Complete description of a generalized Ornstein-Uhlenbeck process related to the nonextensive Gaussian entropy

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Abstract

We consider a generalized Ornstein-Uhlenbeck process described by a nonlinear Fokker-Planck equation related to the one-parametric, nonextensive Gaussian entropy, which is a special case of the two-parametric Sharma-Mittal entropy. We derive the entire hierarchy of distribution functions for that process and, in doing so, derive for the first time the complete description of a stochastic process related to a nonextensive entropy measure.

Key words: nonlinear families of Markov processes; nonlinear Fokker-Planck equations
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A key quantity in the theory of stochastic processes is the probability density $P(x,t;u) = \langle \delta(x - X(t)) \rangle$ of a stochastic process. Here, $u(x)$ is the initial distribution of the process described by the random variable $X(t) \in \Omega$ at $t = t_0$. The relevance of the quantity $P(x,t;u)$ becomes apparent if we bear in mind that all expectation values and in particular all moments and cumulants of the

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process under consideration can be computed from $P(x,t;u)$ [1,2]. Having said this, we realize however that the information that can be drawn from $P(x,t;u)$ is limited. For example, the probability density $P(x,t;u)$ fails to provide us with information about autocorrelation functions of $X(t)$. In general, $P(x,t;u)$ does not describe completely a stochastic process. That is, we cannot compute from $P(x,t;u)$ the hierarchy of joint probability densities

$$P(x,t;u)\ ,\ \ P(x,t;x',t';u)\ ,\ \ P(x,t;x',t';x'',t'';u)\ ,\ldots \tag{1}$$

Recently, this issue has been pointed out in the context of stochastic processes described by nonlinear Fokker-Planck equations [3–5]. In several studies it has been suggested to add information to nonlinear Fokker-Planck equations in terms of Langevin equations in order to obtain complete descriptions of stochastic processes [3,6–8]. From such Langevin equations we can determine all stochastic quantities of interest at least by solving them numerically. However, it is a tedious work to derive analytical expression for the probability densities (1) from Langevin equations. In recognition of this fact, on may alternatively write down evolution equations for Markov transition probability densities $P(x,t|x',t';u)$ [5]. In contrast to transition probability densities of linear Fokker-Planck equations, the transition probability densities $P(x,t|x',t';u)$ depend on initial distributions $u(x)$ and, consequently, describe so-called nonlinear families of Markov processes [5]. From $P(x,t;u)$ and $P(x,t|x',t';u)$ all hierarchy members (1) can be obtained:

$$\begin{aligned}
&P(x,t;x',t') = P(x,t|x',t';u)P(x',t';u)\ ,\\
&P(x,t;x',t';x'',t'') = P(x,t|x',t';u)P(x,t|x'',t'';u)P(x'',t'';u)\ ,\\
&\ldots
\end{aligned} \tag{2}$$

In particular, by means of this approach, we can treat the free energy Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}P\frac{\partial}{\partial x}\frac{\delta F}{\delta P} \tag{3}$$

that involves a free energy functional $F[P] = U[P] - QS[P]$ composed of an internal energy functional U and an entropy and information measure S [9]. Here, Q denotes the noise amplitude of a stochastic process. Note that a variety of nonlinear Fokker-Planck equations can be cast into this form and the multivariate counterpart [10–14]. For a nonlinear Fokker-Planck equation (3) involving a particular nonlinear functional $U[P] \propto P^2$ and the Boltzmann entropy

$$ {}^{\mathrm{B}}S = -\int P \ln P \, \mathrm{d}x \tag{4} $$

exact solutions for $P(x,t;u)$ and $P(x,t|x',t';u)$ have been derived [5]. In contrast, for the Plastino-Plastino Fokker-Planck equation [7,15–17] that involves a linear functional $U[P] \propto P$ and the nonextensive entropy

$$ {}^{\mathrm{T}}S_q = \frac{1}{1-q} \int [P^q - P] \mathrm{d}x \tag{5} $$

proposed by Tsallis [18,19] it seem difficult to derive an analytical expression for $P(x,t|x',t';u)$ although autocorrelation functions have be derived [20]. Therefore, the question arises whether or not in the case of nonextensive systems analytical expression for $P(x,t|x',t';u)$ can be derived at all. We will answer this question in the affirmative. To this end, we will consider in what follows the one-parametric Gaussian entropy

