Fokker-Planck perspective on stochastic delay systems: Exact solutions and data analysis of biological systems

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Stochastic delay systems with additive noise are examined from the perspective of Fokker-Planck equations. For a linear system, the exact stationary probability density is derived by means of a delay Fokker-Planck equation. We show how to determine the delay equation of the linear system from experimental data, and corroborate a fundamental result previously obtained by Küchler and Mensch. We also propose a method to derive delay equations of nonlinear systems from experimental data. To this end, the theory of multivariate Fokker-Planck equations is used. The applicability of this method is demonstrated for stochastic models describing tracking and pointing movements of humans.

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I. INTRODUCTION

In recent years, researchers have become increasingly interested in dynamical systems subjected to delays [1–11]. In particular, many biological systems call for a description by means of differential equations involving time-delayed variables. In many cases, the delay reflects transmission times related to the transport of matter, energy, and information through the system under investigation. Therefore, delay systems can often be regarded as simplified but very useful descriptions of systems involving a reaction chain or a transport process. Some typical examples of biological systems that have been interpreted as delay systems are discussed in Refs. [12–22] and summarized in Table I. For further examples, the reader is referred to Ref. [23].

In recognition of the relevance of delay models, the question arises how to determine the evolution equations of delay systems from experimental data. Since experimental data are usually subjected to fluctuations, we deal with stochastic delay systems rather than deterministic ones. Then, the problem at hand is to map stochastic delay systems to stochastic delay models by means of data analysis techniques that are as unbiased as possible. In what follows, we will consider systems described by a scalar random variable $X(t)$ defined on the real line $\Omega=\mathbb{R}$ and subjected to natural boundary conditions [24]. Taking a general point of view, we may describe such systems by means of stochastic differential equations of the form

$$\frac{d}{dt}X(t)=h_0(X(t),X(t-\tau),\sqrt{Q'}\Gamma'(t))+\sqrt{Q}\Gamma(t) \tag{1}$$

for $t\geqslant 0$, $\tau\geqslant 0$, and $X(t)=\phi(t)$ for $t\in[-\tau,0]$. Here, τ denotes the delay of the system. $\Gamma(t)$ and $\Gamma'(t)$ denote fluctuation forces. The variables Q and Q' denote noise amplitudes. If h_0 includes expressions such as $X(t)\Gamma'(t)$ or $X(t-\tau)\Gamma'(t)$, we deal with parametric noise. A promising approach to analyze systems with parametric noise is to expand h_0 with respect to Γ' and to neglect higher-order terms of Γ' [25]. The equation thus obtained can be cast into the form

$$\frac{d}{dt}X(t)=h(X(t),X(t-\tau))+\sqrt{Q'}g(X(t),X(t-\tau))\Gamma'(t)+\sqrt{Q}\Gamma(t) \tag{2}$$

and describes a system subjected to both additive and multiplicative noise. For examples see Ref. [19] and Sec. II B 2 (note that in Ref. [19] colored noise has been used while we will use white noise). Function $h(x,y)$ corresponds to a deterministic force that depends only on the nondelayed variable $X(t)$ and the delayed variable $X(t-\tau)$. If $Q'g^2\ll Q$ holds, additive noise dominates multiplicative noise. Assuming that $Q'g^2\ll Q$ holds, several biological phenomena have been successfully addressed: stochastic resonance [26,27] (e.g., for time-discrete delay systems [2], delayed phase oscillators [28], and diffusively coupled neural oscillators [29]), postural sway [30], spike train coherence [31,32], brain activity exhibiting $1/f$ noise [33], stimulus-induced synchronization of brain activity [34], critical fluctuations and relaxation times of coordinated finger movements and movement related brain activity [35–39], and bistability of noisy motor control systems [40]. Although additive noise models have been widely applied in the study of biological systems, it has also been demonstrated that multiplicative and parametric noise can play important roles in biological systems, for example, in human motor control systems in general [41] and in the pupil light reflex in particular [19,42]. From the recognition that multiplicative or parametric noise can make essential contributions to system dynamics, noise-induced shifts of bifurcation points related to pointing tasks [21], stimulus-induced synchronization of brain activity [43], and corrective movements on short time scales under delayed feedback [44] have been studied (for further examples, see Ref. [45]). In the present paper, we confine ourselves to systems in which additive noise sources dominate multiplicative and parametric noise sources and, consequently, consider systems that can be described by means of stochastic delay equations of the form

$$\frac{d}{dt}X(t)=h(X(t),X(t-\tau))+\sqrt{Q}\Gamma(t). \tag{3}$$

TABLE I. Examples of biological delay systems.

No.	Type	Phenomenon	Delay τ	τ
1	Reaction chain	Population dynamics [12]	Supply storages, maturation period	
2		HIV infection dynamics [13,14]	Inactive infected phase	1–2 days [14]
3	Transport processes	Neural networks [15,16]	Neural signal transmission times	
4		Breathing [17,18]	Information transmission time	9 s
5		Pupil light reflex [19]	Neural signal transmission times	300 ms
6		Tracking movements [20–22]	Artificial delay of visual feedback	25–50 ms [22]

Moreover, we define $\Gamma(t)$ by a Langevin force with $\langle\Gamma(t)\rangle=0$ and $\langle\Gamma(t)\Gamma(t')\rangle=\delta(t-t')$ [24]. In Sec. II, we will show how to determine the delay equation (3), that is, the unknown quantities h and Q, from experimental data. The focus will be on stationary systems for which the delay τ can either be estimated (see examples 1–5 of Table I) or corresponds to a control variable (see example 6 of Table I) and, consequently, can be fixed (see, however, Ref. [46]). We will distinguish between the linear and nonlinear cases. In the linear case, we will derive the exact stationary solution of the stochastic delay equation (Sec. II A). By means of this solution, the parameters of the linear model can be estimated from experimental data. In the nonlinear case, exact solutions are not available. Therefore, we will propose a data driven method to determine the model equation (3) without any knowledge of the stationary solution (Sec. II B).

