

Reconstruction of Complex Dynamical Systems Affected by Strong Measurement Noise

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(Received 16 June 2005; revised manuscript received 18 July 2006; published 1 September 2006)

This Letter reports on a new approach to properly analyze time series of dynamical systems which are spoilt by the simultaneous presence of dynamical noise and measurement noise. It is shown that even strong external measurement noise as well as dynamical noise which is an intrinsic part of the dynamical process can be quantified correctly, solely on the basis of measured time series and proper data analysis. Finally, real world data sets are presented pointing out the relevance of the new approach.

DOI: [10.1103/PhysRevLett.97.090603](https://doi.org/10.1103/PhysRevLett.97.090603)

PACS numbers: 05.40.Ca, 02.50.Ey, 05.10.Gg

A major challenge in analyzing time series originating from complex systems is to reveal the underlying process dynamics. Typically the simultaneous involvements of nonlinearities, dynamical noise, and measurement noise cause problems for many experimental situations and account for the complexity of this task. The handling of these complications is the central concern of this Letter.

To extract an underlying signal disturbed by noise, linear and nonlinear predictor models or noise reduction schemes are widely used (for discussion see [1] and references therein). Here we choose an alternative approach based on the broad class of Langevin processes, which describes a variety of complex dynamical systems.

Let us consider a one-dimensional Langevin process (the extension to more dimensions is straightforward) that is given by

$$\dot{x} = D^{(1)}(x) + \sqrt{D^{(2)}(x)}\Gamma(t). \quad (1)$$

The term $\Gamma(t)$ represents Gaussian white noise with $\langle \Gamma(t) \rangle = 0$ and $\langle \Gamma(t')\Gamma(t) \rangle = \delta(t - t')$. The terms $D^{(n)}(x)$ are called the drift coefficient ($n = 1$) and the diffusion coefficient ($n = 2$) and reflect the deterministic and the stochastic part, respectively. $\sqrt{D^{(2)}}$ fixes the amplitude of the stochastic part and is referred to as *dynamical noise*. If $D^{(2)}$ depends on x , it is called multiplicative noise; otherwise it is called additive noise.

In recent years a parameter-free reconstruction of the coefficients and thus of the corresponding Langevin process has been achieved [2–6]. It has been successfully demonstrated that traffic flow dynamics [7], the chaotic dynamics of an electronic circuit [8,9], or the human heart beat rhythm [10] can be reconstructed without need of any *a priori* models but just from measured time series and the estimated drift and diffusion coefficients. This estimation is based on the evaluation of the first ($n = 1$) and the

second ($n = 2$) conditional moments:

$$M^{(n)}(x, \tau) = \langle [x(t + \tau) - x(t)]^n \rangle_{x(t)=x}, \quad (2)$$

from which the coefficients are derived according to

$$D^{(n)}(x) = \lim_{\tau \rightarrow 0} \frac{1}{\tau} M^{(n)}(x, \tau). \quad (3)$$

For ideal time series with a sufficient temporal resolution the coefficients $D^{(n)}(x)$ can unambiguously be obtained from Eq. (3). For real data sets, however, the sampling frequency might be too low to resolve the dynamics properly as was pointed out in [11,12]. For small but finite τ the conditional moments are better approximated by an *Ito-Taylor series expansion* (e.g., [12–14]):

$$\begin{aligned} M^{(1)}(x, \tau) &\approx \tau D^{(1)}(x) + \mathcal{O}(\tau^2), \\ M^{(2)}(x, \tau) &\approx \tau D^{(2)}(x) + \mathcal{O}(\tau^2). \end{aligned} \quad (4)$$

Depending on the process it might be necessary to consider further higher order terms for estimating the coefficients. To finally decide whether an obtained set of coefficients represents the real dynamics at least a consistency check between the statistical properties (moments, probability densities, etc.) of the reconstructed and of the original time series has to be performed (cf. [15]).

