A data analysis method for identifying deterministic components of stable and unstable time-delayed systems with colored noise

K. Patanarapeelert[1], T.D. Frank[2], R. Friedrich[2], P.J. Beek[3], I.M. Tang[4]

[1]*Faculty of Science, Department of Mathematics, Mahidol University, Rama VI Road, Bangkok 10400, Thailand*

[2]*Institute for Theoretical Physics, University of Münster, Wilhelm-Klemm-Str. 9, 48149 Münster, Germany*

[3]*Faculty of Human Movement Sciences and Institute for Fundamental and Clinical Human Movement Sciences, Vrije Universiteit, Van der Boechorststraat 9, 1081 BT Amsterdam, The Netherlands*

[4]*Faculty of Science, Department of Physics, Mahidol University, Rama VI Road, Bangkok 10400, Thailand*

Abstract

A method is proposed to identify deterministic components of stable and unstable time-delayed systems subjected to noise sources with finite correlation times (colored noise). Both neutral and retarded delay systems are considered. For vanishing correlation times it is shown how to determine their noise amplitudes by minimizing appropriately defined Kullback measures. The method is illustrated by applying it to simulated data from stochastic time-delayed systems representing delay-induced bifurcations, postural sway and ship rolling.

PACS: 87.19.St, 02.30.Ks, 05.45.Tp

---

[2] e-mail: tdfrank@uni-muenster.de

Preprint submitted to Elsevier Preprint *14 August 2006*

1 Introduction

Fluctuations are often observed in physical systems. They are generally known as thermal noise leading to Brownian motion [1–3] and can be found in various equilibrium and non-equilibrium systems ranging from magnetic systems [4] and lasers [5], chemical systems [6,7], and polymer fluids [8,9] to ecological systems [10–12], neural networks [13–15], financial systems [16], human movement systems [17,18], and other biological systems such as Brownian motors [19,20]. Fluctuations exhibit many properties that may be beneficial to the functioning of a system. In general, however, they obscure the on-going deterministic processes of a system. Therefore, a crucial issue is to reveal these deterministic processes by evaluating noisy data. In particular, systems that are governed by fundamental Newtonian principles, such as human movement systems and a broad class of mechanical systems, will evolve according to second-order differential equations of the form $m\ddot{x}(t) = f + \xi(t)$, where $x(t)$ is a state variable, m reflects a generalized mass, f is a deterministic force and $\xi(t)$ corresponds to a fluctuating force. Equations of the form $m\ddot{x}(t) = f + \xi(t)$ can account for both dissipation and fluctuations. As shown in Fig. 1a), the deterministic force (or drift function) f will depend on the state variable $x(t)$ and a generalized velocity $v(t) = \dot{x}(t)$. In addition, in many instances there will be a feedback subjected to a time delay τ [21–26].

Insert Fig. 1 about here

Feedback components may be classified into delayed feedback via stiffness if f involves the retarded variable $x_\tau = x(t-\tau)$ and delayed feedback via friction if f involves the retarded velocity variable $v_\tau = v(t-\tau)$. In view of Fig. 1a), reconstructing the drift force implies identifying how f depends on x, v, x_τ, v_τ. As a by-product one would also be interested in determining the amplitude of the fluctuating force ξ (see Fig. 1 panel a). In short, we will assume that the dynamics of a system is given by

$$m\ddot{x}(t) = f(x, x_\tau, v, v_\tau) + \xi(t) \ , \tag{1}$$

where the fluctuating force $\xi(t)$ is additive, has vanishing mean, and exhibits a noise amplitude Q (to be defined more precisely below). The objective then is to determine f and if possible Q from noisy data. In contrast to previously proposed data analysis methods for time-delayed processes [27–31], we will exploit a data analysis method that is tailored to address stochastic differential equations such as Langevin equations [32,33], differential equations driven by Levy noise [34] and stable time-delayed equations involving Langevin noise [35–37]. Using this method, we will show that estimates for the drift function f can be derived in various instances that have not been explored before in the literature, see Fig. 1 panel (b). For example, neutral delay systems [1] have not been addressed so far. Such systems, however, have found various applications in physics and engineering sciences [23,38–41] because they correspond to overdamped time-delayed second-order dynamical systems. Similarly, hardly any efforts have been made to develop data analysis methods for time-delayed systems that are unstable (or non-stationary) and/or driven by colored noise. In Sec. 2.1 we will consider overdamped systems that can be described by neutral time-delayed equations including systems that become non-stationary due to a delay-induced bifurcation. In Sec. 2.2 we will address second-order dynamical systems such as self-oscillating systems below and above their Hopf bifurcation points. In addition, we will apply the method to simulated data from stochastic time-delayed systems representing postural sway and rolling motions of ships. In Sec. 2.1 and Sec. 2.2 systems with colored noise and delta-correlated Langevin noise will be studied. In line with an earlier proposal [42], the noise amplitudes of Langevin forces will be determined by means of a Kullback distance measure.

[1] E.g. first-order dynamical systems that involve time-delayed velocity variables, second-order dynamical systems that involve time-delayed acceleration variables, and so on.

2 Data analysis

2.1 Overdamped case/neutral time-delayed systems

First, we are interested in the case that the damping force dominates the system dynamics. That is, we consider the overdamped case and neglect the acceleration term in Eq. (1). We further assume that one can write $f(x, x_\tau, \dot{x}, \dot{x}_\tau) = -\dot{x} + h(x, x_\tau, \dot{x}_\tau)$. The second-order dynamical system (1) then becomes the first-order dynamical system

$$\dot{x}(t) = h(x, x_\tau, \dot{x}_\tau) + \xi(t). \tag{2}$$

Since the deterministic force $h(\cdot)$ involves delayed feedback via both stiffness or friction, Eq. (2) is a stochastic delay differential equation of the neutral type. We now discuss a data analysis method for this type of equation that can be used to determine the drift function $h(\cdot)$ from noisy data. Since the fluctuating force $\xi(t)$ is additive and has zero mean, any conditional ensemble average of $\xi(t)$ vanishes. Consequently, the conditional ensemble average of Eq. (2) yields

$$\langle \dot{x}(t) \rangle |_{x(t)=x, x(t-\tau)=x_\tau, \dot{x}(t-\tau)=\dot{x}_\tau} = h(x, x_\tau, \dot{x}_\tau), \tag{3}$$

where x, x_τ and $\dot{x}_\tau$ can be regarded as coordinates related to the random variables $x(t)$, $x(t-\tau)$ and $\dot{x}(t-\tau)$, respectively. Using a standard discretization of the time derivative of $x(t)$, it follows from Eq. (3) that

$$\left\langle \frac{x(t+\Delta t) - x(t)}{\Delta t} \right\rangle \Bigg|_{x(t)=x, x(t-\tau)=x_\tau, \dot{x}(t-\tau)=\dot{x}_\tau} = h(x, x_\tau, \dot{x}_\tau) + O(\Delta t), \tag{4}$$

where Δt denotes the sample interval of the data at hand. Obviously, Eq. (4) reduces to Eq. (3) in the limit $\Delta t \to 0$. In the following, we exemplify the proposed data analysis method for two systems driven separately by colored noise and a delta-correlated fluctuating force.