II. STATIONARY STATES

A. Linear case: Exact solution and data analysis

In the linear case, Eq. (3) reads

$$\frac{d}{dt}X(t)=-aX(t)-bX(t-\tau)+\sqrt{Q}\Gamma(t). \tag{4}$$

In what follows, we will consider the case: $a\geqslant 0$, $b\geqslant 0$ and $a+b>0$. The probability density of $X(t)$ is given by $P(x,t)=\langle\delta(x-X(t))\rangle$. Similarly, we can define the joint probability density $P(x,t;y,t-\tau)=\langle\delta(x-X(t))\delta(y-X(t-\tau))\rangle$. Then, the delay Fokker-Planck equation associated with Eq. (4) reads

$$\frac{\partial}{\partial t}P(x,t)=\frac{\partial}{\partial x}\left(axP(x,t)+b\int_{\Omega}yP(x,t;y,t-\tau)\mathrm{d}y\right)+\frac{Q}{2}\frac{\partial^2}{\partial x^2}P(x,t), \tag{5}$$

see Refs. [47–49] for details. Since in Eq. (4) the fluctuation force is Gaussian distributed and the drift force is linear, the multivariate probability densities $P_{\mathrm{st}}(x_1,t_1,\ldots,x_N,t_N)=\langle\delta(x_1-X(t_1))\cdots\delta(x_N-X(t_N))\rangle_{\mathrm{st}}$ correspond to multivariate Gaussian distributions (see also Ref. [50]). Therefore, we can substitute the probability densities $P_{\mathrm{st}}(x)$ and $P_{\mathrm{st}}(x,t;y,t-\tau)$ defined by

$$P_{\mathrm{st}}(x)=\frac{1}{\sqrt{2\pi K(\tau)}}\exp\left\{-\frac{x^2}{2K(\tau)}\right\} \tag{6}$$

and

$$P_{\mathrm{st}}(x,t;y,t-\tau)=\frac{c(\tau)\sqrt{1-d^2(\tau)}}{\pi}\exp\{-c(\tau)[x^2+y^2-2d(\tau)xy]\} \tag{7}$$

into Eq. (5) in order to determine the parameters $c(\tau)$ and $d(\tau)$. As shown in Ref. [46] one thus obtains $c(\tau)=2b^2K(\tau)/\{[2bK(\tau)]^2-[Q-2aK(\tau)]^2\}$ and $d(\tau)=[Q-2aK(\tau)]/2bK(\tau)$. Note that $c(\tau)$ and $d(\tau)$ depend on the variance $K(\tau)$ of solution (6). That is, the delay Fokker-Planck equation (7) does not provide us with sufficient information to determine $K(\tau)$ as a function of τ. Therefore, $K(\tau)$ has been derived in two previous studies from the stochastic delay differential equation (4), see Küchler and Mensch [50] and Guillouzic *et al.* [47]. Unfortunately, these studies yield two equations for $K(\tau)$ that are formally different. In addition, both studies do not utilize the delay Fokker-Planck equation (5) and, therefore, are mathematically elaborated. Here, we will discuss an alternative, more compact derivation of $K(\tau)$. This derivation will exploit some computational steps used in both studies and, in this sense, will unify the two approaches discussed in Refs. [47,50]. Moreover, in deriving the variance function $K(\tau)$, we will obtain, as a by-product, the ingredients to determine the parameters a, b, and Q from experimental data.

To begin with, from Eq. (4), we realize that the stationary mean value of $X(t)$ denoted by $\langle X\rangle_{\mathrm{st}}$ satisfies $0=-(a+b)\times\langle X\rangle_{\mathrm{st}}$ and, consequently, equals zero for all $\tau\geqslant 0$. Therefore, the variance $K(\tau)$ is given by the second moment of X: $K(\tau)=\langle X^2\rangle_{\mathrm{st}}$. Taking the boundary conditions of $X(t)\in\Omega=\mathbb{R}$ into account, from the delay Fokker-Planck equation (5) and $d\langle X^2(t)\rangle_{\mathrm{st}}/dt=0$, we obtain

$$2a\langle X^2(t)\rangle_{\rm st}+2b\langle X(t)X(t-\tau)\rangle_{\rm st}=Q \tag{8}$$

by means of partial integration. Following Refs. [47,50] we now examine the stationary autocorrelation function defined by

$$C_\tau(z)=\langle X(t)X(t+z)\rangle_{\rm st}, \tag{9}$$

which satisfies the symmetry relation $C_\tau(z)=C_\tau(-z)$. We assume that $C_\tau(z)$ is a continuous function with respect to z for the stochastic process defined by Eq. (4). In particular, we assume that $C_\tau(z)$ is continuous at $z=0$: $\lim_{z\to 0}C_\tau(z)=C_\tau(0)$. Our objective is to derive $C_\tau(z)$ for all z. Then, $K(\tau)$ can be computed from $C_\tau(z)$ like $K(\tau)=\lim_{z\to 0}C_\tau(z)=C_\tau(0)$. In order to determine $C_\tau(z)$, we will exploit Eq. (8) as obtained from the Fokker-Planck approach, which can be written due to the symmetry relation $C_\tau(-\tau)=C_\tau(\tau)$ in terms of $C_\tau(z)$ as

$$2aC_\tau(0)+2bC_\tau(\tau)=Q. \tag{10}$$

Next, we will derive a differential equation for $C_\tau(z)$, solve the equation, and determine the integration constants by means of the Fokker-Planck approach result, Eq. (10). From Eq. (4), we obtain

$$\begin{aligned}\frac{dC_\tau(z)}{dz}&=\left\langle X(t)\frac{dX(u)}{du}\bigg|_{u=t+z}\right\rangle_{\rm st}\\&=-aC_\tau(z)-bC_\tau(z-\tau)+\sqrt{Q}\langle X(t)\Gamma(t+z)\rangle_{\rm st}.\end{aligned} \tag{11}$$

