Stability analysis of nonequilibrium mean field models by means of self-consistency equations

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Abstract

We consider systems that operator far from thermal equilibrium and can be described by means of nonlinear Fokker-Planck equations using mean field theory. We determine the stability of stationary states by means of Prigogine's Lyapunov functional and self-consistency equations.

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1 Introduction

In various disciplines such as laser physics, chemical physics, biophysics and economics, Fokker-Planck equations and Langevin equations have been proven to be useful descriptions of nonequilibrium systems [1–12]. Using mean field approximations, drift and diffusion coefficients of Fokker-Planck equations can depend on order parameters. Thus, Fokker-Planck equations can become nonlinear with respect to probability densities. Such nonlinear Fokker-Planck

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equations can describe continuous and discontinuous nonequilibrium phase transitions [13–25], reentrant phase transitions [26–31], and pattern formation [32–34]. In particular, wetting transitions and surface growth have been successfully examined in the context of nonlinear Fokker-Planck equations related to the Kadar-Parisi-Zhang equation and the Pikovsky-Kurths model [35–38].

Despite the wide applicability of nonlinear Fokker-Planck equations in the realm of nonequilibrium systems, a stability analysis for systems of this kind has not been developed so far. Usually, one assumes that the concepts that have been developed for equilibrium systems carry over to the nonequilibrium case. More precisely, *Shiino* among others has given rigorous proof that for nonlinear Fokker-Planck equations involving free energy functionals the stability of stationary distributions can be determined by evaluating appropriately defined self-consistency equations [39–43]. For nonequilibrium systems such a proof has not been given so far although self-consistency equations are usually used to determine the stability of stationary states.

The issue of a stability analysis for nonequilibrium mean field models has been discussed recently [44]. However, this study was confined to a special case of a nonlinear Fokker-Planck equation. Moreover, the stability analysis has not been related to nonequilibrium thermodynamics and Prigogine's Lyapunov function $L = -\delta^2 S$, where S denotes the Boltzmann-Gibbs-Shannon entropy functional and $\delta^2 S$ is its second variation. Nonequilibrium thermodynamics states that systems are stable if the excess entropy production defined by $\mathrm{d}\delta^2 S/\mathrm{d}t$ is positive. Consequently, the inequality $\mathrm{d}\delta^2 S/\mathrm{d}t < 0$ is regarded as a necessary condition for instability [45–47].

This manuscript is organized as follows. First, we will prove that self-consistency equation analysis can be applied to nonequilibrium systems. We will derive a control parameter $\tilde{\lambda}$ and show that $\tilde{\lambda} < 0$ is a sufficient condition for instability. In this context, we will discuss the role of $\delta^2 S$. Next, we will introduce an alternative control parameter δ. In this context, we will derive a Lyapunov functional $L = \delta^2 B - \delta^2 S$ and derive sufficient conditions for stable and unstable stationary states and the emergence of nonequilibrium phase transitions.

2 Stability analysis

2.1 Self-consistency equations

Let $X \in \Omega$ denote a random variable of an univariate stochastic process defined on a phase space Ω. Let $P(x,t;u)$ describe the probability density of X for X distributed like $u(x)$ at the initial time $t = t_0$. That is, we have $P(x,t;u) = \langle \delta(x - X(t)) \rangle$ and $P(x,t_0;u) = u(x)$. We assume that for $t \geq t_0$ the stochastic process under consideration can be described by means of the Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,t;u) = -\frac{\partial}{\partial x} D_1(x,\langle A \rangle) P + \frac{\partial^2}{\partial x^2} D_2(x) P \ , \tag{1}$$

which is nonlinear with respect P due to the dependency of D_1 on the order parameter $m = \langle A \rangle = \int_\Omega A(x) P(x,t;u)\,\mathrm{d}x$. Eq. (1) involves the probability current [48]

$$J = D_1(x,\langle A \rangle) P - \frac{\partial}{\partial x} D_2(x) P \ . \tag{2}$$

