Stability analysis of stationary states of mean field models described by Fokker-Planck equations

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Abstract

So far, the stability of stationary solutions of mean field models described by Fokker-Planck equations has only been determined analytically in some special cases. Following two earlier studies (M. Shiino, J. Korean Phys. Soc. 40:2002:1037; T.D. Frank, Prog. Theor. Phys. Supp. 150, in press), we discuss a stability analysis for a large class of mean field models based on Fokker-Planck equations. To this end, we use linear stability analysis in addition to the well-known approaches by means of transcendent equations and Lyapunov's direct method. We demonstrate that all three methods yield consistent results for systems that exhibit free energy functionals (e.g., equilibrium systems). We show that the simple transcendent equation analysis fails for systems that do not exhibit free energy functionals (e.g., nonequilibrium systems) and show how to solve this problem by means of a more sophisticated transcendent equation analysis. Furthermore, we propose a norm for the perturbations of stationary states and illustrate some of our results by a model that exhibits a reentrant noise-induced phase transition.

PACS: 05.20.-y, 05.40.+j

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1 Introduction

Mean field models have been proven to capture at least two essential properties of spatially distributed systems: multistability of stationary states and the emergence of collective behavior. For this reason, they have found applications in various disciplines. For example, they have been used to describe ferromagnetism [1], synchronization [2–4], human motor behavior [5,6], and social behavior [7,8]. Mean field models can describe relaxation processes as well. To this end, evolution equations that are similar to Fokker-Planck equations but are nonlinear with respect to probability densities have frequently been used [4,9–30]. The fact that evolution equations of this kind can indeed describe relaxation processes has been shown by extending the H-theorem for linear Fokker-Planck equations [31,32] to the nonlinear case [10,26,33,34] The nonlinearities typically reflect mean field forces that are produced by the subsystems of a many-body system and impinge on individual subsystems. On account of these nonlinearities, Fokker-Planck equations of this kind can exhibit multiple stationary probability densities [9,35]. In order to discuss bifurcations (e.g., the emergence of a collective behavior or transitions between stationary states) one needs to distinguish between stable and unstable stationary probability densities. In this context, it has been illustrate that stationary distributions that correspond to minima of Lyapunov functionals describe stable stationary probability densities. Likewise, distributions corresponding to maxima or saddle points are unstable ones [10,26]. In sum, Lyapunov's direct method provides us with a powerful tool to analyze the stability of stationary solutions of nonlinear Fokker-Planck equations.

Our understanding of this approach, however, is still incomplete. First, the relationship between linear stability analysis and Lyapunov's direct method has not yet been thoroughly explored. For the special case of a mean field model with a Kuramoto coupling term it has been shown that both methods yield the same result [28]. However, in this study no attempts have been made to pin down the reason for the observed equivalence. Second, until now, Lyapunov's direct method has been developed for two special cases of nonlinear Fokker-Planck equations: the Desai-Zwanzig model [10] and a mean field model proposed by *Kuramoto* and others [26]. Both models involve subsystem-subsystem interaction energies that can be expressed in terms of mean field forces of the form $\langle U_{\mathrm{MF}}(X,X')\rangle_{X,X'}/2$, where the averaging is carried out with respect to X and X'. The functional $\langle U_{\mathrm{MF}}(X,X')\rangle_{X,X'}/2$ is characterized by the fact that its third variational derivative vanishes. It might be worth while to investigate a larger class of nonlinear Fokker-Planck equations involving mean field energy functionals with nonvanishing third variational derivatives. In particular, one may study Fokker-Planck equations that describe systems with mean field energies proportional to $\langle [X^n - \langle X^n\rangle]^2\rangle$. In this case, we deal with drift forces of the form $x^{n-1}\langle X^n\rangle$ (see below). Drift forces of this kind play an important role in the theory of first order phase transitions [15] and may be used in social sciences to describe group behavior [7,8]. Third, the stability of stationary distributions of nonlinear Fokker-Planck equations has also been determined by means of transcendent equations. The relationship between linear stability analysis and Lyapunov's direct method, on the one hand, and transcendent equation analysis, on the other hand, has not yet been explored in detail. In particular, the question arises whether or not transcendent equation analysis can be applied to nonequilibrium systems for which free energy functionals can not be found. Finally, only little attention has been payed to apply Lyapunov's direct method and linear stability analysis to mean field models that involve multiplicative noise sources. This is in contrast to the wide interest in systems of this kind [12,36,37].

This paper is organized as follows. In Sec. 2 we will discuss systems with free energy measures such as equilibrium systems. We will distinguish between systems with additive (Sec. 2.1) and multiplicative (Sec. 2.2) noise sources.

In Sec. 3 we will consider systems for which free energy functionals can not be found or for which the existence of free energy functionals is not obvious (e.g. nonequilibrium systems). Here, we will treat the case of additive and multiplicative noise sources simultaneously. In this context we will discuss a noise-induced reentrant phase transition for which rigorous prove can be given that transcendent equation analysis can be applied.

2 Free energy case

2.1 Additive noise

We consider stochastic processes in M-dimensional phase spaces Ω that are described by a vector $\mathbf{x} = (x_1, \ldots, x_M)$. Let $P(\mathbf{x}, t; u)$ denote the probability density of such a stochastic process, where t is time and $u(\mathbf{x})$ describes the initial distribution of the process given at $t = t_0$: $P(\mathbf{x}, t_0; u) = u(\mathbf{x})$. We assume that $P(\mathbf{x}, t; u)$ satisfies natural boundary conditions or periodic boundary conditions. In the former case we have $\mathbf{x} \in \Omega = \mathbb{R}^M$. In the latter case we have $\mathbf{x} \in \prod_{i=1}^{M} [a_i, b_i]$ with $b_i - a_i = T_i > 0$ and $P(\ldots, x_i + T_i, \ldots) = P(\ldots, x_i, \ldots)$. We assume that the free energy of the system under consideration is given by

$$F[P] = \int_\Omega U_0(\mathbf{x}) P(\mathbf{x}) \, \mathrm{d}^M x + U_{\mathrm{NL}}[P] - Q S[P] \; , \tag{1}$$

where S denotes the Boltzmann-Gibbs-Shannon entropy $S[P] = -\int P \ln P \, \mathrm{d}^M x$ [1] and $U_{\mathrm{NL}}[P]$ is bounded from below like $U_{\mathrm{NL}}[P] \geq U_{\mathrm{NL,min}}$. We further assume that the evolution of P is given by the multivariate Fokker-Planck equation

$$\frac{\partial}{\partial t} P(\mathbf{x}, t; u) = \nabla \cdot \left\{ P \, \nabla \frac{\delta F}{\delta P} \right\} \tag{2}$$

for $t \geq t_0$ and $Q > 0$, which can equivalently be expressed as

$$\frac{\partial}{\partial t} P(\mathbf{x},t;u) = \nabla \cdot \left[\left\{ \nabla U_0(\mathbf{x}) + \nabla \frac{\delta U_{\mathrm{NL}}[P]}{\delta P} \right\} P(\mathbf{x},t;u) \right] + Q\, \Delta P(\mathbf{x},t;u) \tag{3}$$

with $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_M)$ and $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_M^2$. Stationary solutions of Eq. (2) can be obtained from

$$\frac{\delta F[P_{\mathrm{st}}]}{\delta P} = \mu \ , \tag{4}$$

where μ may be regarded as a chemical potential for systems with mass equal to one [1]. Note that Eq. (4) can be written in form of the implicit equation

$$P_{\mathrm{st}}(\mathbf{x}) = \frac{1}{Z} \exp\left\{ -\frac{U_0(\mathbf{x}) + \delta U_{\mathrm{NL}}[P_{\mathrm{st}}]/\delta P}{Q} \right\} \ , \tag{5}$$

where Z denotes a normalization constant given by $\ln Z = 1 - \mu/Q$. Now, let us briefly review the H-theorem and the stability analysis by Lyapunov's direct method for nonlinear free energy Fokker-Planck equations [10,26,38,39].

2.1.1 H-theorem for free energy Fokker-Planck equations

In line with previous work [10,28,40], one can show that F is bounded from below if the Boltzmann distribution $W(\mathbf{x}) \propto \exp\{-U_0(\mathbf{x})/Q\}$ exists (see Appendix A). Using partial integration, from Eq. (2) we obtain the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} F = -\int_{\Omega} P \left[\nabla \frac{\delta F}{\delta P} \right]^2 \mathrm{d}^M x \leq 0 \ . \tag{6}$$

Since, by definition, F does not depend explicitly on time, the implication $\partial P/\partial t = 0 \Rightarrow \mathrm{d}F/\mathrm{d}t = 0$ holds. From Eqs. (2) and (6) we conclude that

[1] In the case of periodic boundary conditions, we assume that the potential U_0 is a $\mathbf{T}$-periodic function $U_0(\ldots, x_i + T_i, \ldots) = U_0(\ldots, x_i, \ldots)$. Likewise, we require that $\delta U_{\mathrm{NL}}[P]/\delta P$ as a function of $\mathbf{x}$ satisfies $\delta U_{\mathrm{NL}}[P]/\delta P(\ldots, x_i + T_i, \ldots) = \delta U_{\mathrm{NL}}[P]/\delta P(\ldots, x_i, \ldots)$.

$\mathrm{d}F/\mathrm{d}t = 0 \Rightarrow \delta F/\delta P = \text{constant} \Rightarrow \partial P/\partial t = 0$. In sum, F satisfies the properties of a Lyapunov functional, which are

$$F \geq F_{\min} \; , \quad \frac{\mathrm{d}}{\mathrm{d}t} F \leq 0 \; , \quad \frac{\mathrm{d}}{\mathrm{d}t} F = 0 \Leftrightarrow \frac{\partial}{\partial t} P = 0 \; . \tag{7}$$

From Eq. (7) we further conclude that the limiting case $\lim_{t\to\infty} \partial P/\partial t = 0$ holds. That is, every transient solution converges to a stationary one in the long time limit (H-theorem [32]).