Using the delay Fokker-Planck equation (5) and the stochastic delay equation (4), we can show that $\langle X(t)\Gamma(t+z)\rangle_{\rm st}$ is given by

$$\sqrt{Q}\langle X(t)\Gamma(t+z)\rangle_{\rm st}=\begin{cases}Q/2 & \text{for } z=0\\ 0 & \text{for } z>0\end{cases}, \tag{12}$$

see the Appendix. Consequently, for $z>0$, Eq. (11) reads

$$\frac{dC_\tau(z)}{dz}=-aC_\tau(z)-bC_\tau(z-\tau) \tag{13}$$

$$=-aC_\tau(z)-bC_\tau(\tau-z). \tag{14}$$

Substituting Eq. (10) into Eq. (14), the right-hand side derivative of $C_\tau(z)$ at $z=0$ can be found as

$$\lim_{z\downarrow 0}\frac{dC_\tau(z)}{dz}=-\frac{Q}{2}. \tag{15}$$

Due to symmetry relation $C_\tau(z)=C_\tau(-z)$, we conclude that the corresponding left-hand side derivative is given by

$$\lim_{z\uparrow 0}\frac{dC_\tau(z)}{dz}=\frac{Q}{2}. \tag{16}$$

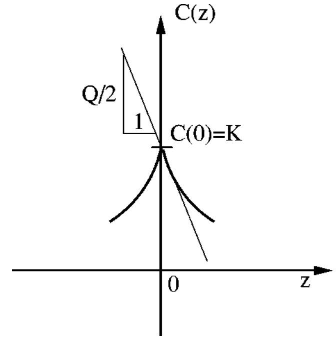

FIG. 1. Shape of the autocorrelation function $C_\tau(z)$ around $z=0$ (thick line). $dC_\tau(z)/dz$ is not continuous at $z=0$. The slope at $z=0$ is given by $Q/2$ (thin line).

Consequently, dC_τ/dz is not continuous at $z=0$. Moreover, if we evaluate Eq. (11) for $z=0$ and take Eq. (10) into account, we see that at $z=0$ the derivative of $C_\tau(z)$ vanishes:

$$\frac{dC_\tau(0)}{dz}=-aC_\tau(z)-bC_\tau(-\tau)+\frac{Q}{2}=0. \tag{17}$$

These results are summarized in Fig. 1.

Now, let us solve Eq. (13) and determine $K(\tau)=C_\tau(0)$ by means of the limiting case $C_\tau(0)=\lim_{z\to 0}C_\tau(z)$. To this end, we differentiate Eq. (14) for $z>0$ with respect to z which gives us

$$\begin{aligned}\frac{d^2C_\tau(z)}{dz^2}&=-a\frac{dC_\tau(z)}{dz}+b\frac{dC_\tau(u)}{du}\Big|_{u=\tau-z}\\&=-a\left[\frac{dC_\tau(z)}{dz}+bC_\tau(\tau-z)\right]-b^2C_\tau\end{aligned} \tag{18}$$

$$=(a^2-b^2)C_\tau(z). \tag{19}$$

Note that in order to derive Eq. (19) from Eq. (18), we have used $C(z)=C(-z)$ and Eq. (13). We put $\omega=\sqrt{|a^2-b^2|}$ and distinguish three cases: (1) $b>a\geqslant 0$, (2) $a>b\geqslant 0$, and (3) $a=b>0$. Then, for $z\geqslant 0$, the solutions of Eq. (19) are described by

$$C_\tau^1(z)=C_\tau^1(0)\cos(\omega z)+e\sin(\omega z), \tag{20}$$

$$C_\tau^2(z)=C_\tau^2(0)\cosh(\omega z)+e'\sinh(\omega z), \tag{21}$$

$$C_\tau^3(z)=C_\tau^3(0)+fz, \tag{22}$$

where $C^i(0)$, e, e', f are so far unknown variables and $C^i(0)$ denotes the limit value $\lim_{z\downarrow 0}C_\tau^i(z)=C_\tau^i(0)$. The variables e,

e', f can be determined by means of Eq. (15). Thus, we get $e=e'=-Q/[2\omega]$ and $f=-Q/2$ and

$$C_\tau^1(z)=C_\tau^1(0)\cos(\omega z)-\frac{Q}{2}\frac{\sin(\omega z)}{\omega}, \tag{23}$$

$$C_\tau^2(z)=C_\tau^2(0)\cosh(\omega z)-\frac{Q}{2}\frac{\sinh(\omega z)}{\omega}, \tag{24}$$

$$C_\tau^3(z)=C_\tau^3(0)-\frac{Q}{2}z \tag{25}$$

for $z\geqslant 0$. We would like to emphasize that these relations hold for $z\geqslant 0$ only. For $z\leqslant 0$, we can compute $C_\tau^i(z)$ from the symmetry condition $C_\tau^i(z)=C_\tau^i(-z)$. Consequently, for $z\in\mathbb{R}$, we obtain

$$C_\tau(z)=\begin{cases} C_\tau^1(z)=C_\tau^1(0)\cos(\omega z)-\frac{Q}{2}\omega^{-1}\sin(\omega|z|), & b>a\geqslant 0\\ C_\tau^2(z)=C_\tau^2(0)\cosh(\omega z)-\frac{Q}{2}\omega^{-1}\sinh(\omega|z|), & a>b\geqslant 0\\ C_\tau^3(z)=C_\tau^3(0)-\frac{Q}{2}|z|, & a=b>0. \end{cases} \tag{26}$$

Finally, let us determine $C_\tau^i(0)$ by means of Eq. (10) obtained from the delay Fokker-Planck equation (5). Substituting Eqs. (23)–(25) into Eq. (10), the variances $C_\tau^i(0)$ are found to be

$$K(\tau)=C_\tau(0)$$
$$=\begin{cases} C_\tau^1(0)=\frac{Q}{2}\left(\frac{1+b\omega^{-1}\sin(\omega\tau)}{a+b\cos(\omega\tau)}\right), & b>a\geqslant 0\\ C_\tau^2(0)=\frac{Q}{2}\left(\frac{1+b\omega^{-1}\sinh(\omega\tau)}{a+b\cosh(\omega\tau)}\right), & a>b\geqslant 0\\ C_\tau^3(0)=\frac{Q}{2}\left(\frac{1+b\tau}{a+b}\right)=\frac{Q}{4a}(1+a\tau), & a=b>0. \end{cases} \tag{27}$$

We realize that the limiting cases $\lim_{b\downarrow a,\omega\to 0}C_\tau^1(0)=C_\tau^3(0)$ and $\lim_{a\downarrow b,\omega\to 0}C_\tau^2(0)=C_\tau^3(0)$ hold. Furthermore, in the limit of vanishing delay, we reobtain the variance of the Ornstein-Uhlenbeck process [24]: we have $C_0^i(0)=Q/[2(a+b)]$ for $i=1,2$ and $C_0^3(0)=Q/[4a]$. Furthermore, our analysis includes the well-known result for the Ornstein-Uhlenbeck process as a special case. For the Ornstein-Uhlenbeck process given by $\dot{v}(t)=-av(t)+\sqrt{Q}\Gamma(t)$, the stationary autocorrelation reads $\langle v(t)v(t+z)\rangle_{\rm st}=Q[2a]^{-1}\exp\{-a|z|\}$ and is continuous with respect to z but not continuously differentiable at $z=0$ (Chap. 3 of Ref. [24]). For $a>b=0$, we reobtain this result by recognizing that $C_\tau^2(0)$ and ω reduce to $C_\tau^2(0)=Q/[2a]$ and $\omega=a$, respectively, and $C_\tau^2(z)$ is given by $C_\tau^2(z)=Q[2a]^{-1}[\cosh(az)-\sinh(a|z|)]$ which leads to $C_\tau^2(z)=Q[2a]^{-1}[\cosh(a|z|)-\sinh(a|z|)]=Q[2a]^{-1}\exp\{-a|z|\}$.

