Exact solution of a generalized Kramers-Fokker-Planck equation retaining retardation effects

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In order to describe non-Gaussian kinetics in weakly damped systems, the concept of continuous time random walks is extended to particles with finite inertia. One thus obtains a generalized Kramers-Fokker-Planck equation, which retains retardation effects, i.e., nonlocal couplings in time and space. It is shown that despite this complexity, exact solutions of this equation can be given in terms of superpositions of Gaussian distributions with varying variances. In particular, the long-time behavior of the respective low-order moments is calculated.

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I. INTRODUCTION

Ever since the famous work by Einstein on Brownian motion in 1905 [1], there has been a continued effort by countless scientists to better understand the dynamics of tracer particles in random environments. In the more recent past, a strong emphasis in this area of research has been placed on the investigation of the origin and the consequences of anomalous (i.e., non-Gaussian) diffusion (see, e.g., Refs. [2–5]). A powerful mathematical framework for the description of non-Gaussian diffusion has been put forward by Montroll and Weiss [6]. In the 1960s, they introduced the notion of continuous time random walks (CTRWs). Here, one considers distribution functions of step size and/or waiting time, which do not possess finite low-order moments. As a consequence, the mean square displacement $\langle x^2 \rangle$ is not proportional to the time t anymore. Instead, one tends to obtain $\langle x^2 \rangle \propto t^{\nu}$ with $\nu \neq 1$. For $\nu > 1$, one speaks of "superdiffusion," whereas the case $\nu < 1$ is called "subdiffusion."

Although most investigations on anomalous diffusion are based on a random-walk-type description in *real* space, there have also been a number of studies which employ a *phase* (position-velocity) space approach (see, e.g., Ref. [7], and references therein). In order to understand under which conditions such an extension is appropriate, it is useful to recall some basic characteristics of Langevin equations. Attempting to relate statistical descriptions of Brownian particles to the underlying dynamical equations, Langevin [8] suggested to extend Newton's law of motion by a fluctuating force $\mathbf{F}$, which is characterized (only) in terms of its statistical properties. Thus, for a specific realization of $\mathbf{F}$, the particle dynamics is determined by

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{u}(t), \quad \frac{d\mathbf{u}(t)}{dt} = \frac{\mathbf{F}(t)}{m}. \tag{1}$$

Kramers [9] considered a special case of Eq. (1) in which the force is separated into a stochastic part and a deterministic part according to

$$\mathbf{F}(t) = m\mathbf{A}(t) - \nabla U[\mathbf{x}(t)] - \lambda m \mathbf{u}(t). \tag{2}$$

Here, $\mathbf{A}(t)$ is usually taken to be a Wiener process and $U(\mathbf{x})$ is a potential. [In the present paper, we will consider the case $U(\mathbf{x}) = 0$.] For strongly damped systems, i.e., for large values of the friction coefficient λ, an adiabatic approximation (see, e.g., Ref. [10]) yields the equation

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{V}(t), \tag{3}$$

where $\mathbf{V}(t) = \lambda^{-1}\mathbf{A}(t)$. Equations (3) and (1) are sometimes called, respectively, V-Langevin equations and A-Langevin equations [11]. This nomenclature indicates whether it is the velocity $\mathbf{V}$ or the acceleration $\mathbf{A}$ that is modeled as a stochastic term.

This discussion highlights the fact that the conventional CTRW approach is linked to V-Langevin equations—implying *overdamped* particle dynamics. In practice, many systems in physics, chemistry, and biology will not satisfy this condition, however. This means, the particles may be *weakly damped*. One is thus led to consider the CTRW analog of A-Langevin equations, describing the dynamics of a particle with finite inertia under the influence of a stochastic force. In the context of such a model, the latter will be described as a series of "random kicks" such that the particles' motion is sometimes changed abruptly, whereas it is ballistic most of the time. Taking such a model as a starting point, we have recently derived a fractional equation of the Kramers-Fokker-Planck (KFP) type [12]. The main goal of the present work is to present the detailed derivation of an exact solution of this equation which could only be briefly mentioned in Ref. [12]. As will become clear in this context, the necessary calculations are highly nontrivial and reveal interesting aspects of the underlying stochastic dynamics.

The remainder of this paper is organized as follows. In Sec. II, which mainly has a pedagogical purpose and sets the stage for the main part of this work, we will revisit and discuss the derivation of a fractional KFP equation (in the force-free limit), which was first presented in Ref. [12]. It retains retardation effects and is characterized by a nonstandard collision operator introducing nonlocality in time *and space*. Several exact results for this fractional KFP equation are then derived in Sec. III. In particular, we show that exact analytical solutions can be given in terms of superpositions of Gaussian distributions with varying variances. Moreover, the long-time behavior of the respective low-order moments is calculated and discussed. A summary of key results is given in Sec. IV, along with some conclusions.

II. FROM CTRWS TO A FRACTIONAL KPF EQUATION

A. Task at hand

In his well-known work from 1905, Einstein [1] provided a link between the microscopic dynamics of a Brownian particle and the macroscopic phenomenon of diffusion. In particular, he was able to show that the probability distribution function (pdf) $g(\mathbf{x},t)$, describing the position of a Brownian particle, obeys the diffusion equation

$$\frac{\partial g(\mathbf{x},t)}{\partial t} = K\Delta g(\mathbf{x},t). \tag{4}$$

Here, K is a constant and $\Delta=\nabla^2$ is the Laplace operator. Some 60 years later, Montroll and Weiss [6] put forward a generalization of Einstein's theory of diffusion that is able to also describe the phenomenon of anomalous diffusion. To this end they introduced the notion of continuous time random walks (CTRWs). They showed that within this framework, the pair distribution functions (PDFs) are determined by fractional diffusion equations, such as

$$\frac{\partial g(\mathbf{x},t)}{\partial t} = D_t^{1-\delta} K\Delta g(\mathbf{x},t), \tag{5}$$

where the so-called Riemann-Liouville operator $D_t^{1-\delta}$ is defined via

$$D_t^{1-\delta} g(\mathbf{x},t) = \frac{1}{\Gamma(\delta)}\frac{\partial}{\partial t}\int_0^t \frac{dt'}{(t-t')^{1-\delta}} g(\mathbf{x},t') \tag{6}$$

for $0<\delta\leqslant 1$. Here, Γ denotes the well-known Γ function. Note that for $\delta=1$, Eq. (5) turns into Eq. (4). For more information on CTRWs and on fractional diffusion equations, we refer the reader to the recent review papers by Metzler and Klafter [4,5].

A generalization of Eq. (4) from real space to phase (position-velocity) space is given by the Kramers-Fokker-Planck (KFP) equation [7,10,14]

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla_x + \mathbf{A}(\mathbf{x})\cdot\nabla_u\right] f(\mathbf{x},\mathbf{u},t) = \mathcal{L}_{\mathrm{FP}} f(\mathbf{x},\mathbf{u},t). \tag{7}$$

Here, we have introduced the Fokker-Planck collision operator

$$\mathcal{L}_{\mathrm{FP}} f = \Gamma\nabla_u\cdot(\mathbf{u}f) + K\Delta_u f. \tag{8}$$

[In the following we will assume, for simplicity, that the acceleration term $\mathbf{A}$ vanishes identically. It can be reincluded later if necessary.] As already mentioned in the Introduction, the main task of the present paper is to revisit the rigorous derivation of a fractional variant of Eq. (7) from first principles (see Ref. [12]) and to provide exact solutions. In this context, it will become clear that retardation effects, which have been neglected in previous publications on fractional KFP equations, are actually an essential ingredient of the theory, leading to significant, even qualitative changes in the system's behavior (see results of the Sec. II B).

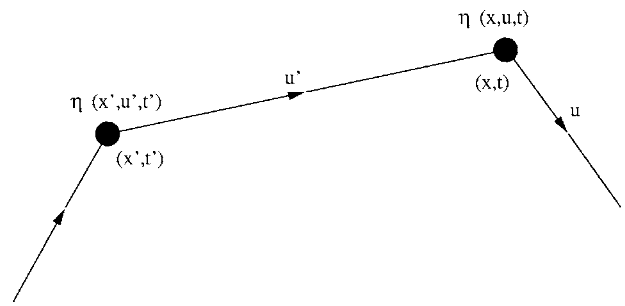

FIG. 1. A ballistic particle whose motion is sometimes changed abruptly by a "random kick." The transition probability $\eta(\mathbf{x},\mathbf{u},t)$ is defined in the text.