We consider boundary conditions and coefficients D_1, D_2 for which stationary distributions can be computed from $J = 0$ and, consequently, are given by

$$P_{\mathrm{st}}(x) = \frac{1}{Z D_2(x)} \exp\left\{ \int^x \frac{D_1(x', \langle A \rangle_{\mathrm{st}})}{D_2(x')}\,\mathrm{d}x' \right\} \ , \tag{3}$$

where Z denotes a normalization constant. The order parameter $\langle A\rangle_{\rm st}$ satisfies the self-consistency equation

$$\langle A\rangle_{\rm st} = R(\langle A\rangle_{\rm st}) \ , \tag{4}$$

where $R(m)$ is defined by

$$R(m) = \frac{\int\limits_{\Omega} \frac{A(x)}{D_2(x)} \exp\left\{ \int^{x} \frac{D_1(x',m)}{D_2(x')}\,{\rm d}x' \right\} {\rm d}x}{\int\limits_{\Omega} \frac{1}{D_2(x)} \exp\left\{ \int^{x} \frac{D_1(x',m)}{D_2(x')}\,{\rm d}x' \right\} {\rm d}x} \ . \tag{5}$$

From Eq. (5) it follows that

$$\left.\frac{{\rm d}R(m)}{{\rm d}m}\right|_{\langle A\rangle_{\rm st}} = C\left[A, \frac{\partial}{\partial\langle A\rangle_{\rm st}} \int^{X} \frac{D_1(x',\langle A\rangle_{\rm st})}{D_2(x')}\,{\rm d}x' \right] \ , \tag{6}$$

where C describes the cross-correlation function

$$C(f,g) = \langle fg\rangle_{\rm st} - \langle f\rangle_{\rm st}\langle g\rangle_{\rm st} = \langle (f-\langle f\rangle_{\rm st})(g-\langle g\rangle_{\rm st})\rangle_{\rm st} \tag{7}$$

for functions $f(x)$ and $g(x)$. In order to apply linear stability analysis, we consider perturbations $\epsilon = P - P_{\rm st}$ with $\int_\Omega \epsilon(x)\,{\rm d}x = 0$ of stationary solutions $P_{\rm st}$. Substituting $P = P_{\rm st} + \epsilon$ into Eq. (1) and neglecting nonlinear terms with respect to ϵ, we obtain the linear evolution equation

$$\begin{aligned} &\frac{\partial}{\partial t}\epsilon(x,t) = \\ &-\frac{\partial}{\partial x}D_1(x,\langle A\rangle_{\rm st})\epsilon + \frac{\partial^2}{\partial x^2}D_2(x)\,\epsilon - \langle A\rangle_\epsilon \frac{\partial}{\partial x}\left[P_{\rm st}(x)\frac{\partial D_1(x,\langle A\rangle_{\rm st})}{\partial\langle A\rangle_{\rm st}} \right] \end{aligned} \tag{8}$$

with $\langle A\rangle_\epsilon = \int_\Omega A(x)\,\epsilon(x,t)\,{\rm d}x$. Using Eq. (3), Eq. (8) can be written as [49]

$$\frac{\partial}{\partial t}\epsilon(x,t) = \frac{\partial}{\partial x}D_2 P_{\rm st}\frac{\partial}{\partial x}\frac{\epsilon}{P_{\rm st}} - \langle A\rangle_\epsilon \frac{\partial}{\partial x}\left[P_{\rm st}(x)\frac{\partial D_1(x,\langle A\rangle_{\rm st})}{\partial\langle A\rangle_{\rm st}} \right] \ . \tag{9}$$

Let us apply Shiino's expansion to the perturbation ϵ [40]. That is, we first write $\epsilon(x,t) = \sqrt{P_{\rm st}}\epsilon'(x,t)$ with $\epsilon'(x,t) = \beta(t)\sqrt{P_{\rm st}}\phi(x) + \chi(x,t)$. Next, we assume that the function χ is orthogonal to the first expansion function $\varphi(x) = \sqrt{P_{\rm st}}(x)\phi(x)$: $\int_\Omega \chi(x,t)\sqrt{P_{\rm st}}(x)\phi(x)\,{\rm d}x = 0$. Furthermore, we require that the relation $\int_\Omega P_{\rm st}(x)\phi(x)\,{\rm d}x = 0$ holds which implies that $\int_\Omega \sqrt{P_{\rm st}(x)}\chi(x,t)\,{\rm d}x = 0$ because of $\int_\Omega \epsilon(x,t)\,{\rm d}x = 0$. We define $\phi(x)$ by