Example 1

In our first example we consider a system for which the following a priori information is available: (i) the deterministic system exhibits a fixed point at the origin (i.e. $x_{\rm st} = 0$), (ii) it involves a feedback via a neutral time-delayed variable $\dot{x}_\tau$, (iii) the stochastic system involves some kind of additive colored noise $\xi(t)$, and (iv) the stochastic system can be described by means of a first-order dynamical differential equation

$$\dot{x}(t) = h(x, \dot{x}_\tau) + \xi(t) \ , \tag{5}$$

where the drift function h is not known. The fixed point may be stable or unstable. The objective is to determine in the vicinity of the fixed point the dependency of h on the feedback variable $\dot{x}_\tau$, both for the stable and unstable case and for arbitrary colored noise. That is, we are interested in reconstructing $h(x = 0, \dot{x}_\tau)$ from the given data. From Eq. (4) it follows that

$$\left\langle \frac{x(t+\Delta t) - x(t)}{\Delta t} \right\rangle \Bigg|_{x(t)=0, \dot{x}(t-\tau)=\dot{x}_\tau} = h(0, \dot{x}_\tau) \tag{6}$$

holds up to terms of the order Δt.

We applied Eq. (6) to the data computed from a linear system with $h(x, \dot{x}_\tau) = -ax - b\dot{x}_\tau$ with $a, b > 0$ driven by a colored noise Ornstein-Uhlenbeck process $\dot{\xi}(t) = -\lambda\xi(t) + A\Gamma(t)$, where $\Gamma(t)$ is a Langevin force with $\langle \Gamma(t)\Gamma(t') \rangle = \delta(t - t')$. The fixed point of this system in the absence of noise is stable for $b < 1$ and unstable for $b > 1$ irrespective of a [23,38]. Consequently, the first moment $\langle x(t) \rangle$ of the stochastic system decays to zero for $b < 1$ and tends to infinity for $b > 1$ when time evolves, see Fig. 2 panel (a). Panel

(b) of Fig. 2 shows the drift function $h(0, \dot{x}_\tau)$ calculated from the conditional average Eq. (6) for the cases $b < 1$ and $b > 1$. Fig. 2 demonstrates that the data analysis method can deal with colored noise and works well both for the stable and unstable case. Qualitatively, we could conclude from Fig. 2 that there is a linear dependency of h on $\dot{x}_\tau$. Quantitatively, we determined the parameter b using linear regression analysis and re-obtained the exact value with an inaccuracy smaller than 1%.

Insert Fig. 2 about here

Example 2

In this example we assume that we are dealing with a linear system

$$\dot{x}(t) = -ax - b\dot{x}_\tau + \sqrt{Q}\Gamma(t) \ , \tag{7}$$

where $a, b > 0$ and $\Gamma(t)$ denotes the Langevin force again. The parameter Q describes the amplitude of the force fluctuations. The system has a stationary distribution $P_{\rm st}(x)$ for $b < 1$ given by [43]

$$P_{\rm st}(x, Q) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \tag{8}$$

where

$$\sigma^2 = \frac{Q}{2a\sqrt{1-b^2}} \left\{ \frac{\sqrt{1-b}\cosh(\omega\tau/2) + \sqrt{1+b}\sinh(\omega\tau/2)}{\sqrt{1+b}\cosh(\omega\tau/2) + \sqrt{1-b}\sinh(\omega\tau/2)} \right\}, \tag{9}$$

is the variance of $P_{\rm st}$ and $\omega = a/\sqrt{1-b^2}$. The objective is to estimate the unknown parameters a, b, Q from the data generated by a linear system of the form (7). To this end, we first computed the drift parameters a and b – as in the previous example. Fig. 3a) shows the reconstruction of the drift function h using Eq. (4). Regression analysis provided us with estimates for a and b. Subsequently, we computed the distribution $P_{\rm st}^{(\rm exp)}(x)$ from the data. We estimated the noise strength Q by fitting $P_{\rm st}(x,Q)$ to $P_{\rm st}^{\rm (exp)}(x)$ on the basis of the previously determined estimates for a and b. To this end, we required that the difference between $P_{\rm st}^{\rm (exp)}(x)$ and $P_{\rm st}(x,Q)$ becomes minimal as measured by the Kullback distance measure [44]

$$K(Q) = \int P_{\rm st}^{\rm (exp)}(x) \ln \left\{ \frac{P_{\rm st}^{\rm (exp)}(x)}{P_{\rm st}(x,Q)} \right\} {\rm d}x \ . \tag{10}$$

Fig. 3b) shows the Kullback distance measure (10) as a function of noise strength Q. We found that the minimum of the Kullback measure corresponds nicely to the value of Q used to produce the data.

Insert Fig. 3 about here

2.2 Time-delayed second-order dynamical systems

Eq. (1) can be normalized by $g = f/m$ and $\zeta = \xi/m$. This then gives us

$$\ddot{x} = g(x, x_\tau, v, v_\tau) + \zeta(t) \ . \tag{11}$$

In order to determine the drift function $g(\cdot)$ we take the conditional average

$$\langle \ddot{x}(t) \rangle|_{x(t)=x, x(t-\tau)=x_\tau, v(t)=v, v(t-\tau)=v_\tau} = g(x, x_\tau, v, v_\tau). \tag{12}$$

We discretize Eq. (12) with respect to time. By means of the velocity one obtains

$$\left\langle \frac{v(t+\Delta t) - v(t)}{\Delta t} \right\rangle \Bigg|_{x(t)=x, x(t-\tau)=x_\tau, v(t)=v, v(t-\tau)=v_\tau} = g(x, x_\tau, v, v_\tau) + O(\Delta t). \tag{13}$$

Note that Eq. (13) can alternatively be derived by separating Eq. (11) into the two first-order dynamical subsystems $\dot{v} = g + \zeta$, $\dot{x} = v$. Subsequently, we take the conditional average with respect to the v dynamics.