Another important effect that complicates a proper estimation of $D^{(n)}$ is the presence of *measurement noise* $\sigma\zeta(t)$ [with $\langle \zeta(t) \rangle = 0$ and $\langle \zeta(t)\zeta(t') \rangle = \delta(t - t')$], which is superimposed on the data. Measurement noise corresponds to a rather unavoidable experimental situation (e.g., [1,16,17]) and means that $y(t) = x(t) + \sigma\zeta(t)$ is examined rather than $x(t)$. For instance, take the measurement of a turbulent velocity time series. The resolution is chosen in such a way that the largest fluctuations (on the largest time scales) are resolved. A certain amount of measurement noise might be negligible for these large scale fluctuations

but can well be significant for the fluctuations on the smallest scales where the fluctuations are much smaller. More generally, the term “measurement noise” refers to any superimposed uncorrelated noise that is present in some complex system; it might even be generated by the complex system itself.

To reconstruct the unknown dynamics $x(t)$ from the accessible $y(t)$ it is thus essential to quantify $\sigma\zeta(t)$ and its influence on the reconstruction of coefficients according to Eqs. (3) and (4), which will be the central concern of this Letter.

In [8,9,18] it has been shown that measurement noise results in an offset term, γ_n , for the conditional moments

$$M^{(n)}(y, \tau) \rightarrow M^{(n)}(x, \tau) + \gamma_n, \quad (5)$$

$$\gamma_1 = 0, \quad \gamma_2 = 2\sigma^2. \quad (6)$$

Any nonzero offset causes a strong overestimation of the coefficients, $D^{(n)}$, because it leads to a divergence of $M^{(n)}(y, \tau)/\tau$ in Eq. (3). In [8] it was therefore proposed to use the offset γ_2 to quantify measurement noise and to take the slope of the conditional moments (as a function of τ) as an estimate of the coefficients.

In this Letter we will show that Eqs. (5) and (6) are restricted to the special case of low measurement noise, that $M^{(1)}(y, \tau)$ must exhibit a τ -independent part (i.e., $\gamma_1 \neq 0$), and that the slope of the conditional moment, $M^{(n)}$, is no longer proportional to the corresponding coefficient, $D^{(n)}$. In Fig. 1 the effect of measurement noise on $M^{(1)}$ is shown for two real world examples, in which a strong offset at $\tau = 0$ causes a divergence of $M^{(1)}/\tau$. Finally, we propose an improved method to quantify measurement noise even for very large noise levels. To this end a general calculation of the conditional moments will be performed. Thus we can explain the measurement noise dependence of the second and the first conditional moment and propose an improved reconstruction of the underlying process. For an Ornstein-Uhlenbeck process the results are analytical; for non-Ornstein-Uhlenbeck processes the corresponding analysis can be performed numerically.

Let us start with the calculation of the conditional moments from the accessible data $y(t)$ spoiled by superimposed measurement noise. Using the definition according to Eq. (2) the following expressions can be derived:

$$\begin{aligned} M^{(1)}(y, \tau) &= \langle y(t+\tau) - y(t) \rangle_{y(t)=y=x(t)+\sigma\zeta(t)} \\ &= \tau \int D^{(1)}(x) f(x|y) dx + \int (x-y) f(x|y) dx \\ &= m^{(1)}(y, \tau) + \gamma_1(y), \end{aligned} \quad (7)$$

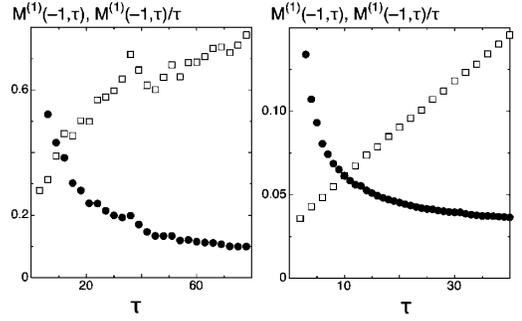


FIG. 1. Both plots show measured $M^{(1)}$ (squares) and $M^{(1)}/\tau$ (circles) as a function of lag τ . The left plot refers to the North Atlantic Oscillation index [23], the right one to an increment time series of financial exchange rates [18]. Units are arbitrary for clarity of presentation. For both systems a significant offset of the first conditional moment is observed causing a divergence of $M^{(1)}/\tau$.