B. Master equation for CTRWs of inertial particles

In order to describe the dynamics of inertial, weakly damped particles, we propose the following model, which may be considered as a generalization of standard CTRW theory [2,3,6]. An individual particle is assumed to be subject to a series of random kicks such that its motion is sometimes changed abruptly, whereas it is ballistic most of the time. Let us consider a particle that at time t' is located in the volume element $d\mathbf{x}'$ about $\mathbf{x}'$, and in the time interval $[t',t'+dt']$ changes its velocity to a new value that lies in the velocity space element $d\mathbf{u}'$ about $\mathbf{u}'$. The probability for such a process shall be denoted by

$$\eta(\mathbf{x}',\mathbf{u}',t')d\mathbf{x}'d\mathbf{u}'dt'. \tag{9}$$

After a (random) time period $\tau=t-t'$, this particle will undergo a further transition to a state with the velocity $\mathbf{u}$ at the position $\mathbf{x}$ (see Fig. 1). The corresponding conditional probability shall be called

$$\xi(\mathbf{x},\mathbf{u},\tau;\mathbf{x}',\mathbf{u}')d\mathbf{x}d\mathbf{u}d\tau. \tag{10}$$

We assume that this PDF can be written in the form

$$\xi(\mathbf{x},\mathbf{u},\tau;\mathbf{x}',\mathbf{u}') = \delta(\mathbf{x}-\mathbf{x}'-\mathbf{u}'\tau)F(\mathbf{u};\mathbf{u}')W(\tau;\mathbf{u}'). \tag{11}$$

Here, $W(\tau;\mathbf{u}')d\tau$ describes the probability that a transition occurs in the time interval $[\tau,\tau+d\tau]$, and $F(\mathbf{u};\mathbf{u}')d\mathbf{u}$ determines the probability that the particle's velocity will end up in the velocity space element $d\mathbf{u}$ about $\mathbf{u}$. Both quantities may depend on the velocity $\mathbf{u}'$ before the transition, whereas they are assumed to be independent of $\mathbf{x}$ and $\mathbf{x}'$. Note that during the time interval $[t',t]$, there is no change of the velocity of the particle. Its motion from point $\mathbf{x}'$ to point $\mathbf{x}=\mathbf{x}'+\mathbf{u}'\tau$ is purely ballistic. Under the assumption that the PDF $\xi(\mathbf{x},\mathbf{u},\tau;\mathbf{x}',\mathbf{u}')$ is statistically independent from the history of the particle path and using Eq. (11), we can relate the PDFs $\eta(\mathbf{x},\mathbf{u},t)$ and $\eta(\mathbf{x}',\mathbf{u}',t')$ via the following equation:

$$\eta(\mathbf{x},\mathbf{u},t) - f_0(\mathbf{x},\mathbf{u})\delta(t) = \int d\mathbf{x}' \int d\mathbf{u}' \int_0^t dt' \xi(\mathbf{x},\mathbf{u},t-t';\mathbf{x}',\mathbf{u}')\eta(\mathbf{x}',\mathbf{u}',t') = \int d\mathbf{u}' \int_0^t dt' F(\mathbf{u};\mathbf{u}') W(t-t';\mathbf{u}') e^{-(t-t')\mathbf{u}'\cdot\nabla_x}\eta(\mathbf{x},\mathbf{u}',t'), \quad (12)$$

where $f_0(\mathbf{x},\mathbf{u})$ denotes the initial condition.

Having established an integral equation that determines the time evolution of $\eta(\mathbf{x},\mathbf{u},t)$, we are now interested in the joint position-velocity distribution function $f(\mathbf{x},\mathbf{u},t)$, which is defined as

$$f(\mathbf{x},\mathbf{u},t) = \int d\mathbf{x}' \int_0^t dt' \delta[\mathbf{x}-\mathbf{x}'-\mathbf{u}(t-t')] \times w(t-t';\mathbf{u})\eta(\mathbf{x}',\mathbf{u},t') = \int_0^t dt' w(t-t';\mathbf{u}) e^{-(t-t')\mathbf{u}\cdot\nabla_x}\eta(\mathbf{x},\mathbf{u},t'). \quad (13)$$

Here, $w(\tau;\mathbf{u})$ denotes the probability that the velocity has taken on the value $\mathbf{u}$ during the entire time period τ. For this quantity, we have the obvious relationship

$$w(\tau;\mathbf{u}) = 1 - \int_0^\tau dt W(t;\mathbf{u}). \quad (14)$$

In order to derive an equation describing the time evolution of $f(\mathbf{x},\mathbf{u},t)$, we may proceed in a way analogous to the standard deviation of the master equation for CTRWs [4]. We start by writing down the Laplace transforms of Eqs. (12) and (13),

$$\eta(\mathbf{x},\mathbf{u},\lambda) = f_0(\mathbf{x},\mathbf{u}) + \int d\mathbf{u}' F(\mathbf{u};\mathbf{u}') \times W(\lambda + \mathbf{u}'\cdot\nabla_x;\mathbf{u}')\eta(\mathbf{x},\mathbf{u}',\lambda) \quad (15)$$

and

$$f(\mathbf{x},\mathbf{u},\lambda) = w(\lambda + \mathbf{u}\cdot\nabla_x;\mathbf{u})\eta(\mathbf{x},\mathbf{u},\lambda). \quad (16)$$

Since in Laplace space, Eq. (14) reads

$$w(\lambda;\mathbf{u}) = \frac{1 - W(\lambda;\mathbf{u})}{\lambda}, \quad (17)$$

Eq. (16) can be rewritten as

$$(\lambda + \mathbf{u}\cdot\nabla_x) f(\mathbf{x},\mathbf{u},\lambda) = [1 - W(\lambda + \mathbf{u}\cdot\nabla_x;\mathbf{u})]\eta(\mathbf{x},\mathbf{u},\lambda). \quad (18)$$

Equations (15) and (18) then yield

$$(\lambda + \mathbf{u}\cdot\nabla_x) f(\mathbf{x},\mathbf{u},\lambda) = -W(\lambda + \mathbf{u}\cdot\nabla_x;\mathbf{u})\eta(\mathbf{x},\mathbf{u},\lambda) + \left[f_0(\mathbf{x},\mathbf{u}) + \int d\mathbf{u}' F(\mathbf{u};\mathbf{u}') \times \Phi(\lambda + \mathbf{u}'\cdot\nabla_x;\mathbf{u}') f(\mathbf{x},\mathbf{u}',\lambda)\right], \quad (19)$$

where we have introduced the quantity

$$\Phi(\lambda;\mathbf{u}) = \frac{\lambda W(\lambda;\mathbf{u})}{1 - W(\lambda;\mathbf{u})} = \frac{1 - \lambda w(\lambda;\mathbf{u})}{w(\lambda;\mathbf{u})}. \quad (20)$$

Expressing f_0 in terms of f as

$$f_0(\mathbf{x},\mathbf{u}) = f(\mathbf{x},\mathbf{u},t=0) \quad (21)$$

and using the identity

$$W(\lambda + \mathbf{u}\cdot\nabla_x;\mathbf{u})\eta(\mathbf{x},\mathbf{u},\lambda) = \Phi(\lambda + \mathbf{u}\cdot\nabla_x;\mathbf{u}) f(\mathbf{x},\mathbf{u},\lambda), \quad (22)$$

which follows from Eqs. (18) and (20), the Laplace inversion of Eq. (19) yields the master equation

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{u},t) = \int_0^t dt' \int d\mathbf{u}' F(\mathbf{u};\mathbf{u}')\Phi(t-t';\mathbf{u}') f(\mathbf{x}-\mathbf{u}'(t-t'),\mathbf{u}',t') - \int_0^t dt' \Phi(t-t';\mathbf{u}) f(\mathbf{x}-\mathbf{u}(t-t'),\mathbf{u},t'), \quad (23)$$

which determines the time evolution of $f(\mathbf{x},\mathbf{u},t)$. Furthermore, the Laplace inversion of Eq. (20) yields, in connection with the initial condition $w(0,\mathbf{u})=1$,

$$\frac{\partial w(t;\mathbf{u})}{\partial t} = -\int_0^t dt' \Phi(t-t';\mathbf{u}) w(t';\mathbf{u}), \quad (24)$$

which identifies $\Phi(t;\mathbf{u})$ as a kernel determining the time evolution of $w(t;\mathbf{u})$. For example, the choice

$$\Phi(t;\mathbf{u}) = \Lambda\delta(t) \quad (25)$$

leads to

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{u},t) = -\Lambda\left[f(\mathbf{x},\mathbf{u},t) - \int d\mathbf{u}' F(\mathbf{u};\mathbf{u}') f(\mathbf{x},\mathbf{u}',t)\right]. \quad (26)$$

Another specific choice of the function $\Phi(t;\mathbf{u})$ leads to a connection with fractional diffusion equations, as will be made explicit below.

In the language of kinetic theory, Eq. (23) can be interpreted as follows. The left-hand side describes a system of particles moving ballistically, i.e., with constant velocity and zero acceleration. The right-hand side represents a collision operator which consists of a source and a sink. The phase space density of particles at $(\mathbf{x},\mathbf{u})$ is increased at time t by particles starting from $\mathbf{x}-\mathbf{u}'(t-t')$ at time t' with a velocity $\mathbf{u}'$ and making a transition to the velocity $\mathbf{u}$ at time t (and position $\mathbf{x}$). $f(\mathbf{x},\mathbf{u},t)$ is decreased, on the other hand, by particles making a transition from the velocity $\mathbf{u}$ to some other velocity. Obviously, the collision operator is nonlocal in time and space, in stark contrast to virtually all expressions commonly used in the kinetic theory of gases or plasmas (a notable exception is, e.g., Ref. [13]). This fact lends a special character to Eq. (23), which describes CTRWs of particles with finite inertia. As will be shown in Sec. II C, this model is able to capture non-Gaussian kinetics in weakly damped systems. Furthermore, we want to point out that the master equation is invariant with respect to Galilean transformations provided $\Phi(t-t',\mathbf{u}')$ is independent on $\mathbf{u}'$ and $F(\mathbf{u},\mathbf{u}')$ depends only on the difference $\mathbf{u}-\mathbf{u}'$. Galilean invariance implies that all probability distributions $f(\mathbf{x}-\mathbf{c}t,\mathbf{u}+\mathbf{c},t)$ with arbitrary velocity $\mathbf{c}$ solve the master equation. This fact sheds light on the occurrence of time retardation in the collision operator.