In general, the time-delayed second-order dynamical system (11) represents a nonlinear oscillatory system that is subjected to feedback control and involves all the components shown in Fig. 1. In particular, Eq. (11) can describe noise-induced oscillations, human movement and ship sway.

Example 1: noise-induced oscillations

We now consider the time-delayed second-order dynamical systems of the form

$$\ddot{x}(t) = g_{\rm osc}(x, \dot{x}) + K(\dot{x}_\tau - \dot{x}) + \zeta(t), \tag{14}$$

where $K > 0$ describes the strength of a linear feedback control loop and $g_{\rm osc}$ is a nonlinear function that describes the intrinsic properties of the oscillator. $\zeta(t)$ is a colored noise satisfying an Ornstein-Uhlenbeck process $\dot{\zeta} = -\gamma\zeta + A\Gamma(t)$. Feedback control via friction as described by the expression $K(\dot{x}_\tau - \dot{x})$ has been successfully applied to various systems, in particular chaotic ones [45]. In what follows, we assume that K has been determined using the conditional average $\langle \ddot{x} \rangle|_{x(t)=v(t)=0, v(t-\tau)=v_\tau} = K v_\tau$ or is known from the experimental setup. Furthermore, we assume that the oscillator has a fixed point at the origin $(x, v) = (0, 0)$ that might be stable or unstable. The question then arises how to determine the intrinsic properties of the oscillator as reflected by the function $g_{\rm osc}$. For example, in the case of a van der Pol oscillator we have $g_{\rm osc}(x, \dot{x}) = (\alpha - \beta x^2)\dot{x} - \omega_0^2 x$ with $\beta > 0$. In this case for $\alpha < 0$ ($\alpha > 0$) we are dealing with an overdamped (self-excited) oscillator. In fact, there has been considerable interest in such time-delayed van der Pol oscillators. It has been found that the correlation time of noise-induced oscillations ($\alpha < 0$) of the van der Pol oscillator depends significantly on the time delay τ involved in the feedback term $K(\dot{x}_\tau - \dot{x})$ [46–48]. In order to reconstruct $g_{\rm osc}$ from experimental data, it follows from Eqs. (13) and (14) that

$$\left\langle \frac{v(t+\Delta t)-v(t)}{\Delta t} \right\rangle\Bigg|_{x(t)=x,v(t)=v,v(t-\tau)=0} + Kv = g_{\rm osc}(x,v) + O(\Delta t). \quad (15)$$

We applied Eq. (15) to the computer generated data of the aforementioned van der Pol oscillator. The results are shown in Fig. 4. Panel (a) gives an example for the analysis of the data for an oscillator with $\alpha < 0$. From the figure in panel (a) we concluded that $g_{\rm osc}$ describes an overdamped system with a stable fixed point at the origin. Moreover, we find that the nonlinearity is of a parabolic type. By means of least square fits we have determined the oscillator parameters. Parameter estimates differed from the original parameters by less than 1%. Panel (b) refers to an oscillator with $\alpha > 0$. Again, from the figure in panel (b) we can draw both qualitative and quantitative conclusions. We concluded that the fixed point is unstable. However, as can be seen in panel (b), if the distance in the x-v plane from the origin becomes large, then the system is decelerated. Consequently, we inferred that $g_{\rm osc}$ describes a self-exited limit oscillator. In addition, we found that the parabolic nature of the nonlinearity was evident and parameter estimates obtained by means of least square fits were found to be close to the exact values.

Insert Fig. 4 about here

Example 2: human movement

In many cases, human multi-joint movements can be decomposed into time-independent eigenvectors and time-dependent amplitudes, see Fig. 5a). To this end, principle component analysis and independent component analysis methods have been used [49–52]. Since Newtonian mechanics applies to human movements, the amplitude dynamics often satisfies second-order dynamical equations. For small amplitudes linearized amplitude equations of the form

$$\ddot{x}(t) = -\gamma\dot{x} - \omega_0^2 x - K_1\dot{x}_\tau - K_2 x_\tau + \zeta(t), \quad (16)$$

have been used, where γ, ω_0, K_1 and K_2 are positive parameters [53,54]. Eq. (16) belongs to the fundamental class of visco-elastic models involving active and passive components. Elastic (spring-like) properties of the amplitude dynamics are accounted for by the terms $-\omega_0^2 x$ and $-K_2 x_\tau$ involving a generalized position variable (feedback via stiffness), whereas viscous properties are reflected by the terms $-\gamma\dot{x}$ and $-K_1\dot{x}_\tau$ involving a generalized velocity variable (feedback via friction). Moreover, it is usually assumed that the non-delayed terms $-\omega_0^2 x$ and $-\gamma\dot{x}$ describe passive components of motor control systems, whereas the time-delayed terms $-K_1\dot{x}_\tau$ and $-K_2 x_\tau$ describe active components [26,37,54]. The time delay τ denotes a neuro-physiological delay that accounts for finite transmission times of the neural signals traveling between muscles, receptors and cortical regions. For large amplitudes and peak velocities, nonlinearities become important. In particular, the output of motor control components is naturally bounded from above. That is, there are maximum values and saturation thresholds. In such cases the linearized model (16) has to be generalized. For example, we may consider an amplitude dynamics given by [55]

$$\ddot{x}(t) = g_\gamma(\dot{x}) + g_0(x) + g_1(\dot{x}_\tau) + g_2(x_\tau) + \zeta(t), \tag{17}$$

where g_γ, g_0, g_1, g_2 are in general nonlinear functions of their arguments (e.g. we may have $g(z) = \tanh(z)$). The function $\zeta(t)$ in Eq. (17) denotes colored noise which will be modeled in terms of an Ornstein-Uhlenbeck process $\dot{\zeta} = -\gamma\zeta + A\Gamma(t)$. Note that Eq. (17) is an additive nonlinear model because the components of the motor control system act together in an additive fashion. In addition, the noise term is additive. Note also that we assume that for small amplitudes Eq. (17) reduces to Eq. (16), which implies that $g_\gamma(0) = g_0(0) = g_1(0) = g_2(0) = 0$.