2.1.2 Stability analysis by means of Lyapunov's direct method

By means of Lyapunov's direct method, we can also analyze the stability of stationary probability densities [10,26,38]. If $\delta^2 F[P_{\mathrm{st}}](\epsilon) > 0$ for all $\epsilon \neq 0$ then P_{st} corresponds to a minimum of F. This implies that for small ϵ there does not exist a probability density $P = P_{\mathrm{st}} + \epsilon$ that corresponds to another stationary point of F and describes a further stationary solution of the Fokker-Planck equation. Since from Eq. (7) we have $\partial P/\partial t \neq 0 \Rightarrow \mathrm{d}F/\mathrm{d}t < 0$ we conclude that every perturbation $P = P_{\mathrm{st}} + \epsilon$ evolves in such a way that the inequality $\mathrm{d}F/\mathrm{d}t < 0$ is satisfied as long as $\epsilon \neq 0$. Since $P = P_{\mathrm{st}} + \epsilon$ is located in a neighborhood of a minimum of F, the time-dependent solution P can not leave this neighborhood (because this would imply an increase of F). The only behavior of the perturbation $P = P_{\mathrm{st}} + \epsilon$ that is consistent with the constraint $\mathrm{d}F/\mathrm{d}t < 0$ for $\epsilon \neq 0$ and the fact the P is located (in the function space of probability densities) in a neighborhood of a minimum of F is the relaxation to the unperturbed state described by P_{st}. Therefore, we get

$$\forall \epsilon \neq 0 : \delta^2 F[P_{\mathrm{st}}](\epsilon) > 0 \Rightarrow P_{\mathrm{st}} = \text{ asymptotically stable } . \tag{8}$$

Let P_{st} correspond to a maximum or a saddle point of F such that there is at least one small perturbation ϵ^* which yields $\delta^2 F[P_{\mathrm{st}}](\epsilon^*) < 0$. Then, the inequality $F[P_{\mathrm{st}} + \epsilon^*] < F[P_{\mathrm{st}}]$ holds and the time-dependent probability density $P(\mathbf{x},t;u)$ with $u = P_{\rm st} + \epsilon^*$ at $t = t_0$ can not return to $P_{\rm st}(\mathbf{x})$ for any $t \geq t_0$ (because F can not increase and $F(t_0) < F[P_{\rm st}]$). In this context, it is usually assumed that a perturbation in the direction of ϵ^* increases with time in the sense that the deviation between P and $P_{\rm st}$ increases with time. Consequently, we obtain:

$$\exists \epsilon^* : \delta^2 F[P_{\rm st}](\epsilon^*) < 0 \Rightarrow P_{\rm st} = \text{unstable} \ . \tag{9}$$

In sum, the sign of the second variation of the free energy F determines the stability of stationary distributions. For F given by Eq. (1) the second variation reads

$$\delta^2 F[P_{\rm st}](\epsilon) = \int_\Omega \int_\Omega \frac{\delta^2 U_{\rm NL}[P_{\rm st}]}{\delta P(\mathbf{x}) \delta P(\mathbf{y})} \, \epsilon(\mathbf{x}) \, \epsilon(\mathbf{y}) \, \mathrm{d}^M x \, \mathrm{d}^M y + Q \int_\Omega \frac{\epsilon^2}{P_{\rm st}} \, \mathrm{d}^M x \ . \tag{10}$$

2.1.3 Linear stability analysis

Let us investigate now the stability of stationary solutions of Eq. (3) by means of linear stability analysis. First, we require that $\delta^2 U_{\rm NL}/\delta P(\mathbf{x}) \delta P(\mathbf{y}) = \chi'(\mathbf{x},\mathbf{y})$ satisfies the symmetry relation $\chi'(\mathbf{x},\mathbf{y}) = \chi'(\mathbf{y},\mathbf{x})$ which implies that the symmetry relation $\chi(\mathbf{x},\mathbf{y}) = \chi(\mathbf{y},\mathbf{x})$ holds for $\delta^2 F/\delta P(\mathbf{x}) \delta P(\mathbf{y}) = \chi(\mathbf{x},\mathbf{y})$. Next, we linearize Eq. (2) at $P_{\rm st}$. That is, we put $P(\mathbf{x},t;u) = P_{\rm st}(\mathbf{x}) + \epsilon(\mathbf{x},t)$. Using Eq. (4) and taking only ϵ-terms of first order into account, we obtain

$$\frac{\partial}{\partial t} \epsilon(\mathbf{x},t) = \nabla \cdot \left\{ P_{\rm st} \nabla \int_\Omega \frac{\delta^2 F[P_{\rm st}]}{\delta P(\mathbf{x}) \delta P(\mathbf{y})} \, \epsilon(\mathbf{y},t) \, \mathrm{d}^M y \right\} \ . \tag{11}$$

Let us consider the functional L defined by the second variation of F like

$$L[P, P_{\rm st}] = \frac{1}{2} \delta^2 F[P_{\rm st}](\epsilon) \tag{12}$$

with $\epsilon = P - P_{\rm st}$. Differentiating L with respect to t, exploiting the symmetry of $\delta^2 F/\delta P(\mathbf{x}) \delta P(\mathbf{y})$, and integrating by parts, from Eq. (11) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}L = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\delta^2 F[P_{\mathrm{st}}](\epsilon) = -\int_\Omega P_{\mathrm{st}} \left[\nabla \int_\Omega \frac{\delta^2 F[P_{\mathrm{st}}]}{\delta P(\mathbf{x})\delta P(\mathbf{y})}\,\epsilon(\mathbf{y},t)\,\mathrm{d}^M y\right]^2 \mathrm{d}^M x \leq 0 \ . \tag{13}$$

It is clear from this result that $\mathrm{d}L/\mathrm{d}t = 0$ implies

$$\int_\Omega \frac{\delta^2 F[P_{\mathrm{st}}]}{\delta P(\mathbf{x})\delta P(\mathbf{y})}\,\epsilon(\mathbf{y},t)\,\mathrm{d}^M y = C \ , \tag{14}$$

where C is a constant. Multiplying Eq. (14) by $\epsilon(\mathbf{x},t)$, integrating the result with respect to $\mathbf{x}$, and taking the normalization constraint $\int_\Omega \epsilon(\mathbf{x},t)\,\mathrm{d}^M x = 0$ into account, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}L = 0 \Rightarrow \delta^2 F[P_{\mathrm{st}}](\epsilon) = 0 \ . \tag{15}$$

In sum, using linear stability analysis, two fundamental results can be found: the inequality (13) and the implication (15). These results can now be used to analysis the stability of stationary probability densities related to stationary (or critical) points of F.

Minima

Let us consider a distribution P_{st} with $\delta^2 F[P_{\mathrm{st}}](\epsilon) > 0$ for all $\epsilon \neq 0$ and $\delta^2 F[P_{\mathrm{st}}](\epsilon) = 0 \Leftrightarrow \epsilon = 0$. Then, from Eqs. (12), (13), and (15) it follows that

$$L \geq 0 \ , \quad \frac{\mathrm{d}}{\mathrm{d}t}L \leq 0 \ , \quad \frac{\mathrm{d}}{\mathrm{d}t}L = 0 \Leftrightarrow \epsilon = 0 \ . \tag{16}$$

Consequently, the limiting cases $\lim_{t\to\infty} \epsilon(\mathbf{x},t) = 0$ and $\lim_{t\to\infty} P(\mathbf{x},t;u) = P_{\mathrm{st}}$ hold for all initial distributions $u \approx P_{\mathrm{st}}$. In other words, by means of linear stability analysis related to the second variation of free energies we reobtain the proposition (8). By means of the second variation $\delta^2 F$, we can determine the stability of stationary distributions in a way consistent with the stability theory for deterministic systems. To this end, we define the norm $||\cdot||$ for positive definite $\delta^2 F$ and functions $\epsilon(\mathbf{x}) \in C^\infty(\Omega)$ with $\int_\Omega \epsilon(\mathbf{x})\,\mathrm{d}^M x = 0$ by

$$||\epsilon|| = \sqrt{\delta^2 F[P_{\rm st}](\epsilon)}\ , \tag{17}$$

see Appendix B. Using $2L(\epsilon) = \delta^2 F[P_{\rm st}](\epsilon) = ||\epsilon||^2 \Rightarrow \mathrm{d}L/\mathrm{d}t = ||\epsilon||\,\mathrm{d}\,||\epsilon||\,/\mathrm{d}t$ and Eq. (16) we obtain

$$\epsilon \neq 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\,||\epsilon|| < 0 \tag{18}$$

for $P_{\rm st}$ with positive definite $\delta^2 F$. Eq. (18) tells us that the norm of every small perturbation of $P_{\rm st}$ decreases as a function of time which means once again that $P_{\rm st}$ is asymptotically stable.

Saddle points and maxima

If there is a ϵ^* such that $\delta^2 F[P_{\rm st}](\epsilon^*) < 0$ then we have $L(t_0) < 0$ for $P(\mathbf{x}, t_0, u = P_{\rm st} + \epsilon^*)$. From Eq. (15) it follows that $\delta^2 F[P_{\rm st}](\epsilon^*) < 0 \Rightarrow \mathrm{d}L/\mathrm{d}t \neq 0$. Finally, from Eq. (13) we obtain $\mathrm{d}L/\mathrm{d}t < 0$ for the perturbation $P(\mathbf{x}, t_0, u = P_{\rm st} + \epsilon^*)$. That is, $|L|$ increases as a function of time for the perturbation related to a saddle point or a maximum of F. The increase of $|L|$ indicates that the deviation between P and $P_{\rm st}$ increase with time and $P_{\rm st}$ corresponds to an unstable distribution. Consequently, linear stability analysis gives us the proposition (9).

Maxima

If $P_{\rm st}$ describes a maximum of F with $\delta^2 F[P_{\rm st}](\epsilon) < 0$ for all $\epsilon \neq 0$ we can introduce a norm defined by

$$||\epsilon|| = \sqrt{-\,\delta^2 F[P_{\rm st}](\epsilon)}\ , \tag{19}$$

see Appendix B. Using $-2L(\epsilon) = -\delta^2 F[P_{\rm st}](\epsilon) = ||\epsilon||^2 \Rightarrow -{\rm d}L/{\rm d}t = ||\epsilon||\,{\rm d}\,||\epsilon||\,/{\rm d}t$ and Eq. (16) we obtain

$$\epsilon \neq 0 \Rightarrow \frac{\rm d}{{\rm d}t}\,||\epsilon|| > 0 \tag{20}$$

for $P_{\rm st}$ with negative definite $\delta^2 F$. Eq. (20) tells us that the norm of every small perturbation of $P_{\rm st}$ increases as a function of time which means that $P_{\rm st}$ is unstable. In sum, we realize that irrespective of the form of the functionals that describe internal energies, Lyapunov's direct method and linear stability analysis yield consistent results. From both methods it follows that stationary probability densities are stable if they correspond to free energy minima with positive definite second variations. In contrast, if there is a perturbation of a stationary probability distribution that involves a decrease of the free energy (1) then the stationary distribution is an unstable one.

