Dynamic mean field models: H-theorem for stochastic processes and basins of attraction of stationary processes

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Abstract

Stochastic processes of dynamic mean field models are studied in terms of nonlinear Fokker-Planck equations. We show that H-theorems for single time-point distributions can be used to derive H-theorems for hierarchies of joint distributions. In doing so, we prove the convergence of transient stochastic processes to stationary ones. For multistable systems we furthermore determine the basins of attraction of these stationary processes. Our results are illustrated by means of a mean field model exhibiting bistability.

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1 Introduction

Dynamic mean field models described by nonlinear Fokker-Planck equations have found applications in various fields such as plasma physics [1–3], surface physics [4,5], astrophysics [6–8], physics of polymer fluids [9] and particle

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beams [10,11], theory of electronic circuitry and laser arrays [12,13], human movement sciences [14,15], biophysics [16–24], and neurophysics [25–29]. The reason for this is that, on the one hand, they illustrate the convergence of many-body systems to stationary states in the long time limit and, on the other hand, they can account for equilibrium and nonequilibrium phase transitions of systems and the emergence of collective phenomena. The approach to stationary states is usually proven by means of H-theorems for particle distribution functions [7,8,30–33]. Phase transitions are often observed in the context of multistability [34]. Taking a microscopic point of view, nonlinear Fokker-Planck equations are often regarded as limiting cases of N-dimensional linear Fokker-Planck equations for $N \to \infty$ [17,27,35,36], mean field approximations of functional Fokker-Planck equations [37], and diffusion equation approximations to nonlinear master equations [38]. Note that the nonlinear Fokker-Planck equations thus obtained typically involve nonlinear order parameter equations.

Although it seems that we have achieved a sound understanding of mean field models given by nonlinear Fokker-Planck equations, there are two key issues that have not been clarified up to now. First, if we are able to prove that particle distribution functions $P(x,t)$ become stationary in the long time limit, does this imply that the underlying stochastic processes become stationary as well? That is, can we conclude from the asymptotic behavior $P(x,t) \to P_{\rm st}(x)$ that joint probability densities $P(x,t;x',t'), P(x,t;x',t';x'',t''),\ldots$ and in particular autocorrelation functions such as $\langle X(t)X(t')\rangle$ become stationary? Second, if there are multiple stationary states described by a set of stationary distributions, say, $P_{\rm st}^1, P_{\rm st}^2, \ldots$, then what are the basins of attraction of these distribution functions? In what follows, we will discuss these two issues both from a general perspective and in application to a bistable mean field model that can be treated analytically.

2 General case

Let $X(t)$ denote an univariate random variable that describes a stochastic process for $t \geq t_0$ defined on a phase space Ω. Let $P(x,t;u) = \langle \delta(x - X(t)) \rangle$ denote the time-dependent probability density of $X(t)$ subjected to the initial condition $P(x,t_0;u) = u(x)$. We assume that P satisfies the nonlinear Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,t;u) = -\frac{\partial}{\partial x} D_1(x,t,P) P(x,t;u) + \frac{\partial^2}{\partial x^2} D_2(x,t,P) P(x,t;u) \ , \quad (1)$$

where D_1 and D_2 are referred to as drift and diffusion coefficients. Eq. (1) can be written as a continuity equation

$$\begin{aligned} &\frac{\partial}{\partial t} P(x,t;u) = -\frac{\partial}{\partial x} J \ , \\ &J = D_1(x,t,P) P(x,t;u) - \frac{\partial}{\partial x} D_2(x,t,P) P(x,t;u) \end{aligned} \quad (2)$$

that involves a probability current J. If for solutions of Eq. (1) the coefficients $D_1'(x,t;u) = D_1(x,t,P)$ and $D_2'(x,t;u) = D_2(x,t,P)$ correspond to Kramers-Moyal coefficients of transition probability densities $P(x,t|x',t';u)$ of Markov diffusion processes, then we can embed the solutions $P(x,t;u)$ of Eq. (1) into these Markov diffusion processes [39]. Accordingly, $P(x,t|x',t';u)$ can be computed from

$$\begin{aligned} \frac{\partial}{\partial t} P(x,t|x',t';u) = & -\frac{\partial}{\partial x} D_1'(x,t;u) P(x,t|x',t';u) \\ & + \frac{\partial^2}{\partial x^2} D_2'(x,t;u) P(x,t|x',t';u) \end{aligned} \quad (3)$$

or

$$\begin{aligned} \frac{\partial}{\partial t} P(x,t|x',t';u) = & -\frac{\partial}{\partial x} D_1(x,t,P(x,t;u)) P(x,t|x',t';u) \\ & + \frac{\partial^2}{\partial x^2} D_2(x,t,P(x,t;u)) P(x,t|x',t';u) \end{aligned} \quad (4)$$

and the Markov diffusion processes can be described in terms of hierarchies of the distribution functions such as

$$\begin{aligned} P(x,t;u) & \quad , \\ P(x,t;x',t';u) &= P(x,t|x',t';u)P(x',t';u) \ , \\ P(x,t;x',t';x'',t'';u) &= P(x,t|x',t';u)P(x',t'|x'',t'';u)P(x'',t'';u) \ , \\ & \ldots \end{aligned} \tag{5}$$

Alternatively, one may consider the Ito-Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = D_1'(X,t;u) + \sqrt{D_2'(X,t;u)}\Gamma(t) \tag{6}$$

with $D_1'(x,t;u) = D_1(x,t,P)$ and $D_2'(x,t;u) = D_2(x,t,P)$ (see above) or the Ito-Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = D_1(x,t,P)|_{x=X(t)} + \sqrt{D_2(x,t,P)}\Big|_{x=X(t)}\Gamma(t) \ , \tag{7}$$

see [39,40]. Here, $\Gamma(t)$ denotes a Langevin force with $\langle\Gamma(t)\Gamma(t')\rangle = 2\delta(t-t')$ [41,42]. While in Eq. (6) the drift and diffusion coefficients D_1' and D_2' are obtained by solving simultaneously the Fokker-Planck equation (1) and substituting P into D_1 and D_2, in Eq. (7) the probability density P is computed from $P(x,t;u) = \langle\delta(x - X(t)\rangle$. Therefore, we refer to Eq. (6) as a two-layered Ito-Langevin equation, whereas Eq. (7) is referred to as a self-consistent Ito-Langevin equation. From the Ito-Langevin equations the distribution functions

$$\begin{aligned} P(x,t;u) &= \langle\delta(x - X(t))\rangle \ , \\ P(x,t;x',t';u) &= \langle\delta(x - X(t))\delta(x' - X(t'))\rangle \ , \\ P(x,t;x',t';x'',t'';u) &= \langle\delta(x - X(t))\delta(x' - X(t'))\delta(x'' - X(t''))\rangle \ , \\ & \ldots \end{aligned} \tag{8}$$

can be computed and, of course, are equivalent to those listed in Eq. (5). We will refer to nonlinear Fokker-Planck equations whose solutions can be embedded in the aforementioned way into Markov diffusion processes as strong nonlinear Fokker-Planck equations [39]. In what follows, we will assume that we deal indeed with strong nonlinear Fokker-Planck equations.

