The Gouy phase shift, the average phase lag of Fourier components of Hermite-Gaussian modes and their application to resonance conditions in optical cavities

T. Ackemann, [a,c] W. Grosse-Nobis [b] and G. L. Lippi [a]

[a] *Institut Non Linéaire de Nice, UMR 6618 CNRS, 1361 Route des Lucioles, F-06560, Valbonne, France.*

[b] *Institut für Angewandte Physik, Westfälische Wilhelms-Universität Münster, Corrensstraße 2/4, D-48149 Münster, Germany*

[c] *Permanent address: Institut für Angewandte Physik, Westfälische Wilhelms-Universität Münster, E-mail: t.ackemann@uni-muenster.de*

Abstract

An interpretation of the Gouy phase shift is proposed which relates the finite Gouy phase reduction of a Gaussian mode to the averaged phase retardation of the tilted waves in its Fourier spectrum compared to an on-axis plane wave. This provides a unified approach to resonance conditions in optical cavities with plane and curved mirrors. The possible relationship to pattern selection in nonlinear cavities is discussed.

Keywords: resonance condition in optical cavities, tilted waves, Gouy phase-shift, optical pattern formation
PACS-numbers: 42.65.Da, 42.65.Sf, 42.65.Jf

1 Introduction

It is very well known that Laguerre- or Hermite-Gaussian modes acquire a shift with respect to an on-axis plane wave if propagating along the optical axis (e.g. [1,2]). This extra phase is usually called Gouy phase shift. It is a special case of the phase anomaly experienced by waves going through a focus ([3], cf. also [4] and Refs. therein). Recently a direct manifestation (sign reversal of the electrical field amplitude after a focus) was observed for terahertz pulses [5]. In the optical region of the spectrum it usually becomes only apparent indirectly by interference effects. The most important and striking example is probably the determination of different resonance frequencies for different Gaussian modes, which are the eigenfunctions of optical resonators

Preprint submitted to Elsevier Science *13 October 2000*

with curved mirrors [1,2]. On the other hand, the resonance properties of plano-planar resonators are typically discussed in terms of plane waves, which might travel on-axis or at an angle with respect to the optical axis (e.g. [6]). The latter, so called *tilted waves*, occur if the optical frequency is blue shifted with respect to a cavity resonance for on-axis radiation since the resonance condition is reestablished in this way (see below Sec. 2.2).

This brief presentation shows two different concepts for selecting the resonance condition for different modes, which appear to be quite incompatible – at least at first sight: the Gouy phase for Gaussian modes and the tilt mechanism for plane waves. Physical reasoning, however, strongly suggests that there should be a smooth transition from a plano-planar resonator to one with a slight mirror curvature. This is exemplified e.g. by the fact that the size of the beam waist of the Gaussian eigenmodes goes to infinity (i.e. converges to a plane wave) if one approaches the plano-planar case. Furthermore the frequency separation between subsequent transverse mode orders diminishes, also approaching the limiting case of a continuous spectrum in the plano-planar case.

We intend to shine some light on this transition by showing that the Gouy phase shift can be obtained by averaging the phase lags of the plane wave components present in the Fourier spectrum of a Gaussian beam. After reviewing some properties of tilted waves and Gaussian modes, we first analyze the Fourier spectrum in the beam waist of a fundamental Gaussian beam. We show that the phase lag which is experienced by the averaged Fourier component exactly coincides with the Gouy shift experienced by the fundamental Gaussian wave going through the beam waist. After a brief consideration of the properties of the intensity moments, the weighted average is done by integrating over the field amplitude since the dephasing acts on the field variable. The analysis is then extended to higher order modes. This treatment yields the correct order of magnitude of the Gouy phase shift as long as one stays within the collimation region (Rayleigh length). Finally, we show that one recovers the finite expression for the phase lag during propagation from minus to plus infinity, if the relative dephasing between modes with different tilt angles which develops during propagation are taken into account. We are aware of the fact that several other interpretations of the Gouy phase shift have been given, e.g. based on the length of propagation paths in real space [7], or on Berry's phase [8,9], or even on a geometrical quantum effect [10]. Although the approach used in [10] is related to the one we are proposing, we think that ours is best suited to understanding the connection between Gaussian modes and tilted waves.