$$\phi(x,\langle A\rangle_{\rm st}) = \frac{\partial}{\partial\langle A\rangle_{\rm st}}\int^x \frac{D_1(x',\langle A\rangle_{\rm st})}{D_2(x')}\,{\rm d}x' - \left\langle \frac{\partial}{\partial\langle A\rangle_{\rm st}}\int^X \frac{D_1(x',\langle A\rangle_{\rm st})}{D_2(x')}\,{\rm d}x'\right\rangle_{\rm st} , \tag{10}$$

which gives us $\epsilon(x,t)$ in form of

$$\begin{aligned}\epsilon(x,t) =& \beta(t)P_{\rm st}\left[\frac{\partial}{\partial\langle A\rangle_{\rm st}}\int^x \frac{D_1(x',\langle A\rangle_{\rm st})}{D_2(x')}\,{\rm d}x' - \left\langle \frac{\partial}{\partial\langle A\rangle_{\rm st}}\int^X \frac{D_1(x',\langle A\rangle_{\rm st})}{D_2(x')}\,{\rm d}x'\right\rangle_{\rm st}\right] \\ &+\chi(x,t)\sqrt{P_{\rm st}}\ .\end{aligned} \tag{11}$$

Note that in what follows we assume that $\phi \neq 0$. Substituting (11) into the right hand side of Eq. (9), we obtain

$$\frac{\partial}{\partial t}\epsilon(x,t) = \beta(t)(1-C)\frac{\partial}{\partial x}\left[P_{\rm st}(x)\frac{\partial D_1(x,\langle A\rangle_{\rm st})}{\partial\langle A\rangle_{\rm st}}\right] + E(\chi)\ , \tag{12}$$

where C takes the same arguments as in Eq. (6) and $E(\chi)$ describes terms that depend on χ and vanish for vanishing functions χ. Multiplying Eq. (12) with $\epsilon/P_{\rm st}$, and performing the integration $\int_\Omega ...{\rm d}x$, one obtains

$$\frac{\rm d}{{\rm d}t}\int_\Omega \frac{\epsilon^2(x,t)}{2P_{\rm st}(x)}{\rm d}x = -\beta^2(1-C)\left\langle \frac{1}{D_2}\left[\frac{\partial D_1}{\partial\langle A\rangle_{\rm st}}\right]^2\right\rangle_{\rm st} + E'(\chi)\ , \tag{13}$$

where $E'(\chi)$ describes terms containing χ that satisfy $E'(\chi) = 0$ for $\chi = 0$. Exploiting the orthogonality of χ and $\sqrt{P_{\rm st}}\phi$, we get

$$\left\langle \phi^2 \right\rangle_{\rm st} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\beta^2(t)}{2} + \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} \frac{\chi^2(x,t)}{2}\mathrm{d}x = -\beta^2(1-C)\left\langle \frac{1}{D_2}\left[\frac{\partial D_1}{\partial \left\langle A \right\rangle_{\rm st}}\right]^2 \right\rangle_{\rm st} + E'(\chi) \ . \tag{14}$$

Note that for perturbations with $\chi(x,t_0)=0$ we have $\mathrm{d}[\int_\Omega \chi^2(x,t)\,\mathrm{d}x]/\mathrm{d}t = 0$ at $t=t_0$. Consequently, taking Eq. (6) into account for perturbations $\epsilon(x,t)$ that satisfy $\chi=0$ and $\beta \neq 0$ at $t=t_0$, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega} \frac{\epsilon^2(x,t_0)}{2P_{\rm st}(x)}\mathrm{d}x \Bigg|_{\beta\neq 0,\chi=0} &= \frac{1}{2}\left\langle \phi^2 \right\rangle_{\rm st} \frac{\mathrm{d}}{\mathrm{d}t}\beta^2(t_0) \\ &= -\beta^2(t_0)\left(1 - \frac{\mathrm{d}R}{\mathrm{d}m}\Bigg|_{\langle A\rangle_{\rm st}}\right) \underbrace{\left\langle \frac{1}{D_2}\left[\frac{\partial D_1}{\partial \left\langle A \right\rangle_{\rm st}}\right]^2 \right\rangle_{\rm st}}_{\mathcal{Y}} \ . \end{aligned} \tag{15}$$