$$ {}^{\mathrm{G}}S_q[P] = \frac{1}{q-1} \left[1 - \exp\{(1-q)\, {}^{\mathrm{B}}S[P]\} \right] \ , \tag{6} $$

which is a special case of the two-parametric Sharma-Mittal entropy [21,22]. The entropy ${}^{\mathrm{G}}S_q[P]$ is nonextensive because for two statistically independent systems A and B we have

$$ {}^{\mathrm{G}}S_q(AB) = {}^{\mathrm{G}}S_q(A) + {}^{\mathrm{G}}S_q(B) + (1-q)\, {}^{\mathrm{G}}S_q(A)\, {}^{\mathrm{G}}S_q(B) \ . \tag{7} $$

From Eq. (7) we read off that q measures the degree of nonextensivity. Substituting $F = U - Q\,{}^{\mathrm{G}}S_q$ with $U = \gamma \left\langle X^2 \right\rangle /2$ into the free energy Fokker-Planck equation (3), we obtain the nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x} \gamma x P + Q \exp\{(1-q)\,{}^{\mathrm{B}}S[P]\} \frac{\partial}{\partial x^2} P \ , \tag{8}$$

which involves the diffusion coefficient $D_2(P) = Q \exp\{(1-q)\,{}^{\mathrm{B}}S[P]\}$. If for a solution $P(x,t;u)$ the coefficient $D_2'(x,t,u) = D_2(P)$ corresponds to the second Kramers-Moyal coefficient of a Markov diffusion process, then this Markov process is described by the solution $P(x,t;u)$ and the transition probability density $P(x,t|x',t';u)$ defined by [5]

$$\frac{\partial}{\partial t} P(x,t|x',t';u) = \\ \frac{\partial}{\partial x} \gamma x P(x,t|x',t';u) + Q \exp\{(1-q)\,{}^{\mathrm{B}}S[P(x,t;u)]\} \frac{\partial}{\partial x^2} P(x,t|x',t';u) \ . \tag{9}$$

Therefore, let us consider an explicit solutions $P(x,t;u)$ of Eq. (8). Such an explicit solution has been derived for the initial distribution $u(x) = \delta(x - x_0)$ in previous studies [21,23] and reads

$$P(x,t;\delta(x-x_0)) = \sqrt{\frac{1}{2\pi K(t)}} \exp\left\{ -\frac{[x - M_1(t)]^2}{2K(t)} \right\} \ , \tag{10}$$

where the first moment $M_1(t)$ is given by

$$M_1(t) = x_0 \exp\{-\gamma(t - t_0)\} \tag{11}$$

and the variance $K(t)$ is defined by

$$K(t) = \left[\frac{Q}{\gamma} [2\pi e]^{(1-q)/2} \left(1 - \exp\{-(1+q)\gamma(t-t_0)\}\right) \right]^{2/(1+q)} \ . \tag{12}$$

Here, we have $e = \exp\{1\}$. From Eqs. (11) and (12) it is clear that in the stationary case the mean value vanishes and the variance reads

$$K_{\mathrm{st}} = \left[\frac{Q}{\gamma}[2\pi e]^{(1-q)/2}\right]^{2/(1+q)} \tag{13}$$

such that Eq. (10) becomes

$$P_{\mathrm{st}}(x) = \sqrt{\frac{1}{2\pi K_{\mathrm{st}}}}\exp\left\{-\frac{x^2}{2K_{\mathrm{st}}}\right\} \ . \tag{14}$$