$$\begin{aligned} M^{(2)}(y, \tau) &= \tau \int [2(x-y)D^{(1)}(x) + D^{(2)}(x)] f(x|y) dx \\ &\quad + \sigma^2 + \int (x-y)^2 f(x|y) dx \\ &= m^{(2)}(y, \tau) + \gamma_2(y). \end{aligned} \quad (8)$$

The coefficients, $D^{(n)}$, are implicitly given by the conditional moments that exhibit a τ dependent as well as a τ -independent part, denoted with $m^{(n)}(y, \tau)$ and $\gamma_n(y)$, respectively. According to *Bayes' theorem* the unknown probability density $f(x|y)$ is given by $\frac{f(y|x)p(x)}{\int f(y|x)p(x)dx}$, where $f(y|x)$ denotes nothing other than the distribution of measurement noise. Here we consider Gaussian distributed measurement noise with variance σ^2 . The distribution of the process $x(t)$, given by Eq. (1), is denoted by $p(x)$. For stationary processes the distribution is known to be

$$p(x) = \frac{\mathcal{N}}{D^{(2)}(x)} \exp\left[2 \int_{-\infty}^x \frac{D^{(1)}(x')}{D^{(2)}(x')} dx'\right], \quad (9)$$

where \mathcal{N} denotes a proper normalization factor; cf. [19]. To extract the coefficients from the four equations of Eqs. (7) and (8) we assume for convenience (but not necessarily) that the coefficients can be modeled as polynomials. For instance, take the case of a multiplicative process with $D^{(1)} = d_{11}x$ and $D^{(2)} = d_{20} + d_{21}x + d_{22}x^2$. Then 5 parameters (σ , d_{11} , d_{20} , d_{21} , and d_{22}) have to be derived by minimizing the distance between the four measured functions, $\hat{\gamma}_n(y)$, $\hat{m}^{(n)}(y)$, and the solutions given by Eqs. (7) and (8), i.e.,

$$\min \left\{ \sum_i [\hat{\gamma}_1(y_i) - \gamma_1(y_i)]^2 + [\hat{\gamma}_2(y_i) - \gamma_2(y_i)]^2 + [\hat{m}^{(1)}(y_i) - m^{(1)}(y_i)]^2 + [\hat{m}^{(2)}(y_i) - m^{(2)}(y_i)]^2 \right\}. \quad (10)$$

For an Ornstein-Uhlenbeck process and for pure noise, Eqs. (7) and (8) can even be solved analytically. For the latter case

[i.e., $y(t) = \sigma \zeta(t)$] the moments are given by

$$M^{(1)}(y, \tau) = -y, \quad M^{(2)}(y, \tau) = y^2 + \sigma^2. \quad (11)$$

This means that for pure noise the moments as a function of τ have vanishing slope but nonzero offset, while for an ideal process according to Eq. (1) the situation is reversed. Data from real processes will generally lead to $M^{(n)}$ values with nonzero offsets and nonzero slopes.

Next we consider an *Ornstein-Uhlenbeck process* (given by $D^{(1)} = -\alpha x$ and $D^{(2)} = \beta$) to which measurement noise is added. In this case $p(x)$ is a Gaussian distribution with zero mean and variance $s^2 = \beta/(2\alpha)$. The offsets

$$\gamma_1(y) = -\frac{\sigma^2}{\lambda^2}y, \quad \gamma_2(y) = \sigma^2 + \frac{\sigma^2 s^2}{\lambda^2} + \frac{\sigma^4}{\lambda^4}y^2 \quad (12)$$

and the $m^{(n)}$ values

$$\begin{aligned} m^{(1)}(y, \tau) &= \tau[-\alpha y - \alpha \gamma_1(y)], \\ m^{(2)}(y, \tau) &= \tau(\beta - 2\alpha\{\gamma_2(y) - \sigma^2\} + y\gamma_1(y)) \end{aligned} \quad (13)$$

can be derived exactly from Eqs. (7) and (8) (see [20] for details). Note that $\lambda^2 := s^2 + \sigma^2$ has been used and that γ_2 approaches $2\sigma^2$ in the small σ limit in accordance with Eq. (6).