Finally, since different modes correspond to different spatial structures or patterns in the transverse plane of the beam, we discuss the implications of our findings on pattern selection in nonlinear resonators, i.e. resonators which contain a nonlinear optical material. The spontaneous appearance of spatial structures due to self-organization phenomena is by now a well-established fact (see e.g. [11,12] for recent reviews). However, patterns in plano-planar cavities (e.g. [13]) and curved mirror cavities (e.g. [14–16]) have been discussed on very different grounds with only a few attempts at establishing a link [17,18]. Even these, however, do not provide an intuitive understanding, probably due to the fact that the mirror curvature mathematically introduces a singular perturbation [18]. Therefore we believe that our considerations might also be helpful in clarifying this issue.

2 Base functions and resonance conditions in optical cavities

2.1 Fourier modes and Hermite-Gaussian functions

In the following, we consider an aligned cavity with infinite extent of the mirrors in the transverse dimensions. For simplicity, we only consider two-mirror Fabry-Perot cavities (or a ring cavity with one curved mirror or lens, and an appropriate number of plane mirrors for closure). This assumption ensures that the beam can be characterized by a single set of beam parameters in the whole resonator. Thus, we avoid the technical difficulties present in more complicated geometries since they do not provide any further insight.

In a plane-plane resonator the eigenmodes are plane waves or Fourier modes. Adopting the convention where $\exp\left(i(\omega t - k_0 z)\right)$ is the carrier wave (ω angular frequency, k_0 wavevector), the transverse part of the optical field in one spatial dimension is given by

$$E(x) = e^{-iqx}, \tag{1}$$

where q denotes the transverse part of the wavevector.

In a resonator with curved mirrors the eigenfunctions are Gauss-Hermite modes. In the beam waist they are given by the real product of a Hermite polynomial H_m and a Gaussian envelope function

$$E_m(x) \sim H_m\Big(\sqrt{2}\frac{x}{w}\Big)\, e^{-(\frac{x}{w})^2}, \tag{2}$$

where w is the beam radius at the beam waist ($1/e^2$-point of the intensity envelope).

2.2 Plano-planar cavity

In a plano-planar cavity, tilted waves are excited when the optical frequency (wavevector) is larger than the resonant frequency (wavevector) for on-axis radiation. The resonance condition is reestablished for a wave emitted off-axis, since the projection of the wavevector on the optical axis has a smaller length (cf. Fig. 1a).

Looking at the problem in real space, tilted waves occur if the cavity is longer than an integer multiple of half wavelengths (full wavelengths in a ring cavity). The resonance condition is reestablished for an inclined wave since the effective wavelength λ_{eff} in the longitudinal direction, i.e. the distance between points of equal phase, increases (cf. Fig. 1b).[1] This is well known from microwave cavities [19].

After a propagation distance δz the phase lag between a tilted wave with transverse wavevector q and the on-axis wave is (in the paraxial approximation)

$$\delta\phi = -(k_{eff} - k_0)\delta z, \tag{3}$$

$$= \frac{q^2}{2k_0}\delta z. \tag{4}$$

Hence, in a Fabry-Perot cavity $\delta z = 2L$ (L resonator length), the mode with the wavevector of magnitude q will be shifted by $\Phi = q^2 L/k_0$ from the longitudinal resonance $q = 0$. This results in a continuous spectrum of Fourier modes on the high frequency side of the longitudinal mode resonance (Fig. 2).

2.3 Resonators with curved mirrors

The equations describing the propagation of Gaussian beams are excellently discussed in [1,2]. Thus, we review only the essential results confining ourselves to the case of one transverse dimension, for simplicity.