For the transient solution (10) the diffusion coefficient reads $D'_2(t,\delta(x-x_0)) = D_2(P) = [2\pi e K(t)]^{(1-q)/2}$ (see also [21]) and indeed can be regarded as the second Kramers-Moyal coefficient of a Markov diffusion process. Consequently, for the transient solution (10) Eq. (9) reads

$$\frac{\partial}{\partial t}P(x,t|x',t';\delta(x-x_0)) = \frac{\partial}{\partial x}\gamma x P + Q[2\pi e K(t)]^{(1-q)/2}\frac{\partial}{\partial x^2}P \tag{15}$$

and defines the transition probability density of a Markov diffusion process. Eq. (15) is solved by

$$P(x,t|x',t';\delta(x-x_0)) = \sqrt{\frac{1}{2\pi K(t,t')}}\exp\left\{-\frac{[x-M_1(t,t')]^2}{2K(t,t')}\right\} \ , \tag{16}$$

where $M_1(t,t')$ and $K(t,t')$ describe the first moment and variance with respect to x. A detailed calculation yields

$$M_1(t,t') = x'\exp\{-\gamma(t-t')\} \tag{17}$$

and

$$\frac{\partial}{\partial t}K(t,t') = -2\gamma\left(K(t,t') - K_{\mathrm{st}}^{(1+q)/2}K^{(1-q)/2}(t)\right) \tag{18}$$

with $Q[2\pi e]^{(1-q)/2}/\gamma = K_{\mathrm{st}}^{(1+q)/2}$, see Eq. (13). Solving Eq. (18) for the initial condition $\lim_{t\to t'} K(t,t') = 0$, we obtain

$$K(t,t') = 2\gamma K_{\mathrm{st}}^{(1+q)/2}\int_{t'}^{t}\exp\{-2\gamma(t-z)\}[K(z)]^{(1-q)/2}\,\mathrm{d}z \ . \tag{19}$$

Eqs. (10), (11), (12), (16), (17), and (19) provide us with a complete description of the generalized Ornstein-Uhlenbeck process related to the nonextensive entropy measure ${}^{\rm G}S_q$ in terms of the analytical expressions $P(x,t;\delta(x-x_0))$ and $P(x,t|x',t';\delta(x-x_0))$ and $M_1(t), K(t), M_1(t,t'), K(t,t')$.

In closing these considerations, let us discuss the approach to the stationary case and some numerical issues. Let us consider the stationary case in terms of the limit $t'\to\infty$ (which implies $t\to\infty$ because of $t\geq t'$). Then, $K(z)$ in Eq. (19) converges to $K_{\rm st}$ and, consequently, Eq. (19) becomes

$$\lim_{t'\to\infty} K(t,t') = K_{\rm st}(t,t') = K_{\rm st}[1-\exp\{-2\gamma(t-t')\}]\ , \tag{20}$$

which means that the transition probability density (16) becomes stationary:

$$\lim_{t'\to\infty} P(x,t|x',t';\delta(x-x_0)) = P_{\rm st}(x,t-t'|x')\ . \tag{21}$$

In order to discuss the asymptotic behavior in this stationary case, we consider the limiting case given by $t'\to\infty$ and $t-t'\to\infty$. Then, we have $M_1(t,t')=0$ and $K(t,t')=K_{\rm st}$, which implies that

$$\lim_{t-t'\to\infty,t'\to\infty} P(x,t|x',t';\delta(x-x_0)) = \lim_{t-t'\to\infty} P_{\rm st}(x,t-t'|x') = P_{\rm st}(x)\ . \tag{22}$$

The Ito-Langevin equation of Eq. (9) reads [3,5]

$$\frac{\rm d}{{\rm d}t}X(t) = -\gamma X(t) + \sqrt{Q\exp\{(1-q)\,{}^{\rm B}S[P]\}}\,\Gamma(t)\ . \tag{23}$$

If we interpret Eq. (23) as ${\rm d}X(t)/{\rm d}t = -\gamma X(t) + \sqrt{D_2'(t)}\,\Gamma(t)$, we may use the Euler forward scheme [2]

$$X^l(t_n+\Delta t) = X^l(t_n) - \Delta t\gamma X^l(t_n) + \sqrt{Q\Delta t\exp\{(1-q)\,{}^{\rm B}S[P]\}}\,w_n^l \tag{24}$$

to compute the realizations X^l of X. Here. $X(t)$ is evaluated at times $t_n=n\Delta t$ for $n=0,1,2,\ldots$ and the variables w_n^l are realizations of Gaussian distributed random variables w_n satisfying $\langle w_n \rangle = 0$ and $\langle w_i w_k \rangle = 2\delta_{ik}$ [2]. Next, we use $\delta(x - x_0) = \exp\{-(x - x_0)^2/(2\Delta x)\}/\sqrt{2\pi\Delta x}$ for $\Delta x \to 0$ and $P(x, t; u) = \langle \delta(x - X(t)) \rangle$ and write