2.1.4 Transcendent equation analysis

We confine ourselves to discuss the univariate case. We consider systems with nonlinear energy functionals of the form

$$U_{\rm NL}[P] = \langle B_0(x)\rangle + B(\langle A\rangle) \geq B_{\rm min} \tag{21}$$

and free energy measures given by

$$F[P] = \int_\Omega V(x)P(x)\,{\rm d}x + \int_\Omega B_0(x)P(x)\,{\rm d}x + B(\langle A\rangle) - QS[P]\ , \tag{22}$$

where $A(x)$ denotes an arbitrary function. From Eq. (21) we read off that the nonlinearity $B(\langle A\rangle)$ is balanced by a potential B_0 such that $\int_\Omega B_0 P {\rm d}x + B(\langle A\rangle)$ is bounded from below. Consequently, we do not require that $B(z)$ itself is bounded from below. In the univariate case, the free energy Fokker-Planck equation (2) reads

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}P\frac{\partial}{\partial x}\frac{\delta F}{\delta P} \ . \tag{23}$$

For F given by Eq. (22) we obtain

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{\mathrm{d}V(x)}{\mathrm{d}x} + \frac{\mathrm{d}B_0(x)}{\mathrm{d}x} + \frac{\mathrm{d}A(x)}{\mathrm{d}x}\left.\frac{\mathrm{d}B(z)}{\mathrm{d}z}\right|_{z=\langle A\rangle}\right]P + Q\frac{\partial^2}{\partial x^2}P \ . \tag{24}$$

By means of the H-theorem derived in Sec. 2.1.1, we can then conclude that the limiting case $\lim_{t\to\infty}\partial P/\partial t = 0$ holds. Using $\delta U_{\mathrm{NL}}[P]/\delta P = B_0(x) + A(x)\mathrm{d}B(\langle A\rangle)/\mathrm{d}z$, Eq. (5) becomes

$$P_{\mathrm{st}}(x) = \frac{1}{Z}\exp\left\{-\frac{V(x) + B_0 + A(x)\mathrm{d}B(\langle A\rangle_{\mathrm{st}})/\mathrm{d}z}{Q}\right\} \ . \tag{25}$$

Let us introduce the order parameter $m = \langle A\rangle_{\mathrm{st}}$ and the functions

$$\begin{aligned} P(x;m) &= \frac{1}{Z(m)}\exp\left\{-\frac{V(x) + B_0(x) + A(x)\mathrm{d}B(m)/\mathrm{d}m}{Q}\right\} \ , \\ R(m) &= \int_\Omega A(x)P(x;m)\,\mathrm{d}x \ , \end{aligned} \tag{26}$$

where $Z(m)$ is the normalization constant of $P(x;m)$. Then, $\langle A\rangle_{\mathrm{st}}$ is given by the solutions of the transcendent equation

$$m = R(m) \ . \tag{27}$$

Not only can we derive from Eq. (27) the order parameter m and thus the explicit form of stationary solutions but we can also determine the stability of stationary solutions. To this end, we examine the intersection points between the function $y_1(m) = R(m)$ and the diagonal $y_2(m) = m$. Inspired by the theory of iterative maps [41], for systems with monotonically increasing $R(m)$, it is assumed that we deal with a stable stationary solution if the slope of $R(m)$

at an intersection point is smaller than one. Likewise, we deal with an unstable stationary solution if the slope of $R(m)$ at $\langle X\rangle_{\rm st}$ is larger than one:

$$\begin{aligned}
&\frac{\mathrm{d}R}{\mathrm{d}m} < 1 \Rightarrow P_{\rm st}(x;m) = \text{asymptotically stable} \;, \\
&\frac{\mathrm{d}R}{\mathrm{d}m} > 1 \Rightarrow P_{\rm st}(x;m) = \text{unstable} \;.
\end{aligned} \tag{28}$$

The slope of $R(m)$ can be computed from Eq. (26):

$$\left.\frac{\mathrm{d}R(m)}{\mathrm{d}m}\right|_{\langle A\rangle_{\rm st}} = -\frac{1}{Q} K_{A,\rm st}(X) \left.\frac{\mathrm{d}^2B(z)}{\mathrm{d}z^2}\right|_{\langle A\rangle_{\rm st}} \;, \tag{29}$$

where K_A denotes the generalized variance defined by $K_A(X) = \langle A(X)^2\rangle - \langle A(X)\rangle^2 = \langle [A - \langle A\rangle]^2\rangle \geq 0$. By means of the stability coefficient

$$\tilde{\lambda} = Q\left(1 - \frac{\mathrm{d}R}{\mathrm{d}m}\right), \tag{30}$$

we conclude that for $\tilde{\lambda} = Q + K_{A,\rm st}\mathrm{d}^2B/\mathrm{d}z^2 > 0$ (< 0) the stationary distributions (25) are asymptotically stable (unstable).

Let us determine now the stability of the distributions (25) by means of Lyapunov's direct method. Since the second variational derivative of $U_{\rm NL}$ reads $\delta^2 U_{\rm NL}[P]/\delta P(x)\delta P(y) = A(x)A(y)\mathrm{d}^2B(\langle A\rangle)/\mathrm{d}z^2$, Eq. (10) can be found as

$$\delta^2 F[P_{\rm st}](\epsilon) = \left.\frac{\mathrm{d}^2B(z)}{\mathrm{d}z^2}\right|_{\langle A\rangle_{\rm st}} \left[\int_\Omega A(x)\,\epsilon(x)\,\mathrm{d}x\right]^2 + Q\int_\Omega \frac{\epsilon^2(x)}{P_{\rm st}(x)}\,\mathrm{d}x \;. \tag{31}$$

Eq. (31) can be evaluated as proposed by *Shiino* [10,38] using

$$\epsilon(x) = \beta\left[A(x) - \langle A(X)\rangle_{\rm st}\right] P_{\rm st}(x) + \chi_\perp(x)\sqrt{P_{\rm st}(x)} \;, \tag{32}$$

where $\chi_\perp$ satisfies the orthogonality relations $\int_\Omega \chi_\perp(x)\sqrt{P_{\rm st}(x)}\,\mathrm{d}x = 0$ and $\int_\Omega A(x)\chi_\perp(x)\sqrt{P_{\rm st}(x)}\,\mathrm{d}x = 0$. Note that analogous to the cases discussed in [10,26,38] we can show that the representation (32) accounts for all possible perturbations ϵ of a stationary probability density. Substituting Eq. (32) into Eq. (31), one can determine the sign of $\delta^2 F$ from

$$\delta^2 F[P_{\rm st}](\epsilon) = \beta^2 K_{A,{\rm st}}(X) \left[Q + \left. \frac{{\rm d}^2 B(z)}{{\rm d}z^2} \right|_{\langle A \rangle_{\rm st}} K_{A,{\rm st}}(X) \right] + Q \int_\Omega [\chi_\perp]^2 \, {\rm d}x \ . \tag{33}$$

For $\tilde{\lambda} = Q + {\rm d}^2 B(m)/{\rm d}m^2 K_{A,{\rm st}}(X) > 0$ with $m = \langle A \rangle_{\rm st}$ we obtain $\delta^2 F > 0$ indicating that the stationary distribution being studied is a stable one. For $\tilde{\lambda} = Q + {\rm d}^2 B(m)/{\rm d}m^2 K_{A,{\rm st}}(X) < 0$ there is a perturbation that yields $\delta^2 F < 0$ which tells us that we deal with an unstable stationary distribution. The critical parameter value of Q can be computed from $\tilde{\lambda} = 0 \Rightarrow Q + {\rm d}^2 B(m)/{\rm d}m^2 K_{A,{\rm st}}(X; Q) = 0$ with $m = \langle A \rangle_{\rm st}$. Consequently, for stochastic processes described by Eqs. (22) and (24) transcendent equation analysis is consistent with Lyapunov's direct method and linear stability analysis. Further special cases in which we can prove the validity of the transcendent equation analysis by means of Lyapunov's direct method are listed in Table 1 and are addressed in Appendix C. In this context the reader is also refereed to [38].

Insert Table 1 about here.

2.2 Multiplicative noise

We consider now systems subjected to multiplicative noise. Our objective is to analyze multiplicative noise systems by means of the methods developed in the previous section. Therefore, we confine ourselves to systems with state-dependent diffusion coefficients and mean field forces that satisfy a particular matching condition. Let us illustrate this issue for the univariate case. To this end, we generalize the Fokker-Planck equation (23) by introducing a state-dependent mobility coefficient $M > 0$ like

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x} M(x) P \frac{\partial}{\partial x} \frac{\delta F}{\delta P} \ . \tag{34}$$

We can proceed as in Sec. 2.1 and show that stationary solutions can be computed from $\delta F/\delta P = \mu$ and the relations $\mathrm{d}F/\mathrm{d}t \leq 0$ and $\mathrm{d}F/\mathrm{d}t = 0 \Leftrightarrow \partial P/\partial t = 0$ hold (see also [39]). Using Lyapunov's direct method, we conclude that stationary distributions P_{st} are asymptotically stable if the inequality $\forall \epsilon \neq 0 : \delta^2 F[P_{\mathrm{st}}](\epsilon) > 0$ holds. If there is at least one perturbation ϵ^* that yields $\delta^2 F[P_{\mathrm{st}}](\epsilon^*) < 0$ then we deal with an unstable stationary distribution. From Eq. (34) it follows that perturbations of stationary distributions satisfy the evolution equation

$$\frac{\partial}{\partial t}\epsilon(x,t) = \frac{\partial}{\partial x} M(x) P_{\mathrm{st}} \frac{\partial}{\partial x} \int\limits_{\Omega} \frac{\delta^2 F}{\delta P(x) \delta P(y)} \epsilon(y,t)\, \mathrm{d}y \ . \tag{35}$$

Just as for systems with $M = 1$, we can then show that if we deal with a free energy minimum distribution P_{st} with $\forall \epsilon \neq 0 : \delta^2 F[P_{\mathrm{st}}](\epsilon) > 0$ then $||\epsilon|| = \sqrt{\delta^2 F[P_{\mathrm{st}}](\epsilon)}$ decreases as a function of time and P_{st} is asymptotically stable. If there is a perturbation ϵ^* with $\delta^2 F[P_{\mathrm{st}}](\epsilon^*) < 0$ then we have $\mathrm{d}|L|/\mathrm{d}t > 0$ for $L = \delta^2 F[P_{\mathrm{st}}](\epsilon^*)$ which indicates that P_{st} describes an unstable stationary distribution. In particular, for free energy maximum distributions with $\forall \epsilon \neq 0 : \delta^2 F[P_{\mathrm{st}}](\epsilon) < 0$ one finds that $||\epsilon|| = \sqrt{-\delta^2 F[P_{\mathrm{st}}](\epsilon)}$ increases as a function of time which means that distributions of this kind describe unstable distributions.