2.1 H-theorem for stochastic processes

Let us assume that for a particular choice of the drift and diffusion coefficients there is a H-theorem that states that solutions of Eq. (1) satisfy the limiting case

$$\lim_{t\to\infty} P(x,t;u) = P_{\rm st}(x) \ . \tag{9}$$

That is, they become stationary in the long time limit. Then the question arises: does a stochastic process described by Eqs. (5) and (8) converge to a stationary one? In fact, we will answer this question in the affirmative and shown that if Eq. (9) holds, then the limiting case

$$\lim_{t\to\infty} P(x,t|x',t';u) = P_{\rm st}(x,\Delta t|x';P_{\rm st}) \tag{10}$$

holds with $\Delta t = t - t'$ which implies that the stochastic process under consideration becomes stationary. Let us consider systems with $D_1(x,t,P) = D_1(x,P)$ and $D_2(x,t,P) = D_2(x,P)$. Then, we first note that on account of Eq. (9) in the limit $t \to \infty$ Eq. (4) can be written as

$$\begin{aligned}\frac{\partial}{\partial t} P(x,t|x',t';u) = & -\frac{\partial}{\partial x} D_1(x,P_{\rm st}) P(x,t|x',t';u) \\ & + \frac{\partial^2}{\partial x^2} D_2(x,P_{\rm st}) P(x,t|x',t';u) \ .\end{aligned} \tag{11}$$

Using the Fokker-Planck operator

$$L_{\rm FP}(x,\partial/\partial x,P_{\rm st}) = -\frac{\partial}{\partial x} D_1(x,P_{\rm st}) + \frac{\partial^2}{\partial x^2} D_2(x,P_{\rm st}) \tag{12}$$

and $t = t' + \Delta t$, the formal solution of Eq. (11) under the initial condition $\lim_{\Delta t \to 0} P(x, t' + \Delta t | x', t'; u) = \delta(x - x')$ reads

$$\begin{aligned} P(x, t' + \Delta t | x', t'; u) &= \exp\{L_{\mathrm{FP}}(x, \partial/\partial x, P_{\mathrm{st}}) \Delta t\} \delta(x - x') \\ &= P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}}) \ . \end{aligned} \tag{13}$$

As indicated, we can read off from this formal solution that $P(x, t' + \Delta t | x', t'; u)$ depends only on the time interval $\Delta t = t - t'$ and, consequently, can be cast into the form $P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}})$.

Our next objective is to show that stationary transition probability densities of strong nonlinear Fokker-Planck equations converge to stationary distribution functions in the limit $\Delta t \to \infty$. That is, we will prove that the limiting case

$$\lim_{\Delta t \to \infty} P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}}) = P_{\mathrm{st}}(x) \tag{14}$$

holds. In order to derive Eq. (14), we confine ourselves to systems with stationary distributions P_{st} that are defined by vanishing probability currents J. For $J = 0$, $D_1(x, t, P) = D_1(x, P)$, $D_2(x, t, P) = D_2(x, P)$, and $P = P_{\mathrm{st}}$ from Eq. (2) it follows

$$D_1(x, P_{\mathrm{st}}) = \frac{\partial}{\partial x} D_2(x, P_{\mathrm{st}}) + D_2(x, P_{\mathrm{st}}) \frac{\partial}{\partial x} \ln P_{\mathrm{st}} \ . \tag{15}$$

In this case, Eq. (11) for $P(x, t | x', t'; u) = P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}})$ can be written as [43]

$$\frac{\partial}{\partial \Delta t} P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}}) = \frac{\partial}{\partial x} D_2(x, P_{\mathrm{st}}) P_{\mathrm{st}}(x) \frac{\partial}{\partial x} \frac{P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}})}{P_{\mathrm{st}}(x)} \ . \tag{16}$$

To see this, note that Eq. (16) can be transformed into

$$\begin{aligned} &\frac{\partial}{\partial \Delta t} P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}}) = \frac{\partial^2}{\partial x^2} D_2(x, P_{\mathrm{st}}) P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}}) \\ &+ \frac{\partial}{\partial x} P_{\mathrm{st}}(x, \Delta t | x'; P_{\mathrm{st}}) \left[\frac{\partial}{\partial x} D_2(x, P_{\mathrm{st}}) + D_2(x, P_{\mathrm{st}}) \frac{\partial}{\partial x} \ln P_{\mathrm{st}} \right] \ . \end{aligned} \tag{17}$$

Using Eq. (15), we obtain Eq. (11). Next, we write Eq. (16) as

$$\frac{\partial}{\partial \Delta t} P_{\rm st}(x, \Delta t | x'; P_{\rm st}) \\ = \frac{\partial}{\partial x} D_2(x, P_{\rm st}) P_{\rm st}(x, \Delta t | x'; P_{\rm st}) \frac{\partial}{\partial x} \ln \frac{P_{\rm st}(x, \Delta t | x'; P_{\rm st})}{P_{\rm st}(x)} \tag{18}$$

and introduce the Kullback measure

$$K(\Delta t, x') = \int_{\Omega} P_{\rm st}(x, \Delta t | x'; P_{\rm st}) \ln \frac{P_{\rm st}(x, \Delta t | x'; P_{\rm st})}{P_{\rm st}(x)} \, \mathrm{d}x \geq 0 \tag{19}$$

which is semi-positive definite [44,45]. Differentiating $K(\Delta t, x')$ with respect to Δt, using Eq. (18), and integrating by parts, we obtain

$$\frac{\partial}{\partial \Delta t} K(\Delta t, x') = \\ -\int_{\Omega} D_2(x.P_{\rm st}) P_{\rm st}(x, \Delta t | x'; P_{\rm st}) \left[\frac{\partial}{\partial x} \ln \frac{P_{\rm st}(x, \Delta t | x'; P_{\rm st})}{P_{\rm st}(x)} \right]^2 \mathrm{d}x \leq 0 \ . \tag{20}$$