From Eq. (13) it is seen that the slopes of the moments, $m^{(n)}/\tau$, are affected by γ_n . Thus simply taking the slope as an estimate of the coefficients—as suggested by Eq. (6)—is not appropriate in the presence of larger measurement noise, even for rather simple cases such as the Ornstein-Uhlenbeck process. Estimates according to Eq. (3) will be increasingly in error as $\gamma_n(y)$ dominates the conditional moments $M^{(n)}(y, \tau)$ for large σ .

For illustration let us consider a numerical realization of an Ornstein-Uhlenbeck process with $\alpha = \beta = 1$. Figure 2(a) shows the pure process ($\sigma = 0$), and Fig. 2(b) shows the process with strong superimposed measurement noise ($\sigma = 1$), corresponding to a negative signal-to-noise ratio of approximately $S/N = 20 \log_{10}(s/\sigma) = -3 \text{ dB}$. Without measurement noise the coefficients are directly obtained either from the slopes of the conditional moments or by using Eq. (3) as shown in previous works. The reconstructed drift coefficient of Fig. 2(e) is found to be $\alpha = 1 \pm 0.01$, and analogously $D^{(2)} = \beta = 1 \pm 0.01$ is well reconstructed [see Fig. 2(f)].

In presence of measurement noise the moments $M^{(n)}(y, \tau)$ are still linear functions of τ but, in agreement with Eqs. (7) and (8), exhibit an additional offset term as can be seen in Fig. 2(d). From the measured $M^{(n)}(y, \tau)$ the terms $m^{(n)}(y, \tau)$ and $\gamma_n(y)$ are obtained as follows:

$$\begin{aligned} m^{(1)}/\tau &= -(0.34 \pm 0.02)y, \\ m^{(2)}/\tau &= (0.33 \pm 0.02) + (0.42 \pm 0.01)y^2, \\ \gamma_1 &= (0.667 \pm 0.001)y, \\ \gamma_2 &= (1.33 \pm 0.02) + (0.445 \pm 0.002)y^2. \end{aligned} \quad (14)$$

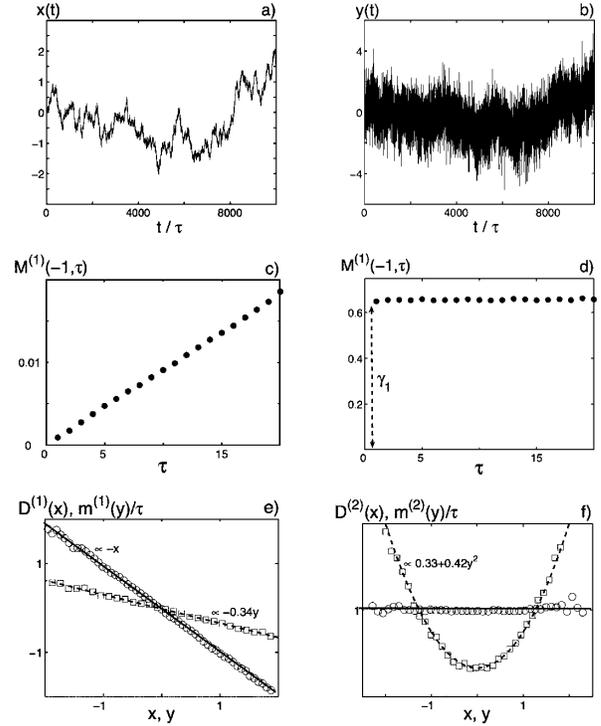


FIG. 2. (a) and (b) show an excerpt of an Ornstein-Uhlenbeck process ($\tau = 10^{-3}$, $\alpha = 1$, and $\beta = 1$) with $\sigma = 0$ in (a) and $\sigma = 1$ in (b). In (c) and (d) the derived $M^{(1)}$ are shown, where (c) refers to $x = -1$ and $\sigma = 0$, and (d) to $y = -1$ and $\sigma = 1$. In (e) and (f) the symbols represent the measured $D^{(n)}$, fitted by the solid lines ($\sigma = 0$), and $m^{(n)}/\tau$, fitted by the dashed lines ($\sigma = 1$), according to Eq. (13).