$$P(x, t; u) = \frac{1}{L\sqrt{2\pi\Delta x}} \sum_{l=1}^{L} \exp\left\{-\frac{[x - X^l(t)]^2}{2\Delta x}\right\} \tag{25}$$

for Δx small and L large. Then, from ${}^{\mathrm{B}}S[P] = -\langle \ln P \rangle$ it follows that

$${}^{\mathrm{B}}S[P] = -\frac{1}{L} \sum_{k=1}^{L} \ln\left[\frac{1}{L\sqrt{2\pi\Delta x}} \sum_{l=1}^{L} \exp\left\{-\frac{[X^k(t) - X^l(t)]^2}{2\Delta x}\right\}\right] . \tag{26}$$

In sum, Eq. (24) becomes

$$X^l(t_n + \Delta t) = X^l(t_n) - \Delta t \gamma X^l(t_n)$$
$$+ w_n^l \sqrt{Q\Delta t \exp\left\{\frac{(q-1)}{L} \sum_{k=1}^{L} \ln\left[\frac{1}{L\sqrt{2\pi\Delta x}} \sum_{l=1}^{L} \exp\left\{-\frac{[X^k(t_n) - X^l(t_n)]^2}{2\Delta x}\right\}\right]\right\}} . \tag{27}$$

In figures 1 and 2 we have plotted $M_1(t)$, $K(t)$ and $K(t, t')$ for $q > 1$ and $q < 1$, respectively. We have computed these quantities both from the analytical expressions (11), (12), and (19) (solid lines) and from the Langevin equation (23) by solving Eq. (27) iteratively (diamonds). We see that there is an excellent agreement between theory and numerics.

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Figure captions:

Fig. 1: $M_1(t)$, $K(t)$, and $K(t,t')$ for $q = 1.5$. For $t \le t'$ we have put $K(t,t') = 0$. Parameters: $\gamma = 0.5$, $x_0 = -2$, $t_0 = 0$, $Q = 1$, $t' = 0.7$, $x' = -1$.

Fig. 2: $M_1(t)$, $K(t)$, and $K(t,t')$ for $q = 0.5$; other parameters as in Fig. 1.

Figures:

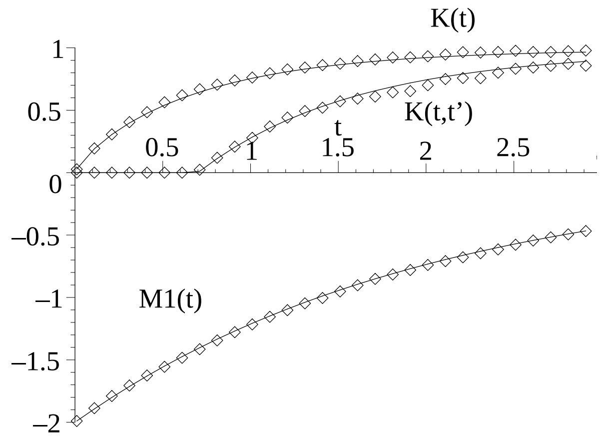

Fig. 1.

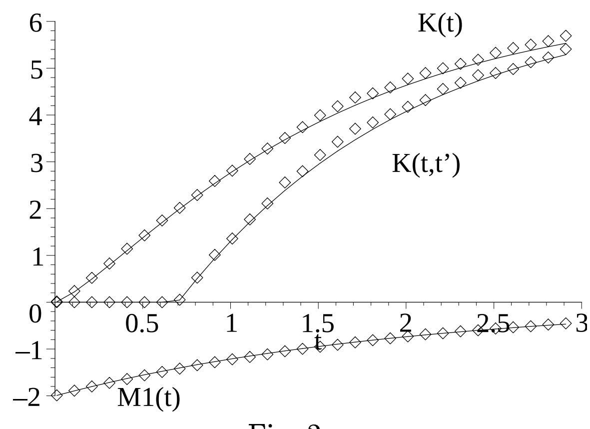

Fig. 2.