It is clear from Eq. (20) that $\partial K / \partial \Delta t = 0$ implies $P_{\rm st}(x, \Delta t | x'; P_{\rm st}) / P_{\rm st}(x) = C(x', \Delta t)$, where C is independent of x. From Eq. (16) it then follows that $\partial P_{\rm st}(x, \Delta t | x'; P_{\rm st}) / \partial \Delta t = 0$ holds which means that $P_{\rm st}(x, \Delta t | x'; P_{\rm st})$ is independent of Δt. This in turn implies that $C(x', \Delta t)$ is independent of Δt. Thus, we obtain the intermediate result: $\partial K / \partial \Delta t = 0 \Rightarrow P_{\rm st}(x, \Delta t | x'; P_{\rm st}) = C(x') P_{\rm st}(x)$. Integrating this result with respect to x and taking the normalization condition into account, we see that $C(x') = 1$ holds. In sum, $K(\Delta t, x')$ satisfies the relations:

$$K(\Delta t, x') \geq 0 \ , \\ \frac{\partial}{\partial \Delta t} K(\Delta t, x') \leq 0 \ , \\ \frac{\partial}{\partial \Delta t} K(\Delta t, x') = 0 \Leftrightarrow P_{\rm st}(x, \Delta t | x'; P_{\rm st}) = P_{\rm st}(x) \ . \tag{21}$$

Consequently, $K(\Delta t, x')$ is a Lyapunov functional for $P_{\rm st}(x, \Delta t | x'; P_{\rm st})$ and we conclude that Eq. (14) is satisfied.

2.2 Basins of attraction

In order to determine the basins of attraction for stationary distributions, we assume the nonlinear Fokker-Planck equation (1) exhibits multiple stationary distributions that can be written in terms of $P_{\rm st} = P_{\rm st}(x; q_{\rm st})$, where $q_{\rm st}$ denotes the stationary value of an order parameter q. Then the question arises to which one of the stationary solutions does a transient solution converge. In other words: what are the basins of attraction of the stationary distributions. We will not answer this question in general. However, there is a simple answer to this question for Fokker-Planck equations for which closed evolution equations for order parameters can be derived that read

$$\frac{\rm d}{{\rm d}t} q(t) = f(q) \ . \tag{22}$$

For every time-dependent distribution $P(x,t;u)$ with initial distribution $u(x)$ and order parameter value $q(t_0)$ we obtain the corresponding stationary value $q_{\rm st}$ by solving Eq. (22), which, in turn, gives us the corresponding stationary distribution. In other words, from $P(x,t_0;u) = u(x)$ we get $q(t_0)$. From $q(t_0)$ we get $q_{\rm st}$ by means of Eq. (22) and from $q_{\rm st}$ we get $P_{\rm st}(x, q_{\rm st})$. Thus, we can determine the basins of attraction of stationary distributions and write down a mapping $u(x) \to q_{\rm st} \to P_{\rm st}(x, q_{\rm st})$. In the context of nonlinear Fokker-Planck equations nonlinear order parameter equations of the form (22) have been derived in many fields such as chemistry [38], electronic circuitry [46], synchronization [17,47–49], surface physics [4], population dynamics [23], neurophysics [50], and have been used to study basic properties of many-body systems with long-range interactions [14,35,51–54], see Table 1.

Insert Table 1 about here

Now, let us assume that the coefficients $D_1(x,t,P) = D_1(x,P)$ and $D_2(x,t,P) = D_2(x,P)$ in Eq. (1) depend on P by means of an order parameter $q = \langle A \rangle$. In this case, Eq. (1) reads

$$\frac{\partial}{\partial t}P(x,t;u) = -\frac{\partial}{\partial x}D_1(x,\langle A\rangle)P(x,t;u) + \frac{\partial^2}{\partial x^2}D_2(x,\langle A\rangle)P(x,t;u)\ . \quad (23)$$

We may further assume that there are multiple stationary distributions of the form $P_{\rm st} = P_{\rm st}(x;\langle A\rangle_{\rm st})$ and that the order parameter $\langle A\rangle$ evolves like

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle(t) = f(\langle A\rangle)\ . \quad (24)$$

Then, the initial value of $\langle A\rangle$ determines completely which stationary distribution out of all possible stationary distributions is selected in the long time limit.

2.3 Free energy Fokker-Planck equation with Boltzmann statistics

The evolution of probability densities $P(x,t;u)$ of systems with free energy measures $F[P] = U[P] - QS[P]$ may be described by the free energy Fokker-Planck equation

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}P\frac{\partial}{\partial x}\frac{\delta F}{\delta P} \quad (25)$$

that can, for example, derived from linear nonequilibrium thermodynamics [33,55], the kinetical interaction principle [31] or alternative methods [8]. Here, $U[P]$ is a measure for the internal energy and in general is nonlinear with respect to P. The variable S denotes a general entropy measure. If F is bounded from below, then there is a H-theorem that states that $P(x,t;u)$ becomes stationary in the long time limit [33] (see also [7,8,30–32]). If S is given by the Boltzmann entropy $S[P] = -\int P\ln P\,\mathrm{d}x$, then Eq. (25) can equivalently be expressed as

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}P\frac{\partial}{\partial x}\frac{\delta U}{\delta P} + Q\frac{\partial^2}{\partial x^2}P\ . \quad (26)$$

By comparison with Eq. (1), we find that $D_1(x,P) = -(\partial/\partial x)\delta U/\delta P$ and $D_2(x,P) = Q$. In the special case $U[P] = \int_\Omega V(x)P(x)\,\mathrm{d}x + \int_\Omega B_0(x)P(x)\,\mathrm{d}x + B(\langle A\rangle)$ Eq. (26) reads

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}P\left[\frac{\mathrm{d}}{\mathrm{d}x}[V(x)+B_0(x)] + \frac{\mathrm{d}B(\langle A\rangle)}{\mathrm{d}\langle A\rangle}\frac{\mathrm{d}A(x)}{\mathrm{d}x}\right] + Q\frac{\partial^2}{\partial x^2}P \tag{27}$$

and involves a drift coefficient $D_1(x,P) = D_1(x,\langle A\rangle)$. If the expression given by $\int_\Omega B_0(x)P(x)\,\mathrm{d}x + B(\langle A\rangle)$ is bounded from below, one can show that F is bounded from below [56]. Note that in this case $B(z)$ must not be bounded from below. In sum, nonlinear Fokker-Planck equations that can be cast into the form (27) are globally stable in the sense that transient solutions converge to stationary ones. In addition, Eq. (27) can exhibit multiple stationary states because of the nonlinear drift term $D_1(x,\langle A\rangle))P(x,t;u)$. Therefore, Eq. (27) can be used to illustrate the issues discussed in sections 2.1 and 2.2 — as we will show next.

3 Example

The evolution of the magnetization $M(t)$ of an Ising ferromagnet is often described by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}M(t) = -a_1 M(t) + a_2\tanh(a_3 M(t)) \tag{28}$$

with $a_1, a_2, a_3 > 0$ [57–59]. Our objective now is to illustrate the results of the previous sections by means of a nonlinear Fokker-Planck equations whose solutions satisfy Eq. (28). Just as in the case of the Desai-Zwanzig model, we can then regard symmetric stationary solutions with $M = 0$ as some kind of paramagnetic phases and asymmetric stationary solutions with $M \neq 0$ as some kind of ferromagnetic phases.