Using Eq. (12) we obtain the drift coefficient $D^{(1)}(x) = -\alpha x$ with $\alpha = 1.01 \pm 0.02$ in good agreement with the expected value of $\alpha = 1$.

To reconstruct the diffusion term $D^{(2)} = \beta$ the knowledge of $\gamma_1(y)$ and $\gamma_2(y)$ even at a single position y is sufficient when α is known. For instance, for $y = -1$ the measured offsets are $\gamma_1 = 0.65$ and $\gamma_2 = 1.74$, leading to $s = 0.73$ and $\sigma = 0.99$. With $s = \sqrt{\beta/(2\alpha)}$ it follows that $\beta = 1.01 \pm 0.04$. To improve the accuracy of the parameters a least squares algorithm is applied.

Based on the foregoing discussion of an Ornstein-Uhlenbeck process, two important new results can already be given. First, we see that the estimation of the magnitude of measurement noise by the simple approach according to Eqs. (5) and (6) is misleading. For instance, from the offset $\gamma_2(y = 0) = 2\sigma^2 \approx 1.34 \pm 0.02$ a σ^2 value of 0.67, which is about 67% of the real value, would be extracted. This underestimation has already been reported in [8] and can now be understood quantitatively. Second, if the small- τ -estimate $m^{(2)}/\tau$ is taken as an approximation of $D^{(2)}$, as it is commonly done, then an artificial quadratic diffusion term [see Fig. 2(f)] is obtained, masquerading as multiplicative noise or a bad temporal resolution [12].

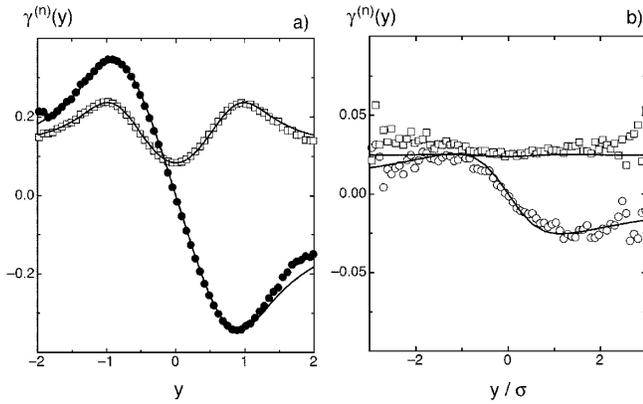


FIG. 3. Symbols represent measured offsets and solid lines the fits according to Eqs. (7) and (8). In both plots circles refer to γ_1 and squares to γ_2 . (a) Iterated multiplicative process with $D^{(1)} = -x$, $D^{(2)} = 0.1 + 0.5x^2$, and $\sigma = 0.25$. (b) Normalized financial increment time series.

Finally, we consider a *general non-Ornstein-Uhlenbeck process* where the coefficients, $D^{(n)}$, are implicitly given only by $m^{(n)}$ and γ_n according to Eqs. (7)–(9). As an example, let us consider a process with multiplicative noise ($D^{(2)} = b + cx^2$) and linear drift ($D^{(1)} = -ax$) which is observed in various systems ranging from finance to turbulence (cf. [15,21,22]). Here we take $a = 1$, $b = 0.1$, and $c = 0.5$ and iterate numerically. The measurement noise amplitude is $\sigma = 0.25$, which corresponds to a signal-to-noise ratio of $S/N = 0$. Figure 3(a) shows the observed offsets, γ_n , together with the (numerical) solutions according to Eqs. (7) and (8) for the iterated multiplicative process. Figure 3(b) refers to the financial increment time series for which the reconstructed coefficients are found to be $D^{(1)}(x) = -0.0012x$ and $D^{(2)}(x) = 0.0014 + 0.001x^2$ and the measurement noise amplitude is $\sigma = 0.11$. This result shows that multiplicative dynamical noise causes intermittent heavy-tailed volatility statistics in financial data as was proposed, and this is not due to a spurious effect of measurement noise, as for the case of Fig. 2(f)].