It is clear that due to the state-dependency of M we deal with stochastic processes with multiplicative noise. However, M also occurs in the drift term. Let us illustrate this issue for the free energy measure $F = \langle V(X) \rangle + \kappa K_A(X)/2 - QS$ (see Table 1, second row) for which Eq. (34) becomes

$$\frac{\partial}{\partial t} P(x,t;u) =$$
$$\frac{\partial}{\partial x} M(x) \left[\frac{\mathrm{d}V(x)}{\mathrm{d}x} + \kappa \left(A(x) - \frac{\mathrm{d}A(x)}{\mathrm{d}x} \langle A(X) \rangle_P \right) \right] P + Q \frac{\partial}{\partial x} M(x) \frac{\partial}{\partial x} P. \tag{36}$$

Using the transformation $(M, A, V) \rightarrow (D, A', V')$ given by

$$\frac{\mathrm{d}V'}{\mathrm{d}x} = M\frac{\mathrm{d}V}{\mathrm{d}x} - Q\frac{\mathrm{d}M}{\mathrm{d}x} + \kappa M A \ ,$$
$$D(x) = QM(x) \ ,$$
$$A'(x) = \sqrt{\frac{\kappa}{Q}} A(x) \tag{37}$$

we can transform Eq. (36) into

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{\mathrm{d}V'(x)}{\mathrm{d}x} - \underbrace{D(x)\frac{\mathrm{d}A'(x)}{\mathrm{d}x}\left\langle A'(X)\right\rangle_P}_{A_{\mathrm{eff}}(x,P)}\right] P + \frac{\partial^2}{\partial x^2} D(x) P \ . \tag{38}$$

Conversely, every mean field Fokker-Planck equation of the form (38) can be transformed into Eq. (36) (via the backwards transformation $(D', A', V') \rightarrow (M, A, V)$: $\mathrm{d}V/\mathrm{d}x = QD^{-1}[\mathrm{d}V'/\mathrm{d}x + \mathrm{d}D/\mathrm{d}x] - \sqrt{\kappa Q} A'$, $M(x) = D(x)/Q$, $A(x) = \sqrt{Q/\kappa} A'(x)$) and, subsequently, the stability of stationary distributions of Eq. (38) can be determined using the free energy approach. In other words, from Eq. (38) it follows that multiplicative noise systems can be investigated by means of the free energy approach if the effective mean field force $A_{\mathrm{eff}}(x, P)$ can be decomposed into $A_{\mathrm{eff}}(x, P) = D(x)\left\langle A'\right\rangle \mathrm{d}A'(x)/\mathrm{d}x$, that is, if the diffusion coefficient $D(x)$ and $A_{\mathrm{eff}}(x, P)$ satisfy a matching condition.

3 General case

3.1 Failure of the simple transcendent equation analysis

We study next systems for which free energy measures do not exist or for which it is not obvious whether or not free energy measures can be derived. Our departure point is the nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{\mathrm{d}V(x)}{\mathrm{d}x} + \kappa\left(A(x) - \left\langle A(X)\right\rangle_P\right)\right] P + \frac{\partial^2}{\partial x^2} D(x) P \tag{39}$$

with $\kappa > 0$ and $D > 0$. If $D(x)$ depends on x we deal with multiplicative noise. For $D(x) = Q > 0$ we deal with additive noise. By means of $h(x) = -\mathrm{d}V/\mathrm{d}x$, stationary solutions of Eq. (39) can be written as

$$P_{\mathrm{st}}(x) = \frac{1}{ZD(x)} \exp\left\{ \int^{x} \frac{h(x') - \kappa A(x')}{D(x')} \,\mathrm{d}x' + \kappa \left\langle A \right\rangle_{\mathrm{st}} \int^{x} \frac{1}{D(x')} \mathrm{d}x' \right\} , \quad (40)$$

where Z denotes a normalization constant [2]. By virtue of

$$P(x,m) = \frac{1}{Z(m)D(x)} \exp\left\{ \int^{x} \frac{h(x') - \kappa[A(x') - m]}{D(x')} \,\mathrm{d}x' \right\} ,$$
$$R(m) = \int_{\Omega} A(x) P(x;m) \,\mathrm{d}x , \quad (41)$$

we can define the transcendent equation $m = R(m)$ whose solutions correspond to the expectation values $\langle A_{\mathrm{st}} \rangle$ occurring in Eq. (40). Differentiation of $R(m)$ with respect to m gives us

$$\left. \frac{\mathrm{d}R(m)}{\mathrm{d}m} \right|_{\langle A \rangle_{\mathrm{st}}} = \kappa C_{\mathrm{st}} \left(A(x), \int^{x} \frac{1}{D(x')} \,\mathrm{d}x' \right) , \quad (42)$$

where $C_{\mathrm{st}}(f,g) = \langle [f(X) - \langle f(X) \rangle_{\mathrm{st}}][g(X) - \langle g(X) \rangle_{\mathrm{st}}] \rangle_{\mathrm{st}}$ describes the cross correlation coefficient of the functions f and g. Frequently, the hypothesis is made that for systems described by Eq. (39) the same relationship between the slope $\mathrm{d}R/\mathrm{d}m$ and the stability of stationary distributions holds as for free energy systems. Let us investigate the validity of this hypothesis.

Substituting $P = P_{\mathrm{st}} + \epsilon$ into Eq. (39), the evolution equation for the perturbation ϵ is found as

$$\frac{\partial}{\partial t} \epsilon(x,t) = -\frac{\partial}{\partial x} \left[h(x) - \kappa \left(A(x) - \langle A \rangle_{\mathrm{st}} \right) \right] \epsilon + \frac{\partial^2}{\partial x^2} D(x) \, \epsilon$$

[2] This result holds both for natural and periodic boundary conditions. In the latter case, however, we need to require that h, A, D are defined in such as way that $P(x+T) = P(x)$ holds, where T is the period of the processes under consideration.

$$-\kappa \int_\Omega A(x')\epsilon(x')\,\mathrm{d}x' \frac{\mathrm{d}P_{\mathrm{st}}}{\mathrm{d}x} \; . \qquad (43)$$

Using Eq. (40), we can transform Eq. (43) into

$$\frac{\partial}{\partial t}\epsilon(x,t) = \frac{\partial}{\partial x}\left[D(x)P_{\mathrm{st}}\frac{\partial}{\partial x}\frac{\epsilon}{P_{\mathrm{st}}}\right] - \kappa\int_\Omega A(x')\epsilon(x',t)\,\mathrm{d}x'\frac{\mathrm{d}P_{\mathrm{st}}}{\mathrm{d}x} \; . \qquad (44)$$

Eq. (44) can be evaluated for perturbations described by

$$\epsilon(x,t) = \beta(t)\left[\int^x \frac{1}{D(x')}\,\mathrm{d}x' - \left\langle \int^X \frac{1}{D(x')}\,\mathrm{d}x'\right\rangle_{\mathrm{st}}\right] P_{\mathrm{st}}(x) \; . \qquad (45)$$

The integral $\int_\Omega A(x)\epsilon(x,t)\,\mathrm{d}x$ can be expressed in terms of the cross correlation coefficient C_{st}: $\int_\Omega A(x)\epsilon(x,t)\,\mathrm{d}x = \beta(t)C_{\mathrm{st}}(A, \int^x D^{-1}(x')\,\mathrm{d}x')$. Therefore, if we substitute Eq. (45) into Eq. (44), multiply with $A(x)$ and integrate with respect to x, we obtain $\mathrm{d}\beta(t)/\mathrm{d}t\, C_{\mathrm{st}} = \beta[1-\kappa C_{\mathrm{st}}]\int \mathrm{d}P_{\mathrm{st}}(x)/\mathrm{d}x A(x)\,\mathrm{d}x$. Using $\int \mathrm{d}P_{\mathrm{st}}(x)/\mathrm{d}x A(x)\,\mathrm{d}x = -\langle \mathrm{d}A(X)/\mathrm{d}x\rangle_{\mathrm{st}}$ and assuming that $C_{\mathrm{st}} \neq 0$ holds, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t) = \kappa\left\langle\frac{\mathrm{d}A(X)}{\mathrm{d}x}\right\rangle_{\mathrm{st}}\left(1-\left[\left.\frac{\mathrm{d}R}{\mathrm{d}m}\right|_{\langle A\rangle_{\mathrm{st}}}\right]^{-1}\right)\beta(t) \; . \qquad (46)$$

Since Eq. (45) describes a particular perturbation, from Eq. (46) we can read off sufficient conditions for unstable stationary distributions and necessary conditions for asymptotically stable distributions. For example, if the inequality $\langle \mathrm{d}A(X)/\mathrm{d}x\rangle_{\mathrm{st}} > 0$ holds, stationary distributions are unstable for $\mathrm{d}R(\langle A\rangle_{\mathrm{st}})/\mathrm{d}m > 1$, which is the prediction of transcendent equation analysis. In addition, we see that the inequality $\mathrm{d}R(\langle A\rangle_{\mathrm{st}})/\mathrm{d}m < 1$ is a necessary conditions for P_{st} being asymptotically stable. Eq. (46) also tells us that nonequilibrium systems with mean field forces can exhibit a counter-intuitive behavior: for $\langle \mathrm{d}A(X)/\mathrm{d}x\rangle_{\mathrm{st}} < 0$ they may exhibit unstable stationary distributions with $\mathrm{d}R/\mathrm{d}m < 1$ and asymptotically stable stationary distribution with $\mathrm{d}R/\mathrm{d}m > 1$. In other words, the simple transcendent equation analysis that evaluates only $R(m)$ fails because for systems described by Eq. (39) the stability of the stationary states is determined by two stability criteria: the slope of $R(m)$ at $m = \langle A \rangle_{\mathrm{st}}$ and the sign of $\langle \mathrm{d}A/\mathrm{d}x \rangle_{\mathrm{st}}$.

3.2 Validity of the simple transcendent equation analysis

Here, we briefly address two special cases in which the impact of the additional stability parameter $\langle \mathrm{d}A/\mathrm{d}x \rangle_{\mathrm{st}}$ can be neglected.

3.2.1 Linear mean field forces

For $A(x) = x$ we have

$$\frac{\partial}{\partial t} P(x, t; u) = \frac{\partial}{\partial x} \left[\frac{\mathrm{d}V(x)}{\mathrm{d}x} + \kappa \left(x - \langle X \rangle_P \right) \right] P + \frac{\partial^2}{\partial x^2} D(x) P \qquad (47)$$

and $\mathrm{d}\beta(t)/\mathrm{d}t = \lambda \beta(t)$ with

$$\lambda = \kappa \left(1 - \left[\left. \frac{\mathrm{d}R}{\mathrm{d}m} \right|_{\langle X \rangle_{\mathrm{st}}} \right]^{-1} \right) . \qquad (48)$$

Consequently, stationary solutions of Eq. (47) become unstable if the slope $\mathrm{d}R/\mathrm{d}m$ becomes larger than unity at intersection points with the diagonal in the (m, R)-plane.