To begin with, let us consider Eq. (27) with

$$B(z) = -\ln\cosh(\sqrt{c}\, z) \tag{29}$$

for $c > 0$, $A(x) = x$, $V(x) = \gamma x^2/2$, and $B_0(x) = \gamma x^2/2$:

$$\frac{\partial}{\partial t} P(x,t;u) = \frac{\partial}{\partial x}\left[(\gamma + c)x - \sqrt{c}\, \tanh(\sqrt{c}\,\langle X\rangle)\right] P + Q\frac{\partial^2}{\partial x^2} P \ . \tag{30}$$

The first moment of the solutions $P(x,t;u)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} M_1 = -(\gamma + c)M_1 + \sqrt{c}\tanh(\sqrt{c}M_1) = -\frac{\mathrm{d}V_M}{\mathrm{d}M_1} \ , \tag{31}$$

where V is defined by

$$V_M(z) = \frac{\gamma}{2} z^2 + \underbrace{\frac{c}{2} z^2 - \ln\cosh(\sqrt{c}z)}_{V_c(z)} \ . \tag{32}$$

We realize now that for $a_1 = \gamma + c$, $a_2 = \sqrt{c}$ and $a_3 = \sqrt{c}$ Eq. (31) corresponds to Eq. (28). In addition, Eq. (30) can be regarded as a free energy Fokker-Planck equation (25) with F defined by

$$F[P] = \frac{\gamma}{2}\left\langle X^2\right\rangle + \frac{c}{2}\left\langle X^2\right\rangle + B(\langle X\rangle) - Q\, S[P] \ . \tag{33}$$

We can verify that the expression $Y = B(z) + (c + \gamma)z^2/2$ is bounded from below for $\gamma > -c$ (hint: for large $|z|$ we have $B \propto -|z|$). Consequently, it can be shown that F is bounded from below [56] and that the H-theorem for free energy Fokker-Planck equations applies which means that we have $\lim_{t\to\infty} \partial P(x,t;u)/\partial t = 0$. Moreover, one can show that $V_c(z)$ vanishes at $z = 0$ and increases monotonically with $|z|$ both for $z > 0$ and $z < 0$. Consequently, the potential (32) is monostable for $\gamma > 0$ and bistable for $\gamma < 0$ (and $\gamma > -c$, see below), see Fig. 1. That is, for $\gamma \in (-c, 0)$ the potential V_M describes a double-well potential and there are multiple stationary solutions of M_1.

Insert Fig. 1 about here

Eq. (30) belongs to a class of nonlinear Fokker-Planck equations (27) that has been studied in detail in [60]. Accordingly, stationary distributions satisfy the implicit equation

$$P_{\rm st}(x) = \sqrt{\frac{\gamma+c}{2\pi Q}} \exp\left\{ -\frac{(\gamma+c)}{2Q} \left[x - \frac{\sqrt{c}}{c+\gamma} \tanh(\sqrt{c}\, \langle X \rangle_{\rm st}) \right]^2 \right\} \ , \quad (34)$$

(see Eq. (25) in [60]) and the stationary values of the order parameter $\langle X \rangle$ can be obtained from the self-consistency equation

$$\langle X \rangle_{\rm st} = R(\langle X \rangle_{\rm st}) \quad (35)$$

with $R(m)$ given by

$$R(m) = \frac{\sqrt{c}}{c+\gamma} \tanh(\sqrt{c}\, m) \quad (36)$$

(see also Eq. (31) for the stationary case). It is clear that $R(0) = 0$ holds and, consequently, a stationary solution is described by

$$P_{\rm st}(x) = \sqrt{\frac{\gamma+c}{2\pi Q}} \exp\left\{ -\frac{(\gamma+c) x^2}{2Q} \right\} \ . \quad (37)$$

In view of the antisymmetry property $R(m) = -R(-m)$, we conclude that if $m \neq 0$ is a solution of the self-consistency equation $m = R(m)$ then $-m$ corresponds to a solution as well. Now, let us discuss under which conditions the self-consistency equation has nonvanishing solutions. First, we note that $R(m)$ is a monotonically increasing function with

$$\frac{{\rm d}R(m)}{{\rm d}m} = \frac{c}{c+\gamma} \left[1 - \tanh^2(\sqrt{c}\, m) \right] \geq 0 \quad (38)$$

because of $1 - [\tanh(z)]^2 \geq 0$. It is clear that if ${\rm d}R/{\rm d}m < 1$ holds for all m, then $m = R(m)$ is only solved by $m = 0$. The reason for this is that if $R(m)$

with $R(0) = 0$ is smaller than m for $m > 0$ and larger than m for $m < 0$, then $m = R(m)$ is only solved by $m = 0$. Since $0 \leq 1 - \tanh^2(z') \leq 1$ for $z' \in \mathbb{R}$, for $\gamma > 0$ we have $\forall m : \mathrm{d}R/\mathrm{d}m < 1$ and the self-consistency equation has a unique solution given by $m = 0$. In contrast, for $-c < \gamma < 0$ we have $\mathrm{d}R(0)/\mathrm{d}m = c/(c+\gamma) > 1$ and we find that the self-consistency equation exhibits solutions with $m \neq 0$, see Fig. 2.

The stability of stationary distributions can be determined by means of self-consistency equations, Lyapunov's direct method, and linear stability analysis [30,60]. For $\gamma > 0$ we have $\mathrm{d}R(0)/\mathrm{d}m = c/(c+\gamma)$ which implies that the symmetric probability density (37) is asymptotically stable (because of $\mathrm{d}R(0)/\mathrm{d}m < 1$), whereas for $\gamma < 0$ the distribution (37) becomes unstable (because of $\mathrm{d}R(0)/\mathrm{d}m > 1$ for $\gamma < 0$). Moreover, from Fig. 2 we read off that for $-c < \gamma < 0$ we have $\mathrm{d}R(m)/\mathrm{d}m < 1$ at solutions $m \neq 0$ of $m = R(m)$. Consequently, the asymmetric probability densities (34) with $\langle X \rangle_{\mathrm{st}} \neq 0$ are asymptotically stable if they exist. The mean values $M_{1,\mathrm{st}} = m$ computed from $m = R(m)$ for several values of γ are shown in Fig. 3. For $\gamma \downarrow -c$ the stationary mean values $M_{1,\mathrm{st}} \neq 0$ behave like $M_{1,\mathrm{st}} \to \pm\infty$. For $\gamma \leq -c$ stationary solutions cease to exist.