C. Derivation of a fractional KFP equation

For concreteness, we now consider an important class of processes for which $\Phi(\mathbf{u},t)$ can be replaced by the velocity-independent function $\Lambda Q(t)$, and $F(\mathbf{u};\mathbf{u}')$ can be represented by the Gaussian

$$F(\mathbf{u};\mathbf{u}') = \left(\frac{\Lambda}{4\pi K}\right)^{3/2} \exp\left[-\frac{(\mathbf{u}-\mathbf{u}'+\Gamma\mathbf{u}'/\Lambda)^2}{4K/\Lambda}\right], \quad (27)$$

which satisfies the constraint

$$\int d\mathbf{u} F(\mathbf{u};\mathbf{u}') = 1. \quad (28)$$

This expression can be derived in the spirit of the Rayleigh model for Brownian motion [14] by considering a heavy test particle of mass M embedded in a thermal bath of light particles of mass m (for details, see Ref. [15]). For large values of the parameter Λ, which is proportional to the mass ratio M/m, one obtains

$$\int d\mathbf{u}' F(\mathbf{u};\mathbf{u}')g(\mathbf{u}') - g(\mathbf{u}) = \Lambda^{-1}\mathcal{L}_{\mathrm{FP}}g(\mathbf{u}) \quad (29)$$

to leading order in Λ^{-1} where the Fokker-Planck collision operator $\mathcal{L}_{\mathrm{FP}}$ is defined by Eq. (8). For details, the reader is referred to Appendix A. Demanding at the same time

$$\lim_{\Lambda\to\infty} \frac{\Phi(t)}{\Lambda} = Q(t), \quad (30)$$

Eq. (23) then takes the form

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{u},t) = \mathcal{L}_{\mathrm{FP}}\int_0^t dt' Q(t-t') \times f(\mathbf{x}-\mathbf{u}(t-t'),\mathbf{u},t') \quad (31)$$

for $\Lambda\to\infty$.

We would like to point out that Eq. (31) is a rigorous result based on a consideration of CTRWs in phase space. It represents a highly nontrivial extension of the usual (force-free) KFP equation,

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{u},t) = \mathcal{L}_{\mathrm{FP}} f(\mathbf{x},\mathbf{u},t), \quad (32)$$

which is recovered for $Q(t-t')=\delta(t-t')$. This choice for $Q(t)$ corresponds to the waiting time distribution

$$W(t) = \Lambda \exp(-\Lambda t), \quad (33)$$

which satisfies

$$\int_0^\infty dt W(t) = 1, \quad \bar{t} \equiv \int_0^\infty dt t W(t) = 1/\Lambda. \quad (34)$$

Equation (30) thus implies that the mean time between collisions gets smaller as Λ gets larger, such that $\Lambda\bar{t}=1$.

In comparison to Eq. (32), Eq. (31) contains two different features: temporal memory (i.e., the stochastic process is not Markovian) and retardation (due to the particle's finite inertia). Although the former effect has been considered before by many authors (see, e.g., Ref. [15]), the latter has never been included. It is important, however, in order to ensure Galilean invariance, as pointed out before. Retardation enters quite naturally in the present CTRW framework, and its physical origin is evident, given the mixed nature of the underlying physical process—a particle being subject to a deterministic acceleration and a series of random kicks. It introduces qualitatively different characteristics as will be shown in Sec. III.

As discussed in Ref. [4], a fractional diffusion equation for the pdf $g(\mathbf{x},t)$ of an overdamped particle is obtained from the CTRW master equation

$$\frac{\partial g(\mathbf{x},t)}{\partial t} = K\int_0^t dt' Q(t-t')\Delta g(\mathbf{x},t') \quad (35)$$

by the formal substitution [cmp. Eq. (6)]

$$\int_0^t dt' Q(t-t')g(\mathbf{x},t') \to D_t^{1-\delta} g(\mathbf{x},t), \quad (36)$$

which then yields Eq. (5). The underlying physical reason is the assumed limiting behavior

$$Q(t-t') \propto -\frac{1}{(t-t')^{2-\delta}} \quad (37)$$

of $Q(t-t')$ for large values of $(t-t')$. The substitution (36) can be considered as a regularization of the function (37), which leads to divergent terms for $(t-t')\to 0$ [11]. With regard to Eq. (31), we now have to find suitable regularizations of the expression

$$\int_0^t dt' Q(t-t') e^{-(t-t')\mathbf{u}\cdot\nabla} f(\mathbf{x},\mathbf{u},t), \quad (38)$$

where $Q(t-t')$ again is given by Eq. (37). As is shown in Appendix B, expression (38) is to be replaced by the "fractional substantial derivative"

$$\mathcal{D}_t^{1-\delta}f(\mathbf{x},\mathbf{u},t) = \frac{1}{\Gamma(\delta)}\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] \times \int_0^t \frac{dt'}{(t-t')^{1-\delta}} e^{-(t-t')\mathbf{u}\cdot\nabla} f(\mathbf{x},\mathbf{u},t'), \quad (39)$$

where $\Gamma(\delta)$ denotes the well-known Γ function, and the master equation then reads

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] f(\mathbf{x},\mathbf{u},t) = \mathcal{L}_{\mathrm{FP}}\mathcal{D}_t^{1-\delta}f(\mathbf{x},\mathbf{u},t). \quad (40)$$

For $\delta=1$, this fractional KFP equation turns into Eq. (32). We note in passing that the fractional substantial derivative (39) may also be defined by means of its representation in Laplace space, $\mathcal{D}_t^{1-\delta}\leftrightarrow[\lambda+\mathbf{u}\cdot\nabla]^{1-\delta}$, and that the operators $\mathcal{L}_{\mathrm{FP}}$ and $\mathcal{D}_t^{1-\delta}$ do not commute. Moreover, in Eqs. (27) and (8), the replacements $\Lambda\to\Lambda_\delta$, $\Gamma\to\Gamma_\delta$, and $K\to K_\delta$ (with units $[\Lambda_\delta]=[\Gamma_\delta]=\mathrm{s}^{-\delta}$ and $[K_\delta]=\mathrm{m}^2\,\mathrm{s}^{-2-\delta}$) have to be made. For $\delta<1$ and $\lambda\to 0$, one then has $\Phi(\lambda)=\Lambda_\delta\lambda^{1-\delta}$ and

$$W(\lambda) = \frac{\Phi(\lambda)}{\Phi(\lambda)+\lambda} \approx 1 - \frac{\lambda^\delta}{\Lambda_\delta}. \quad (41)$$

Defining a characteristic time scale τ via the (nonanalytic) low-λ behavior of the waiting time distribution, $W(\lambda)=1-(\lambda\tau)^\delta+\ldots$, we find that in the fractional case, Eq. (30) implies that

$$\Lambda_\delta\tau^\delta = 1, \quad (42)$$

i.e., $\tau\to 0$ for $\Lambda_\delta\to\infty$. In this context, τ is not to be confused with $\bar{t}$, however. The latter quantity is infinite here, due to $W(t)\propto t^{-1-\delta}$ for $t\to\infty$.

D. Comparison to other KFP-type equations

In the following, we compare Eq. (40), which may be considered a key result of the present section, with other KFP-type equations that can be found in the literature. In fairly recent work by Barkai and Silbey [15], an equation similar to Eq. (40) was proposed. In the force-free case, it reads

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] f(\mathbf{x},\mathbf{u},t) = \mathcal{L}_{\mathrm{FP}}D_t^{1-\delta}f(\mathbf{x},\mathbf{u},t), \quad (43)$$

where the usual Riemann-Liouville operator $D_t^{1-\delta}$ [see Eq. (6)] has been used in lieu of the fractional substantial derivative $\mathcal{D}_t^{1-\delta}$. Obviously, retardation effects are not taken into account in the collision operator of Eq. (43). This has two main consequences. First, Eq. (40) exhibits Galilean invariance for $\Gamma=0$, whereas Eq. (43) does not. Second, the long-time behavior of the low-order moments differs in these two cases, as will be demonstrated in Sec. III. Interestingly, both Eqs. (40) and (43) reduce to the same fractional Fokker-Planck equation

$$\frac{\partial F(\mathbf{u},t)}{\partial t} = \mathcal{L}_{\mathrm{FP}}D_t^{1-\delta}F(\mathbf{u},t) \quad (44)$$

for the quantity

$$F(\mathbf{u},t) = \int d\mathbf{x} f(\mathbf{x},\mathbf{u},t) \quad (45)$$

if they are integrated over space. In Ref. [15], Eq. (43) was introduced as a generalization of Eq. (44), but the same can be said about Eq. (40). However, only the latter equation can be derived rigorously from a microscopic dynamical model and preserves Galilean invariance in the appropriate limit.

Another kind of fractional KFP equation has been proposed in Refs. [16–18]. Using a non-Markovian generalization of the Chapman-Kolmogorov equation, the authors obtained

$$\frac{\partial f(\mathbf{x},\mathbf{u},t)}{\partial t} = \theta D_t^{1-\delta}[-\mathbf{u}\cdot\nabla + \mathcal{L}_{\mathrm{FP}}]f(\mathbf{x},\mathbf{u},t), \quad (46)$$

where the factor θ is defined as the ratio $\theta\equiv\tau^*/\tau^\delta$ of an intertrapping time scale τ^* and the waiting time τ taken to the power δ. Equation (46) describes subdiffusive transport and reduces again to Eq. (44) with $\theta\Gamma\to\Gamma_\delta$ and $\theta K\to K_\delta$ if the spatial coordinates are integrated out. Moreover, as a variant of this approach, the fractional KFP equation

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] f(\mathbf{x},\mathbf{u},t) = \theta D_t^{1-\delta}\mathcal{L}_{\mathrm{FP}}f(\mathbf{x},\mathbf{u},t) \quad (47)$$

has been put forward [19]. It describes sub-ballistic superdiffusion and turns out to be identical to Eq. (43) if the substitutions $\theta\Gamma\to\Gamma_\delta$ and $\theta K\to K_\delta$ are made.

