2.
A Model for Pattern Formation

(references refer to the list of publications given in chapter 12)

2.1 General remarks

An important aspect of the described work is the claim that many of the experimentally observed patterns in electrical transport systems of the chapters Electrical Networks: Experiment and Theory, DC Gas-Discharge Systems: Experiment, AC Gas-Discharge Systems: Experiment and Semiconductors: Experiment can qualitatively be described in terms of a 3-component reaction-diffusion equation. This equation serves as a kind of "normal form" reflecting the experimentally observed universal behaviour of self-organized pattern in a relatively large class of nonlinear systems. Since frequently we will refer to this equation, in what follows we give some insight of how the 3-component reaction-diffusion equation can be motivated physically.

A schematic representation of the model for the envisaged electrical transport system is given in fig. 2. 1. The corresponding stationary local (current density)-(voltage) characteristics of the linear (or monotonous)

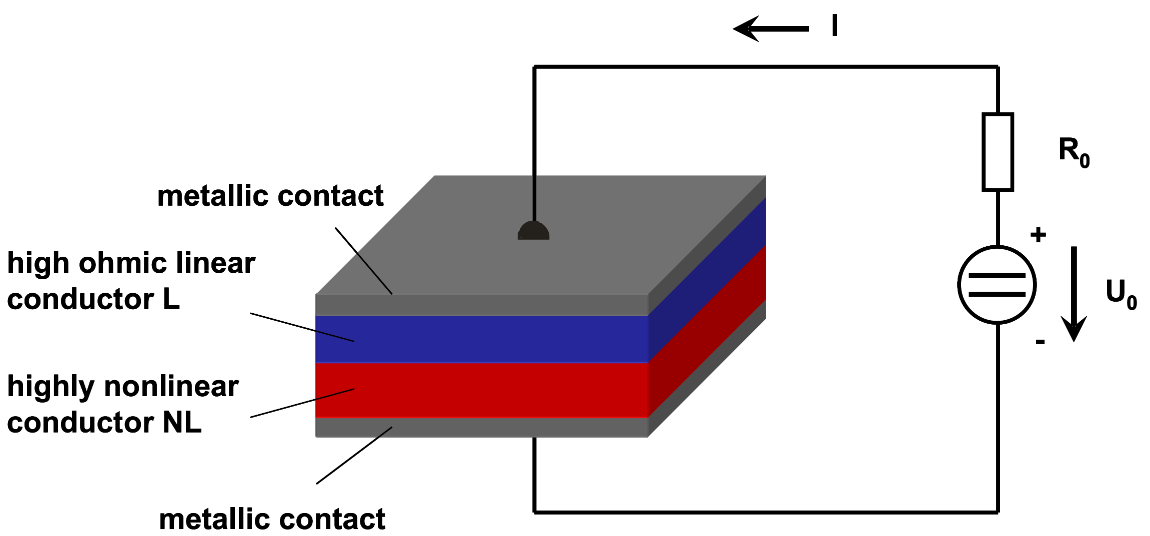

Fig. 2.1

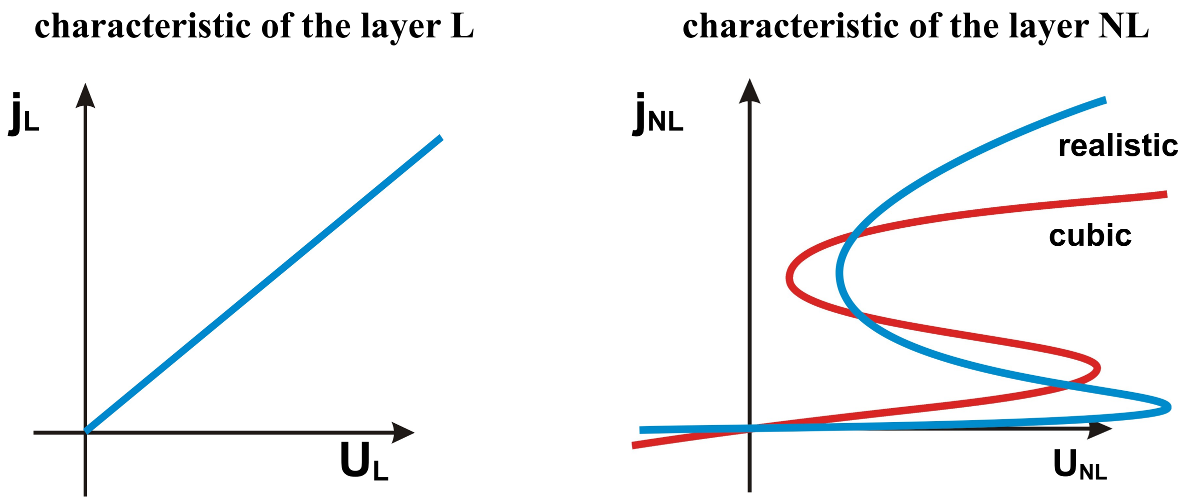

Fig. 2.2

layer L and the strongly nonlinear layer NL can bee seen in fig. 2. 2. The form of the (current density)-(voltage) characteristics of NL is also referred to as S shaped. In theoretical discussions, it is convenient to approximate the real characteristic of NL e.g. by an inverted cubic polynomial. Patterns of interest show up in the plane parallel to the metallic contacts e.g. in the current density of NL or in the voltage drop at L. Devices with lateral extension being small with respect to the characteristic length of the observed pattern in one direction and large in the other one, can be considered as quasi 1-dimension. If the extension is large in both directions they are referred to as quasi-2-dimensional.