In order to demonstrate how to determine the structure (e.g. the functions g_1 and g_2) of the active motor control components from movement data, we analyzed the data generated by the tanh movement model proposed in [35,55–57] using the central relationship (13). That is, we computed g_1 and g_2 from

$$\left\langle \frac{v(t+\Delta t)-v(t)}{\Delta t} \right\rangle\Bigg|_{x(t)=x(t-\tau)=v(t)=0, v(t-\tau)=\dot{x}_\tau} = g_1(\dot{x}_\tau) + O(\Delta t) \ ,$$
$$\left\langle \frac{v(t+\Delta t)-v(t)}{\Delta t} \right\rangle\Bigg|_{x(t)=0, x(t-\tau)=x_\tau, v(t)=v(t-\tau)=0} = g_2(x_\tau) + O(\Delta t) \ . \qquad (18)$$

Fig. 5b) shows the reconstructions of the drift functions $g_1(\dot{x}_\tau)$ and $g_2(x_\tau)$. We estimated the feedback parameters K_1 and K_2 involved in the tanh model using least square fits and found estimation errors smaller than 1%.

Insert Fig. 5 about here

2.2.1 Example 3: ship rolling

Our last example concerns ships rolling on the waves. Ship motions can be described by the angle of tilt $x(t)$ measured with respect to the upright position of the ship body, see Fig. 6a). The ship body is subjected to gravity and buoyancy. In addition, there is friction between the ship body and the water. For small angles of tilt all forces can be linearized such that the ship motions satisfy the equation of a damped pendulum: $\ddot{x}(t) = -\gamma\dot{x}(t) - \omega_0^2 x(t)$, where $\gamma > 0$ describes an effective damping constant and $\omega_0 > 0$ the eigenfrequency of the rolling motions [58,59]. The classical engineering problem is to avoid underdamped (i.e. oscillatory) rolling motions. In order to avoid this kind of motions typically the friction constant is increased with the help of roll tanks. That is, tanks are installed on both sides of the ship body that are partially filled with water as indicated in Fig. 6a). If the ship body swings to the right (left), the water is pumped from the left to the right tank (right to the left tank) such that (in the linearized case) there is an additional friction term of the form $-K\dot{x}$ with $K > 0$. The pumping mechanism cannot respond instantaneously which implies that $-K\dot{x}$ has to be replaced by $-K\dot{x}(t-\tau)$, where τ is the characteristic time delay of the roll tank device. In sum, ship motions under the impact of roll tanks are assumed to satisfy the second-order delay differential equation $\ddot{x}(t) = -\gamma\dot{x}(t) - K\dot{x}(t-\tau) - \omega_0^2 x(t)$ for small angles of tilt. This is the classical Minorsky problem [58,60]. We will consider this problem in a stochastic framework and show how to analyze data from the ship rolling related to the stochastic Minorsky model

$$\ddot{x}(t) = -\gamma\dot{x} - K\dot{x}_\tau - \omega_0^2 x + \sqrt{Q}\Gamma(t), \tag{19}$$

where $\Gamma(t)$ is the Langevin force used in the previous examples and Q is the noise amplitude of the stochastic driving force. We assume that γ and ω_0 are known from the construction of the ship body or have been determined experimentally as shown in example 1 of this section (in this context note also that γ and ω_0 can be derived independently using the conditional averages $\langle\ddot{x}\rangle|_{\dot{x}(t)=v,\dot{x}(t-\tau)=0,x(t)=0} = -\gamma v$ and $\langle\ddot{x}\rangle|_{\dot{x}(t)=0,\dot{x}(t-\tau)=0,x(t)=x} = -\omega_0^2 x$). Then, the problem that remains to be solved is to determine K and Q from ship rolling data.

In order to determine K and Q, we basically proceeded in the same manner as in example 2 of Sec. 2.1. We first determined the dependency of the drift function g of $\dot{x}(t-\tau)$ using $\langle\ddot{x}\rangle|_{\dot{x}(t)=0,\dot{x}(t-\tau)=v_\tau,x(t)=0} = g(v_\tau, x = v = 0) = -Kv_\tau$, see Fig. 6b. Linear regression analysis yielded an estimate of K. Second, we estimated Q by requiring that the analytical solution $P_{\rm st}(v)$ of the Minorsky problem (19) approximates best the distribution $P_{\rm st}^{(\rm exp)}(v)$ obtained from the data. The analytical solution is [61]

$$P_{\rm st}(v) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{v^2}{2\sigma^2}\right\}, \tag{20}$$

where the variance can be calculated from $\sigma^2 = a_1 + a_2$ with

$$a_1 = \frac{Q\omega_1\eta}{2(\omega_1^2 - \omega_2^2)} \left(\frac{1 + K\,\eta\sinh(\omega_1\tau)}{\gamma + K\cosh(\omega_1\tau)}\right),$$

$$a_2 = \frac{KQ\beta/2 - (\gamma + K\cosh(\omega_1\tau))a_1)}{\gamma + K\cosh(\omega_2\tau)},$$

$$\beta = \frac{1}{K} + \frac{\omega_2\sinh(\omega_2\tau) - \omega_1\sinh(\omega_1\tau)}{\omega_2^2 - \omega_1^2},$$

$$\omega_{1,2} = \frac{\sqrt{\gamma^2 - K^2}}{2} \pm \sqrt{\frac{\gamma^2 - K^2}{4} - \omega_0^2}, \tag{21}$$

and $\eta = \sqrt{|K^2 - \gamma^2|}$. Note that we have here $K < \gamma, K^2 < \gamma^2 - 4\omega_0^2$. For other parameter cases similar expressions are available [61]. As shown in Fig. 6c, we computed the Kullback measure

$$K(Q) = \int P_{\rm st}^{(\rm exp)}(v)\ln\left\{\frac{P_{\rm st}^{(\rm exp)}(v)}{P_{\rm st}(v,Q)}\right\}\mathrm{d}v, \tag{22}$$

as a function Q. We obtained a minimum of the Kullback measure close to the exact value $Q = 1.0$ of the original model.

Insert Fig. 6 about here

3 Conclusions

In the present study, we generalized a previous data analysis method [32,35–37] to deal with stable and unstable colored noise time-delayed systems of the neutral and retarded type. Since both stable and unstable systems can be analyzed with this proposed method, it represents a useful tool for studying bifurcations in time-delayed systems. For example, we showed how delay-induced bifurcations leading to unstable systems (see Sec. 2.1) and the onset of self-excited oscillations in oscillators with non-invasive feedback control (see Sec. 2.2) can be studied. Since delay systems of both the neutral and retarded type can be addressed, the proposed method allows one to delineate feedback involving friction from feedback involving stiffness. In Sec. 2.2 we showed, using simulated data, that it is possible in principle to apply this method to gain insight into the nature of feedback loops underlying human motor control. In general, one can say that, because the proposed method can resolve nonlinearities both qualitatively and quantitatively, it can be usefully applied in the life sciences (e.g. to determine saturation functions and estimate feedback parameters, see example 2 of Sec. 2.2) as well as the engineering sciences (e.g. in the context of ship rolling and non-invasively controlled nonlinear oscillators, see examples 1 and 3 of Sec. 2.2). However, when considering or seeking application of the method to actual experimental data, researchers will be confronted with three important issues that each require a solution, namely the estimation of time delays, the determination of time derivatives and the use of averages, which we will therefore discuss in turn, drawing the article to a close.