Obviously, neither of the fractional KFP-type equations presented in this section do retain retardation effects. Hence, neither of them possesses Galilean invariance (for $\Gamma=0$). However, as will be shown next, the inclusion of retardation alters the particle dynamics even qualitatively and should therefore be taken into account.

III. EXACT SOLUTIONS OF THE GENERALIZED KFP EQUATIONS

A. A generic ansatz

In the following, we shall look for exact solutions of Eqs. (31) and (40), where the initial condition is chosen as

$$f(\mathbf{x},\mathbf{u},t=0) = \delta(\mathbf{x})\delta(\mathbf{u}). \quad (48)$$

(Note that the general solution of Eq. (31) *without retardation* and for $\Gamma=0$ has been given in Ref. [20].) Ignoring its right-hand side, Eq. (31) reads

$$\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla_x\right] f(\mathbf{x},\mathbf{u},t) = 0. \quad (49)$$

This equation has the exact solution

$$f(\mathbf{x},\mathbf{u},t) = G(\boldsymbol{\xi},\mathbf{u}), \quad (50)$$

where

$$\boldsymbol{\xi} = \mathbf{x} - \mathbf{u}t, \quad G(\mathbf{x},\mathbf{u}) = f(\mathbf{x},\mathbf{u},t=0). \quad (51)$$

In the general case (i.e., if the right-hand side is finite), we are thus led to look for a solution of the form

$$f(\mathbf{x},\mathbf{u},t) = H(\boldsymbol{\xi},\mathbf{u},t). \quad (52)$$

Using Eq. (31), we find that $H(\boldsymbol{\xi},\mathbf{u},t)$ is determined by

$$\frac{\partial H(\boldsymbol{\xi},\mathbf{u},t)}{\partial t} = \int_0^t dt' Q(t-t')[\Gamma(\nabla_u - t\nabla_\xi)\cdot\mathbf{u} + K(\nabla_u - t\nabla_\xi)^2]H(\boldsymbol{\xi},\mathbf{u},t'). \quad (53)$$

Introducing the generating function $Z(\mathbf{k},\boldsymbol{\eta},t)$ via

$$Z(\mathbf{k},\boldsymbol{\eta},t) = \int d\boldsymbol{\xi}\int d\mathbf{u}\,\exp(i\mathbf{k}\cdot\boldsymbol{\xi} + i\boldsymbol{\eta}\cdot\mathbf{u})H(\boldsymbol{\xi},\mathbf{u},t), \quad (54)$$

we obtain

$$\frac{\partial Z(\mathbf{k},\boldsymbol{\eta},t)}{\partial t} = \int_0^t dt' Q(t-t')[-\Gamma(\boldsymbol{\eta}-\mathbf{k}t)\cdot\nabla_\eta - K(\boldsymbol{\eta}-\mathbf{k}t)^2]Z(\mathbf{k},\boldsymbol{\eta},t'). \quad (55)$$

Equation (55) can be solved by the integral transform

$$Z(\mathbf{k},\boldsymbol{\eta},t) = \int d\alpha\int d\beta\int d\gamma W(\alpha,\beta,\gamma,t) \times\exp\left(\frac{-\alpha\boldsymbol{\eta}^2}{2} - \beta\mathbf{k}\cdot\boldsymbol{\eta} - \frac{\gamma\mathbf{k}^2}{2}\right), \quad (56)$$

where the unknown function $W(\alpha,\beta,\gamma,t)$ is to be determined. Equation (56) represents the general solution of Eq. (40) for the initial condition $Z(\mathbf{k},\boldsymbol{\eta},t=0)=1$, provided suitable boundary conditions for $W(\alpha,\beta,\gamma,t)$ are specified as discussed below. The corresponding distribution function then satisfies Eq. (48). As a consequence, we obtain the characteristic function $Z(\mathbf{k},\boldsymbol{\eta},t)$ as a superposition of characteristic functions corresponding to a Gaussian process with correlations $\langle\mathbf{u}^2\rangle=\alpha$, $\langle\mathbf{u}\cdot\boldsymbol{\xi}\rangle=\beta$, and $\langle\boldsymbol{\xi}^2\rangle=\gamma$. In Eq. (56), the integral is over all positive values of α and γ, and the value of β is restricted to $|\beta|<(\alpha\gamma)^{1/2}$. Fourier inversion yields the corresponding probability distribution. The result is an extension of the well-known ansatz for the solution of the time fractional diffusion equation for a single variable, Eq. (5), discussed in [21]. For more information, refer the reader to the review papers [4,5]. Furthermore, we note that such representations have also been used in order to model non-Gaussian distribution functions of velocity increments in fluid turbulence [22–24] and seem to be a good starting point for dealing with the phenomenon of intermittency. For a detailed treatment, it turns out to be useful to distinguish between the cases $\Gamma=0$ and $\Gamma\neq 0$. We remind the reader that the case $\Gamma=0$ may be considered as a generalization of Obukhov's model [25] for the motion of a Lagrangian particle in fluid turbulence using an accelerating force with the characteristic statistics of a CTRW.

B. Case $\Gamma=0$

1. Introducing the functions $w_u(\alpha,t)$ and $w_\xi(\gamma,t)$

In the limit $\Gamma\to 0$, it is straightforward to determine the probability distributions of the variables $\mathbf{u}(t)$ and $\boldsymbol{\xi}(t)$, governed, respectively, by the characteristic functions $Z(0,\boldsymbol{\eta},t)$ and $Z(\mathbf{k},0,t)$. In fact, from Eq. (55) we obtain the following two equations:

$$\frac{\partial Z(0,\boldsymbol{\eta},t)}{\partial t} = -\int_0^t dt' Q(t-t')K\boldsymbol{\eta}^2 Z(0,\boldsymbol{\eta},t') + \delta(t),$$

$$\frac{\partial Z(\mathbf{k},0t)}{\partial t} = -t^2\int_0^t dt' Q(t-t')K\mathbf{k}^2 Z(\mathbf{k},0,t') + \delta(t). \quad (57)$$

Here, the δ functions have been added in order to take into account the initial condition $Z(\mathbf{k},\boldsymbol{\eta},0)=1$. These equations can be solved by the ansatz

$$Z(0,\boldsymbol{\eta},t) = \int d\alpha w_u(\alpha,t)e^{-\alpha\boldsymbol{\eta}^2/2},$$

$$Z(\mathbf{k},0,t) = \int d\alpha w_\xi(\gamma,t)e^{-\gamma\mathbf{k}^2/2}. \quad (58)$$

Using the relationships

$$\boldsymbol{\eta}^2 Z(0,\boldsymbol{\eta},t) = -2\int d\alpha w_u(\alpha,t)\frac{\partial}{\partial\alpha}e^{-\alpha\boldsymbol{\eta}^2/2} = 2\int d\alpha\frac{\partial}{\partial\alpha}w_u(\alpha,t)e^{-\alpha\boldsymbol{\eta}^2/2} - 2w_u(0,t),$$

$$\mathbf{k}^2 Z(0,\boldsymbol{\eta},t) = -2\int d\alpha w_\xi(\alpha,t)\frac{\partial}{\partial\gamma}e^{-\gamma\mathbf{k}^2/2} = 2\int d\alpha\frac{\partial}{\partial\gamma}w_\xi(\alpha,t)e^{-\alpha\boldsymbol{\eta}^2/2} - 2w_\xi(0,t), \quad (59)$$

we obtain the following two equations determining the functions $w_u(\alpha,t)$ and $w_\xi(\gamma,t)$:

$$\frac{\partial w_u(\alpha,t)}{\partial t} = -2K\int_0^t dt' Q(t-t')\frac{\partial w_u(\alpha,t')}{\partial\alpha},$$

$$\frac{\partial w_\xi(\gamma,t)}{\partial t} = -2Kt^2\int_0^t dt' Q(t-t')\frac{\partial w_\xi(\gamma,t')}{\partial\gamma}. \quad (60)$$

Furthermore, in order to ensure that the initial condition $Z(\mathbf{k},\boldsymbol{\eta},0)=1$ is satisfied, these functions have to obey the boundary conditions [see also Eq. (91)]

$$2K\int_0^t dt' Q(t-t')w_u(0,t') = \delta(t),$$

$$2Kt^2\int_0^t dt' Q(t-t')w_\xi(0,t') = \delta(t). \quad (61)$$

These equations can be solved in Laplace space. The boundary conditions read as follows:

$$w_u(0,\lambda) = \frac{1}{2KQ(\lambda)},$$

$$w_\xi(0,\lambda) = \frac{1}{2KQ(\lambda)}\left[\frac{\lambda^2}{2} + b\lambda + c\right]. \tag{62}$$

The solution for $w_u(\alpha,\lambda)$ reads

$$w_u(\alpha,\lambda) = \frac{1}{2KQ(\lambda)} e^{-\alpha\lambda/[2KQ(\lambda)]}. \tag{63}$$

No closed expression could be obtained for $w_\xi(\gamma,\lambda)$. In Laplace space, $w_\xi(\gamma,\lambda)$ is determined by the differential equation

$$\lambda w_\xi(\gamma,\lambda) = -2K\frac{\partial^2}{\partial\lambda^2}\frac{\partial}{\partial\gamma}Q(\lambda)w_\xi(\gamma,\lambda). \tag{64}$$

Using the ansatz

$$w_\xi(\gamma,\lambda) = u(\xi,\lambda)\frac{\lambda^2}{Q(\lambda)}, \tag{65}$$

one obtains the differential equation

$$\frac{\lambda^3}{Q(\lambda)}u_\xi(\gamma,\lambda) = -2K\frac{\partial}{\partial\gamma}\frac{\partial^2}{\partial\lambda^2}\lambda^2 u_\xi(\gamma,\lambda). \tag{66}$$

We will return to this equation later.