The Gouy phase shift acquired by the Hermite-Gaussian mode of order m

[1] In most real devices there are also resonances on the low frequency side of a longitudinal resonance due to very high order modes belonging to the previous longitudinal order. However, they are included in our treatment since they occur on the high frequency side of their longitudinal resonance. For simplicity, we restrict ourselves to a single-longitudinal mode situation.

traveling from position z_1 to position z_2 is given by

$$\Delta\phi_m(z_1 \to z_2) = (m + \frac{1}{2})\phi_{Gouy}(z_1 \to z_2), \tag{5}$$

with

$$\phi_{Gouy}(z_1 \to z_2) = \arctan(\frac{z_2}{z_r}) - \arctan(\frac{z_1}{z_r}). \tag{6}$$

z_1 and z_2 are measured with respect to the location of the beam waist. $z_r = \pi w^2/\lambda$ is the Rayleigh length of the beam, w the beam radius at the beam waist ($1/e$ point of the field amplitude). In particular, this means that

$$\Delta\phi_m = (2m + 1)\,\Delta\phi_0, \tag{7}$$

where $\Delta\phi_0$ denotes the value of the Gouy phase for the fundamental mode.

The resonances of the Gauss-Hermite modes are shifted from those of a plane on-axis carrier wave by an amount

$$\Phi = 2\,(m + \frac{1}{2})\phi_{Gouy}(z_1 \to z_2). \tag{8}$$

In a Fabry-Perot cavity z_1 and z_2 denote the longitudinal positions of the left and right resonator mirror with respect to the beam waist. In a ring cavity the factor 2 is missing and z_1 and z_2 denote the position of one mirror (chosen as reference) with respect to the beam waist at the beginning and the end of the round trip, respectively.

The arguments presented in this article are valid for arbitrary stable resonators, although they are expected to be most useful in the close-to-planar ones. In this case, the mode spectrum consists of a series of higher order Hermite-Gaussian modes on the high frequency side of the longitudinal resonance (Fig. 2). Note the similarity to the plano-planar case. However, due to the additional constraints imposed by the boundary conditions, the spectrum is discrete.

Since we will focus first on the situation in the vicinity of the beam waist, we calculate now the Gouy phase shift acquired by the Gaussian-Hermite mode of order m in the vicinity of the beam waist, i.e. from a position $-\delta z/2$ to $\delta z/2$:

$$\delta\phi_m = (m + \frac{1}{2})\left(\arctan(\frac{\delta z}{2z_r}) - \arctan(-\frac{\delta z}{2z_r})\right), \tag{9}$$

which, for small δz, reduces to:

$$\delta\phi_0 = (m + \frac{1}{2})\,\frac{1}{z_r}\,\delta z. \tag{10}$$

This is the value to which we have to compare the result that will be obtained by averaging over the contributions of the different spectral components in Fourier space.

3 The effective phase lag of Hermite-Gaussian modes

3.1 Fourier spectrum of a Hermite-Gaussian mode in the beam waist and intensity moments

We first consider the situation in the vicinity of the beam waist of a fundamental Gaussian beam since, due to the plane wavefront, one can expect this situation to resemble the plane wave case most closely. Up to scaling factors, the field is given by

$$E_m(x) = H_m(x)\, e^{-(\frac{x}{w})^2}, \tag{11}$$

where H_m denotes the Hermite polynomial of order m. The Fourier spectrum is given by (again up to scaling factors):

$$\tilde{E_m}(q) \sim H_m(\frac{w}{\sqrt{2}}q)\, e^{-(\frac{wq}{2})^2}. \tag{12}$$