At this junction, it is useful to briefly discuss the domains of definition of the stationary probability densities (6) related to the three cases $b>a\geqslant 0$, $a>b\geqslant 0$, and $a=b>0$. We first consider $b>a\geqslant 0$, that is, $C_\tau^1(0)$. For $b>a\geqslant 0$ and $\tau\geqslant 0$, we have $b/\omega\leqslant 1\Rightarrow 1+b\omega^{-1}\sin(\omega\tau)\geqslant 0$. That is, the numerator of $C_\tau^1(0)$ does not vanish. As far as the denominator is concerned, we realize that for $b>a\geqslant 0$, there exists a τ^* with $\tau^*\omega\in[0,\pi]$ such that $a+b\cos(\omega\tau^*)=0$. For $\tau\in[0,\tau^*)$ we have $a+b\cos(\omega\tau^*)>0$. The upper limit τ^* is explicitly defined by $\tau^*\omega=\arccos\{-a/b\}<\pi$. In view of these considerations, the variance $K(\tau)$ becomes infinite in the limit $\tau\to\tau^*$ and the stationary solution (6) exists only for $\tau\in[0,\tau^*)$. Next, let us consider $C_\tau^i(0)$ with $i=2,3$. From the definitions of $C_\tau^i(0)$ in Eq. (27), we read off that $C_\tau^i(0)<\infty$ for all $0\leqslant\tau<\infty$ and $i=2,3$. Consequently, for $a>b\geqslant 0$ and $a=b>0$ the Gaussian solution (6) exists for every delay τ. Table II summarizes these results.

Equations (26) and (27) have previously been derived by Küchler and Mensch [50] using an approach different from the one presented here. Actually, our departure point (11) is the one used by Guillouzic *et al.* [47]. Therefore, we have shown here that the approaches by Küchler and Mensch, on the one hand, and Guillouzic *et al.*, on the other hand, lead to the same result. Moreover, we are now in the position to analyze stochastic delay systems with respect to the linear model (4).

Our departure point is the hypothesis that we deal with a system that can be described by the linear stochastic delay equation (4). First of all, we can corroborate our hypothesis by determining the stationary probability density $P_{\rm st}(x)=\langle\delta(x-X(t))\rangle_{\rm st}$ from experimental data $X(t)$. If $P_{\rm st}$ corre-

TABLE II. Definition domains of the Gaussian probability density (6).

$b>a\geqslant 0$	$a>b\geqslant 0$	$a=b>0$
$\tau\in[0,\omega^{-1}\arccos(-a/b)]$	$\tau\geqslant 0$	$\tau\geqslant 0$

sponds to a Gaussian distribution, we have found support for our hypothesis. If P_{st} differs substantially from a Gaussian distribution, our hypothesis is falsified. Alternatively, we introduce a pseudoforce $\tilde{h}(x)$ defined by

$$\tilde{h}(x) = -\frac{d}{dx}\ln P_{st}(x). \tag{28}$$

From Eq. (6) we appreciate that for a linear stochastic delay system, $\tilde{h}(x)$ is linear with respect to x. Consequently, the observation of a linear pseudoforce $\tilde{h}(x)$ would support our hypothesis that we deal with a linear delay system. Next, we can determine the parameters a, b, and Q for model (4) from experimental data. Q can be derived from Eq. (15):

$$Q = -2\frac{dC_\tau(z)}{dz}\bigg|_{z=0^+}. \tag{29}$$

To determine the parameters a and b, we need two equations involving these parameters. On the basis of the delay Fokker-Planck equation (5) we can derive various equations involving a and b and measurable correlation functions of the form $\langle X^n(t)X^m(t-\tau)\rangle_{st}$. One example is given by Eq. (8). Another example can be derived by multiplying Eq. (5) with x^4 and by integrating the equation thus obtained with respect to x. Then, integration by parts leads to

$$2a\langle X^4(t)\rangle - 2b\langle X^3(t)X(t-\tau)\rangle = 3Q. \tag{30}$$

From Eqs. (8) and (30) and Q given by Eq. (29), the parameters a and b can be computed from

$$b = \frac{Q}{2}\,\frac{3K^2 - \langle X^4(t)\rangle_{st}}{K\langle X^3(t)X(t-\tau)\rangle_{st} - C_\tau(\tau)\langle X^4(t)\rangle_{st}},$$

$$a = \frac{Q - 2bC_\tau(\tau)}{2K} \tag{31}$$

with $K = K(\tau) = \langle X^2(t)\rangle_{st}$ and $C_\tau(\tau) = C_\tau(-\tau) = \langle X(t)X(t-\tau)\rangle_{st}$ (see above). Having determined the parameters a, b, and Q from experimental data, we can check once again whether or not the data are consistent with our hypothesis that we deal with a linear delay system. To this end, we may plot $P_{st}(x,t;y,t-\tau;a,b,Q)$ as predicted by Eq. (7) versus $P_{st}(x,t;y,t-\tau)$ as obtained from the experimental data (for the sake of convenience, we may restrict our attention to some moments of the form $\langle X^n(t)X^m(t-\tau)\rangle_{st}$ which can easily be derived from experimental data, on the one hand, and computed from Eq. (7), on the other hand). If the theoretical result agrees well with the empirical one, we have a strong indication that our hypothesis was correct. In particular, the data analysis based on the exact stationary solution given by Eqs. (6) and (27) is tailored to delay systems for which τ is a control variable (e.g., see example 6 of Table I). If such a delay system is linear or can be linearized (e.g., for weak noise sources, See Ref. [46]), we can plot $K(\tau)$ as obtained from the experiment against $K(\tau)$ as predicted by the theory and, in doing so, demonstrate that the linear delay equation (4) can be regarded as an appropriate model for the system under consideration.