2. Calculating the moments of $w_u(\alpha,t)$ and $w_\xi(\gamma,t)$

We now turn to Eq. (58). It allows for a recursive calculation of the moments $\langle u(t)^{2n}\rangle$ and $\langle \xi(t)^{2n}\rangle$ obeying the initial conditions $\langle u(t=0)^{2n}\rangle = 0$ and $\langle \xi(t=0)^{2n}\rangle = 0$. For the sake of simplicity, we resort to the one-dimensional case. Using Eq. (54), we obtain

$$\langle u(t)^{2n}\rangle = \frac{(2n)!}{n!2^n}\int_0^\infty d\alpha\,\alpha^n w_u(\alpha,t) = \frac{(2n)!}{n!2^n}\langle\alpha^n\rangle(t),$$

$$\langle \xi(t)^{2n}\rangle = \frac{(2n)!}{n!2^n}\int_0^\infty d\gamma\,\gamma^n w_\xi(\gamma,t) = \frac{(2n)!}{n!2^n}\langle\gamma^n\rangle(t). \tag{67}$$

These relations are a direct consequence of the representation of the generating function (54) as a superposition of Gaussian generating functions with moments α, β, γ, and with weights $w(\alpha,\beta,\gamma,t)$. In order to determine the moments of $u(t)$ and $x(t)$, one thus has to determine the moments $\langle\alpha^k\rangle \times (t)$ and $\langle\gamma^k\rangle(t)$. The latter can be determined recursively

$$\frac{d}{dt}\langle\alpha^n\rangle(t) = 2Kn\int_0^t dt'\,Q(t-t')\langle\alpha^{n-1}\rangle(t'),$$

$$\frac{d}{dt}\langle\gamma^n\rangle(t) = 2Knt^2\int_0^t dt'\,Q(t-t')\langle\gamma^{n-1}\rangle(t'). \tag{68}$$

These equations can again be solved in Laplace space. The transformed recursion relations read

$$\langle\alpha^n\rangle(\lambda) = 2Kn\frac{Q(\lambda)}{\lambda}\langle\alpha^{n-1}\rangle(\lambda),$$

$$\langle\gamma^n\rangle(\lambda) = 2Kn\frac{1}{\lambda}\frac{\partial^2}{\partial\lambda^2}Q(\lambda)\langle\gamma^{n-1}\rangle(\lambda). \tag{69}$$

The first of these equations can be solved explicitly. One obtains

$$\langle\alpha^n\rangle(\lambda) = n!\left[\frac{2KQ(\lambda)}{\lambda}\right]^n = n![\langle\alpha\rangle(\lambda)]^n. \tag{70}$$

3. Moments of $w_u(\alpha,t)$ and $w_\xi(\gamma,t)$ in the fractional case

Next, we consider the fractional case, Eq. (40). Here, we have $Q(\lambda)=\lambda^{1-\delta}$, as is known from fractional calculus [4,26]. This will lead us to the scaling laws

$$\langle u(t)^{2n}\rangle \propto t^{n\delta}, \quad \langle x(t)^{2n}\rangle \propto t^{n(\delta+2)}. \tag{71}$$

As indicated above and outlined in Appendix B, we make the replacement

$$\int dt'\,Q(t-t') \to \frac{1}{\Gamma(\delta)}\frac{\partial}{\partial t}\int_0^t \frac{dt'}{(t-t')^{1-\delta}}. \tag{72}$$

Because of the initial conditions $\langle\alpha^n\rangle(t=0)=0$ and $\langle\gamma^n\rangle(t=0)=0$, we have the recursion relations

$$\langle\alpha^n\rangle(t) = 2Kn\frac{1}{\Gamma(\delta)}\int_0^t \frac{dt'}{(t-t')^{1-\delta}}\langle\alpha^{n-1}\rangle(t') = 2KnD_t^{-\delta}\langle\alpha^{n-1}\rangle(t),$$

$$\langle\gamma^n\rangle(t) = 2Knt^2\frac{1}{\Gamma(\delta)}\int_0^t \frac{dt'}{(t-t')^{1-\delta}}\langle\gamma^{n-1}\rangle(t') = 2Knt^2D_t^{-\delta}\langle\gamma^{n-1}\rangle(t). \tag{73}$$

This chain of equations can be solved by the ansatz

$$\langle\alpha^n\rangle(t) = \alpha_n t^{n\delta}, \quad \langle\gamma^n\rangle(t) = \gamma_n t^{n(\delta+2)}. \tag{74}$$

Using the identity

$$D_t^{-\delta}t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\delta+\mu+1)}t^{\mu+\delta}, \tag{75}$$

we obtain the recursion relations

$$\alpha_n = 2Kn\frac{\Gamma[(n-1)\delta+1]}{\Gamma(n\delta+1)}\alpha_{n-1},$$

$$\gamma_n = 2Kn\frac{\Gamma[(n-1)(2+\delta)+1]}{\Gamma[n(2+\delta)-1]}\gamma_{n-1}. \tag{76}$$

It follows that

$$\alpha_n = (2K)^n n!\frac{1}{\Gamma(n\delta+1)},$$

$$\gamma_n = (2K)^n n!\frac{(2+\delta)(2+\delta-1)}{\Gamma[n(2+\delta-1)]}\Pi_{i=1}^{n-1}(i-1). \tag{77}$$

4. Determining $w_u(\alpha,t)$ and $w_\xi(\gamma,t)$ in the fractional case

In the fractional case, the probability distribution $w_u(\alpha,t)$ can be written in terms of the function L_δ, which is the left-sided Lévy distribution [4,21] (often denoted as inverse Lévy distribution) of order δ

$$w_u(\alpha,t) = \frac{Xt}{\alpha\delta} L_\delta(Xt), \quad X = \left[\frac{\alpha\Gamma(\delta)}{2K}\right]^{-1/\delta}. \tag{78}$$

As mentioned before, $w_\xi(\gamma,t)$ can be expressed by means of $u_\xi(\gamma,t)$ whose Laplace transform is determined by Eq. (66). In the present context, this equation reads

$$\lambda^{2+\delta} u_\xi(\gamma,\lambda) = -2K \frac{\partial}{\partial\gamma}\frac{\partial^2}{\partial\lambda^2}\lambda^2 u_\xi(\gamma,\lambda). \tag{79}$$

Using the ansatz

$$u(\gamma,\lambda) = U(\lambda\gamma^{1/(2+\delta)}) = U(y), \tag{80}$$

one obtains the differential equation

$$y^{1+\delta}U(y) = -K\frac{\partial^3}{\partial y^3}[2y^2 U(y)]. \tag{81}$$

Thus far, no analytical expression for $U(y)$ has been found. However, one can establish the existence of a scaling solution of the form

$$w_\xi(\gamma,t) = \tilde{W}\left(\frac{t}{\gamma^{1/(2+\delta)}}\right)\frac{1}{\gamma^{1+1/(2+\delta)}}. \tag{82}$$

5. Low-order moments of $u(t)$ and $x(t)$

To complete our discussion of the solution of Eq. (40) for $\Gamma=0$, we now provide explicit expressions for the low order moments. They read

$$\langle u(t)^2\rangle = \frac{2K}{\Gamma(1+\delta)}t^\delta,$$

$$\langle x(t)^2\rangle = \frac{4K}{\Gamma(3+\delta)}t^{\delta+2},$$

$$\langle u(t)^4\rangle = \frac{24K^2}{\Gamma(1+2\delta)}t^{2\delta},$$

$$\langle x(t)^4\rangle = \frac{12K^2}{\Gamma(5+2\delta)}[80+32(1-\delta)-4\delta(1-\delta)]t^{2\delta+4}. \tag{83}$$

In deriving these expressions, we have made use of relations of the form

$$\langle \boldsymbol{\xi}(t)^2\rangle = \langle \mathbf{x}(t)^2\rangle - 2\langle \mathbf{x}(t)\cdot\mathbf{u}(t)\rangle t + \langle \mathbf{u}(t)^2\rangle t^2. \tag{84}$$

Finally, to quantify the deviation from normal statistics in the case $\Gamma=0$, we calculate the flatness of the PDFs for the stochastic variables $u(t)$ and $x(t)$. One obtains

$$F_u = \frac{\langle u^4\rangle - 3\langle u^2\rangle^2}{\langle u^2\rangle^2} = 6\frac{\Gamma(1+\delta)^2}{\Gamma(1+2\delta)} - 3. \tag{85}$$

Since δ is taken from the interval $0<\delta\leqslant 1$, the flatness F_u satisfies $0\leqslant F_u<3$. Similarly, one gets

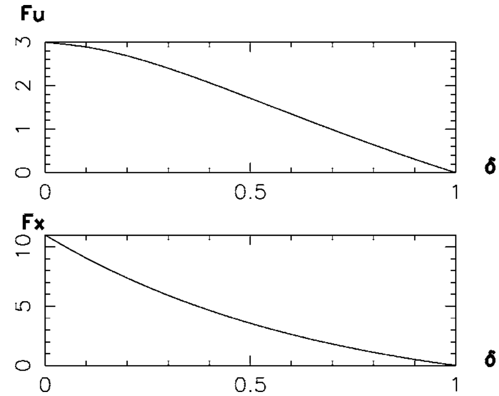

FIG. 2. The flatness of the PDFs for the stochastic variables $u(t)$ and $x(t)$ [denoted, respectively, by F_u and F_x] as a function of the parameter δ.

$$F_x = \frac{\langle x^4\rangle - 3\langle x^2\rangle^2}{\langle x^2\rangle^2} = 3\frac{\Gamma(3+\delta)^2}{\Gamma(5+2\delta)} \times[20+8(1-\delta)-\delta(1-\delta)]-3. \tag{86}$$

Thus, the flatness of the process $x(t)$ lies in the interval $0\leqslant F_x<11$. Both F_u and F_x are displayed in Fig. 2.