Insert Figures 2 and 3 about here

3.1 Illustration of the H-theorem for stochastic processes

As shown in Sec. 2.1 the limiting case (10) holds. Our aim now is to illustrate this asymptotic behavior by means of the autocorrelation function $C(t,t') = \langle X(t)X(t') \rangle$ defined for $t \geq t' \geq t_0$. From Eq. (31) it follows that for every initial distribution $u(x)$ the first moment M_1 corresponds to a continuous function of time: $M_1 = M_1(t;u)$ Therefore, Eq. (30) can equivalently be expressed as

$$\frac{\partial}{\partial t}P(x,t;u) = \frac{\partial}{\partial x}\left[(\gamma+c)x - \sqrt{c}\,\tanh(\sqrt{c}M_1(t;u))\right]P + Q\frac{\partial^2}{\partial x^2}P \quad (39)$$

involving the drift coefficient $D_1'(x,t;u) = -(\gamma+c)x + \sqrt{c}\,\tanh(\sqrt{c}M_1(t;u))$. Since D_1' can be regarded as the first Kramers-Moyal coefficient of a Markov diffusion process, Eq. (30) is a strong nonlinear Fokker-Planck equation and the transition probability density of the Markov diffusion process of interest satisfies

$$\begin{aligned}\frac{\partial}{\partial t}P(x,t|x',t';u) = &\frac{\partial}{\partial x}\left[(\gamma+c)x - \sqrt{c}\,\tanh(\sqrt{c}\,\langle X\rangle_{P(x,t;u)})\right]P(x,t|x',t';u)\\ &+Q\frac{\partial^2}{\partial x^2}P(x,t|x',t';u)\ . \end{aligned} \quad (40)$$

From Eq. (40) we obtain the evolution equation

$$\frac{\partial}{\partial t}C(t,t') = -(\gamma+c)C(t,t') + \sqrt{c}M_1(t')\tanh(\sqrt{c}M_1(t)) \quad (41)$$

which has to be solved under the initial condition $C(t',t') = M_2(t') = \langle X^2(t')\rangle$. Thus, we get

$$\begin{aligned}C(t,t') = &M_2(t')\exp\{-(\gamma+c)(t-t')\}\\ &+\sqrt{c}M_1(t')\int_{t'}^{t}\exp\{-(\gamma+c)(t-z)\}\tanh(\sqrt{c}M_1(z))\,\mathrm{d}z\ . \end{aligned} \quad (42)$$

We need to determine now the unknown second moment M_2. From Eq. (30) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}M_2(t) = -2(\gamma+c)M_2(t) + 2Q + 2\sqrt{c}M_1(t)\tanh(\sqrt{c}M_1(t))\ . \quad (43)$$

Note that in the stationary case this equation reduces to $M_{2,\mathrm{st}} = Q/(\gamma+c) - M_{1,\mathrm{st}}^2$ which gives us the variance $K_{\mathrm{st}} = Q/(\gamma+c)$ of the stationary distribution (34) again. In sum, solving the evolution equations for $M_1(t)$ and $M_2(t)$ given by Eqs. (31) and (43) and substituting the result into Eq. (42) we can obtain for all pairs (t,t') the autocorrelation function $C(t,t')$.

Now let us show that the autocorrelation function becomes stationary in the stationary case: $\lim_{t\to\infty,t'\to\infty} C(t,t') = C_{\rm st}(t-t')$. To this end, let us first note that if $z \in [t',t]$ and $t,t' \to \infty$ holds then we can draw the (somewhat trivial) conclusion that $z \to \infty$. Consequently, the integral in Eq. (42) becomes $\sqrt{c}M_{1,\rm st}\tanh(\sqrt{c}M_{1,\rm st})\int_{t'}^{t}\exp\{-(\gamma+c)(t-z)\}\,{\rm d}z$. If we use $\sqrt{c}\tanh(\sqrt{c}M_{1,\rm st}) = (\gamma+c)M_{1,\rm st}$ (see Eq. 31)), we obtain

$$\lim_{t\to\infty,t'\to\infty} C(t,t') = M_{1,\rm st}^2 + K_{\rm st}\exp\{-(\gamma+c)(t-t')\} = C_{\rm st}(t-t') \ . \quad (44)$$

That is, $C(t,t')$ indeed becomes stationary in the stationary case. Furthermore, Eq. (44) satisfies the special cases $C_{\rm st}(0) = M_{2,\rm st}$ and $\lim_{\Delta t\to\infty} C_{\rm st}(\Delta t) = M_{1,\rm st}^2$.

3.2 Basins of attraction

As mentioned earlier the potential $V_M(z)$ given by Eq. (32) is bounded from below for $\gamma > -c$. It is monostable for $\gamma \geq 0$ and bistable for $-c < \gamma < 0$, see Fig. 1. For $-c < \gamma < 0$ we conclude from Eq. (31) that probability densities $P(x,t;u)$ with $M_1(0) > 0$ converge to the asymptotically stable, asymmetric stationary distribution (34) with $M_{1,\rm st} > 0$, whereas $P(x,t;u)$ with $M_1(0) < 0$ converge to Eq. (34) with $M_{1,\rm st} < 0$. Probability densities $P(x,t;u)$ with a vanishing first moment converge to the unstable solution (37). Consequently, in the function space of probability densities the distributions $u(x)$ with $\int_\Omega xu(x)\,{\rm d}x = 0$ describe some kind of separatrix. We would like to point out that not only converge all distribution functions $P(x,t;u)$ with $\int_\Omega xu(x)\,{\rm d}x \neq 0$ to the stationary distribution (34) with $M_{1,\rm st} > 0$ or $M_{1,\rm st} < 0$ but also all stochastic processes described by Eqs. (5) and (8) converge to two kinds of stationary stochastic processes exhibiting either $M_{1,\rm st} > 0$ or

$M_{1,\mathrm{st}} < 0$.

3.3 Langevin equation

The self-consistent Langevin equation (7) that corresponds to the strong nonlinear Fokker-Planck equation given by Eqs. (30) and (40) reads

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t) = -(\gamma + c)X + \sqrt{c}\tanh(\sqrt{c}\,\langle X \rangle) + \sqrt{Q}\Gamma(t) \ . \tag{45}$$

We can compute now the autocorrelation function $C(t, t')$ from Eqs. (31), (42), and (43) obtained from the Fokker-Planck description or alternatively from the Langevin equation (45). As shown in figures 4 and 5 both methods yield the same results. In particular, Fig. 5 illustrates the convergence of the stationary autocorrelation function $C_{\mathrm{st}}(\Delta t)$ to the limit value $M_{1,\mathrm{st}}^2$ in the limit $\Delta t \to \infty$. In order to illustrate the basins of attraction, we solved the Langevin equation (45) for several initial values of M_1, see Fig. (6). Again, we solved also the corresponding evolution equation (31) obtained from the Fokker-Planck description and found consistent results.