B. Nonlinear case: Multivariate Markov process and data analysis

1. Data analysis based on multivariate Markov processes

We now consider the nonlinear stochastic delay equation (3). Before we show how to derive $h(x,y)$ and Q from experimental data, we briefly review some fundamental aspects of the data analysis of Markov processes [51–55]. Let $X(t) \in \Omega = \mathbb{R}$ describe a Markov process defined by the Langevin equation $\dot{X}(t) = h(X) + \sqrt{Q}\,\Gamma(t)$ for $t \geqslant 0$ with $X(0) = x_0$. Then, $h(x)$ corresponds to the first Kramers-Moyal coefficient $D(x)$ and, by definition of D, can be computed from $h(x) = D(x) = \lim_{\Delta t\to 0}\langle X(t+\Delta t) - X(t)\rangle_{X(t)=x}/\Delta t$ which can be expressed alternatively as $h(x) = D(x) = \lim_{\Delta t\to 0}[\Delta t]^{-1}\int_\Omega (y-x)P(y;t+\Delta t|x,t)dy$ using the conditional probability density $P(x;t|x',t') = P(x,t;x',t')/\int_\Omega P(x,t;x',t')dx$ with $P(x,t;x',t') = \langle \delta(x - X(t))\delta(x' - X(t'))\rangle$. Similarly, we may consider a multivariate Markov process given by the random vector $\mathbf{X}(t) = (X_1, \ldots, X_N) \in \mathbb{R}^N$ satisfying

$$\frac{d}{dt}X_k(t) = h_k(\mathbf{X}) + \sqrt{Q}\Gamma_k(t) \tag{32}$$

for $k = 1, \ldots, N$ with $\langle \Gamma_k(t)\rangle = 0$ and $\langle \Gamma_k(t)\Gamma_l(t')\rangle = \delta_{kl}\delta(t-t')$ [24]. Then, the force vector $\mathbf{h} = (h_1, \ldots, h_k)$ can be computed from the first Kramers-Moyal coefficients as

$$h_k(\mathbf{x}) = D_k(\mathbf{x}) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle X_k(t+\Delta t) - X_k(t)\rangle_{\mathbf{X}(t)=\mathbf{x}}. \tag{33}$$

In the multivariate case, the computation of $\mathbf{h}(\mathbf{x})$ can be simplified if there is additional information available regarding the arguments $\mathbf{x} = (x_1, \ldots, x_N)$ of $\mathbf{h}(\mathbf{x})$. If we know *a priori* that $h_k(\mathbf{x})$ depends only on M variables with $M < N$, say, on $x_{k_1}, \ldots, x_{k_M}$, then Eq. (33) becomes

$$h_k(x_{k_1}, \ldots, x_{k_M}) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\langle X_k(t+\Delta t) - X_k(t)\rangle_{X_{k_1}(t)=x_{k_1}, \ldots, X_{k_M}(t)=x_{k_M}}. \tag{34}$$

So far, this fundamental technique for the analysis of Markov processes has been applied to fluid dynamics, traffic flows, engineering problems, tremor data, economics, and so on (for references see the preceding). We would like to point out that this procedure can also be applied to non-Markov processes described by Eq. (32) where Γ_k denote general fluctuation forces with $\langle \Gamma_k\rangle = 0$. In this case, the limiting procedures $\Delta t \to 0$ and $\langle \cdots \rangle$ are interchanged (i.e., in Eqs. (33) and (34), we have $\langle \lim_{\Delta t\to 0}\cdots\rangle$ instead of $\lim_{\Delta t\to 0}\langle\cdots\rangle$).

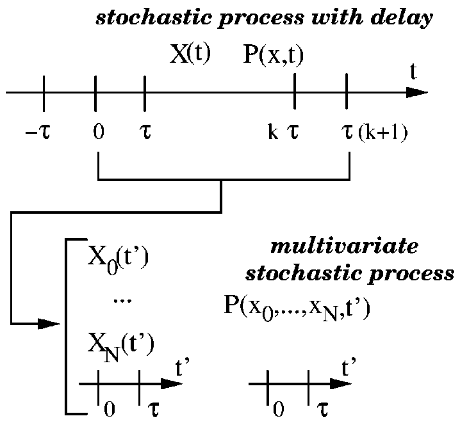

FIG. 2. Description of a stochastic process with delay in terms of a multivariate stochastic process without delay.

Since we eventually use a small but finite Δt to carry out the limit $\Delta t \rightarrow 0$ and a large but finite number of samples to compute the mean $\langle \cdots \rangle$, the order of the limiting procedures is irrelevant.

In the following, we will show how to apply the data analysis technique to data obtained from stochastic delay systems. To this end, we build on a recent study in which it was demonstrated that stochastic delay systems satisfying Eq. (3) can alternatively be described by multivariate Markov processes [49]. Let $X(t)$ be defined by Eq. (3). Then, for $t \in [k\tau,(k+1)\tau]$ and $k=0,1,2,\ldots$, we can assign to $X(t)$ a $(k+1)$-dimensional Markov process described by the random variables $X_i(t')$ satisfying

$$\frac{d}{dt'}X_i(t') = h(X_i(t'),X_{i-1}(t')) + \sqrt{Q}\Gamma_i(t') \qquad (35)$$

for $i=0,\ldots,k$ and $t' \in [0,\tau]$, where Γ_i denote again statistically independent Langevin forces defined by $\Gamma_i(t-i\tau) = \Gamma(t)$ and X_{-1} is defined by $X_{-1}(z) = \phi(z-\tau)$, see Fig. 2.