C. The case $\Gamma\neq 0$

1. Distribution function as a superposition of Gaussians

For $\Gamma\neq 0$, the function $W(\alpha,\beta,\gamma,t)$ has to obey the evolution equation

$$\frac{\partial W(\alpha,\beta,\gamma,t)}{\partial t} = -\int_0^t dt' Q(t-t')[2\partial_\alpha(K-\alpha\Gamma) - \partial_\beta(2Kt+\Gamma\beta-\Gamma\alpha t) + 2\partial_\gamma(Kt^2-\Gamma\beta t)] \times W(\alpha,\beta,\gamma,t'). \tag{87}$$

In order to prove this relationship, we note that using the ansatz (56), Eq. (55) reads

$$\int d\alpha d\beta d\gamma \frac{\partial W(\alpha,\beta,\gamma,t)}{\partial t} e^{-\alpha\boldsymbol{\eta}^2/2-\beta\boldsymbol{\eta}\cdot\mathbf{k}-\gamma\mathbf{k}^2/2} = \int d\alpha d\beta d\gamma \int_0^t dt' Q(t-t')W(\alpha,\beta,\gamma,t')[(-2\Gamma\alpha+2K)\partial_\alpha + (\Gamma\alpha t-2Kt-\Gamma\beta)\partial_\beta + (2Kt^2+2\Gamma\beta t)\partial_\gamma] \times e^{-\alpha\boldsymbol{\eta}^2/2-\beta\boldsymbol{\eta}\cdot\mathbf{k}-\gamma\mathbf{k}^2/2} + \delta(t). \tag{88}$$

The δ function has been added in order to take into account the initial condition

$$Z(\mathbf{k},\boldsymbol{\eta},0)=1, \quad f(\boldsymbol{\xi},\mathbf{u},t=0)=\delta(\boldsymbol{\xi})\delta(\mathbf{u}). \tag{89}$$

Moreover, we have made use of relationships of the form

$$\boldsymbol{\eta}^2 e^{-\boldsymbol{\eta}^2/2-\beta\boldsymbol{\eta}\cdot\mathbf{k}-\gamma\mathbf{k}^2/2} = -2\partial_\alpha e^{-\alpha\boldsymbol{\eta}^2/2-\beta\boldsymbol{\eta}\cdot\mathbf{k}-\gamma\mathbf{k}^2/2}. \tag{90}$$

Equation (87) is then obtained by partial integration. We would like to point out that there are contributions from the boundaries $\alpha=\beta=0$, $\gamma=\beta=0$, and $\beta=\pm(\alpha\gamma)^{1/2}$ that can be fixed in order to guarantee the initial condition $Z(\mathbf{k},\boldsymbol{\eta},0)=1$ This leads to the following boundary conditions [using the shorthand notation $W(t)\equiv W(0,0,0,t)$]:

$$W(\alpha,\beta=\pm(\alpha\gamma)^{1/2},\gamma,t)=\delta(\alpha)\delta(\gamma)W(t),$$

$$W(0,0,\gamma,t)=\delta(\gamma)W(t),$$

$$W(\alpha,0,0,t)=\delta(\alpha)W(t),$$

$$W(\alpha,\beta,\gamma,0)=\delta(\alpha)\delta(\beta)\delta(a). \tag{91}$$

Consequently, the function $W(t)$ is determined by the condition

$$2K\int_0^t dt' Q(t-t')[(1+t^2)W(t')]=\delta(t). \tag{92}$$

As a result, we obtain the following representation of $f(\mathbf{x},\mathbf{u},t)$:

$$f(\mathbf{x},\mathbf{u},t)=\int d\alpha\int d\beta\int d\gamma W(\alpha,\beta,\gamma,t)[(2\pi)^N \det A]^{-1/2} \times \exp\left[\frac{-Q_{11}u^2}{2}-Q_{12}u(x-ut)-\frac{Q_{22}(x-ut)^2}{2}\right]. \tag{93}$$

Here, N denotes the spatial dimension of the system, and Q is the inverse of the matrix A with the elements $A_{11}=\alpha$, $A_{22}=\gamma$, and $A_{12}=A_{21}=\beta$, i.e.,

$$Q=\begin{pmatrix} Q_{11} & Q_{12}\\ Q_{21} & Q_{22}\end{pmatrix}=\frac{1}{\alpha\gamma-\beta^2}\begin{pmatrix} +\gamma & -\beta\\ -\beta & +\alpha\end{pmatrix}. \tag{94}$$

This means that the position-velocity distribution function $f(\mathbf{x},\mathbf{u},t)$ can be written as a superposition of Gaussians with varying variances. The function $W(\alpha,\beta,\gamma,t)$ can be viewed as a time-dependent variance distribution.

2. Distribution function in the Obukhov case

It is quite interesting to see how the ordinary Obukhov model (31), which is defined by $\Gamma=0$, $Q(t-t')=\delta(t-t')$, can be solved by an application of Eq. (93). In this case, the function $W(\alpha,\beta,\gamma,t)$ obeys [see Eq. (87)]

$$\frac{\partial W(\alpha,\beta,\gamma,t)}{\partial t}=-2K[\partial_\alpha - t\partial_\beta + t^2\partial_\gamma]W(\alpha,\beta,\gamma,t). \tag{95}$$

This type of linear first-order partial differential equation can be solved by the method of characteristics. Here, the latter are given by

$$\alpha(t)=2Kt+\alpha_0,$$

$$\beta(t)=-Kt^2+\beta_0,$$

$$\gamma(t)=\frac{2Kt^3}{3}+\gamma_0. \tag{96}$$

Using the initial condition,

$$W(\alpha(t),\beta(t),\gamma(t),t)=W(\alpha_0,\beta_0,\gamma_0)=\delta(\alpha_0)\delta(\beta_0)\delta(\gamma_0), \tag{97}$$

the function $W(\alpha,\beta,\gamma,t)$ reads

$$W(\alpha,\beta,\gamma,t)=\delta(\alpha-2Kt)\delta(\beta+Kt^2)\delta\left(\gamma-\frac{2Kt^3}{3}\right), \tag{98}$$

leading in a straightforward way to

$$f(\mathbf{x},\mathbf{u},t)=N(t)\exp\left[-\frac{1}{Kt}\mathbf{u}^2+\frac{3}{Kt^2}\mathbf{x}\cdot\mathbf{u}-\frac{3}{Kt^3}\mathbf{x}^2\right]. \tag{99}$$

3. Calculation of $w_u(\alpha,t)$

Let us now turn back to the general case, considering the quantity $w_u(\alpha,t)$. Its time dependence is determined by

$$\frac{\partial w_u(\alpha,t)}{\partial t}=-\int_0^t dt' Q(t-t')\frac{\partial}{\partial\alpha}[2K-2\Gamma\alpha]w_u(\alpha,t'). \tag{100}$$

This equation can be solved in a straightforward manner in Laplace space

$$w_u(\alpha,\lambda)=\frac{1}{2KQ(\lambda)}\left(1-\frac{\Gamma\alpha}{K}\right)^{\lambda/[2Q(\lambda)\Gamma]-1}. \tag{101}$$

Here, we have used the initial condition

$$w_u(0,\lambda)=\frac{1}{2KQ(\lambda)}. \tag{102}$$

For $Q(\lambda)=\lambda^{1-\delta}$, we obtain

$$w_u(\alpha,\lambda)=\frac{\lambda^{\delta-1}}{2K(1-\Gamma\alpha/K)}\exp\left[(2\Gamma)^{-1}\ln\left(1-\frac{\Gamma\alpha}{K}\right)\lambda^\delta\right]. \tag{103}$$

Using the shorthand notation

$$A=\left[2K\left(1-\frac{\Gamma\alpha}{K}\right)\right]^{-1} \tag{104}$$

and

$$s=-\frac{K}{\Gamma}\left(1-\frac{\Gamma\alpha}{K}\right)\ln\left(1-\frac{\Gamma\alpha}{K}\right), \tag{105}$$

the inverse Laplace transform can be determined with the help of the relationship

$$w_u(\alpha,\lambda)=A\lambda^{\delta-1}e^{-As\lambda^\delta}\to\frac{1}{\delta A^{1/\delta}}\frac{t}{s^{1+1/\delta}}L_\delta\left[\frac{t}{(As)^{1/\delta}}\right]=w_u(\alpha,t), \tag{106}$$

where $L_\delta(x)$ denotes the half sided Lévy distribution of order δ.