Insert Figures 4,5, and 6 about here

4 Top-down versus bottom-up approach

In the introductory part of Sec. 2 we have briefly addressed the embedding of solutions of strong nonlinear Fokker-Planck equations into Markov diffusion processes as suggested in [39]. This embedding procedure should be regarded as a top-down approach because our departure point is a nonlinear Fokker-Planck equation of the form (1) and we assume that we do not have any further information at our disposal. In contrast, various studies have been concerned with the derivation of nonlinear Fokker-Planck equations by means of bottom-up approaches, see, for example, [17,27,35,36]. We would like to show next that for nonlinear Fokker-Planck equations with drift and diffusion coefficients that depend on an order parameter both approaches yield the same result.

We start off with a many-body system composed of subsystems that can be described by the state variables $X_k(t)$ with $k = 1, \ldots, N_0$. We consider the thermodynamic limit: $N_0 \to \infty$. In order to deal with the limit $N_0 \to \infty$, in what follows, we will focus on finite subsets of the subsystem ensemble. Let $I_L = \{i_1, \ldots, i_L\}$ denote a set of L different indices with $i_l \geq 1$. Then, we assume that for all $L \geq 1$ and all possible index-sets I_L the dynamics of the subsystems can be described by the L-dimensional Ito-Langevin equation

$$\frac{\mathrm{d}}{\mathrm{d}t} X_{i_k}(t) = D_1\left(X_{i_k}, t, \lim_{N_0\to\infty} \frac{1}{N_0} \sum_{i=1}^{N_0} A(X_i)\right) = \sqrt{D_2\left(X_{i_k}, t, \lim_{N_0\to\infty} \frac{1}{N_0} \sum_{i=1}^{N_0} A(X_i)\right)}\, \Gamma_{i_k}(t) \tag{46}$$

for $i_k \in I_L$ and $\langle \Gamma_{i_v}(t')\Gamma_{i_w}(t)\rangle = 2\delta_{vw}\delta(t-t')$. Note that Eq. (46) accounts for the impact of all ensemble members on an individual subsystem i_k (i.e., the sum runs from one to infinity and does not run from 1 to L). Let us describe the sub-population of ensemble members I_L by means of the state vector $\mathbf{X}_L = (X_{i_1}, \ldots, X_{i_L})$. We will prove now that the subsystems are statistically-independent for all times $t \geq t^*$ if they are statistically-independent at time t^* and are distributed according to the same law. Consequently, we assume that there is a time t^* for which

$$P(\mathbf{x}_L, t^*) = \prod_{k=1}^{L} P(x, t^*)|_{x_{i_k}} \tag{47}$$

holds for all $L \geq 1$ and all index-sets I_L. Here, $P(\mathbf{x}_L, t^*)$ denotes the multivariable probability density $P(\mathbf{x}_L, t^*) = \langle \delta(\mathbf{x}_L - \mathbf{X}_L(t^*))\rangle$, while $P(x,t)$ denotes a single subsystem probability density. From Eq. (47) it is clear that $\lim_{N\to\infty} N^{-1} \sum_{i=1}^{N} A(X_i(t^*)) = \langle A\rangle (t^*) = \int A(x)P(x,t^*)\,\mathrm{d}x$ holds. To proceed further, we assume (just as in the top-down approach) that the coefficients

$$D_1'(x,t) = D_1(x,t,\langle A\rangle(t)) \ , \quad D_2'(x,t) = D_2(x,t,\langle A\rangle(t)) \tag{48}$$

denote first and second order Kramers-Moyal coefficients of a Markov diffusion process at time t^*. Consequently, from Eq. (46) we obtain the L-dimensional Fokker-Planck equation

$$\frac{\partial}{\partial t}P(\mathbf{x}_L,t^*) = \\ \sum_{k=1}^{L}\left[-\frac{\partial}{\partial x}D_1(x,t^*,\langle A\rangle(t^*)) + \frac{\partial^2}{\partial x^2}D_2(x,t^*,\langle A\rangle(t^*))\right]_{x_{i_k}} P(\mathbf{x}_L,t^*) \tag{49}$$

for all $L \geq 1$ and all sub-populations I_L (note that the partial derivatives also act beyond the squared bracket). From Eqs. (47) and (49) we read off that P at time t^* evolves like

$$\frac{\partial}{\partial t}P(\mathbf{x}_L,t^*) = \sum_{k=1}^{L}\left\{\left(\prod_{l=1,l\neq k}^{L} P(x,t^*)|_{x_{i_l}}\right)\frac{\partial}{\partial t}\,P(x,t^*)|_{x_{i_k}}\right\} \tag{50}$$

with

$$\frac{\partial}{\partial t}P(x,t) = \left[-\frac{\partial}{\partial x}D_1(x,t,\langle A\rangle(t)) + \frac{\partial^2}{\partial x^2}D_2(x,t,\langle A\rangle(t))\right]P(x,t) \ . \tag{51}$$

As a result, for infinitesimal small time steps Δt we find that

$$P(\mathbf{x}_L,t^*+\Delta t) = \prod_{k=1}^{L} P(x,t^*+\Delta t)|_{x_{i_k}} \tag{52}$$

holds for all $L \geq 1$ and all index-sets I_L. Eq. (52) states that the many-body system described by Eq. (46) evolves from a state of statistically-independent subsystems in such way that we obtain again a state of statistically-independent subsystems. Therefore, we draw the conclusion that if for solutions $P(x,t)$ of Eq. (51) the drift and diffusion coefficients defined by Eq. (48) correspond to Kramers-Moyal coefficients and if all subsystems at time t_0 are statistically-independent and distributed according to the same law, then Eq. (46) can equivalently be described by means of the strong nonlinear Fokker-Planck equation (51). Likewise, in this case $P(\mathbf{x}_L,t)$ is given by $P(\mathbf{x}_L,t) = \prod_{k=1}^{L} P(x_{i_k},t)$. Moreover, from Eqs. (46) and (48) it follows that the transition probability density $P(\mathbf{x}_L,t|\mathbf{x}'_L,t')$ satisfies the evolution equation

$$\frac{\partial}{\partial t}P(\mathbf{x}_L,t|\mathbf{x}'_L,t') = \sum_{k=1}^{L}\left[-\frac{\partial}{\partial x}D_1(x,t,\langle A\rangle_{P(x,t)}) + \frac{\partial^2}{\partial x^2}D_2(x,t,\langle A\rangle_{P(x,t)})\right]_{x_{i_k}} P(\mathbf{x}_L,t|\mathbf{x}'_L,t') \ , \quad (53)$$

which, in turn, can be solved by $P(\mathbf{x}_L,t|\mathbf{x}'_L,t') = \prod_{k=1}^{L} P(x,t|x',t')|_{x_{i_k}}$ and

$$\frac{\partial}{\partial t}P(x,t|x',t') = \left[-\frac{\partial}{\partial x}D_1(x,t,\langle A\rangle_{P(x,t)}) + \frac{\partial^2}{\partial x^2}D_2(x,t,\langle A\rangle_{P(x,t)})\right] P(x,t|x',t') \ . \quad (54)$$

