p\\ \frac{\mathrm{d}}{\mathrm{d}t}p(t) &= -\kappa \int\limits_{\Omega_\phi} \sin[\phi(t)-\phi] P(\phi,t)\,\mathrm{d}\phi + f(t) - \gamma p + \sqrt{Q}\,\Gamma(t) \ .\end{aligned} \tag{50}$$

Using $w(z) = -\kappa[e^{iz} - e^{-iz}]/[2i]$ and the corresponding Fourier transform $w(k) = -\kappa[\delta(k-1) - \delta(k+1)]/[2i]$, from Eq. (46) the evolution equations for for $\epsilon(k=1,t)$ and $\epsilon(k=-1,t)$ can be found:

$$\begin{aligned}&\epsilon(k=\pm 1,t) = \\ &-\frac{\gamma}{2\pi Q}\int\limits_{t_0}^{t}\left[f(t')\frac{\partial}{\partial t}\left\langle e^{\mp i\phi(t)}\phi(t')\right\rangle_{0,\mathrm{st}} \pm \frac{\pi\kappa}{i}\epsilon(k=\mp 1,t')\frac{\partial}{\partial t}\left\langle e^{\mp 2i\phi(t)}\phi(t')\right\rangle_{0,\mathrm{st}}\right]\mathrm{d}t' \ .\end{aligned} \tag{51}$$

Note that $\epsilon(1,t)$ is explicitly defined by

$$\epsilon(1,t) = \frac{1}{2\pi}\left[\int\limits_{\Omega_\phi}\cos(\phi)\,\epsilon(\phi,t)\,\mathrm{d}\phi - i\int\limits_{\Omega_\phi}\sin(\phi)\,\epsilon(\phi,t)\,\mathrm{d}\phi\right] \ . \tag{52}$$

Since in general $\epsilon(-k,t)$ is the conjugate complex of $\epsilon(k,t)$ and in particular we have $\epsilon(-1,t) = \epsilon^*(1,t)$, we can also write

$$\begin{aligned}&\epsilon(k=1,t) = \\ &-\frac{\gamma}{2\pi Q}\int\limits_{t_0}^{t}\left[f(t')\frac{\partial}{\partial t}\left\langle e^{-i\phi(t)}\phi(t')\right\rangle_{0,\mathrm{st}} + \frac{\pi\kappa}{i}\epsilon^*(k=1,t')\frac{\partial}{\partial t}\left\langle e^{-2i\phi(t)}\phi(t')\right\rangle_{0,\mathrm{st}}\right]\mathrm{d}t' \ .\end{aligned} \tag{53}$$

We see that Eq. (53) provides us with a closed description for the evolution of the modes $\epsilon(k=1,t)$ and $\epsilon(k=-1,t) = \epsilon^*(k=1,t)$. From Eq. (46) it further follows that all other modes $\epsilon(k,t)$ with $k \neq \pm 1$ are coupled to the modes $\epsilon(k=\pm 1,t)$.

3 Conclusions

We have generalized a fluctuation-dissipation theorem for ordinary linear Fokker-Planck equations to nonlinear Fokker-Planck equations of the Desai-Zwanzig type and Vlasov-Fokker-Planck equations. According to this theorem, the response of a dynamical mean field system does not only depend on the driving force that acts on the system but depends also on the history of the response. That is, while the original fluctuation-dissipation theorem can be cast into the form $R(t) = \int G(t-t')f(t')\,\mathrm{d}t'$, where $f(t)$ and $R(t)$ denote driving force and response, respectively, the fluctuation-dissipation theorem for dynamical mean field systems basically reads $R(t) = \int G(t-t')[f(t') + \kappa R(t')]\,\mathrm{d}t'$, where κ is a coupling constant. For small coupling parameters κ, however, this memory effect can be eliminated because in this case the integral equation $R(t) = \int G(t-t')[f(t')+\kappa R(t')]\,\mathrm{d}t'$ can be solved iteratively. In particular, from Eq. (17) we obtain

$$\langle X \rangle_\epsilon (t) = \\ -\frac{1}{Q}\int\limits_{t_0}^{t} \left[f(t') - \frac{\kappa}{Q}\int\limits_{t_0}^{t'} f(t'')\frac{\partial}{\partial t'}C_0(t'-t'')\,\mathrm{d}t' \right] \frac{\partial}{\partial t}C_0(t-t')\,\mathrm{d}t' + O(\kappa^2)\ . \quad (54)$$

We see that eliminating the memory effect implies that the response $\langle X \rangle_\epsilon (t)$ now depends not only on the driving force $f(t)$ but also on higher order terms like $f(t)f(t')$.

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