4. Second-order moments of u(t) and x(t)

In order to compute the second-order moments of $u(t)$ and $x(t)$, Eq. (87) can used to obtain the following relations in Laplace space:

$$\langle\alpha\rangle(\lambda) = \frac{2KQ(\lambda)}{\lambda[\lambda + 2\Gamma Q(\lambda)]},$$

$$\langle\beta\rangle(\lambda) = -\frac{1}{\lambda + \Gamma Q(\lambda)}\frac{\partial}{\partial\lambda}Q(\lambda)\left[\frac{2K}{\lambda} - \Gamma\langle\alpha\rangle(\lambda)\right],$$

$$\langle\gamma\rangle(\lambda) = \frac{2K}{\lambda}\frac{\partial^2}{\partial\lambda^2}\frac{Q(\lambda)}{\lambda} + \frac{2\Gamma}{\lambda}\frac{\partial}{\partial\lambda}Q(\lambda)\langle\beta\rangle(\lambda). \tag{107}$$

The time-dependent quantities $\langle\alpha\rangle(t)$, $\langle\beta\rangle(t)$, and $\langle\gamma\rangle(t)$ can be evaluated from Eq. (107) by means of the relation

$$\int_0^\infty \frac{d\alpha}{Q(\lambda)} e^{-\alpha[\lambda/Q(\lambda)+n\Gamma]} = -\frac{1}{\lambda + n\Gamma Q(\lambda)} \tag{108}$$

and the corresponding Laplace inversion

$$G(n,t) = \int_0^\infty d\alpha e^{-n\Gamma\alpha} w(\alpha,t). \tag{109}$$

Alternatively, one can evaluate the second-order moments in the following way. Starting from Eq. (31), one obtains

$$G(2,t)^{-1} * \langle u^2(t)\rangle = 2Q(t) * K,$$

$$G(1,t)^{-1} * \langle u(t)x(t)\rangle = \langle u^2(t)\rangle - \Gamma Q(t) * \langle u^2(t)\rangle,$$

$$G(0,t)^{-1}\langle x^2(t)\rangle = 2\langle u(t)x(t)\rangle. \tag{110}$$

Here, we have introduced the following notation for the convolution integral:

$$G(n,t) * h(t) = \int_0^t dt' G(n,t-t')h(t'). \tag{111}$$

We would like to point out that the term $-\Gamma Q(t)*\langle u^2(t)\rangle$ in Eq. (110) is due to the time retardation effect in the collision term of the master equation (31). The solutions in Laplace space read

$$\langle u^2\rangle(\lambda) = \frac{2K}{\lambda[2\Gamma + \lambda/Q(\lambda)]},$$

$$\langle xu\rangle(\lambda) = \frac{2K}{\lambda[2\Gamma + \lambda/Q(\lambda)]}\frac{1}{\Gamma + \lambda/Q(\lambda)}\left[\frac{1}{Q(\lambda)} + \Gamma\frac{Q'(\lambda)}{Q(\lambda)}\right],$$

$$\langle x^2\rangle(\lambda) = \frac{2}{\lambda}\langle xu\rangle(\lambda). \tag{112}$$

It is interesting to consider the long-time limit of these expressions, i.e., the behavior for small values of λ. Assuming that $\lambda/Q(\lambda)\to 0$ for $\lambda\to 0$, we obtain

$$\lim_{\lambda\to 0}\langle u^2\rangle(\lambda) = \frac{K}{\Gamma}\frac{1}{\lambda},$$

$$\lim_{\lambda\to 0}\langle xu\rangle(\lambda) = \frac{K}{\Gamma^2}\frac{1}{\lambda}\left[\frac{1}{Q(\lambda)} + \Gamma\frac{Q'(\lambda)}{Q(\lambda)}\right], \tag{113}$$

while the last equation, which reads

$$\frac{1}{2}\frac{d}{dt}\langle x(t)^2\rangle = \langle x(t)u(t)\rangle \tag{114}$$

in the time domain, remains unaltered. This limit is obtained by approximating $G(n,\lambda)\approx n\Gamma$. The term $Q'(\lambda)/Q(\lambda)$ is due to the retardation effect in the collision operator. From this, it follows that $\langle u(t)^2\rangle$ approaches a constant in the long-time limit,

$$\lim_{t\to\infty}\langle u(t)^2\rangle = K/\Gamma, \tag{115}$$

independent of the choice of $Q(t)$. The long-time behavior of $\langle x(t)u(t)\rangle$ and $\langle x(t)^2\rangle$, on the other hand, depends on the long-time behavior of $Q(t)$. Taking $Q(\lambda)=\lambda^{1-\delta}$ for $\lambda\to 0$, the second part of Eq. (113) turns into

$$\lim_{\lambda\to 0}\langle xu\rangle(\lambda) = \frac{K}{\Gamma^2}\lambda^{\delta-2} + (1-\delta)\frac{K}{\Gamma}\lambda^{-2}, \tag{116}$$

which corresponds to

$$\lim_{t\to\infty}\langle x(t)u(t)\rangle = \frac{1}{\Gamma(2-\delta)}\frac{K}{\Gamma^2}t^{1-\delta} + (1-\delta)\frac{K}{\Gamma}t. \tag{117}$$

Using Eq. (114), one finally obtains

$$\lim_{t\to\infty}\langle x(t)^2\rangle = \frac{2}{\Gamma(3-\delta)}\frac{K}{\Gamma^2}t^{2-\delta} + (1-\delta)\frac{K}{\Gamma}t^2. \tag{118}$$

Thus, for $\delta=1$, we recover the classical result of normal diffusion of an overdamped particle,

$$\langle u(t)^2\rangle = \frac{K}{\Gamma}, \quad \langle x(t)u(t)\rangle = \frac{K}{\Gamma^2}, \quad \langle x(t)^2\rangle = \frac{2K}{\Gamma^2}t \quad (\text{for } t\to\infty). \tag{119}$$

In contrast, for $0<\delta<1$, we obtain

$$\langle u(t)^2\rangle = \frac{K}{\Gamma}, \quad \langle x(t)u(t)\rangle = (1-\delta)\frac{K}{\Gamma}t,$$

$$\langle x(t)^2\rangle = (1-\delta)\frac{K}{\Gamma}t^2 \quad (\text{for } t\to\infty), \tag{120}$$

revealing the ballistic nature of the associated diffusion processes for any $0<\delta<1$ as long as Γ and K are both nonzero. This universality is a direct consequence of the temporal retardation in the collision term of Eq. (31). If this effect were absent, such as in the master equation considered by Barkai and Silbey [15], the dominant scaling behavior would be given by $\langle x(t)^2\rangle\propto t^{2-\delta}$, corresponding to sub-ballistic superdiffusion.

IV. SUMMARY AND CONCLUSIONS

In the present paper, we provided (a rederivation and) an exact solution of a fractional (KFP equation, which was first proposed in Ref. [12], assuming the absence of external forces. This equation may be viewed as a model for anomalous diffusion and non-Gaussian statistics of inertial, weakly damped (as opposed to overdamped) particles. In contrast to other fractional KFP-type equations found in the literature, the version presented here can be based rigorously on a statistical model and exhibits Galilean invariance in the appropriate limiting case ($\Gamma=0$). The corresponding Fokker-Planck collision operator exhibits a retardation effect, associated with nonlocal couplings in time *and space*.

Despite this complexity, several rigorous results concerning the behavior of the position-velocity distribution function could be derived. The latter tends to exhibit strongly non-Gaussian characteristics. This is in stark contrast to the solutions of the conventional KFP equation, where the PDFs are known to become Gaussian in the long-time limit. Interestingly, these non-Gaussian distributions can be described in terms of suitable superpositions of Gaussian distributions. As a consequence, the types of CTRWs proposed in this paper can be used to model stochastic processes with non-normal statistics and anomalous scaling behavior in time. Furthermore, we were able to calculate the long-time behavior of the second-order moments, revealing the ballistic nature of the associated diffusion processes for any $0<\delta<1$ as long as Γ and K are both nonzero. This universality could be identified as a direct consequence of retardation effects. Neglecting the latter, the system would exhibit sub-ballistic superdiffusion instead.

Although the main thrust of the present work has been on the further development of the basic concepts of anomalous diffusion rather than on particular applications, we are confident that it will help shed light on the physics of inertial, weakly damped particles exhibiting superdiffusion. The list of physical systems for which the present approach may provide useful include, e.g., turbulent or sheared flows in fluids and plasmas, complex liquids, porous glasses, and various biological systems (see, e.g., Ref. [4], and references therein).