This multivariate Markov process is equivalent to the stochastic process $X(t)$ defined by the delay equation (3) in the sense that for every k, we have

$$X(t) = X_k(t-k\tau) \qquad (36)$$

or $X(t)=X_k(t')$ with $t'=t-k\tau$. The drift function $h(x,y)$ in Eq. (35) can now be computed from the random variables X_i and X_{i-1}. From Eq. (34), it follows that

$$h(x,y) = \lim_{\Delta t' \rightarrow 0}\frac{1}{\Delta t'}\langle X_k(t'+\Delta t') - X_k(t')\rangle_{X_k(t')=x,X_{k-1}(t')=y}. \qquad (37)$$

Equations (36) and (37) can be used to determine $h(x,y)$ within an interval $t \in [k\tau,(k+1)\tau]$ of length τ based on the experimental data $X(t)$. In order to improve the accuracy of the average $\langle \cdots \rangle$ occurring in Eq. (37), we need to circumvent the constraint $t \in [k\tau,(k+1)\tau]$. To this end, we consider the joint probability density $P_k(x,y) = \langle \delta(x - X_k(t'))\delta(y - X_{k-1}(t'))\rangle$ and recognize that in the stationary case $t \rightarrow \infty$ there exists a limit distribution $P_{st}(x,y)$ of the form $\lim_{k\rightarrow\infty}P_k(x,y) = P_{st}(x,y)$. Consequently, if the system under investigation exhibits a stationary behavior for this stationary condition, index k becomes arbitrary. Using $X_k(t') = X(t)$ and $X_{k-1}(t') = X(t-\tau)$ and $P_{st}(x,y) = \langle \delta(x - X(t))\delta(y - X(t-\tau))\rangle$, from Eq. (37) we obtain

$$h(x,y) = \lim_{\Delta t \rightarrow 0}\frac{1}{\Delta t}\langle X(t+\Delta t) - X(t)\rangle_{st;X(t)=x,X(t-\tau)=y}, \qquad (38)$$

where $X(t)$ and $X(t-\tau)$ denote stationary random variables of the process under consideration—as indicated. Finally, at issue is to determine the fluctuation strength Q of a stochastic delay system with additive noise. To this end, consider the delay Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\int_{\Omega} h(x,y)P(x,t;y,t-\tau)dy + \frac{Q}{2}\frac{\partial^2}{\partial x^2}P(x,t) \qquad (39)$$

that corresponds to Eq. (3). Multiplying Eq. (39) by x^2, integrating with respect to x, we obtain for the stationary case the equivalence

$$Q = 2\langle X(t)h(X(t),X(t-\tau))\rangle_{st}. \qquad (40)$$

That is, having obtained $h(x,y)$ from Eq. (38), the average (40) gives us the fluctuation strength Q.

2. Examples

Our next objective is to demonstrate that Eq. (38) can also be used under the hypothesis that we deal with ergodic systems. Then, the ensemble average in Eq. (38) can be replaced by the time average

$$h(x,y) = \lim_{\Delta t \rightarrow 0}\frac{1}{\Delta t}\lim_{N_0 \rightarrow \infty}\frac{1}{N_0} \times \sum_{i=1}^{N_0}[X(t_i+\Delta t) - X(t_i)]_{X(t_i)=x,X(t_i-\tau)=y}. \qquad (41)$$

We illustrate the power of Eq. (41) by evaluating artificially generated trajectories $X(t)$. To this end, we simulate stochastic delay equations of form (3) by means of an Euler forward scheme, where $\Gamma(t)$ is obtained from a Box-Müller algorithm and time is discretized in steps δt. The limiting case $\Delta t \rightarrow 0$ is then realized by putting $\Delta t = \delta t$.

As a first example, we use the linear stochastic delay equation (4). First, we have computed $X(t_i)$ from Eq. (4). Subsequently, we have evaluated the trajectory $X(t_i)$ by means of Eq. (41). Figure 3 illustrates that the model equation (4) can indeed be reproduced by the proposed data analysis technique. Second, we consider a model proposed by Tass *et al.* for oscillatory tracking movements under delayed visual feedback [22]. The model equation reads

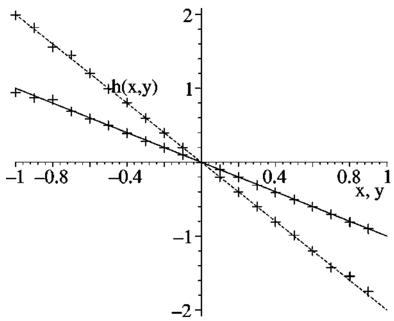

FIG. 3. Data analysis for the linear model (4). Drift $h(x,y)$ plotted versus x and y for $h(x,0)$ (dashed line) and $h(0,y)$ (solid line). Crosses indicate results obtained from the data analysis. Lines describe the model equation. Evaluation of h in 20 bins of width $\Delta x = \Delta y = 0.1$. Parameters: $a=2$, $b=1$, $\tau=0.2$, $Q=1$, $N_0=10^8$, and $\delta t = 0.02$.

$$\frac{d}{dt}X(t) = -a\sin([\Omega X(t)-\phi_0] - b\sin[\Omega X(t-\tau)] + \sqrt{Q}\Gamma(t), \tag{42}$$

where Ω describes the oscillation frequency. $X(t)$ describes the relative phase between the tracking movement and the target motion and denotes a periodic variable defined on $[0,2\pi]$. For example, $X=0$ corresponds to a perfect synchronization of tracking and target movements. According to Eq. (42), tracking movements involve two kinds of control mechanisms. Movement control due to proprioceptive feedback is described by the term $a\sin[\Omega X(t)-\phi_0]$ with $a \geqslant 0$, whereas the impact of the visual system is modeled by the expression $b\sin[\Omega X(t-\tau)]$ with $b \geqslant 0$. We assume that the artificially introduced delay $\tau_{\rm ext}$ dominates intrinsic delays of the visual system such that the delay τ in Eq. (42) is approximatively given by $\tau_{\rm ext}$. The variable ϕ_0 accounts in a crude manner for intrinsic delays of the visual and proprioceptive systems [see Eq. (39) in Ref. [46] and put $\phi(t)+\Omega\tau'_{vis}/2 = X(t)$]. We have simulated Eq. (42) and evaluated a stationary trajectory by means of Eq. (41). The result is shown in Fig. 4. Finally, we consider the tanh model given by

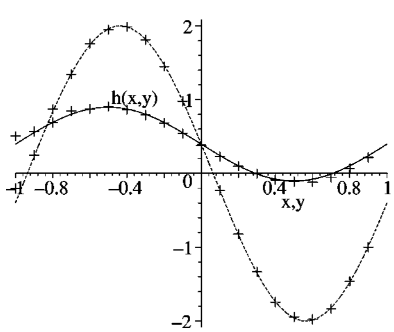

FIG. 4. Data analysis for the delay model (42). Drift $h(x,y)$ as a function of x and y for $h(x,0)$ (dashed line) and $h(0,y)$ (solid line). Crosses represent results obtained from the data analysis. Lines describe the model equation. Evaluation of h in 20 bins of width $\Delta x = \Delta y = 0.1$. Parameters: $a=2$, $b=0.5$, $\tau=0.2$, $Q=1$, $\phi_0=-0.2$, $\Omega=\pi$, $N_0=10^8$, and $\delta t=0.02$.