Each of the different Fourier components would experience a phase lag given by Eq. (4). We average over the transverse wavevectors by calculating

$$\bar{q_m^2} = \frac{1}{N}\left(\int_{-\infty}^{\infty} dq\, q^2 \cdot \left[H_m(\frac{w}{\sqrt{2}}q)\, e^{-(\frac{wq}{2})^2}\right]^2\right), \tag{13}$$

where $N = \int_{-\infty}^{\infty} dq[H_m(\frac{w}{\sqrt{2}}q)\ \exp\left(-(wq/2)^2\right)]^2$ provides the normalization. The intensity moments of Hermite-Gaussian modes are tabulated (e.g. in [20]), and provide:

$$\bar{q_m^2} = \frac{2m+1}{w^2}. \tag{14}$$

According to Eq. (4) this amounts to a phase shift of

$$\delta\bar{\phi}_m = \frac{1}{2}\,(m + \frac{1}{2})\,\frac{1}{z_r}\,\delta z. \tag{15}$$

Comparing this to the Gouy phase shift acquired by the Gaussian beam in the vicinity of the beam waist given in Eq. (10), we find that the averaged phase shift is exactly half the value of the Gouy phase shift. This is true for *all* mode orders. This already indicates that there is a close connection between the Gouy phase and the average phase lag of the tilted wave spectral components, though the agreement is not quantitative.

3.2 Field moments and the Gouy phase of the fundamental Gaussian mode

In the preceding paragraph an expression for an averaged phase was obtained which nearly, but not exactly, agreed with the Gouy phase. Since, however, the beam shaping with propagation is essentially an interference phenomenon, it should be more appropriate to perform the weighting by field amplitudes rather than intensity (energy). This choice can be also motivated in a more formal way as follows: Any field $E(x,z)$ can be considered to result from a superposition of its Fourier components $\tilde{E}(q,z)$, $E(x,z) = \int_{-\infty}^{\infty} dq\, \tilde{E}(q,z)\exp(iqx)$. Then the field at the position $E(x, z+\delta z)$ is given by the superposition of the Fourier components $\exp(iq^2\delta z/2k_0)\tilde{E}(q,z)$. The Gouy phase is the additional phase acquired on axis ($x = 0$, for points off-axis there are the additional contributions due to curvature). We are looking for an effective $< q^2 >$ which captures this phase shift. Therefore one has to introduce the ansatz:

$$\begin{aligned} E(0, z+\delta z) &= e^{i<q^2>\delta z/2k_0} \cdot E(0,z), \\ \left(\int_{-\infty}^{\infty} dq\, e^{iq^2\delta z/2k_0} \cdot \tilde{E}(q,z)\right) &= e^{i<q^2>\delta z/2k_0} \cdot \left(\int_{-\infty}^{\infty} dq\, \tilde{E}(q,z)\right). \end{aligned} \tag{16}$$

For paraxial beams (small q) and small propagation distances δz close to the beam waist, one can expand the exponentials in first order and, after subtracting $E(0,z)$ and removing common prefactors one arrives at

$$\int_{-\infty}^{\infty} dq\, q^2 \cdot \tilde{E}(q,z) = < q^2 > \int_{-\infty}^{\infty} dq\, \tilde{E}(q,z). \tag{17}$$

This equation might be used to calculate an effective $< q^2 >$ for an arbitrary light field [2] but we will concentrate on the Hermite-Gaussian functions since they are the relevant eigenfunctions for curved-mirror resonators. Setting $\tilde{E}(q,0) \sim \exp\left(-(\frac{wq}{2})^2\right)$ for the fundamental mode one obtains

$$< q_0^2 >= \frac{2}{w^2}. \tag{18}$$

According to Eq. (4) this amounts to a phase shift of

$$< \delta\phi_0 >= \frac{1}{2z_r}\,\delta z, \tag{19}$$

which coincides with the value for the Gouy phase as given in Eq. (10) for $m = 0$. This means that close to the beam waist the Gouy phase shift per unit length agrees *exactly* with the averaged phase lag of the plane waves present in its Fourier spectrum, if the weighting is done accordingly to the interfering amplitudes.

Although we do not indent to discuss the 2D case in this paper, we emphasize that if the averaging is repeated in two transverse spatial dimensions one obtains twice the value obtained in 1D (Eq. (18)), as it should be, since the Gouy phase of a 2D Gaussian is twice the one of a 1D beam [2].