In our study we assumed that the relevant time delays are known a priori. Indeed, in physics and engineering sciences they are likely to be known from the construction of the devices and machines being studied. In life sciences estimates for the time delays may be available in the literature. Alternatively, time delays can be estimated in combination with data analysis methods, in general, and in combination with our proposed method, in particular. For example, a so-called optimal time delay can be defined such that it minimizes the error of the reconstruction with respect to the data [53,54]. This optimal time delay may be interpreted as an estimate for the actual time delay.

In Sec. 2 we approximated time derivatives by means of first-order difference formulas (e.g. we used $\dot{x}(t) = [x(t+\Delta t) - x(t)]/\Delta t + O(\Delta t)$). Estimates of this kind can result in relatively large errors for experimental data. Therefore, the question arises whether the accuracy can be improved by using higher-order difference formulas based on polynomial approximations of time series [62]. Unfortunately, in our context such polynomial approximations should be regarded as weighted time averaging procedures. Consequently, in general they destroy the correlation properties of time series that are crucial for our understanding of the underlying stochastic processes. For example, such polynomial approximations would destroy the Markov property of data obtained from an Ornstein-Uhlenbeck process. In sum, the application of higher-order difference formulas to experimental data that exhibit a high degree of fluctuations needs to be justified by detailed calculations. For the sake of simplicity, we will discuss such a justification only for overdamped systems satisfying Eq. (2). Let $M(x, x_\tau, \dot{x}_\tau, z)$ denote the conditional average $M(\cdot, z) = \langle x(t+z) - x(t)\rangle|_{x(t)=x, x(t-\tau)=x_\tau, \dot{x}(t-\tau)=\dot{x}_\tau}$ with $M(\cdot, 0) = 0$. Then, Eq. (3) can equivalently be expressed as $\mathrm{d}M(\cdot, 0)/\mathrm{d}z = h$. Since $M(\cdot, z)$ is assumed to be a smooth function, we can compute the (right-hand side) derivative of M by means of various (forward) higher-order difference formulas. That is, using the smooth function $M(\cdot, z)$ instead of the stochastic trajectory $x(t)$ we can circumvent the problem that higher-order interpolation procedures will affect time series correlations. For example, the first-order difference scheme $L^{(1)}[M]_{z=0} = [M(\cdot, \Delta t) - M(\cdot, 0)]/\Delta t$ [62] gives us $\mathrm{d}M(\cdot, 0)/\mathrm{d}z = L^{(1)}[M]_{z=0} + O(\Delta t) = h$ and is equivalent to the approach discussed in Sec. 2.1. That is, we have $L^{(1)}[M]_{z=0} = [M(\cdot, \Delta t) - M(\cdot, 0)]/\Delta t = (\Delta t)^{-1} \langle x(t+\Delta t) - x(t)\rangle|_{x(t)=x, x(t-\tau)=x_\tau, \dot{x}(t-\tau)=\dot{x}_\tau}$. In contrast, for the second-order difference scheme $L^{(2)}[M]_{z=0} = [-3M(\cdot, 0) + 4M(\cdot, \Delta t) - M(\cdot, 2\Delta t)]/(2\Delta t)$ [62] we obtain $\mathrm{d}M(\cdot, 0)/\mathrm{d}z = L^{(2)}[M]_{z=0} + O(\Delta t^2) = h$, which improves the accuracy of our estimates for time derivatives by one order. For example, for an Ornstein-Uhlenbeck process given by $\dot{x}(t) = -\gamma x(t) + A\Gamma(t)$ we find that $M = x(\exp\{-\gamma z\} - 1)$ and can explicitly verify that $L^{(1)}[M]_{z=0} = -\gamma x + O(\Delta t)$ and $L^{(2)}[M]_{z=0} = -\gamma x + O(\Delta t^2)$. Next, we note that for the conditional average M and any nth-order difference scheme the following properties hold: (i) M is linear with respect to x, (ii) $L^{(n)}$ is linear with respect to M, and (iii) we have $L^{(n)}[M + c] = L^{(n)}[M]$ for arbitrary real values c. Substituting $c = x$, one can then show that $L^{(n)}[M(\cdot, z)]\Big|_{z=0} = \left\langle L^{(n)}[x(t)]\right\rangle\Big|_{x(t)=x, x(t-\tau)=x_\tau, \dot{x}(t-\tau)=\dot{x}_\tau}$ holds. That is, the time derivative $\dot{x}(t)$ can be replaced by any nth-order difference scheme $L^{(n)}[x(t)]$ such that Eq. (4) becomes $\left\langle L^{(n)}[x(t)]\right\rangle\Big|_{x(t)=x, x(t-\tau)=x_\tau, \dot{x}(t-\tau)=\dot{x}_\tau} = h + O(\Delta t^n)$. As a result, we have explicitly demonstrated that as far as the derivation of the deterministic force h is concerned higher-order difference schemes can be applied not only on the level of conditional ensemble averages (such as M) but also on the level of single trial data. By similar reasonings, one can show that this result also holds for the second-order dynamical systems discussed in Sec. 2.2.

Finally, we would like to mention that in our study (conditional) ensemble averages have been used. For autonomous systems (i.e. for systems that do not depend explicitly on time) these ensemble averages can be replaced by time averages in the stationary case. The situation, however, is fundamentally different for non-autonomous, non-stationary systems. For example, the proposed data analysis method should be treated with caution in the context of time-delayed systems that are driven by periodic external driving forces [63].

Acknowledgements

This work was supported by the Thailand Research Fund, Contract No. PHD/0241/2545, 3. M. MU/45/B.1.

References

[1] A. Einstein, Annalen der Physik **17**, 549 (1905), in German.

[2] M. Haw, Phys. World **18(1)**, 19 (2005).

[3] G. Parisi, Nature **433**, 221 (2005).

[4] F. Servant, S. Raymond, B. Fak, P. Lejay, and J. Flouquet, Solid State Commun. **116**, 489 (2000).

[5] H. Haken, *Synergetics: Itroduction and Advanced topics* (Springer, Berlin, 2004).

[6] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62**, 251 (1990).