Eq. (15) states that if $\mathrm{d}R(\langle A\rangle_{\rm st}/\mathrm{d}m > 1$ holds then the amplitude $|\beta|$ of the perturbation (11) increases at $t=t_0$. Likewise, the norm of the perturbation $\|\epsilon\|$ with $\|f\| = \int_\Omega f^2/P_{\rm st}\,\mathrm{d}x$ increases at t_0. We assume that the spectrum of Lyapunov exponents is real valued. That is, we focus on fixed points $P_{\rm st}$ exhibiting a non-oscillatory decay or growth of perturbations. (E.g., if $\Omega = \mathbb{R}$ we consider stationary solutions $P_{\rm st}$ for which the Fourier expansion $\epsilon(x,t) = \int [a(k,t)\cos(kx) + b(k,t)\sin(kx)]\,\mathrm{d}k$ with $a(k,t) = a(k,0)\exp\{\lambda(k)t\}$, $b(k,t) = b(k,0)\exp\{\lambda'(k)t\}$ involves only real valued exponents λ, λ'.) For this fundamental class of fixed points $P_{\rm st}$ the increase of $\|\epsilon\|$ means that $P_{\rm st}$ is an unstable distribution. In contrast, if $P_{\rm st}$ describes an asymptotically stable stationary distribution, every perturbation and in particular the perturbation (11) for $\beta \neq 0$ and $\chi = 0$ vanishes, which implies that the inequality $\mathrm{d}R(\langle A\rangle_{\rm st})/\mathrm{d}m < 1$ holds. We conclude:

$$P_{\rm st} \text{ asymptotically stable} \Rightarrow \frac{\mathrm{d}R(m)}{\mathrm{d}m}\Bigg|_{\langle A\rangle_{\rm st}} < 1 \ , \tag{16}$$

$$\frac{\mathrm{d}R(m)}{\mathrm{d}m}\Bigg|_{\langle A\rangle_{\rm st}} > 1 \Rightarrow P_{\rm st} \text{ unstable} \ . \tag{17}$$

Thus, we have found rigorous proof that the stability of stationary distributions can be determined by means of self-consistency equation analysis.

2.2 The role of $\delta^2 S$

Let us dwell now on the role of $\delta^2 S$ with $S = -\int_\Omega P \ln P \, \mathrm{d}x$. By means of the relation $\delta^2 S[P_{\rm st}](\epsilon) = -\int_\Omega \epsilon^2(x)/P_{\rm st}(x) \, \mathrm{d}x$, we can write Eq. (15) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left. \frac{\delta^2 S[P_{\rm st}](\epsilon)}{2} \right|_{\beta\neq 0, \chi=0, t=t_0} = \beta^2(t_0) \left(1 - \left. \frac{\mathrm{d}R}{\mathrm{d}m} \right|_{\langle A \rangle_{\rm st}} \right) \left\langle \frac{1}{D_2} \left[\frac{\partial D_1}{\partial \langle A \rangle_{\rm st}} \right]^2 \right\rangle \quad (18)$$

and arrive at the implication

$$\frac{\mathrm{d}}{\mathrm{d}t} \left. \frac{\delta^2 S[P_{\rm st}](\epsilon)}{2} \right|_{\beta\neq 0, \chi=0, t=t_0} < 0 \Leftrightarrow \left. \frac{\mathrm{d}R(m)}{\mathrm{d}m} \right|_{\langle A \rangle_{\rm st}} > 1 \Rightarrow P_{\rm st} \text{ unstable} \; . \quad (19)$$

That is, if the excess entropy production described by $\mathrm{d}\delta^2 S/\mathrm{d}t$ is negative for perturbations of the form (11), then the stationary distribution is an unstable one. This is in line with the claim made by nonequilibrium thermodynamics that $\mathrm{d}\delta^2 S/\mathrm{d}t < 0$ is a necessary condition for instability [45–47]. For the systems described by Eq. (1) with monotonically decaying or growing perturbations we can draw the conclusion that the inequality $\mathrm{d}\delta^2 S/\mathrm{d}t < 0$ is a sufficient condition for instability.