APPENDIX A: FOKKER-PLANCK COLLISION OPERATOR

In this appendix, we will show that the Fokker-Planck collision operator

$$\mathcal{L}_{\mathrm{FP}}g(\mathbf{u}) = \Gamma\nabla_u \cdot [\mathbf{u}g(\mathbf{u})] + K\Delta_u g(\mathbf{u}) \tag{A1}$$

follows naturally from a suitable ansatz for the quantity $F(\mathbf{u},\mathbf{u}')$. To this aim, we take the latter to be a Gaussian of the form

$$F(\mathbf{u};\mathbf{u}') = \left(\frac{\Lambda}{4\pi K}\right)^{3/2} \exp\left[-\frac{(\mathbf{u}-\mathbf{u}'+\Gamma\mathbf{u}'/\Lambda)^2}{4K/\Lambda}\right]. \tag{A2}$$

We thus get

$$\begin{aligned}
&\int d\mathbf{u}' F(\mathbf{u};\mathbf{u}')g(\mathbf{u}') - g(\mathbf{u}) \\
&\quad = \int d\mathbf{u}' \left(\frac{\Lambda}{4\pi K}\right)^{3/2} \exp\left[-\frac{(\mathbf{u}-\mathbf{u}'+\Gamma\mathbf{u}'/\Lambda)^2}{4K/\Lambda}\right] \\
&\qquad \times g(\mathbf{u}') - g(\mathbf{u}) \\
&\quad = \int \frac{d\mathbf{w}}{(1-\Gamma/\Lambda)^3}\left(\frac{\Lambda}{4\pi K}\right)^{3/2} \exp\left[-\frac{\mathbf{w}^2}{4K/\Lambda}\right] \\
&\qquad \times g\left(\frac{\mathbf{u}+\mathbf{w}}{1-\Gamma/\Lambda}\right) - g(\mathbf{u}).
\end{aligned} \tag{A3}$$

In the limit $\Lambda\to\infty$, the Gaussian is very narrow and the function g may be Taylor expanded about $\mathbf{u}$. Using the relationships

$$\left(\frac{\Lambda}{4\pi K}\right)^{3/2} \int d\mathbf{w} \exp\left[-\frac{\mathbf{w}^2}{4K/\Lambda}\right] = 1, \tag{A4}$$

$$\left(\frac{\Lambda}{4\pi K}\right)^{3/2} \int d\mathbf{w} w_j^2 \exp\left[-\frac{\mathbf{w}^2}{4K/\Lambda}\right] = \frac{2K}{\Lambda}, \tag{A5}$$

and

$$(1-\Gamma/\Lambda)^{-n} = 1 + \frac{n\Gamma}{\Lambda} + \mathcal{O}(\Lambda^{-2}), \tag{A6}$$

one thus obtains

$$\int d\mathbf{u}' F(\mathbf{u};\mathbf{u}')g(\mathbf{u}') - g(\mathbf{u}) = \Lambda^{-1}\mathcal{L}_{\mathrm{FP}}g(\mathbf{u}) \tag{A7}$$

to leading order in Λ^{-1}.

APPENDIX B: FRACTIONAL SUBSTANTIAL DERIVATIVE

In this appendix, we shall discuss a physical justification of the formal substitution

$$\int_0^t dt' Q(t-t') \to \frac{1}{\Gamma(\delta)}\frac{d}{dt}\int_0^t \frac{dt'}{(t-t')^{1-\delta}} = D_t^{1-\delta}, \tag{B1}$$

where δ is taken from the range $0<\delta\leqslant 1$. This substitution leads to fractional Fokker-Planck equations. For the present case, it has to be generalized and will lead us to the definition of fractional substantial derivatives.

We assume that the function $Q(t)$ is characterized by the existence of two ranges exhibiting different types of behavior. An example we have in mind would be

$$Q(t) = \begin{cases} Q_1(t) = \text{const} & \text{for } t<\Delta \\ Q_0(t) \propto t^{\delta-2} & \text{for } t>\Delta \end{cases} \tag{B2}$$

When evaluating convolution integrals involving $Q(t-t')$, we will therefore separate them into two parts,

$$\int_0^t dt' Q(t-t')H(t') = \int_0^{t-\Delta} dt' Q_0(t-t')H(t') + \int_{t-\Delta}^t dt' Q_1(t-t')H(t'). \tag{B3}$$

Furthermore, we assume that the long-time behavior is given by

$$Q_0(t-t') = -\frac{1-\delta}{\Gamma(\delta)}\frac{1}{(t-t')^{2-\delta}} = \frac{1}{\Gamma(\delta)}\frac{d}{dt}\frac{1}{(t-t')^{1-\delta}}. \tag{B4}$$

As a consequence, we can rearrange the terms according to

$$\begin{aligned}\int_0^t dt' Q(t-t')H(t') &= \frac{1}{\Gamma(\delta)}\frac{d}{dt}\int_0^{t-\Delta} dt' \frac{H(t')}{(t-t')^{1-\delta}} - \frac{1}{\Gamma(\delta)}\frac{H(t-\Delta)}{\Delta^{1-\delta}} + \int_{t-\Delta}^t dt' Q_1(t-t')H(t') \\ &= D_t^{1-\delta}H(t) + \int_{t-\Delta}^t dt' Q_1(t-t')H(t') - \frac{1}{\Gamma(\delta)}\frac{d}{dt}\int_{t-\Delta}^t dt' \frac{H(t')}{(t-t')^{1-\delta}} - \frac{1}{\Gamma(\delta)}\frac{H(t-\Delta)}{\Delta^{1-\delta}}.\end{aligned} \tag{B5}$$

Provided that the function H is smooth, we may replace $H(t')$ and $H(t-\Delta)$ by $H(t)$ in the limit $\Delta \to 0$. The last three terms then read

$$H(t)\int_{t-\Delta}^t dt' Q_1(t-t') - \frac{\Delta^\delta}{\delta\Gamma(\delta)}\frac{d}{dt}H(t) - \frac{1}{\Gamma(\delta)}\frac{H(t)}{\Delta^{1-\delta}}. \tag{B6}$$

The second of these terms vanishes as Δ tends to zero, and the remaining two terms cancel provided we choose $Q_1(t)$ such that

$$\int_0^\Delta dt Q_1(t) = \frac{1}{\Gamma(\delta)\Delta^{1-\delta}}. \tag{B7}$$

In this case, we recover Eq. (B1) in the limit $\Delta \to 0$. Assuming that $Q_1(t)=C=$const, it follows from Eq. (B7) that $C = C(\Delta) \propto \Delta^{\delta-2}$, which is consistent with $Q_0(\Delta) \propto \Delta^{\delta-2}$.

Next, we would like to extend these considerations to the regularization of the expression

$$\int_0^t dt' Q(t-t')e^{-(t-t')\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t'). \tag{B8}$$

Like before, we split the integral into several parts,

$$\begin{aligned}&\int_0^t dt' Q(t-t')e^{-(t-t')\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t') \\ &= \int_{t-\Delta}^t dt' Q_1(t-t')e^{-(t-t')\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t') - \frac{1-\delta}{\Gamma(\delta)} \times \int_0^{t-\Delta}\frac{dt'}{(t-t')^{2-\delta}}e^{-(t-t')\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t') \\ &= \mathcal{D}_t^{1-\delta}f(\mathbf{x},\mathbf{u},t) - \frac{1}{\Gamma(\delta)}\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] \times \int_{t-\Delta}^t \frac{dt'}{(t-t')^{1-\delta}}e^{-(t-t')\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t') \\ &\quad + \int_{t-\Delta}^t dt' Q_1(t-t')e^{-(t-t')\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t') - \frac{1}{\Gamma(\delta)}\frac{1}{\Delta^{1-\delta}}e^{-\Delta\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t-\Delta),\end{aligned} \tag{B9}$$

where

$$\mathcal{D}_t^{1-\delta}f(\mathbf{x},\mathbf{u},t) = \frac{1}{\Gamma(\delta)}\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] \times \int_0^t \frac{dt'}{(t-t')^{1-\delta}}e^{-(t-t')\mathbf{u}\cdot\nabla}f(\mathbf{x},\mathbf{u},t'). \tag{B10}$$

Here, it is crucial to note that the *substantial* derivative enters during the last step. Now, we use again the fact that $Q_1(t)$ can be chosen such that the last three terms on the right-hand side of Eq. (B9) vanish in the limit $\Delta \to 0$, provided $f(\mathbf{x},\mathbf{u},t)$ is a smooth function of t and of $\mathbf{x}$ such that $f(\mathbf{x}-\mathbf{u}(t-t'),\mathbf{u},t') = f(\mathbf{x},\mathbf{u},t) + \mathcal{O}(\Delta)$ for $t \approx t'$. Thus in summary, we find that Eq. (B1) can be generalized to

$$\int_0^t dt' Q(t-t')e^{-(t-t')\mathbf{u}\cdot\nabla} \to \frac{1}{\Gamma(\delta)}\left[\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right] \times \int_0^t \frac{dt'}{(t-t')^{1-\delta}}e^{-(t-t')\mathbf{u}\cdot\nabla} = \mathcal{D}_t^{1-\delta}. \tag{B11}$$

We note that, alternatively, one may define this fractional substantial derivative by means of its representation in Laplace space,

$$\mathcal{D}_t^{1-\delta} \leftrightarrow [\lambda + \mathbf{u}\cdot\nabla]^{1-\delta}. \tag{B12}$$

This operator, describing retardation effects, is nonlocal in time and space. For $\delta \to 1$, it becomes the identity operator. It should be pointed out that the "naive" definition

$$\int_0^t dt' Q(t-t')e^{-(t-t')\mathbf{u}\cdot\nabla} \to \frac{1}{\Gamma(\delta)}\frac{\partial}{\partial t}\int_0^t \frac{dt'}{(t-t')^{1-\delta}}e^{-(t-t')\mathbf{u}\cdot\nabla} \tag{B13}$$

would not allow to perform this limit.

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