[7] N. G. van Kampen, *Stochastic processes in physics and chemistry* (North-Holland Publ. Company, Amsterdam, 1981).

[8] P. de Gennes and J. Prost, *The physics of liquid crystals* (Clarendon Press, Oxford, 1993).

[9] H. C. Öttinger, *Stochastic processes in polymeric fluids* (Springer, Berlin, 1996).

[10] W. Ebeling and I. M. Sokolov, *Statistical thermodynamics and stochastic theory of nonequilibrium systems* (World Scientific, Singapore, 2004).

[11] A. Okubo and S. A. Levin, *Diffusion and ecological problems: modern perspectives* (Springer, Berlin, 2001).

[12] F. Schweitzer, *Brownian agents and active particles* (Springer, Berlin, 2003).

[13] M. Abeles, *Corticonics - Neural circuits of the cerebral cortex*, vol. I + II (Cambridge University Press, Cambridge, 1991).

[14] H. Haken, *Brain dynamics* (Springer, Berlin, 2002).

[15] P. A. Tass, *Phase resetting in medicine and biology - Stochastic modelling and data analysis* (Springer, Berlin, 1999).

[16] R. N. Mantegna and H. E. Stanley, *An introduction to econophysics* (Cambridge University Press, Cambridge, 2000).

[17] J. A. S. Kelso, *Dynamic patterns - The self-organization of brain and behavior* (MIT Press, Cambridge, 1995).

[18] K. M. Newell and D. M. Corcos, *Variability and motor control* (Human Kinetics Publishers, Champaign, 1993).

[19] P. Hänggi, F. Marchesoni, and F. Nori, Annalen der Physik **14**, 51 (2005).

[20] P. Reimann, Phys. Rep. **361**, 57 (2002).

[21] J. M. Cushing, *Integrodifferential equations and delay models in population dynamics* (Springer, Berlin, 1977).

[22] M. C. Mackey and L. Glass, Science **197**, 287 (1977).

[23] G. Stepan, *Retarded dynamical systems: stability and characteristic functions* (Longman Scientific & Technical, New York, 1989).

[24] A. Longtin, J. G. Milton, J. E. Bos, and M. C. Mackey, Phys. Rev. A **41**, 6992 (1990).

[25] G. A. Bocharov and F. A. Rihan, J. Comput. Appl. Math. **125**, 183 (2000).

[26] J. L. Cabrera and J. G. Milton, Phys. Rev. Lett. **89**, 158702 (2002).

[27] M. J. Bünner, M. Popp, T. Meyer, A. Kittel, and J. Parisi, Phys. Rev. E **54**, R3082 (1996).

[28] H. Voss and J. Kurths, Chaos, solitons & fractals **10**, 805 (1999).

[29] W. Horbelt, J. Timmer, and H. U. Voss, Phys. Lett. A **299**, 513 (2002).

[30] V. I. Ponomarenko and M. D. Prokhorov, Phys. Rev. E **66**, 026215 (2002).

[31] S. Ortin, J. M. Gutierrez, L. Pesquera, and H. Vasquez, Physica A **351**, 133 (2005).

[32] R. Friedrich and J. Peinke, Phys. Rev. Lett. **78**, 863 (1997).

[33] S. Siegert, R. Friedrich, and J. Peinke, Phys. Lett. A **243**, 275 (1998).

[34] S. Siegert and R. Friedrich, Phys. Rev. E **64**, 041107 (2001).

[35] T. D. Frank, P. J. Beek, and R. Friedrich, Phys. Rev. E **68**, 021912 (2003).

[36] T. D. Frank, P. J. Beek, and R. Friedrich, Phys. Lett. A **328**, 219 (2004).

[37] T. D. Frank, P. J. Beek, and R. Friedrich, Phys. Lett. A **338**, 74 (2005).

[38] J. Hale, *Theory of functional differential equations* (Springer, Berlin, 1977).

[39] C. T. H. Baker, G. Bocharov, C. A. H. Paul, and F. A. Rihan, J. Math. Biol. **37**, 341 (1998).

[40] M. Courtemanche, L. Glass, and J. P. Keener, Phys. Rev. Lett. **70**, 2182 (1993).

[41] Y. Kuang, *Delay differential equations with application in population dynamics* (Academic Press, San Diego, 1993).

[42] D. Kleinhans, R. Friedrich, A. Nawroth, and J. Peinke, Phys. Lett. A **346**, 42 (2005).

[43] T. D. Frank, J. Phys. A **38**, L485 (2005).

[44] S. Kullback, *Information theory and statistics* (Dover Publications, New York, 1968).

[45] K. Pyragas, Phys. Lett. A **170**, 421 (1992).

[46] A. G. Balanov, N. B. Janson, and E. Schöll, Physica D **199**, 1 (2004).

[47] N. B. Janson, A. G. Balanov, and E. Schöll, Phys. Rev. Lett. **93**, 010601 (2004).

[48] J. Pomplun, A. Amann, and E. Schöll, Europhysics Letters **71**, 366 (2005).

[49] V. K. Jirsa, R. Friedrich, and H. Haken, Physica D **89**, 100 (1995).

[50] R. Friedrich and C. Uhl, Physica D **98**, 171 (1996).

[51] H. Haken, *Principles of brain functioning* (Springer, Berlin, 1996).

[52] A. Daffertshofer, C. J. C. Lamoth, O. G. Meier, and P. J. Beek, Clinical Biomechanics **19**, 415 (2004).

[53] R. J. Peterka, J. Neurophysiol **88**, 1097 (2001).

[54] A. V. Alexandrov, A. A. Frolov, F. B. Horak, P. Carlson-Kuhta, and S. Park, Biol. Cybern. **93**, 309 (2005).

[55] K. Vasilakov and A. Beuter, J. Theo. Biol. **165**, 389 (1993).

[56] R. Engbert, C. Scheffczyk, R. T. Krampe, M. Rosenblum, J. Kurths, and R. Kliegl, Phys. Rev. E **56**, 5823 (1997).

[57] T. Ohira and Y. Sato, Phys. Rev. Lett. **82**, 2811 (1999).

[58] R. D. Driver, *Ordinary and delay differential equations — Applied mathematical sciences Vol. 20* (Springer, New York, 1977).

[59] R. C. Dorf and R. H. Bishop, *Modern control systems* (Addison Wesley, Reading MA, 1995).

[60] N. Minorsky, Trans. ASME **69**, 735 (1947).

[61] K. Patanarapeelert, T. D. Frank, P. J. Beek, R. Friedrich, and I. M. Tang, Phys. Rev. E **73**, 021901 (2006).