3.2.2 Local free energy functionals

Ito case

As we will show next, systems may exhibit some kind of local free energy functional. Let us write Eq. (44) as

$$\frac{\partial}{\partial t}\epsilon(x,t) = \frac{\partial}{\partial x}\left[D(x)P_{\rm st}\frac{\partial}{\partial x}\left\{\frac{\epsilon}{P_{\rm st}} - \kappa\int_{\Omega} A(x')\,\epsilon(x',t)\,\mathrm{d}x' \int^{x} \frac{1}{D(x')}\mathrm{d}x'\right\}\right] . \tag{49}$$

If we put now

$$A(x) = c\int^{x} \frac{1}{D(x')}\,\mathrm{d}x' \tag{50}$$

(where c is a constant) and introduce the local free energy functional

$$\Psi[P] = \frac{\kappa}{2c}K_A(X) - S[P] , \tag{51}$$

then Eq. (49) can equivalently be expressed as

$$\frac{\partial}{\partial t}\epsilon(x,t) = \frac{\partial}{\partial x}D(x)P_{\rm st}\frac{\partial}{\partial x}\int_{\Omega} \frac{\delta^2\Psi[P_{\rm st}]}{\delta P(x)\delta P(y)}\epsilon(y,t)\,\mathrm{d}y . \tag{52}$$

Having obtained Eq. (52), we can proceed as in Sec. 2.1.4. For $L = \delta^2\Psi[P_{\rm st}](\epsilon)/2$ we can show that analogous to Eqs. (13) and (15) the relations $\mathrm{d}L/\mathrm{d}t \leq 0$ and $\mathrm{d}L/\mathrm{d}t = 0 \Leftrightarrow \delta^2\Psi[P_{\rm st}](\epsilon) = 0$ hold. The second variation $\delta^2\Psi$ can be evaluated as demonstrated in Sec. 2.1.4 (see also the Appendix C). Thus, we obtain a stability condition similar to the one listed in the second row of Table 1: if

$$\tilde{\lambda} = c - \kappa K_{A,\rm st}(X) \tag{53}$$

is positive then we have $\forall\epsilon \neq 0 : \delta^2\Psi > 0$ and $\forall\epsilon \neq 0 : L > 0$ and, consequently, $P_{\rm st}$ corresponds to an asymptotically stable distribution. Conversely, if $\tilde{\lambda} < 0$ then there exists a ϵ^* such that $\delta^2\Psi[P_{\rm st}](\epsilon^*) < 0 \Rightarrow L < 0$ and, consequently, $P_{\rm st}$ is unstable. Using the matching condition (50), Eqs. (40) and (41) read

$$P_{\rm st}(x) = \frac{1}{ZD(x)}\exp\left\{\int^{x} \frac{h(x')}{D(x')}\,\mathrm{d}x' - \frac{\kappa}{2c}\Big[A(x) - \langle A\rangle_{\rm st}\Big]^2\right\} \tag{54}$$

and

$$
\begin{aligned}
P(x;m) &= \frac{1}{Z(m)D(x)} \exp\left\{ \int^x \frac{h(x')}{D(x')}\,\mathrm{d}x' - \frac{\kappa}{2c}\left[A(x) - m\right]^2 \right\} \ , \\
R(m) &= \int_\Omega A(x)P(x;m)\,\mathrm{d}x \ ,
\end{aligned} \tag{55}
$$

which eventually leads to $\mathrm{d}R/\mathrm{d}m = \kappa K_{A,\mathrm{st}}(X)/c$ at $m = \langle A\rangle_{\mathrm{st}}$ and $A(x) = c\int^x D^{-1}(x')\,\mathrm{d}x'$. As a result, in line with transcendent equation analysis we conclude that for $\mathrm{d}R(\langle A\rangle_{\mathrm{st}})/\mathrm{d}m < 1$ stationary distributions are asymptotically stable and for $\mathrm{d}R(\langle A\rangle_{\mathrm{st}})/\mathrm{d}m > 1$ stationary distributions are unstable. Note that Eq. (50) implies that the inequality $\langle \mathrm{d}A(X)/\mathrm{d}x\rangle_{\mathrm{st}} > 0$ is satisfied. Therefore, the result derived here is consistent with Eq. (46).

Stratonovich case

In order to study noise induced phenomena it is often more appropriate to consider Fokker-Planck equations such as

$$
\frac{\partial}{\partial t}P(x,t;u) = -\frac{\partial}{\partial x}\left[h(x) - \kappa\left(A(x) - \langle A(X)\rangle_P\right)\right]P + \frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}g(x)P \tag{56}
$$

with $g(x) > 0$. For $\kappa = 0$ Eq. (56) describes a stochastic process defined by the Stratonovich Langevin equation [32]

$$
\frac{\mathrm{d}}{\mathrm{d}t}X(t) = h(X) + \underbrace{g(X)\Gamma(t)}_{\text{Stratonovich}} \ . \tag{57}
$$

Using $D(x) = g^2(x)$, we can cast Eq. (56) into the form of Eq. (39):

$$
\frac{\partial}{\partial t}P(x,t;u) = -\frac{\partial}{\partial x}\left[h(x) + \frac{1}{2}\frac{\mathrm{d}D(x)}{\mathrm{d}x} - \kappa\left(A(x) - \langle A\rangle_P\right)\right]P + \frac{\partial^2}{\partial x^2}D(x)P \ . \tag{58}
$$

As shown in pervious studies [6,42–45], the Ito Langevin equation corresponding to Eq. (58) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = h(X) + \frac{1}{2}\frac{\mathrm{d}D(X)}{\mathrm{d}x} - \kappa\left[A(X) - \langle A\rangle\right] + \underbrace{\sqrt{D(X)}\Gamma(t)}_{\text{Ito}} \ . \tag{59}$$

As a consequence, Eq. (59) also corresponds to the stochastic differential equation of Eq. (56). From Eq. (58) it follows that Eqs. (54) and (55) become

$$P_{\mathrm{st}}(x) = \frac{1}{Z\sqrt{D(x)}} \exp\left\{\int^{x} \frac{h(x')}{D(x')}\,\mathrm{d}x' - \frac{\kappa}{2c}\left[A(x) - \langle A\rangle_{\mathrm{st}}\right]^2\right\} \tag{60}$$

and

$$\begin{aligned} P(x;m) &= \frac{1}{Z(m)\sqrt{D(x)}} \exp\left\{\int^{x} \frac{h(x')}{D(x')}\,\mathrm{d}x' - \frac{\kappa}{2c}\left[A(x) - m\right]^2\right\} \ , \\ R(m) &= \int_{\Omega} A(x)P(x;m)\,\mathrm{d}x \ . \end{aligned} \tag{61}$$

Comparing Eqs. (54) and (60), we realize that the factor $1/D$ is replaced by $1/\sqrt{D}$.

3.3 *Example: noise-induced reentrant bifurcation*

Inspired by several studies on reentrant bifurcations [13,16,46–49], we use $g(x) = \sqrt{Q}\,(1 + bx^2)$ and $D(x) = Q(1 + bx^2)^2$ with $b > 0$ which means that we deal with a mean field force involving $A(x) = c\int_0^x D^{-1}(x')\,\mathrm{d}x' = c[\arctan(\sqrt{b}x)/\sqrt{b}+x/(1+bx^2)]/(2Q)$. Since the expression $\arctan(x)+x/(1+x^2)$ behaves like an arctan-function [3], we introduce the modified arctan-function $\arctan'(x) = \arctan(x) + x/(1+x^2)$. For $c = 2Q$ we then obtain from Eq. (58) the mean field model

[3] Both $\arctan(x)$ and $\arctan(x) + x/(1+x^2)$ are symmetric and bounded functions that increase strictly monotonically: $\mathrm{d}[\arctan(x)+x/(1+x^2)]/\mathrm{d}x = 2/(1+x^2)^2 > 0$.

$$\frac{\partial}{\partial t}P(x,t;u) = -\frac{\partial}{\partial x}\left[h(x) - \kappa\left(\frac{\arctan'(\sqrt{b}x)}{\sqrt{b}} - \left\langle\frac{\arctan'(\sqrt{b}X)}{\sqrt{b}}\right\rangle_P\right)\right]P$$
$$+Q\frac{\partial}{\partial x}\left(1+bx^2\right)\frac{\partial}{\partial x}\left(1+bx^2\right)P\ . \tag{62}$$

For $h(x) = -ax(1+bx^2)^2$ with $a > 0$ [13] stationary distributions of Eq. (62) satisfy the implicit equation

$$P_{\rm st}(x) =$$
$$\frac{1}{Z(1+bx^2)}\exp\left\{-\frac{1}{2Q}\left(ax^2 + \frac{\kappa}{2}\left[\frac{\arctan'(\sqrt{b}x)}{\sqrt{b}} - \left\langle\frac{\arctan'(\sqrt{b}X)}{\sqrt{b}}\right\rangle_{\rm st}\right]^2\right)\right\} \tag{63}$$

(see Eq. (60) and replaced $Z\sqrt{Q}$ by Z). Accordingly, the transcendent equation $m = R(m)$ for $m = \langle\arctan'(X)\rangle$ involves the functions

$$P(x;m) = \frac{1}{Z(m)}\frac{1}{1+bx^2}\exp\left\{-\frac{1}{2Q}\left(ax^2 + \frac{\kappa}{2}\left[\frac{\arctan'(\sqrt{b}x)}{\sqrt{b}} - m\right]^2\right)\right\}\ ,$$
$$R(m) = \int_\Omega \arctan'(x)P(x;m)\,{\rm d}x\ . \tag{64}$$

Since $\arctan'(x)$ describes an antisymmetric function, the relation $R(0) = 0$ holds which in turn implies that a symmetric solution $P_{\rm st}$ with $\langle X\rangle = 0$ and $\left\langle\arctan'(\sqrt{b}X)/\sqrt{b}\right\rangle = 0$ exists for all parameters κ and Q. We refer to this solution as the paramagnetic solution describing a state of disorder. In contrast, we refer to stationary distributions with $\langle X\rangle \neq 0$ and $\left\langle\arctan(\sqrt{b}X)/\sqrt{b}\right\rangle \neq 0$ as ferromagnetic solutions that describe systems featuring some kind of order. Note that for stationary distributions of the form (63) the implication $\langle X\rangle_{\rm st} \neq 0 \Leftrightarrow \left\langle\arctan'(\sqrt{b}X)/\sqrt{b}\right\rangle_{\rm st} \neq 0$ holds. Let us examine the emergence of ferromagnetic solutions. To this end, we first determine the qualitative behavior of $R'(m) = R(m) - m$. It is clear that the relation $R'(m) = 0$ defines solutions of the transcendent equation $m = R(m)$. Moreover, if the inequality ${\rm d}R'(m)/{\rm d}m < 0$ (> 0) holds for $R'(m) = 0$, we deal with solutions of $m = R(m)$ with ${\rm d}R/{\rm d}m < 1$ (> 1) and asymptotically stable (unstable) stationary distributions. $R'(m)$ is shown in Fig. 1 for three values of Q. We realize that when ferromagnetic solutions with $\left\langle \arctan'(\sqrt{b}X)\right\rangle/\sqrt{b} \neq 0$ and $\langle X\rangle \neq 0$ emerge then the paramagnetic solution becomes unstable (because the slope $\mathrm{d}R'(0)/\mathrm{d}m$ becomes positive). Furthermore, we realize that the bifurcation is reentrant [13]. The bifurcation line can be computed from $\mathrm{d}R(m)/\mathrm{d}m = \kappa K_{A,\mathrm{st}}/c = 1$ with $c = 2Q$ and is depicted in Fig. 2. The order parameter $m = \langle A\rangle_{\mathrm{st}}$ as a function of Q is given in Fig. 3. Note that in Fig. 3 only the stable solutions are shown.