2.3 Sufficient conditions for stability

From Eq. (12) we have derived the evolution equation (15) that completely describes the evolution of $\beta(t)$ at $t = t_0$ in combination with the control parameter $\tilde{\lambda} = 1 - \mathrm{d}R(\langle A \rangle_{\rm st})/\mathrm{d}m$. For $\tilde{\lambda} > 0$ the amplitude $|\beta(t)|$ decreases at $t = t_0$, whereas for $\tilde{\lambda} < 0$ the amplitude $|\beta(t)|$ increases at $t = t_0$. From Eq. (12) we may obtain alternative control parameters. These control parameters will yield the same result but they may reveal some additional information about the stability problem. Multiplying Eq. (12) with $A(x)$, integration with respect to x and partial integration, gives us

$$C\frac{\mathrm{d}}{\mathrm{d}t}\beta(t_0)+\underbrace{\int\limits_{\Omega}\sqrt{P_{\mathrm{st}}(x)}A(x)\frac{\partial\chi(x,t_0)}{\partial t}\,\mathrm{d}x}_{\mathcal{Y}}=\left\langle\frac{\mathrm{d}A}{\mathrm{d}X}\frac{\partial D_1(X,\langle A\rangle_{\mathrm{st}})}{\partial\langle A\rangle_{\mathrm{st}}}\right\rangle_{\mathrm{st}}(C-1)\beta(t_0) \tag{20}$$

for perturbations with $\chi(x,t)=0$ and $\beta\neq 0$ at $t=t_0$. If there are perturbations (11) for which the contribution $\chi(x,t)$ evolves in a subspace of the function space which is orthogonal to $A(x)\sqrt{P_{\mathrm{st}}}$ (and orthogonal to $\phi(x)\sqrt{P_{\mathrm{st}}}$; see the preceding), then we have

$$\int\limits_{\Omega}\chi(x,t)A(x)\sqrt{P_{\mathrm{st}}(x)}\,\mathrm{d}x=0 \tag{21}$$

for $t\geq t_0$ and, consequently, the term $\mathcal{Y}$ in Eq. (20) vanishes. Considering $C=\mathrm{d}R(\langle A\rangle_{\mathrm{st}})/\mathrm{d}m\neq 0$ and using Eq. (6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\beta(t_0)=\left\langle\frac{\mathrm{d}A}{\mathrm{d}X}\frac{\partial D_1(X,\langle A\rangle_{\mathrm{st}})}{\partial\langle A\rangle_{\mathrm{st}}}\right\rangle_{\mathrm{st}}\left(1-\left[\left.\frac{\mathrm{d}R(m)}{\mathrm{d}m}\right|_{\langle A\rangle_{\mathrm{st}}}\right]^{-1}\right)\beta(t_0)\ , \tag{22}$$

which suggests to introduce the control parameter δ defined by

$$\delta=\left\langle\frac{\mathrm{d}A}{\mathrm{d}X}\;\frac{\partial D_1(X,\langle A\rangle_{\mathrm{st}})}{\partial\langle A\rangle_{\mathrm{st}}}\right\rangle_{\mathrm{st}}\ . \tag{23}$$

Note that δ has been derived in a previous study for a special case of Eq. (1) with $D_1(x,\langle A\rangle)\propto(A(x)-\langle A\rangle)$ [44]. Multiplying Eq. (22) with β and comparing the result with Eq. (15) one finds

$$\delta=\underbrace{\frac{1}{\langle\phi^2\rangle_{\mathrm{st}}}\left\langle\frac{1}{D_2}\left[\frac{\partial D_1}{\partial\langle A\rangle_{\mathrm{st}}}\right]^2\right\rangle_{\mathrm{st}}}_{>0}\left.\frac{\mathrm{d}R(m)}{\mathrm{d}m}\right|_{\langle A\rangle_{\mathrm{st}}}\ , \tag{24}$$

which implies that if the constraint (21) can be satisfied, then the control parameters δ and $\mathrm{d}R(\langle A\rangle_{\mathrm{st}})/\mathrm{d}m$ may differ in magnitude but they have the same sign. An issue that seems to be of interest with regard to the parameter δ are the conditions under which δ becomes relevant at all. For example, on account of the orthogonality properties of χ with respect to ϕ, the constraint (21) is satisfied if there is a function $f(z)$ such that

$$A(x) f(\langle A \rangle_{\mathrm{st}}) = \frac{\partial}{\partial \langle A \rangle_{\mathrm{st}}} \int^{x} \frac{D_1(x', \langle A \rangle_{\mathrm{st}})}{D_2(x')} \, \mathrm{d}x' \; . \tag{25}$$