Now let us write down the self-consistent Ito-Langevin equation (7) for $D_1(x,t,P) = D_1(x,t,\langle A\rangle)$ and $D_2(x,t,P) = D_2(x,t,\langle A\rangle)$ in terms of the realizations X^k of $X(t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}X^k(t) = D_1\left(X^k,t,\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}A(X^i)\right) = \sqrt{D_2\left(X^k,t,\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}A(X^i)\right)}\,\Gamma^k(t) \ . \quad (55)$$

Since realizations of random variables are by definition statistically-independent quantities, by comparison of Eqs. (46) and (55) we find that we may interpret the realizations of random variables defined by nonlinear Fokker-Planck equations as the state variables of the statistically-independent (but interacting) subsystems of many-body systems. In symbols, we may put $N_0 = N$ and $X_k = X^k$. In this sense, the bottom-up approach is consistent with the top-down approach proposed in [39]. Let us conclude these considerations with two remarks.

First, in the derivation above we have required that the coefficients $D_1'(x,t;u) = D_1(x,t,P)$ and $D_2'(x,t;u) = D_2(x,t,P)$ in general and the coefficients (48) in particular correspond to Kramers-Moyal coefficients. This requirement should be regarded as a severe one. For example, we may consider systems with order parameters like $\langle X^2 \rangle$ (which e.g. denote intensities of electric or magnetic field variables). If we study such systems for initial distributions that decay like a power law $1/|x|^\delta$ with $\delta \leq 3$ (e.g. Lévy flights) then the order parameter $\langle X^2 \rangle$ is infinite which means that the coefficients $D_1(x,t,\langle X^2 \rangle)$ and $D_2(x,t,\langle X^2 \rangle)$ are not well-defined and do not correspond to Kramers-Moyal coefficients.

Second, we have shown above that if Eq. (47) holds then the subsystems are statistically-independent for all times $t \geq t^*$. This phenomenon is also called the propagation of chaos and has, for example, proven in previous studies by means of the path integral representation of solutions of Fokker-Planck equations [36]. In this context, it should be pointed out that the propagation of chaos occurs in an even larger class of systems as discussed here, for example, in many-body systems that involve fluctuating system parameters [36].

5 Conclusions

We have shown that stochastic processes defined by strong nonlinear Fokker-Planck equations converge to stationary stochastic processes in the long time limit when the corresponding single time-point distributions converge to stationary distributions. This implies that several H-theorems that have been proposed for nonlinear Fokker-Planck equations can now in fact be applied to describe the asymptotic behavior of complete stochastic processes. Our result also holds for multistable systems. The stochastic trajectories converge to different kinds of stochastic trajectories depending on the initial distribution of the systems under consideration. In this context, for simple systems we have shown how to identify the basins of attraction of stationary stochastic processes or stationary distributions. Most interesting we have found for a particular bistable mean field model that the separatrix in the function space of distribution functions is given by those probability densities that converge to unstable stationary distributions related to free energy maxima. We also have elucidated the power of the embedding of solutions of strong nonlinear Fokker-Planck equations into Markov diffusion processes as proposed in [39]. We have been able to determine autocorrelation functions in terms of semi-analytical solutions of nonlinear Fokker-Planck equations, one the one hand, and full numerical computations involving Langevin equations, on the other hand. In addition, we have shown that the proposed Markov embedding is consistent with conventional bottom-up approaches to nonlinear Fokker-Planck equations.

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Tables:

Table 1

Nonlinear order parameter equations related to nonlinear Fokker-Planck equations

Systems/Phenomena	$f(q)$	Ref.
Chemical reactions	$\alpha(N-(1+k_0\exp\{-\eta\, q/N\})$	[38]
Josephson junctions	$-2r_0 q\cosh(\lambda q+\sigma(t))+2r_0\sinh((\lambda q+\sigma(t))$	[46]
Synchronization	$\lambda\, q-g\, q\lvert q\rvert^2$	[17,47–49]
Roughening	$\exp\{-q^3/t_c^3\}$	[4]
Collective motion of swarms	$(1-3\sigma-q^2)q$	[23]
Noise-induced first-order phase transition	$h(q)+0.5\sigma^2 g(q)\mathrm{d}g(q)/\mathrm{d}q$	[51,52]
Pitchfork bifurcation of variance	$\lambda(q-C)-2\beta(q-C)^3$	[14]
Coupled van der Pol oscillators	$c_1 q-c_2 q^3$	[53]
Desai-Zwanzig model	$\mathrm{d}q_i/\mathrm{d}t=f_i(q_1,\ldots,q_N)$ for $i=1,\cdots,N$	[35]
Desai-Zwanzig model	$\mathrm{d}q_1/\mathrm{d}t=(1-q_1^2)q_1-3q_1q_2$, $\mathrm{d}q_2/\mathrm{d}t=2(1-\theta-3q_1^2)q_2+2D$	[54]
Coupled FitzHugh-Nagumo neurons	$\mathrm{d}q_1/\mathrm{d}t=h(q_1)-cq_2+I(t)$, $\mathrm{d}q_2/\mathrm{d}t=bq_1-dq_1+e$	[50]

Figure caption:

Fig. 1: $V_M(z)$ for $c = 1$ and several values of the parameter γ: $\gamma = 0.5$ (dotted line) and $\gamma = -0.5$ (solid line).

Fig. 2: Illustration of the self-consistency equation $m = R(m)$ with $R(m)$ given by Eq. (36). Solid lines: $R(m)$ for $c = 1$ and $\gamma = -0.5$ and $\gamma > 0.5$. Dashed line: diagonal of the (m, R)-plane.

Fig. 3: Bifurcation diagram for the order parameter $\langle X \rangle_{\mathrm{st}}$ of the dynamic mean field model (30).

Fig. 4: $M_1(t)$, $M_2(t)$, and $C(t, t') = \langle X(t)X(t') \rangle$ as functions of t for a stochastic processes with $u(x) = \delta(x - x_0)$. Solid lines represent results obtained from the Fokker-Planck approach (31), (42), and (43). Diamonds represent results obtained from a simulation of the Langevin equation (45). $\langle X(t)X(t') \rangle$ is given for $t \geq t'$. For $t < t'$ we have put $\langle X(t)X(t') \rangle = 0$. Parameters: $Q = 1$, $x_0 = -3$, $\gamma = -0.5$, $c = 1$, and $t' = 2$, $t_0 = 0$.

Fig. 5: As in Fig. 4 but for $t' = 12$. As shown by the first and second moments in this case we are in the stationary regime. The thin horizontal line describes $M_{1,\mathrm{st}}^2$.