Insert figures 1, 2, and 3 about here.

4 Conclusions

Using a Fokker-Planck approach, we have discussed the stability of stationary states of mean field models that involve several kinds of internal and free energy functionals and describe stochastic processes subjected to additive and multiplicative noise. We have shown how to determine the stability of stationary distributions by means of linear stability analysis, Lyapunov's direct method, and transcendent equation analysis. We have found that the three methods yield consistent results for systems with free energy functionals (Sec. 2), linear mean field forces (Sec. 3.2.1), and local free energy functionals (Sec. 3.2.2). The consistence between Lyapunov's direct method and linear stability analysis arises due to the fact that both methods exploit the same quantity, namely, the second variation of free energy measures. For systems that can not be described by means of a free energy functional, in general, the simple transcendent equation analysis that only evaluates the slope of $R(m)$ (right hand side of the transcendent equation) fails. The reason for this is that in the general case there is another stability parameter that has to be taken into account (Sec. 3.1). Consequently, there might be nonequilibrium systems that exhibit a somewhat counter-intuitive behavior in the sense that stationary distributions become unstable when the slope of $R(m)$ at intersection points with the diagonal $y(m) = m$ becomes smaller than unity.

We have introduced a norm for the perturbations of stationary states related to free energy minima and maxima. By means of this norm, one can rigorously prove that perturbations of free-energy-minimum-distributions vanish in the long time limit whereas perturbations of free-energy-maximum-distributions grow as a function of time. The existence of the norm functional is closely related to the fact that second variational derivatives are positive (negative) definite at free energy minima (maxima) with nonvanishing second variational derivatives. We would like to point out that we deal with a norm which is locally defined in the function space of probability densities. That is, for different stationary probability densities we usually obtain different norm functionals.

Moreover, we have studied special cases for which local free energy functionals can be used to determine the stability of stationary distributions (Sec. 3.2.1). Future studies may elucidate the relevance of these special cases. For example, in the context of a reentrant bifurcation we have considered a system that involves a mean field coupling given by a modified arctan-function. Close to the origin this modified arctan-function corresponds to a linear mean field coupling frequently used in literature. Therefore, we speculate that for weak fluctuation forces and paramagnetic states the arctan-coupling is a good approximation for a linear coupling. In other words, it might be the case that in general in the weak noise limit multiplicative noise systems described by mean field Fokker-Planck equations can be evaluated by means of local free energy functionals.

We have required that free energy functionals are bounded from below. However, as far as the stability analysis of stationary states is concerned, we can dispense with this requirement. We only need the boundedness of free energy measures to prove that systems are globally stable in the sense that every transient solution eventually converges to a stationary one. Consequently, the stability analysis carried out in Sec. 2 also applies to mean field models with free energy measures that are not bounded from below.

The H-theorem for free energy Fokker-Planck equations states that transient probability densities converge to stationary ones. In this context, the question arises whether or not stochastic differential equations that can be assigned to free energy Fokker-Planck equations exhibit the same convergence property. We can not answer this question in general because there are many ways to assign stochastic differential equations to nonlinear Fokker-Planck equations. Following an idea by *McKean Jr.*, the time-dependent solution of a nonlinear Fokker-Planck equation can be interpreted as a virtual Markov process described by a suitably defined linear Fokker-Planck equation and thus Langevin equations can be assigned to nonlinear Fokker-Planck equations [44,45] (see also Sec. 3.2.2). In doing so, the realizations of the random variables described by the Langevin equations can be regarded as the state variables of particles (or subsystems) of a many-body system. Consequently, if we assume that a free energy Fokker-Planck equation describes an ensemble of statistically independent particles (subsystems), then we can immediately write down a stochastic differential equation for the Fokker-Planck equation. In this case, the Fokker-Planck equation and the Langevin equation yield the same time-dependent probability density. In particular, if a free energy Fokker-Planck equation is multistable then the corresponding Langevin equation is multistable, too (see e.g. [6,42]). Taking this particular perspective, the convergence to stationary probability densities can be regarded as the limiting case $t \to \infty$ $N \to \infty$, where N is the number of particles and the limiting case $N \to \infty$ is carried out before the limiting case $t \to \infty$ is carried out. Alternatively, nonlinear Fokker-Planck equations may be regarded as the limiting cases of multivariate linear Fokker-Planck equations [2,3]. In line with this approach, we can write down N-dimensional Langevin equations for nonlinear Fokker-Planck equations and we can carry out the limiting procedure $N \to \infty$ $t \to \infty$, where the limiting case $t \to \infty$ is carried out before the limiting case $N \to \infty$ is carried out. Since finite dimensional multivariate Fokker-Planck equations are usually monostable, the corresponding Langevin equations are monostable, too. If the nonlinear Fokker-Planck equation is multistable we deal with a problem. It might be possible that we deal with a Langevin equation that is monostable in the limit $N \to \infty$ $t \to \infty$. Then, the stationary probability density of the Langevin equation would not reflect the stationary behavior of the nonlinear Fokker-Planck equation. Alternatively, in the limit $N \to \infty$ $t \to \infty$ there might be a degeneration of the eigenvalue zero of the linear multivariate Fokker-Planck operator indicating that there are several eigenfunctions with eigenvalue zero [50]. Then, in the limit $N \to \infty$ $t \to \infty$ the Langevin equation could exhibit multiple stationary probability densities. Finally, it has been suggested to interpret nonlinear Fokker-Planck equations as nonlinear evolution equations for transition probability densities [51]. The relevance of multistability for this kind of nonlinear Fokker-Planck equations, however, has not yet been addressed.

Appendix

A Boundedness of the free energy functional (1)

In what follows, we will show that the functional (1) is bounded from below for infinitely differentiable potentials $U_0(\mathbf{x})$ and functionals U_{NL} with infinitely differentiable integral kernels provided that particular constraints are satisfied. Let us first discuss a stochastic process subjected to periodic boundary conditions. In this case we require that the potentials U_0 and U_{NL} are periodic (see Sec. 2.1). Since Ω is finite and the integral kernels of $\int_\Omega U_0 P \mathrm{d}^M x$ (which is U_0) and U_{NL} are assumed to be continuous, the integral kernels assume finite minimum values on Ω. Consequently, the integrals $\int_\Omega U_0 P \mathrm{d}^M x$ and U_{NL} are bounded from below, say, by $\int_\Omega U_0 P \mathrm{d}^M x \geq U_{0,\mathrm{min}}$ and $U_{\mathrm{NL}} \geq U_{\mathrm{NL,min}}$. For finite phase spaces, the entropy S becomes maximal for the uniform distribution. Consequently, S is bounded by $S \leq S_{\max}$. Therefore, we obtain $F \geq U_{0,\min} + U_{\mathrm{NL},\min} - QS_{\max}$. Next, let us consider a stochastic process subjected to natural boundary conditions. In this case, we assume that the Boltzmann distribution $W(\mathbf{x}) = \exp\{-U_0(\mathbf{x})/Q\}/Z_B$ with $Z_B = \int_\Omega \exp\{-U_0(\mathbf{x})/Q\}\mathrm{d}^M x$ exists and that U_{NL} is bounded from below by $U_{\mathrm{NL}} \geq U_{\mathrm{NL,min}}$. By means of $U_0(\mathbf{x}) = -Q \ln Z_B + Q \ln[Z_B/ \exp\{-U_0(\mathbf{x})/Q\}]$ the free energy (1) can be written as

$$F[P] = -Q \ln Z_B + U_{\mathrm{NL}} + Q \underbrace{\int_\Omega P \ln \frac{P}{W} \, \mathrm{d}^M x}_{L_{\mathrm{K}} \geq 0} \ . \tag{A.1}$$

As indicated above, the expression labeled L_{K} is larger than zero or equals zero because it represents the Kullback distance measure [52]. From $U_{\mathrm{NL}} \geq U_{\mathrm{NL,min}}$ we obtain $F \geq -Q \ln Z_B + U_{\mathrm{NL,min}}$.

B On a norm for the deviations of probability densities

Let $C_\epsilon(\Omega) = \left\{ f(\mathbf{x}) \mid f \in C^\infty(\Omega) \ \wedge \ \int_\Omega f(\mathbf{x}) \, \mathrm{d}^M x = 0 \right\}$ denote the function space of the deviations of probability densities. Then the functions $\epsilon \in C_\epsilon$ are the vectors of a linear vector space and satisfy for $a_1, a_2 \in \mathbb{R}$ and $\epsilon_1, \epsilon_2 \in C_\epsilon$ the relation $a_1\epsilon_1 + a_2\epsilon_2 = \epsilon_3 \in C_\epsilon$. Consider a free energy functional F with second variational derivatives that are symmetric at a stationary point P_{st} of F (see Sec. 2.1.3) and second variations that are positive definite at P_{st}:

$$\epsilon \neq 0 \Leftrightarrow \delta^2 F[P_{\mathrm{st}}](\epsilon) > 0 \ , \ \epsilon = 0 \Leftrightarrow \delta^2 F[P_{\mathrm{st}}](\epsilon) = 0 \ . \tag{B.1}$$

Now, we introduce the functional

$$(\epsilon_1, \epsilon_2) = \delta^2 F[P_{\mathrm{st}}](\epsilon_1, \epsilon_2) = \int_\Omega \frac{\delta^2 F[P_{\mathrm{st}}]}{\delta P_1(\mathbf{x}) \delta P_2(\mathbf{x})} \epsilon_1(\mathbf{x}) \epsilon_2(\mathbf{x}) \, \mathrm{d}^M x \mathrm{d}^M y \tag{B.2}$$

which is akin to a scalar product. This expression is a bilinear form satisfying

$$
\begin{aligned}
&(\epsilon_1, \epsilon_2) = (\epsilon_2, \epsilon_1) \ , \\
&(a_1\epsilon_1 + a_2\epsilon_2, \epsilon_3) = a_1(\epsilon_1, \epsilon_3) + a_2(\epsilon_2, \epsilon_3)
\end{aligned}
\tag{B.3}
$$

for $a_1, a_2 \in \mathbb{R}$. Then, we introduce the functional $||\cdot||$ defined by

$$
||\epsilon|| = \sqrt{\delta^2 F[P_{\rm st}](\epsilon)} = \sqrt{(\epsilon, \epsilon)} \ . \tag{B.4}
$$

Our objective is to show that the relations

$$
\begin{aligned}
&||\epsilon|| \geq 0 \ , \\
&||\epsilon|| = 0 \Leftrightarrow \epsilon = 0 \ , \\
&||\epsilon_1 + \epsilon_2|| \leq ||\epsilon_1|| + ||\epsilon_2|| \ \ , \\
&||a\epsilon|| = |a| \ ||\epsilon||
\end{aligned}
\tag{B.5}
$$

are satisfied for $a \in \mathbb{R}$ which means that Eq. (B.4) is a norm related to a free energy minimum. We can be read off from Eqs. (B.1,...,B.4) that the first two and the last property in Eq. (B.5) are satisfied. The triangle inequality can be proven in line with the proof of the triangle inequality for vectors of the Euclidean space (e.g., prove first that the Cauchy Schwarz inequality holds and use the Cauchy Schwarz inequality to derive the triangle inequality). By analogy one can show that Eq. (19) describes a norm related to a free energy maximum.