In our preceding analysis we have derived sufficient conditions for the instability of stationary states. Next, we will derive sufficient conditions for stable stationary states. To this end, we will restrict our attention to systems for which the matching condition (25) holds. First, we write Eq. (9) as

$$\frac{\partial}{\partial t} \epsilon(x,t) = \frac{\partial}{\partial x} D_2 P_{\mathrm{st}} \frac{\partial}{\partial x} \left[\frac{\epsilon}{P_{\mathrm{st}}} - \langle A \rangle_{\epsilon} \int^{x} \frac{\partial}{\partial \langle A \rangle_{\mathrm{st}}} \frac{D_1(x', \langle A \rangle_{\mathrm{st}})}{D_2(x')} \, \mathrm{d}x' \right] \; . \tag{26}$$

Using Eq. (25), we can write Eq. (26) as

$$\frac{\partial}{\partial t} \epsilon(x,t) = \frac{\partial}{\partial x} D_2 P_{\mathrm{st}} \frac{\partial}{\partial x} \left[\frac{\epsilon}{P_{\mathrm{st}}} - f(\langle A \rangle_{\mathrm{st}}) A(x) \int_{\Omega} A(y) \, \epsilon(y,t) \, \mathrm{d}y \right] \; . \tag{27}$$

Using $B(z)$ with $f(z) = -\mathrm{d}^2 B / \mathrm{d}z^2$ and

$$\Psi[P] = B(\langle A \rangle) - S[P] \; , \tag{28}$$

we can transform Eq. (27) into

$$\frac{\partial}{\partial t} \epsilon(x,t) = \frac{\partial}{\partial x} D_2(x) P_{\mathrm{st}} \frac{\partial}{\partial x} \int_{\Omega} \frac{\delta^2 \Psi[P_{\mathrm{st}}]}{\delta P(x) \delta P(y)} \epsilon(y,t) \, \mathrm{d}y \; . \tag{29}$$

Linear evolution equations similar to Eq. (29) have been derived for models of equilibrium systems [44]. By the same reasoning as for these equilibrium models, one can show that the relations

$$\frac{\mathrm{d}}{\mathrm{d}t} \delta^2 \Psi \leq 0 \; , \; \frac{\mathrm{d}}{\mathrm{d}t} \delta^2 \Psi = 0 \Leftrightarrow \delta^2 \Psi = 0 \tag{30}$$

hold. Furthermore, generalizing Shiino's expansion one can show that the control parameter

$$\tilde{\lambda} = 1 + K_{A,\mathrm{st}}(X) \left. \frac{\mathrm{d}^2 B}{\mathrm{d}m^2} \right|_{\langle A \rangle_{\mathrm{st}}} \tag{31}$$

with $K_{A,\mathrm{st}}(X) = \langle [A - \langle A \rangle_{\mathrm{st}}]^2 \rangle_{\mathrm{st}}$ determines whether $\delta^2 \Psi$ is a positive or negative definite function. For stationary solutions with $\tilde{\lambda} > 0$ one finds that $\delta^2 \Psi$ is positive for $\epsilon \neq 0$ and that $L = \delta^2 \Psi$ satisfies the relations

$$\frac{\mathrm{d}}{\mathrm{d}t} L \leq 0 \ , \ \frac{\mathrm{d}}{\mathrm{d}t} L = 0 \Leftrightarrow L = 0 \Leftrightarrow P = P_{\mathrm{st}} \ , \ L > 0 \ . \tag{32}$$