Fig. 6: $M_1(t)$ as a function of t as obtained from a simulation of Eq. (45) (diamonds) and from solving numerically Eq. (31) (solid lines) for delta distributed $u(x)$ and different $M_1(0)$ ($t_0 = 0$). For distributions with $M_1(0) > 0$ (< 0) the stochastic processes converge to $M_{1,\mathrm{st}} \approx 2$ (-2). Parameters: $Q = 1$, $c = 1$, $\gamma = -0.5$, and $M_1(0) = 3$, $M_1(0) = 0.2$, $M_1(0) = -0.2$, and $M_1(0) = -3$ (from top to bottom).

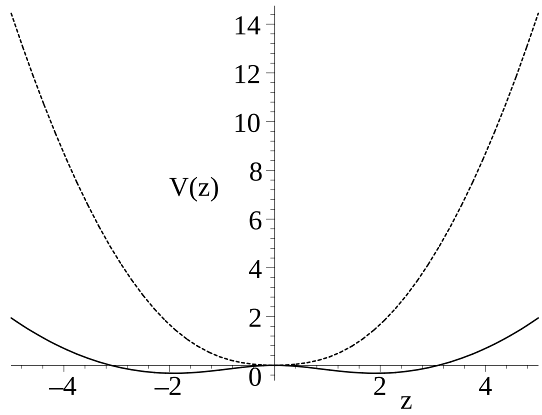

Fig. 1.

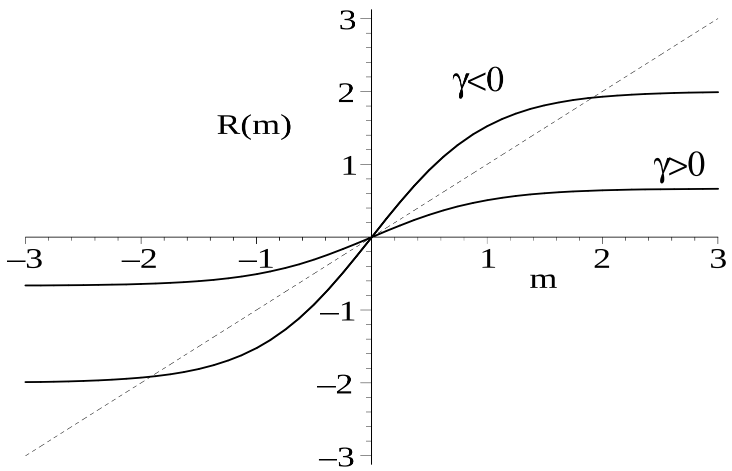

Fig. 2.

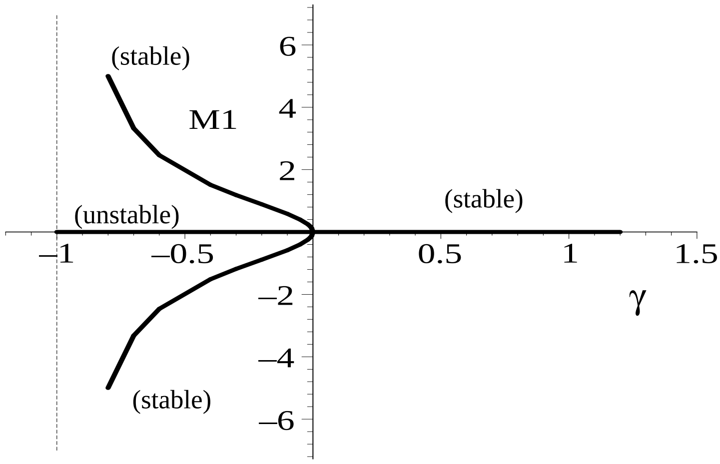
(stable)
M1
(unstable)
(stable)
(stable)
γ
6
4
2
–2
–4
–6
–1
–0.5
0.5
1
1.5

Fig. 3.

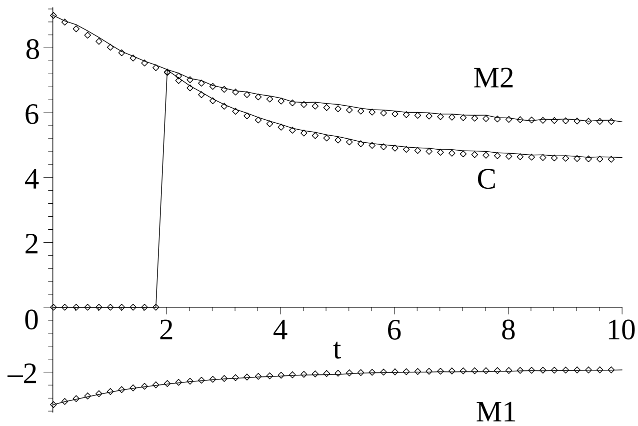
M2
C
M1
t
8
6
4
2
0
–2
2
4
6
8
10

Fig. 4.

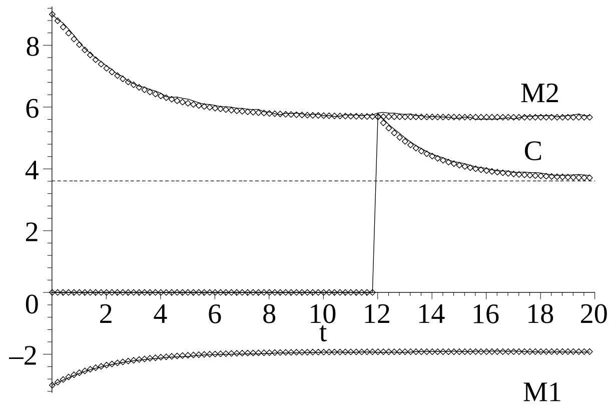

Fig. 5.

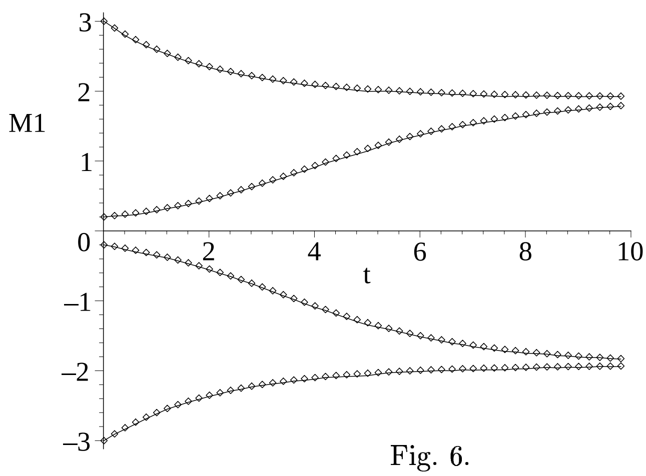

Fig. 6.