[62] R. W. Southworth and S. L. Deleeuw, *Digital computation and numerical methods* (McGraw-Hill, New York, 1965).

[63] J. Gradisek, R. Friedrich, E. Govekar, and I. Grabec, Phys. Lett. A **294**, 234 (2002).

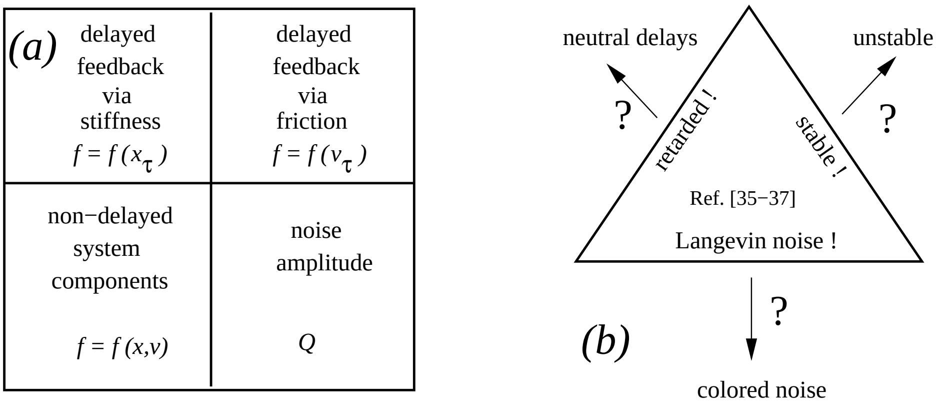

Fig. 1. Panel (a): basic components of time-delayed dynamical systems addressed in this study. Panel (b): solved (!) and unsolved (?) problems. See text for details.

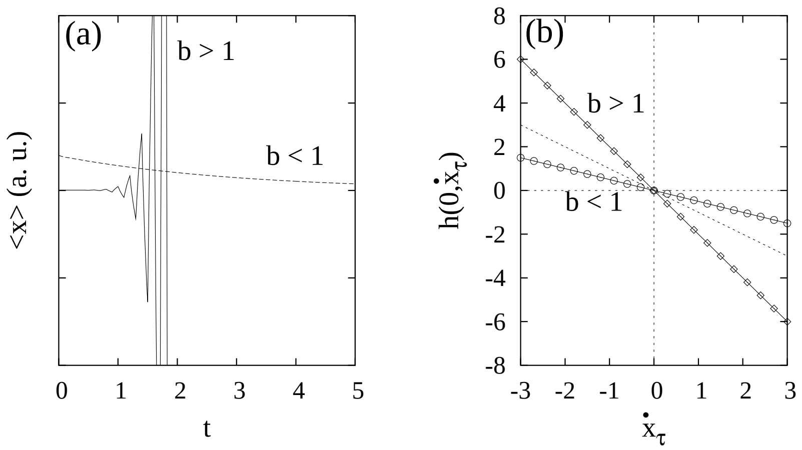

Fig. 2. Panel (a): first moment $\langle x(t)\rangle$ as a function of time for the stable and unstable case. On the vertical axis different scales were used in order to allow for a comparison of both cases. Panel (b): reconstruction of the drift function $h(0, \dot{x}_\tau)$ of a neutrally delayed linear system. Solid lines represent the exact drift function $h(0, \dot{x}_\tau) = -b\dot{x}_\tau$ of the system. The points describe the reconstruction of $h(0, \dot{x}_\tau)$. The dashed line describes the instability point $b = 1$. Using regression analysis we found $b = 0.500 \pm 0.004$ (stable case) and $b = 2.002 \pm 0.004$ (unstable case). Parameters: $\gamma = 1.0$, $A = 1.0$, $a = 0.5$, $b = 0.5$ (stable case) and $b = 2.0$ (unstable case).

.

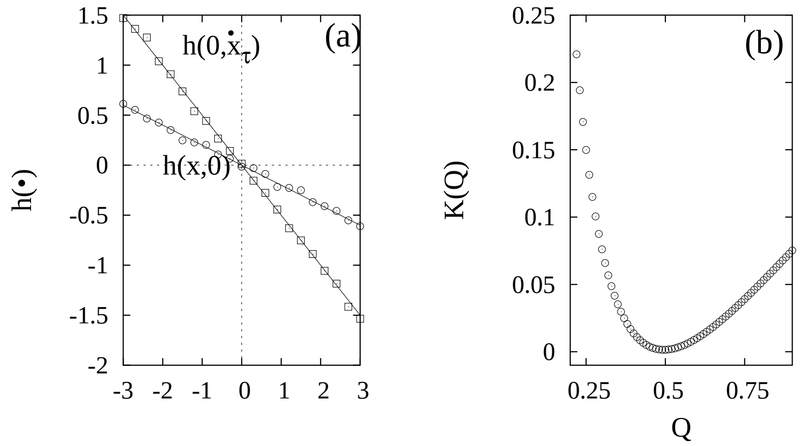

Fig. 3. Panel (a): reconstruction of the drift function h with respect to $\dot{x}_\tau$ (squares) and with respect to x (circles) for the linear system (7). A regression analysis yielded the parameter estimates $a = 0.198 \pm 0.003$ and $b = 0.504 \pm 0.004$. Solid lines represent the exact drift function as used to produce the data (parameters: $a = 0.2$, $b = 0.5$ and $Q = 0.5$). Panel (b): Kullback measure calculated from Eq. (10) using the estimates $a = 0.198$, $b = 0.504$. The minimum of the Kullback measure can be observed at $Q \approx 0.5$ in agreement with the noise amplitude $Q = 0.5$ of the original model.