For details see [44, Sec. 2.1.4]. In this case, $L = \delta^2 \Psi = \delta^2 B - \delta^2 S$ is a local Lyapunov functional (i.e., a Lyapunov functional for perturbations) and the stationary solutions correspond to asymptotically stable ones. In contrast, for a stationary solution P_{st} with $\tilde{\lambda} < 0$ we have $\delta^2 \Psi < 0$ for $\epsilon \neq 0$. Since we have $\mathrm{d}\delta^2 \Psi / \mathrm{d}t \leq 0$ the function $|\delta^2 \Psi|$ increases at $t = t_0$, which indicates that P_{st} describes an unstable stationary distribution. The local Lyapunov functional $\delta^2 \Psi$ can, for example, be used to examine the stability of stationary states of systems exhibiting noise-induced reentrant phase transitions [44]. Finally, note that using Eqs. (7) and (25), one can show that Eq. (31) can equivalently be expressed as $\tilde{\lambda} = 1 - \mathrm{d}R(\langle A \rangle_{\mathrm{st}})/\mathrm{d}m$.

3 Conclusions

We have shown that self-consistency equation analysis can be used to determine the stability of stationary states of nonequilibrium mean field models described by nonlinear Fokker-Planck equations. In particular, if $\mathrm{d}\delta^2 S'/\mathrm{d}t$ denotes the change of $\delta^2 S$ with time for perturbations (11) for $\beta \neq 0$ and $\chi = 0$, then we can draw the following conclusions. First, if systems occupy a stable stationary state then they exhibit a positive excess entropy production $\mathrm{d}\delta^2 S'/\mathrm{d}t$. Second, stationary states are unstable for which the excess entropy production $\mathrm{d}\delta^2 S'/\mathrm{d}t$ is negative. Consequently, if the control parameter $\tilde{\lambda} = 1 - \mathrm{d}R(\langle A \rangle_{\mathrm{st}})/\mathrm{d}m$ is changed from $\tilde{\lambda} > 0$ to $\tilde{\lambda} < 0$ which implies that $\mathrm{d}\delta^2 S'/\mathrm{d}t$ changes its sign from $\mathrm{d}\delta^2 S'/\mathrm{d}t > 0$ to $\mathrm{d}\delta^2 S'/\mathrm{d}t < 0$, then we either have a phase transition point at $\tilde{\lambda} > 0$ or at $\tilde{\lambda} = 0$. If systems satisfy a particular matching condition (see Eq. (25) above), we can draw stronger conclusions. Then, the bifurcation point is at $\tilde{\lambda} = 0$. Moreover, stationary states with $\mathrm{d}\delta^2\Psi/\mathrm{d}t < 0$ (> 0) are stable (unstable), where $\delta^2\Psi$ corresponds to the expression $-\delta^2 S$ shifted by a term $\delta^2 B$ and B may be interpreted as some kind of internal energy.

References

[1] R. Graham, in *Quantum statistics in optics and solid-state physics. Springer tracts in modern physics, Vol. 66*, edited by G. Höhler (Springer, Berlin, 1973), pp. 1–97.

[2] N. S. Goel and N. R. Dyn, *Stochastic models in biology* (Academic Press, New York, 1974).

[3] H. Haken, *Synergetics. An introduction* (Springer, Berlin, 1977).

[4] E. W. Montroll and B. J. West, in *Studies in statistical mechanics*, edited by E. W. Montroll and J. L. Lebowitz (North-Holland Publ. Company, Amsterdam, 1979), pp. 61–175.

[5] P. Hänggi and H. Thomas, Phys. Rep. **88**, 207 (1982).

[6] Y. Kuramoto, *Chemical oscillations, waves, and turbulence* (Springer, Berlin, 1984).

[7] W. Horsthemke and R. Lefever, *Noise-induced transitions* (Springer, Berlin, 1984).

[8] C. R. Doering, Phys. Lett. A **122**, 133 (1987).

[9] P. A. Tass, *Phase resetting in medicine and biology - Stochastic modelling and data analysis* (Springer, Berlin, 1999).

[10] S. H. Strogatz, Physica D **143**, 1 (2000).

[11] L. Borland, Phys. Rev. Lett. **89**, 098701 (2002).

[12] F. Schweitzer, *Brownian agents and active particles* (Springer, Berlin, 2003).

[13] R. C. Desai and R. Zwanzig, J. Stat. Phys. **19**, 1 (1978).

[14] L. L. Bonilla, J. Stat. Phys. **46**, 659 (1987).

[15] A. R. Bulsara, T. C. Elston, C. R. Doering, S. B. Lowen, and K. Lindenberg, Phys. Rev. E **53**, 3958 (1996).