In particular, bright (dark) localize structures (LSs) are generated in the plane parallel to the metallic electrodes by solitary high (low) current filaments on a stationary homogeneous low (high) current background that intersect the plane vertically. In this way a 3-(2-)dimensional filament generates a LS in 2-(1-)dimensional subspace. We also note that in the absence of global coupling ($R_0 = 0$). In real physical systems any number of LSs in the form of DSs may exist, provided the domain is large enough.

Most experiments are performed with a series resistor R_0 in order to protect the device under investigation or because a real voltage source is used. Usually this has the consequence that the background of the LSs depends on their number. Therefore in the strict sense of the definition given in the Introduction the presence of the global coupling does not allow for the existence of DSs. However in most of the investigated systems in particular bright filaments correspond to pronounced local breakdown with a background of which the current density is orders of magnitude smaller than the peak current density of the filaments, as can be seen from the realistic non-cubic (current density)-(voltage) characteristic of fig. 2.2. Therefore in real systems in general the background change due to an additional filament is negligible and a given stable solitary LS can be looked upon as a DS to very good approximation.

With respect to the formation of lateral self-organized patterns in the device figs. 2.1 a model equation has been derived by starting from the discrete electronic equivalent circuit fig. 3.1 of the chapter Electrical Networks: Experiment and Theory [for references see section 2.2]. This equation reads as

$$\partial_t u = d_u^2 \Delta u + f(u) - \kappa_3 v - \kappa_4 w + \kappa_1 - \kappa_2 \int_\Omega u \, d\Omega + \mu (\nabla u)(\nabla u), \qquad (1)$$

$$\partial_t v = d_v^2 \Delta v + u - v - \kappa_1' + \kappa_2' \int_\Omega v \, d\Omega, \qquad (2)$$

$$\partial_t w = d_v^2 \Delta w + u - w, \qquad (3)$$

$$u = u(x, y; t); \quad v = v(x, y; t); \quad w = w(x, y; t)$$

with appropriate boundary conditions usually being of Neumann zero flux or periodic type. Thereby most of the symbols represent normalized quantities with the following meaning:

x, y	space coordinates in the plane of the layers
t	normalize time
u	current density in the nonlinear layer (activator)
v	voltage drop at the high ohmic layer (1. inhibitor)
w	additional dependent variable (e.g. surface charge) which allows for stable propagation of isolated DSs also in $R^{2,3}$
$f(u)$	related to the local (current density)-(voltage) characteristic of NL of fig. 2.2, usually approximated by $\lambda u - u^3$
$\lambda > 0$	autocatalytic constant
$\tau \geq 0$	overall voltage (dielectric) relaxation time normalized to the overall current density relaxation time
$\Theta \geq 0$	overall relaxation times of w normalized to the overall current density relaxation time
$d_u, d_v, d_w \geq 0$	diffusion lengths of u, v, w
κ_1, κ_1'	driving voltage (ether $\kappa_1, \kappa_2 = 0$ or $\kappa_1', \kappa_2' = 0$)
$\kappa_2, \kappa_2' \geq 0$	global couplings (ether $\kappa_1, \kappa_2 = 0$ or $\kappa_1', \kappa_2' = 0$)
$\kappa_3, \kappa_4 \geq 0$	strength of action of inhibitors v, w
$\mu \geq 0$	occasionally introduced to generate DS splitting while increasing κ_2 or κ_2'

The part of the above field equation being written in black (equs. (1,2) with $\kappa_4, \mu = 0$) comes from the discrete electronic equivalent circuit fig. 3.1 of the chapter Electrical Networks: Experiment and Theory by applying Kirchhoff´s rules and performing the transition to the continuum. We have $\kappa'_1, \kappa'_2 = 0$ if $C_{NL} = 0$ and $\kappa_1, \kappa_2 = 0$ if $C_U = 0$. The red terms in the field equation (the term with κ_4 in equ. (1) and equ. (3)) are added in particular in order to allow for stable travelling DSs in more than one spatial dimension. In gas-discharge systems these terms can qualitatively be justified by the influence of surface charges. Occasionally, the blue term (with $\mu \neq 0$ in equ. (1)) is used to support the formation of LSs due to splitting of LSs when increasing the drivers κ_1 or κ'_1.

Essentially *f(u)* is the inversion of the local (current density)-(voltage) characteristic. Usually *f(u)* is a cubic polynomial and results from the characteristic of NL in fig. 2.2. *f(u)* consists of three monotonic braches that decrease for small and large *u* and increase for intermediated *u*. Because of mathematical convenience, in some theoretical investigations *f(u)* has been chosen to be piece-wise linear. Also occasionally *f(u)* has been taken from the experiment in the form of the measured (voltage)-(current density) characteristic. In other cases the cubic form has been modified in order to get a better qualitative agreement with the experimental characteristic.

A graphic description of how self-organized solitary filaments can be stabilized in the device fig. 2.1 is given in [Pu127; Pu131].

In the chapter Reaction-Diffusion Equations we will refer to the equations (1 - 3) as the generalized FitzHugh-Nagumo (FHN) equation and we will discuss analytical and numerical solutions.

2.Listing of main results

Pu002: **Berkemeier, Dirksmeyer, Klempt, Purwins (1986)**
proof that analogue electronic circuits can be described quantitatively by r-d equations with respect to the formation of self-organized patterns - see also: Electrical Networks: Experiment and Theory, Reaction-Diffusion Equations

Pu003: **Radehaus, Kardell, Baumann, Jäger, Purwins (1987)**
electrodynamic derivation of the 2-component reaction-diffusion equation including global coupling - claim that many planar semiconductor devices can be considered as r-d systems with respect to lateral patter formation - claim that the high ohmic resistively layer with monotonic current voltage characteristic supports spatially inhomogeneous patterns - stressing the importance of the activator inhibitor principle with local activation and lateral inhibition - see also: Semiconductors: Experiment, Reaction-Diffusion Equations

Pu005: **Radehaus, Dirksmeyer, Willebrand, Purwins (1987)**
first claim, that these planar dc gas-discharge devices can be considered as r-d systems with respect to lateral patter formation - see also: DC Gas-Discharge Systems: Experiment, Reaction-Diffusion Equations

Pu007: **Purwins, Klempt, Berkemeier (1987)**
2-component reaction-diffusion system as the basic model for the universal behaviour of a relatively large class biological, chemical and physical systems - see also: Electrical Networks: Experiment and Theory, DC Gas-Discharge Systems: Experiment, Semiconductors: Theory, Reaction-Diffusion Equations