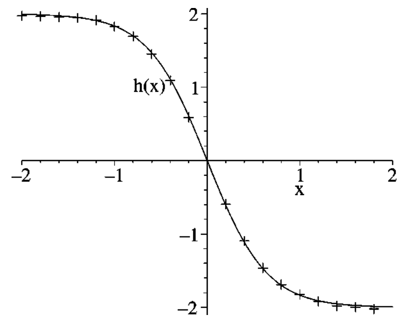

FIG. 5. Data analysis for the tanh model (43). Results obtained from Eq. (43) (crosses) are plotted against the drift function $h(y) = -a\tanh(cy)$ (solid line). Parameters: $a=2$, $c=\pi/2$, $\tau=0.1$, $Q=1$, $N_0=10^8$, $\delta t=0.01$, and $\Delta x=0.2$.

$$\frac{d}{dt}X(t) = -a\tanh[cX(t-\tau)] + \sqrt{Q}\Gamma(t) \tag{43}$$

for $X(t)\in\mathbb{R}$ and $a>0$, $c>0$. Similar models with drift forces that vanish in the asymptotic limit $x\to\pm\infty$ have been used to describe coordinated human finger movements [56] and have been discussed in the context of stochastic resonance [2]. In particular, human motor performance during pointing tasks has been modeled in terms of a tanh model with parametric noise that reads like $dX/dt = -a\tanh[cX(t-\tau)+\sqrt{Q'}\Gamma'(t)]$, where $\Gamma'(t)$ denotes a fluctuation force and Q' describes a fluctuation strength [21]. As pointed out in Ref. [21], such a model, however, does not take fluctuations into account that directly affect finger positions. Such fluctuations may be described by extending the original model to obtain $dX/dt = -a\tanh[cX(t-\tau)+\sqrt{Q'}\Gamma'(t)] + \sqrt{Q}\Gamma(t)$. In line with our remarks in Sec. I, we may transform the parametric model into a model including a multiplicative noise source like $dX/dt = -a\tanh[cX(t-\tau)] - a\sqrt{Q'}\Gamma'(t)/\cosh^2[cX(t-\tau)] + \sqrt{Q}\Gamma(t)$. If additive noise dominates multiplicative noise (e.g., for $Q' \ll Q$), we obtain Eq. (43). Equation (43) corresponds to a nonlinear stochastic delay equation of the form

$$\frac{d}{dt}X(t) = h(X(t-\tau)) + \sqrt{Q}\Gamma(t). \tag{44}$$

For stochastic processes of this kind, Eq. (38) reduces to

$$h(y) = \frac{1}{\Delta t}\langle X(t+\Delta t) - X(t)\rangle|_{{\rm st};X(t-\tau)=y} \tag{45}$$

and Eq. (41) becomes

$$h(y) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\lim_{N_0\to\infty}\frac{1}{N_0}\sum_{i=1}^{N_0}[X(t_i+\Delta t) - X(t_i)]_{X(t_i-\tau)=y}. \tag{46}$$

We have numerically solved Eq. (43) and analyzed the stationary solution by means of Eq. (46). The result of the data analysis versus the model equation is shown in Fig. 5.

III. CONCLUSIONS

For nondelayed Markov diffusion processes $X(t) \in \Omega = \mathbb{R}$ with additive noise satisfying the Langevin equation $\dot{X}(t) = h(X) + \sqrt{Q}\,\Gamma(t)$, the fluctuation strength Q can be determined using the definition of the second Kramers-Moyal coefficient and the drift force h can be derived from the experimentally observed stationary probability density $P_{\rm st}$ according to $h(x) = -Q^{-1} d \ln P_{\rm st}(x)/dx$. The reason for this is that there is an analytical relationship between $P_{\rm st}$ and h, namely, $P_{\rm st}(x) = Z^{-1}\exp\{-V(x)/Q\}$ with $Z^{-1} = \int_\Omega \exp\{-V(x)/Q\}dx$ and $V(x) = -\int^x h(x')dx'$, which is invertible.

For stochastic delay systems subjected to additive noise, we proposed two methods to determine in the stationary case from experimental data the drift forces acting on the systems. If the drift forces are linear, we can exploit the aforementioned approach for Markov processes because in this case the analytical form of the stationary probability density $P_{\rm st}$ can be determined. We have shown that from autocorrelation functions [such as $K = \langle X^2(t)\rangle_{\rm st}$, $\langle X^4(t)\rangle_{\rm st}$, $\langle X(t)X(t-\tau)\rangle_{\rm st}$, and $\langle X^3(t)X(t-\tau)\rangle_{\rm st}$], we can derive all parameters (i.e., a, b, and Q) of linear stochastic delay systems. Having obtained these parameters by means of the exact solutions for the stationary probability density $P_{\rm st}(x;K)$ and the stationary joint probability density $P_{\rm st}(x,t;y,t-\tau;a,b,Q)$, we can in turn check our hypothesis that we are dealing with a linear delay system. In order to illustrate this approach, we have derived the stationary probability density $P_{\rm st}$ for all parameters τ using an approach different from previous approaches by Küchler and Mensch, on the one hand, and Guillouzic *et al.*, on the other hand. In particular, we have demonstrated how to take advantage of the delay Fokker-Planck equation proposed by Guillouzic *et al.* [47]. For nonlinear stochastic delay systems, analytical descriptions of stationary probability densities are not available. Consequently, a counterpart to the relation $h(x) = -Q^{-1} d \ln P_{\rm st}(x)/dx$ mentioned in the preceding is not available. We are confronted with a situation known from the theory of multivariate Markov processes described by Fokker-Planck equations. There are a numerous multivariate stochastic processes for which analytical expressions for stationary distributions are not available (e.g., Lorenz attractor with white noise). In such systems, drift forces can be determined from experimental data using a direct approach that does not require the knowledge about the stationary probability density of the system under investigation, e.g., Refs. [53,54]. Using this direct approach in combination with a reinterpretation of delay systems as multivariate nondelayed systems, we developed a data driven method in order to determine the drift forces of nonlinear stochastic delay systems.