[16] M. Morillo, J. Gomez-Ordonez, and J. M. Casado, Phys. Rev. E **52**, 316 (1995).

[17] H. Daido, Phys. Rev. Lett. **77**, 1406 (1996).

[18] K. Wiesenfeld, P. Colet, and S. H. Strogatz, Phys. Rev. Lett. **76**, 404 (1996).

[19] S. Kim, S. H. Park, and C. S. Ryu, Phys. Rev. Lett. **78**, 1616 (1997).

[20] J. H. Li, Z. Q. Huang, and D. Y. Xing, Phys. Rev. E **58**, 2838 (1998).

[21] T. D. Frank, A. Daffertshofer, C. E. Peper, P. J. Beek, and H. Haken, Physica D **144**, 62 (2000).

[22] T. D. Frank, A. Daffertshofer, and P. J. Beek, Phys. Rev. E **63**, 011905 (2001).

[23] J. H. Li and P. Hänggi, Phys. Rev. E **64**, 011106 (2001).

[24] J. T. Ariaratnam and S. H. Strogatz, Phys. Rev. Lett. **86**, 4278 (2001).

[25] T. Birner, K. Lippert, R. Müller, A. Kühnel, and U. Behn, Phys. Rev. E **65**, 046110 (2002).

[26] C. van den Broeck, J. M. R. Parrondo, and R. Toral, Phys. Rev. Lett. **73**, 3395 (1994).

[27] J. Garcia-Ojalvo, J. M. R. Parrondo, J. M. Sancho, and C. van den Broeck, Phys. Rev. E **54**, 6918 (1996).

[28] R. Müller, K. Lippert, A. Kühnel, and U. Behn, Phys. Rev. E **56**, 2658 (1997).

[29] C. van den Broeck, J. M. R. Parrondo, R. Toral, and R. Kawai, Phys. Rev. E **55**, 4084 (1997).

[30] W. Genovese, M. A. Munoz, and J. M. Sancho, Phys. Rev. E **57**, R2495 (1998).

[31] A. A. Zaikin, J. Garcia-Ojalvo, and L. Schimansky-Geier, Phys. Rev. E **60**, R6275 (1999).

[32] J. M. R. Parrondo, C. van den Broeck, J. Buceta, and F. J. de la Rubia, Physica A **224**, 153 (1996).

[33] A. A. Zaikin and L. Schimansky-Geier, Phys. Rev. E **58**, 4355 (1998).

[34] J. Garcia-Ojalvo and J. M. Sancho, *Noise in spatially extended systems* (Springer, New York, 1999).

[35] M. A. Munoz and R. Pastor-Satorras, Phys. Rev. Lett **90**, 204101 (2003).

[36] F. de los Santos, M. M. Telo da Gamma, and M. A. Munoz, Phys. Rev. E **67**, 021607 (2003).

[37] M. Marsili and A. J. Bray, Phys. Rev. Lett. **76**, 2750 (1996).

[38] L. Giada and M. Marsili, Phys. Rev. E **62**, 6015 (2000).

[39] M. Shiino, Phys. Lett. A **112**, 302 (1985).

[40] M. Shiino, Phys. Rev. A **36**, 2393 (1987).

[41] M. Shiino, Phys. Rev. E **67**, 056118 (2003).

[42] T. D. Frank, A. Daffertshofer, C. E. Peper, P. J. Beek, and H. Haken, Physica D **150**, 219 (2001).

[43] T. D. Frank, Annalen der Physik **11**, 707 (2002).

[44] T. D. Frank, Physica D **189**, 199 (2004).

[45] P. Glansdorff and I. Prigogine, *Thermodynamic theory of structure, stability, and fluctuations* (John Wiley and Sons, New York, 1971).

[46] G. Nicolis and I. Prigogine, *Self-organization in nonequilibrium system* (John Wiley and Sons, New York, 1977).

[47] D. Kondepudi and I. Prigogine, *Modern thermodynamics* (John Wiley and Sons, New York, 1998).

[48] H. Risken, *The Fokker-Planck equation — Methods of solution and applications* (Springer, Berlin, 1989).

[49] T. D. Frank, Prog. theor. phys. supp. **150**, 48 (2003).