C Special cases of free energy models

C.1 Linear mean field forces (Desai-Zwanzig model)

We consider the Desai-Zwanzig model for a general pinning force $-\mathrm{d}V/\mathrm{d}x$ [9,10] and assume that the evolution of $P(x,t;u)$ with $x \in \Omega = \mathbb{R}$ is defined by

$$
\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{\mathrm{d}V(x)}{\mathrm{d}x} + \kappa\left(x - \langle X \rangle_P\right)\right] P + Q\frac{\partial^2}{\partial x^2} P \tag{C.1}
$$

with $\kappa \geq 0$. We can verify that Eq. (C.1) can be written as Eq. (23) with F given by

$$F[P] = \int_\Omega V(x)P(x)\,\mathrm{d}x + \frac{\kappa}{2}K(X) - QS\ . \tag{C.2}$$

The free energy F may alternatively be expressed as

$$F[P] = \int_\Omega V(x)P(x)\,\mathrm{d}x + \frac{\kappa}{4}\int_\Omega\int_\Omega [x-y]^2 P(x)P(y)\,\mathrm{d}x\,\mathrm{d}y - QS \tag{C.3}$$

because of $2K(X) = \int_\Omega [x-y]^2 P(x)P(y)\,\mathrm{d}x\,\mathrm{d}y$. Since the nonlinear energy functional U_{NL} is bounded from below, F is bounded form below for all potentials V for which the Boltzmann distribution of V exists (see Appendix A). Let us re-arrange the terms of the free energy (C.2) like

$$F[P] = \int_\Omega [V(x) + \frac{\kappa}{2}x^2]P(x)\,\mathrm{d}x - \frac{\kappa}{2}\int_\Omega\int_\Omega xyP(x)P(y)\,\mathrm{d}x\,\mathrm{d}y - QS\ . \tag{C.4}$$

In this case, comparing Eqs. (22) and (C.4), we obtain

$$U_0(x) = V(x) + \frac{\kappa}{2}x^2\ , \quad B(z) = -\frac{\kappa}{2}z^2\ , \quad A(x) = x\ . \tag{C.5}$$

As a result, Eq. (25) reads

$$P_{\mathrm{st}}(x) = \frac{1}{Z}\exp\left\{-\frac{V(x) + \kappa x^2/2 - \kappa x\,\langle X\rangle_{\mathrm{st}}}{Q}\right\}\ , \tag{C.6}$$

which can be solved by means of the corresponding transcendent equation for the expectation value $m = \langle X\rangle_{\mathrm{st}}$ [10,39]. That is, we define $P(m;x)$ and $R(m)$ by

$$\begin{aligned} P(x;m) &= \frac{1}{Z(m)}\exp\left\{-\frac{V(x) + \kappa x^2/2 - \kappa x m}{Q}\right\}\ , \\ R(m) &= \int_\Omega xP(x;m)\,\mathrm{d}x\ , \end{aligned} \tag{C.7}$$

where $Z(m)$ is a normalization constant. Then, m is given by the solutions of $m = R(m)$. For the Desai-Zwanzig model the second variation of F reads

$$
\begin{aligned}
\delta^2 F[P_{\mathrm{st}}](\epsilon) &= -\kappa \int_{\Omega} x\,\epsilon(x)\,\mathrm{d}x \int_{\Omega} y\,\epsilon(y)\,\mathrm{d}y + Q \int_{\Omega} \frac{\epsilon^2(x)}{P_{\mathrm{st}}(x)}\,\mathrm{d}x \\
&= -\kappa \left[\int_{\Omega} x\,\epsilon(x)\,\mathrm{d}x\right]^2 + Q \int_{\Omega} \frac{\epsilon^2(x)}{P_{\mathrm{st}}(x)}\,\mathrm{d}x
\end{aligned}
\tag{C.8}
$$

and can be evaluated by considering ϵ in form of

$$
\epsilon(x) = \beta\,[x - \langle X\rangle_{\mathrm{st}}]\,P_{\mathrm{st}}(x) + \chi_{\perp}(x)\sqrt{P_{\mathrm{st}}(x)}\ , \tag{C.9}
$$

where $\chi_{\perp}$ satisfies the orthogonality relations $\int_{\Omega} \chi_{\perp}(x)\sqrt{P_{\mathrm{st}}(x)}\,\mathrm{d}x = 0$ and $\int_{\Omega} x\chi_{\perp}(x)\sqrt{P_{\mathrm{st}}(x)}\,\mathrm{d}x = 0$ [10]. Then, one obtains

$$
\delta^2 F[P_{\mathrm{st}}](\epsilon) = \beta^2 K_{\mathrm{st}}(X)\,[Q - \kappa\,K_{\mathrm{st}}(X)] + Q\int_{\Omega} [\chi_{\perp}]^2\,\mathrm{d}x\ , \tag{C.10}
$$

where $K_{\mathrm{st}}(X)$ denotes the variance of a stationary distribution (C.6). It can be seen now that for a distribution P_{st} with $\tilde{\lambda} = Q - \kappa K_{\mathrm{st}}(X) > 0$ we have $\delta^2 F > 0$ for all $\epsilon \neq 0$. Consequently, from the stability analysis carried out in sections 2.1.2 and 2.1.3, we conclude that in this case P_{st} correspond to a stable stationary probability density. In contrast, for every distribution P_{st} that yields $\tilde{\lambda} < 0$ there exists a ϵ^* such that $\delta^2 F[P_{\mathrm{st}}(\epsilon^*) < 0$ and we deal with an unstable stationary distribution. Critical parameter values of κ and Q define a bifurcation line which can be computed from $\tilde{\lambda} = 0 \Rightarrow Q = \kappa K_{\mathrm{st}}(X;\kappa,Q)$. Since from Eq. (C.7) it follows that the relation

$$
\left.\frac{\mathrm{d}R}{\mathrm{d}m}\right|_{\langle A\rangle_{\mathrm{st}}} = \frac{\kappa}{Q} K_{\mathrm{st}}(X) \tag{C.11}
$$

holds, transcendent equation analysis yields the same results as Lyapunov's direct method and linear stability analysis.

C.2 Generalized variance: $U_{\rm NL} = \kappa K_A/2 = \kappa \langle [A-\langle A\rangle]^2\rangle /2$

Here we consider the Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{\mathrm{d}V(x)}{\mathrm{d}x} + \kappa\left(A(x) - \frac{\mathrm{d}A(x)}{\mathrm{d}x}\langle A(X)\rangle_P\right)\right]P + Q\frac{\partial^2}{\partial x^2}P \ , \tag{C.12}$$

for $t \geq t_0$ and $\kappa \geq 0$. Then, by means of

$$F[P] = \int_\Omega V(x)P(x)\,\mathrm{d}x + \frac{\kappa}{2}K_A(X) - QS[P] \ , \tag{C.13}$$

Eq. (C.12) can be transformed into Eq. (23). Comparing the free energy functionals (1) and (C.13) we may put $U_0 = V$ and $U_{\rm NL} = 0.5\kappa K_A(X) \geq 0$. Due to the boundedness of $U_{\rm NL}$, we have $F \geq F_{\rm min}$ and can further conclude that the limiting case $\lim_{t\to\infty}\partial P/\partial t = 0$ holds. Decomposing F like $U_0 = V(x)+\kappa A^2(x)/2$ and $U_{\rm NL}[P] = -\kappa\langle A\rangle^2/2$ with $\delta U_{\rm NL}/\delta P = -\kappa A(x)\langle A\rangle$, from Eq. (5) we obtain

$$P_{\rm st}(x) = \frac{1}{Z}\exp\left\{-\frac{V(x)+\kappa A^2(x)/2 - \kappa A(x)\langle A\rangle_{\rm st}}{Q}\right\} \ . \tag{C.14}$$

Introducing the order parameter $m = \langle A\rangle_{\rm st}$ in combination with the functions

$$\begin{aligned} P(x;m) &= A(x)\exp\left\{-\frac{V(x)+\kappa A(x)^2/2 - \kappa A(x)m}{Q}\right\} \ , \\ R(m) &= \frac{1}{Z(m)}\int_\Omega A(x)P(x;m)\,\mathrm{d}x \end{aligned} \tag{C.15}$$

we can determine $\langle A\rangle_{\rm st}$ by solving the transcendent equation $m = R(m)$. In order to discuss the stability of these stationary distributions, we first note that $\delta^2 U_{\rm NL}/\delta P(x)\delta P(y) = -\kappa A(x)A(y)$. Then, Eq. (10) reads

$$\delta^2 F[P_{\rm st}](\epsilon) = -\kappa\left[\int_\Omega A(x)\,\epsilon(x)\,\mathrm{d}x\right]^2 + Q\int_\Omega \frac{\epsilon^2(x)}{P_{\rm st}(x)}\,\mathrm{d}x \ . \tag{C.16}$$

Substituting Eq. (32) into Eq. (C.16), gives us

$$\delta^2 F[P_{\rm st}](\epsilon) = \beta^2 K_{A,\rm st}(X)\,[Q - \kappa\, K_{A,\rm st}(X)] + Q \int_\Omega [\chi_\perp]^2\,{\rm d}x\ . \qquad \text{(C.17)}$$

Consequently, for $\tilde{\lambda} = Q - \kappa\, K_{A,\rm st}(X) > 0$ ($\tilde{\lambda} < 0$) we deal with a stable (unstable) stationary probability density and critical parameter values of κ and Q can be computed from $Q - \kappa\, K_{A,\rm st}(X; Q, \kappa) = 0$. Transcendent equation analysis based on the hypothesis (28) involves the derivative

$$\left.\frac{{\rm d}R(m)}{{\rm d}m}\right|_{\langle A\rangle_{\rm st}} = \frac{\kappa}{Q} K_{A,\rm st}(X) \qquad \text{(C.18)}$$

and leads to the same result as Lyapunov's direct method and linear stability analysis. Note that for $A(x) = x^n$ we obtain the example mentioned in the introduction.