From the examples of Fig. 3 we see again that the understanding of the influence of measurement noise on the conditional moments, i.e., on γ_n and $m^{(n)}/\tau$, is the key to achieving a proper reconstruction of the underlying dynamical process (including the contribution of dynamical noise and measurement noise) from pure data analysis. Naively applying the definition according to Eq. (3) will no longer be appropriate as soon as measurement noise is present.

To conclude, we have shown (for numerical as well as real world data) that adding measurement noise to signals generated from a Langevin process leads to a fundamental modification of the data analysis via the conditional moments. A general equation describing this modification has been presented and for the class of Ornstein-Uhlenbeck

processes analytical results are given. This makes it possible to extract the strength of measurement noise, σ , the standard deviation of the underlying process, s , as well as the drift and diffusion coefficients, $D^{(1)}$ and $D^{(2)}$, rather precisely even in presence of very strong measurement noise. It is noteworthy that the evaluation of the process' coefficients is solely based on analyzing the conditional moments, which are directly obtained from the time series without any need of premanipulating (e.g., filtering, modeling) the data.

We wish to acknowledge stimulating discussions with M. Siefert, St. Barth, M. Hölling, A. Nawroth, and S.G. Bradley.

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- [1] H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis* (Cambridge University Press, Cambridge, England, 1997).
- [2] R. Friedrich and J. Peinke, Phys. Rev. Lett. **78**, 863 (1997).
- [3] S. Siebert, R. Friedrich, and J. Peinke, Phys. Lett. A **243**, 275 (1998).
- [4] G. Ryskin, Phys. Rev. E **56**, 5123 (1997).
- [5] R. Friedrich *et al.*, Phys. Lett. A **271**, 217 (2000).
- [6] J. Gradisek, S. Siebert, R. Friedrich, and I. Grabec, Phys. Rev. E **62**, 3146 (2000).
- [7] S. Kriso, J. Peinke, R. Friedrich, and P. Wagner, Phys. Lett. A **299**, 287 (2002).
- [8] M. Siefert, A. Kittel, R. Friedrich, and J. Peinke, Europhys. Lett. **61**, 466 (2003).
- [9] M. Siefert and J. Peinke, Int. J. Bifurcation Chaos Appl. Sci. Eng. **14**, 2005 (2004).
- [10] T. Kuusela, Phys. Rev. E **69**, 031916 (2004).
- [11] M. Ragwitz and H. Kantz, Phys. Rev. Lett. **87**, 254501 (2001).
- [12] P. Sura and J. Barsugli, Phys. Lett. A **305**, 304 (2002).
- [13] R. Friedrich, C. Renner, M. Siefert, and J. Peinke, Phys. Rev. Lett. **89**, 149401 (2002).
- [14] P.E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer, New York, 1999).
- [15] C. Renner, J. Peinke, and R. Friedrich, J. Fluid Mech. **433**, 383 (2001).
- [16] J.P.M. Heald and J. Stark, Phys. Rev. Lett. **84**, 2366 (2000).
- [17] E.J. Kostelich and T. Schreiber, Phys. Rev. E **48**, 1752 (1993).
- [18] C. Renner, J. Peinke, and R. Friedrich, Physica (Amsterdam) **298A**, 499 (2001).
- [19] H. Risken, *The Fokker-Planck Equation* (Springer, New York, 1989).
- [20] F. Böttcher, Ph.D. thesis, Oldenburg, 2005.
- [21] M. Waechter, F. Riess, T. Schimmel, U. Wendt, and J. Peinke, Eur. Phys. J. B **41**, 259 (2004).
- [22] M. Tutkun and L. Mydlarski, New J. Phys. **6**, 49 (2004).
- [23] P.G. Lind, A. Mora, J.A.C. Gallas, and M. Haase, Int. J. Bifurcation Chaos Appl. Sci. Eng. (to be published).