Pu008: **Purwins, Radehaus C. Radehaus, and J. Berkemeier (1987)**
see [Pu007] - detailed derivation of the 2-component reaction-diffusion equation for the electronic network and the corresponding generalized 2-layer model - extended discussion of the basic reaction-diffusion model - see also: Electrical Networks: Experiment and Theory, DC Gas-Discharge Systems: Experiment, Reaction-Diffusion Equations

Pu009: **Purwins, Radehaus (1987)**
the quantitative theoretical description of 1-dimensional electrical networks without any fitting in [R. Schmeling, Thesis, University of Münster (1994); Purwins, Amiranshvili, Bödeker (2009)], by solving the corresponding 2-component reaction equation - see also: Electrical Networks: Experiment and Theory, Reaction-Diffusion Equations

Pu011: **Purwins, Radehaus, Dirksmeyer, Dohmen, Schmeling, Willebrand (1989)**
focus on the importance of the activator-inhibitor principle for the investigated reaction-diffusion equations and the formation of localized structures in physical systems - splitting of filamente while increasing the external driver is achieved by choosing $\mu \neq 0$ - see also: Electrical Networks: Experiment and Theory, DC Gas-Discharge Systems: Experiment, Reaction-Diffusion Equations

Pu013: **Dirksmeyer, Schmeling, Berkemeier, Purwins (1990)**
discussion of the electrical network with respect to the introduction of "current diffusion" - first quantitative description of the experimentally observed stat dark (invers) DSs for 1-dimensional electrical networks without any fitting in [R. Schmeling, Thesis, University of Münster (1994)], by solving the corresponding 2-component reaction equation - see remarks [Pu009] - see also: Electrical Networks: Experiment and Theory, Reaction-Diffusion Equations

Pu053: **Schenk, Or-Guil , Bode, Purwins (1997)**
for the first time a 3-component reaction-diffusion system is introduced in order to realize an arbitrary number of stable isolated travelling DSs or ensembles of DSs with mutual interaction in $R^{1,2,3}$ - individual stationary and travelling DSs as well as scattering,annihilation and generation are observed numerically - see also: Reaction-Diffusion Equations

Pu060: **Or-Guil, Bode, Schenk, Purwins (1998)**
analytical investigation of the 3-component reaction diffusion equation in $R^{1,2,3}$ near to the bifurcation from of stationary DS to a travelling one using multiple scale perturbation theory - stability analysis for stationary and travelling DSs - derivation of the normal form of the bifurcation containing the centre coordinate of the involved DS as dependent variable; this equation represents also a dynamical equation for a single DS in the particle picture - analytical expression for the speed of propagation - an essential ingredient of the presented theory is the fact that in the treated bifurcation scenario a discrete real eigenvalue crosses the imaginary axis; together with the symmetry modes this leads to the generation of generalized eigenvectors at the bifurcation point which in turn are responsible for the onset of propagation - see also: Reaction-Diffusion Equations

Pu118: **Purwins, Bödeker, Liehr (2005)**
review of previous works on DSs in reaction diffusion systems - mechanisms of pattern formation in 2-k systems: Turing patterns, ocalized structures, activator-inhibitor principle - derivation of the 2- and 3-component reaction-diffusion equation by starting from an electronic equivalent circuit - numerical solutions of the 3-component reaction-diffusion equation for isolated DS: DSs with non-oscillatory and oscillatory tails - numerical solutions of the 3-component reaction-diffusion equation for two interacting DSs: scattering, formation of rotating molecules, annihilation, generation - anal treatment of the 3-component reaction-

diffusion equation: normal form for the drift bifurcation, reduction of the field equation to a particle equation for a single particle and for mutually interacting particles, calculation of the particle interaction law - num, an: rotating molecule in R^3 propagating along the axis of rotation - experimental results on DSs in dc gas-discharge: isolated DSs with non-oscillatory and oscillatory tails, drift bifurcation, interaction law, generation and annihilation - see also: Electrical Networks: Experiment and Theory, DC Gas-Discharge Systems: Experiment , Reaction-Diffusion Equations