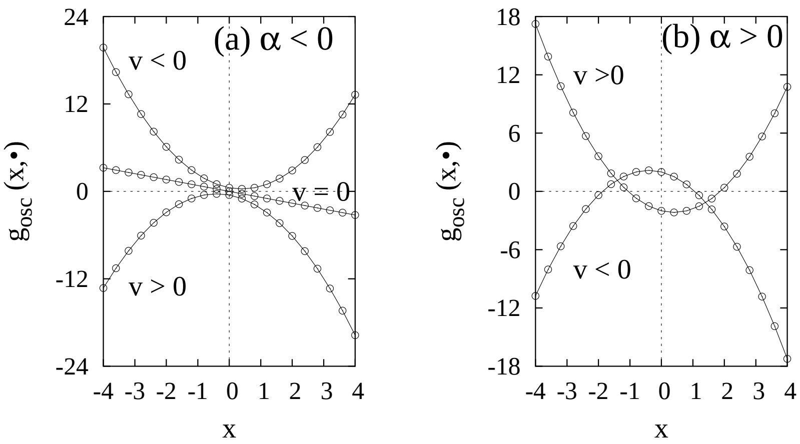

Fig. 4. Reconstruction of the drift function $g_{\rm osc}(x, v)$ of a van der Pol system with delay and colored noise (14) for the overdamped (panel a) and self-excited case (panel b). Solid lines represent $g_{\rm osc}$ of the two original oscillators. Points describe reconstructions of $g_{\rm osc}$ from computer generated data. The functions $g_{\rm osc}(x, v = 1)$, $g_{\rm osc}(x, v = 0)$, and $g_{\rm osc}(x, v = -1)$ are shown ($g_{\rm osc}(x, v = 0)$ is not shown in panel b). Least square fits were performed on the functions $g_{\rm osc}(v = 1) = \alpha - \beta x^2 - \omega_0^2 x$ and $g_{\rm osc}(v = -1) = -\alpha + \beta x^2 - \omega_0^2 x$. Parameter estimates thus obtained differed by less than 1% from the original parameters. Original oscillator parameters: $\omega_0 = 0.9$, $\beta = 1.0$, $\alpha = -0.5$ for panel (a) and $\alpha = 2.0$ for panel (b). Other parameters: $K = 1.0$, $\gamma = 1.0$ and $A = 1.0$.

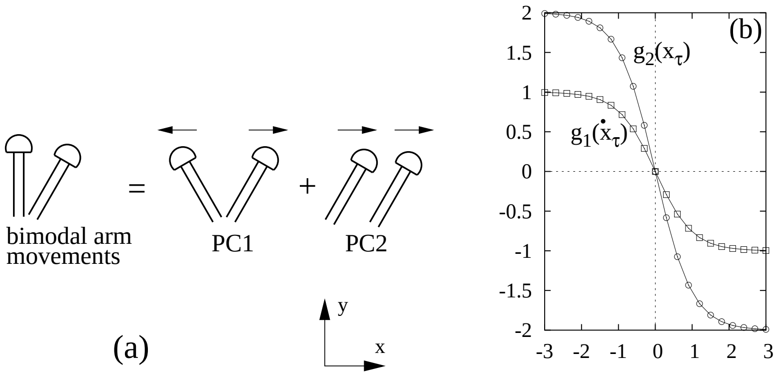

Fig. 5. Panel (a): Example of a principle component decomposition: bimodal arm movements performed in the xy-plane are regarded as superpositions of anti-parallel and parallel eigenvector movements (labeled PC1 and PC2, respectively). Panel (b): Drift functions $g_1(\dot{x}_\tau)$ (squares) and $g_2(x_\tau)$ (circles) of active motor control components as derived from data of a tanh model for the amplitude dynamics of an independently controlled eigenvector. Solid lines represent the exact drift functions of the model. Feedback parameters K_1 and K_2 were estimated from data by means of least square fits: $K_1 = 1.004 \pm 0.009$, $K_2 = 1.998 \pm 0.009$. Original tanh model: $g_\gamma(\dot{x}) = -\gamma\dot{x}$, $g_0 = -\omega_0^2 x$, $g_1(\dot{x}_\tau) = -K_1 \tanh(\dot{x}_\tau)$, $g_2(x_\tau) = -K_2 \tanh(x_\tau)$. Original parameters: $\gamma = 1.0$, $\omega_0 = 0.9$, $A = 1.0$, $K_1 = 1.0$, $K_2 = 2.0$.

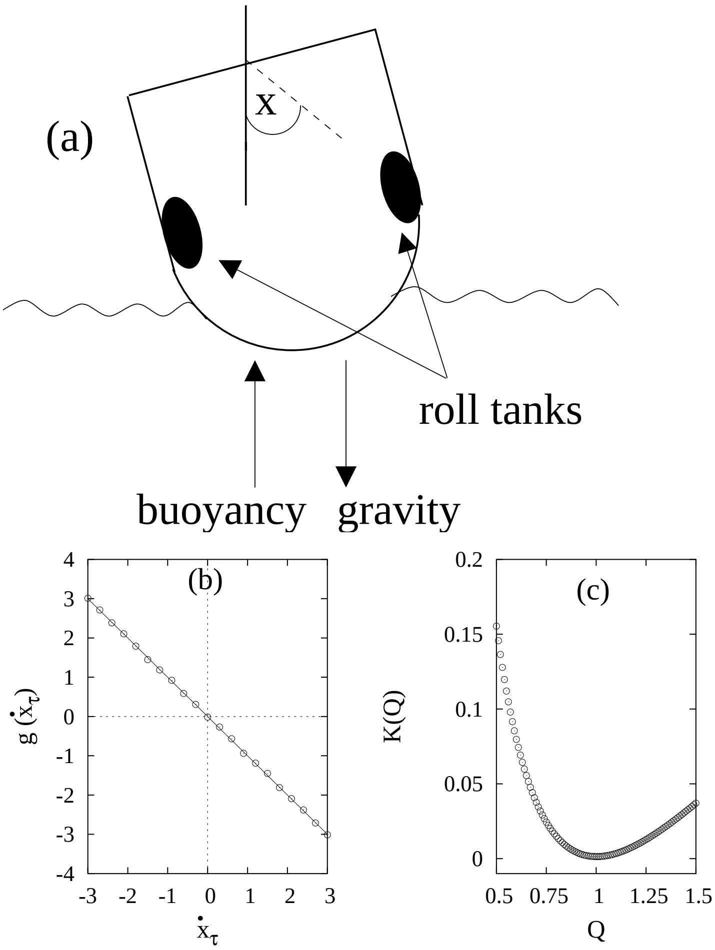

Fig. 6. Analysis of ship rolling data as obtained from the Minorsky model. Panel (a): illustration of the Minorsky model (19). Panel (b): reconstruction of the drift function g with respect to $\dot{x}_T$ (circles). The solid line represents the exact function $g(\dot{x}_T, x = v = 0)$ of the model (19). Using regression analysis, we obtained an estimate for K given by $K = 0.998 \pm 0.003$. Panel (c): Kullback measure calculated from Eq. (22) as a function of Q. A minimum of the Kullback measure was observed at $Q \approx 1.0$. Original parameter of the Minorsky model (19) used to produce the ship rolling data: $\gamma = 2.0$, $\omega_0 = 0.8$, $K = 1.0$ and $Q = 1.0$.