Our considerations were centered around stochastic systems involving additive noise sources. A more general class of stochastic delay systems is described by Eq. (1). However, for systems of this kind, it is not at all clear that how to interpret nonlinear expressions of $\Gamma(t)$ such as $(X+\Gamma)^2$ or $\tanh(X+\Gamma)$ [25]. For example, if $\Gamma(t)$ denotes the random variable of an Ornstein-Uhlenbeck process, a stochastic differential equation as simple as $dX/dt = -aX + \Gamma^2(t)$ requires a tedious and mathematically involved analysis [57]. Therefore, as pointed out in the Introduction, a useful strategic approach is to consider only weak parametric noise sources and to transform parametric models into models with multiplicative noise sources. Consequently, in order to generalize the results derived in the present paper, one may use Eq. (2) rather than Eq. (1) as a departure point.

APPENDIX A: DERIVATION OF EQ. (12)

For a stochastic process defined by the linear delay equation (4), the evolution of $\langle X^2(t)\rangle$ can be obtained from Eqs. (4) and (5):

$$\frac{d}{dt}\langle X^2(t)\rangle = -2a\langle X^2(t)\rangle - 2b\langle X(t)X(t-\tau)\rangle + 2\sqrt{Q}\langle X(t)\Gamma(t)\rangle \tag{A1}$$

and

$$\frac{d}{dt}\langle X^2(t)\rangle = -2a\langle X^2(t)\rangle - 2b\langle X(t)X(t-\tau)\rangle + Q \tag{A2}$$

(see also Ref. [6]). By comparing these results, we obtain

$$\langle X(t)\Gamma(t)\rangle = \frac{\sqrt{Q}}{2}. \tag{A3}$$

At issue is now to determine the expression $\langle X(t)\Gamma(t+z)\rangle$ for $z>0$. To this end, we will make use of the solution of the initial value problem $\dot{z}(t) = -az(t) + f(t)$ and $z(t_0) = z_0$ which reads $z(t) = z_0\exp\{-a(t-t_0)\} + \int_{t_0}^t \exp\{-a(t-t')\}f(t')\,dt'$. In recognition of this solution, Eq. (4) can be solved using the method of steps. First, we solve Eq. (4) for $t\in[0,\tau]$ which gives us

$$X(t) = Y_1(t) = \phi(0)e^{-at} + \int_0^t e^{-a(t-t')}[\sqrt{Q}\Gamma(t') - b\phi(t'-\tau)]dt' \tag{A4}$$

where we have introduced the function Y_1. In particular, we have

$$X(\tau) = Y_1(\tau) = \phi(0)e^{-a\tau} + \int_0^\tau e^{-a(\tau-t')}[\sqrt{Q}\Gamma(t') - b\phi(t'-\tau)]dt'. \tag{A5}$$

Second, we solve Eq. (4) for $t\in[\tau,2\tau]$ which gives us

$$X(t) = Y_2(t) = Y_1(t)e^{-a(t-\tau)} + \int_\tau^t e^{-a(t-t')} \times[\sqrt{Q}\Gamma(t') - bY_1(t'-\tau)]dt', \tag{A6}$$

where we have defined the function Y_2. In detail, Eq. (A6) reads

$$X(t)=\phi(0)e^{-at}+\int_0^{\tau}e^{-a(\tau-t')}[\sqrt{Q}\Gamma(t')-b\phi(t'-\tau)]dt'$$
$$+\int_{\tau}^{t}e^{-a(t-t')}\Bigg[\sqrt{Q}\Gamma(t')-b\phi(0)e^{-a(t'-\tau)}$$
$$-b\int_0^{t'-\tau}e^{-a(t'-\tau-t'')}\{\sqrt{Q}\Gamma(t'')$$
$$-b\phi(t''-\tau)\}dt''\Bigg]dt'. \qquad \text{(A7)}$$

We can proceed with this approach introducing step by step auxiliary functions Y_n in order to solve Eq. (4) for $t \in [n\tau,(n+1)\tau]$. In doing so, we realize that $X(t)$ can be written as a superposition of terms L_i like

$$X(t)=\sum_{i=1}^{M(n)} L_i[\phi,\Gamma(t'\leq t)] \qquad \text{(A8)}$$

with $M(n)<\infty$ for a finite time t. It is clear from Eqs. (A4)–(A7) that the fluctuation term $\Gamma(t')$ occurs in L_i with argument $t'\geq t$ as indicated in Eq. (A8). This fact basically reflects the causality of the stochastic process described by Eq. (4). Furthermore, from Eqs. (A4)–(A7), we can deduce that the terms L_i are linear with respect to Γ: $L_i[\phi,c_1f_1(t')+c_2f_2(t')]=c_1L_i[\phi,f_1(t')]+c_2L_i[\phi,f_2(t')]$. In particular, we have $L_i[\phi,0]=0$. With Eq. (A8) at hand, we can show that the average $\langle X(t)\Gamma(t+z)\rangle$ vanishes for $z>0$:

$$\langle X(t)\Gamma(t+z)\rangle=\sum_i \langle \Gamma(t+z)L_i[\phi,\Gamma(t'\leq t)]\rangle$$
$$=\sum_i L_i[\phi,\langle\Gamma(t+z)\Gamma(t'\leq t)\rangle]$$
$$=\sum_i L_i[\phi,0]=0. \qquad \text{(A9)}$$

Taking Eqs. (A3) and (A9) together, we get

$$\sqrt{Q}\langle X(t)\Gamma(t+z)\rangle=\begin{cases}Q/2 & \text{for } z=0\\ 0 & \text{for } z>0\end{cases}, \qquad \text{(A10)}$$

for the nonstationary case. In the limit $t\to\infty$, the arguments developed so far still hold. In particular Eqs. (A1) and (A2) hold, and Eq. (A9) holds with $M(n)\to\infty$ for $t\to\infty$. Consequently, in the stationary case, Eq. (A10) becomes Eq. (12).

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