C.3 Couplings via bounded potentials $U_{\rm NL}[P] = B(\langle A\rangle) \geq B_{\rm min}$

Our final example is concerned with the mean field Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{{\rm d}V(x)}{{\rm d}x} + \frac{{\rm d}A(x)}{{\rm d}x}\left.\frac{{\rm d}B(z)}{{\rm d}z}\right|_{z=\langle A\rangle}\right] P + Q\frac{\partial^2}{\partial x^2}P \qquad \text{(C.19)}$$

for $t \geq t_0$ and $B(z) \geq B_{\rm min}$. For example, if we put $A(x) = x^n$ with $n \geq 1$ we get

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x}\left[\frac{{\rm d}V(x)}{{\rm d}x} + n x^{n-1}\frac{{\rm d}B(\langle X^n\rangle)}{{\rm d}z}\right] P + Q\frac{\partial^2}{\partial x^2}P\ , \qquad \text{(C.20)}$$

which has previously been discussed [53]. By virtue of

$$F[P] = \int_\Omega V(x)P(x)\,{\rm d}x + B(\langle A\rangle) - QS[P]\ , \qquad \text{(C.21)}$$

Eq. (C.19) can be written like Eq. (23). From $U_{\rm NL}[P] = B(\langle A \rangle) \geq B_{\rm min}$ it follows that $F \geq F_{\rm min}$ and, consequently, the H-theorem derived in Sec. 2.1.1 gives us $\lim_{t\to\infty} \partial P/\partial t = 0$. Using $\delta U_{\rm NL}[P]/\delta P = A(x){\rm d}B(\langle A \rangle)/{\rm d}z$, Eq. (5) becomes

$$P_{\rm st}(\mathbf{x}) = \frac{1}{Z} \exp\left\{ -\frac{V(x) + A(x){\rm d}B(\langle A \rangle_{\rm st})/{\rm d}z}{Q} \right\} . \tag{C.22}$$

The stationary expectation value $m = \langle A \rangle_{\rm st}$ can be obtained in a self-consistent fashion by introducing the functions

$$\begin{aligned} P(x;m) &= \frac{1}{Z(m)} \exp\left\{ -\frac{V(x) + A(x){\rm d}B(m)/{\rm d}m}{Q} \right\} , \\ R(m) &= \int_\Omega A(x) P(x;m)\, {\rm d}x , \end{aligned} \tag{C.23}$$

and solving the transcendent equation $m = R(m)$. In order to carry out a stability analysis of the stationary probability distributions thus obtained, we can proceed as in Sec. 2.1.4 and obtain the results reported in Table 1 (third row). The reason for this is that the functionals (22) and (C.21) have the same second variations. Moreover, the slope of $R(m)$ is given by Eq. (29) which means that transcendent equation analysis and Lyapunov's direct method yield the same results.

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Table 1

Free energy cases

U_{NL}	F	$\tilde{\lambda}$
$\propto K(X) = \langle X^2 \rangle - \langle X \rangle^2$	$\langle V(X) \rangle + \kappa K(X)/2 - QS$	$Q - \kappa K_{\mathrm{st}}$
$\propto K_A(X) = \langle A^2 \rangle - \langle A \rangle^2$	$\langle V(X) \rangle + \kappa K_A(X)/2 - QS$	$Q - \kappa K_{A,\mathrm{st}}$
$B(\langle A \rangle) \geq B_{\min}$	$\langle V(X) \rangle + B(\langle A \rangle) - QS$	$Q + K_{A,\mathrm{st}} \mathrm{d}^2 B / \mathrm{d}m^2$
$\langle B_0 \rangle + B(\langle A \rangle) \geq B_{\min}$	$\langle V + B_0 \rangle + B(\langle A \rangle) - QS$	$Q + K_{A,\mathrm{st}} \mathrm{d}^2 B / \mathrm{d}m^2$

Figure caption:

Fig. 1: Illustration of the transcendent equation $R'(m) = R(m) - m = 0$ for the mean field Fokker-Planck equation (62). $R'(m)$ for $\kappa = 20$ and $a = 1$ and several values of Q: $Q = 1$, $Q = 10$, $Q = 40$. Solutions of $R'(m) = 0$ describe stationary distributions. Solutions of $R'(m) = 0$ with negative (positive) slopes correspond to asymptotically stable (unstable) distributions.

Fig. 2: Bifurcation lines of reentrant bifurcations described by the mean field model (62). Bifurcation lines are computed from $\kappa K_{A,\mathrm{st}}/c = 1$ for $a = 1$ (solid line) and $a = 1.5$ (thick dashed line), $b = 0.2$ and $c = 2Q$.

Fig. 3: Order parameter $m = \left\langle \arctan'(\sqrt{b}X)/\sqrt{b} \right\rangle_{\mathrm{st}}$ of a reentrant bifurcation described by Eq. (62). Solid lines are computed from $m = R(m)$ and Eq. (64). Crosses and diamonds are obtained from simulations of the Langevin equation (59) using an Euler forward discretization scheme [32]. Q is gradually increased along the horizontal line depicted in Fig. 2. Parameters: $a = 1$, $b = 0.2$, $\kappa = 20$.

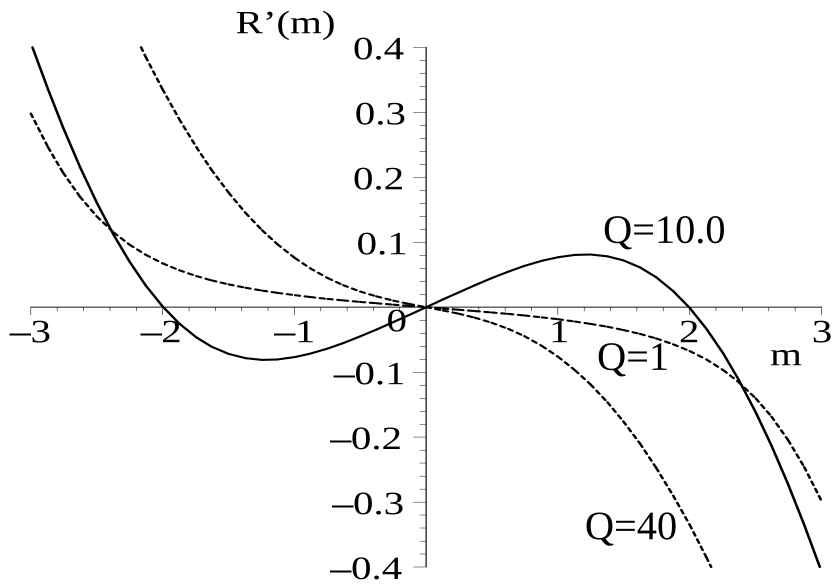

Fig. 1.

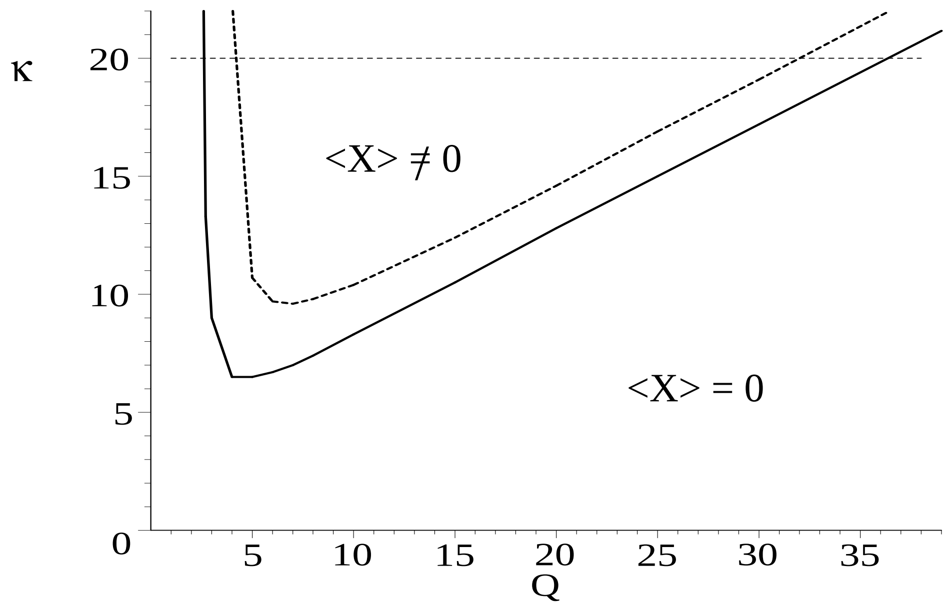

Fig. 2.

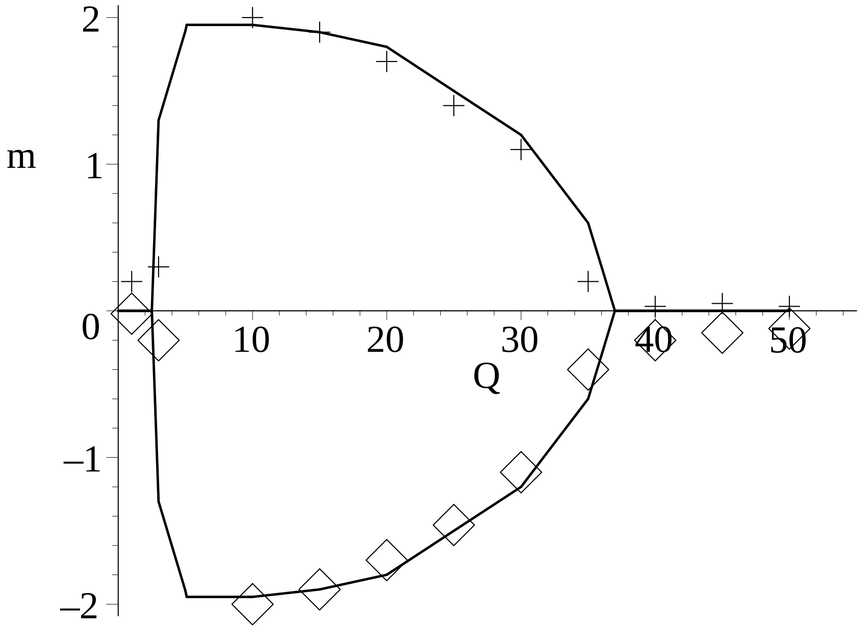

Fig. 3.