3.3 Higher order modes

A first approach to analyze the spectrum and the resulting phase lag of higher order modes of arbitrary indices is to consider the asymptotic representations for a Hermite polynomial H_m of order m [21]. For even m one obtains

$$H_m(u)e^{-\frac{u^2}{2}} \sim \left[\cos\left(\sqrt{2m+1}u\right) + O\Big(\frac{1}{\sqrt[4]{m/2}}\Big)\right], \tag{20}$$

for odd m

$$H_m(u)e^{-\frac{u^2}{2}} \sim \left[\sin\left(\sqrt{2m+1}u\right) + O\Big(\frac{1}{\sqrt[4]{(m-1)/2}}\Big)\right], \tag{21}$$

[2] As stated in the introduction any beam going through a focus experiences some phase anomaly.

where $u = \sqrt{2}x/w$. Taking the wavenumber of the trigonometric functions as the asymptotic estimate for q, one obtains

$$q^2_{asymp,m} = (2m+1)\frac{2}{w^2}. \tag{22}$$

This reproduces not only the scaling behaviour as a function of the mode order m (cf. Eq. (7)), but also the absolute value of the Gouy phase shift. The latter is surprising since we used an expression which is only valid in the asymptotic limit.

In order to investigate this behaviour further we calculated the moments

$$< q^2_m >= \frac{1}{N}\left(\int_0^\infty dq\, q^2 \cdot H_m(\frac{w}{\sqrt{2}}q)\, e^{-(\frac{wq}{2})^2}\right) \tag{23}$$

numerically up to the order $m = 100$. For the fundamental mode analyzed here and all even higher order modes the integration interval could also be chosen from $-\infty$ to $+\infty$. For odd modes, however, the second moment defined in this way would be zero without the average tilt being zero. Therefore we integrate from 0 until $+\infty$ (see also the remark below).

We find an exact agreement between the asymptotic expression and the expected values for the Gouy phase for the even modes (Fig. 3). For the odd modes the agreement is not exact, but the order of magnitude is correct and there is a slow oscillatory convergence towards the asymptotic limit, thus confirming the validity of the asymptotic result. The reason for the different behaviour of the odd and even modes is not exactly clear at the moment, but we think that it is due to the principal dichotomy of odd and even functions. In the translational symmetric system odd and even functions are not really different since there is no preferential axis: cosine and sine waves are converted into each other by a simple translation and the power spectrum is also the same. If the translational symmetry is broken by curved mirrors there is a preferred axis and odd and even modes are really distinguished (the Fourier spectra are also different). Furthermore, odd-order modes will be only excited in the cavity if the symmetry is broken by odd components in the input beam or if the input beam is tilted with respect to the axis [22]. Therefore it is our opinion that the even modes are the more 'proper' ones to compare with the translational invariant case. Since we can still qualitatively reproduce the behaviour of the odd modes correctly for finite m, and exactly in the asymptotic limit, we are quite confident that our approach captures the essential underlying physics for all modes.

For an interpretation of the previous result let us consider the standing wave pattern generated by the interference of two tilted waves with wavevectors

$$\pm q_{av,m} := \pm\sqrt{< q_m^2 >} = \pm\frac{2}{w}\sqrt{m+\frac{1}{2}}. \tag{24}$$

The resulting field structure is strongly reminiscent of the situation found in a microwave waveguide consisting of two perfectly conducting, infinitely extended planes. In these waveguides the dispersion relation for modes of order m ($m \geq 0$, contrary to the terminology conventionally used) is given by (boundary conditions $E = 0$, distance of plates $2a$) [23]:

$$q = \frac{\pi}{2a}\,(m+1). \tag{25}$$

Comparing eqs. (24) and (25) it is apparent that the Gaussian beam behaves as if it were propagating in a waveguide with the effective radius

$$a_{eff} = \frac{\pi}{4}\,w\,\frac{m+1}{\sqrt{m+\frac{1}{2}}} \tag{26}$$

$$\propto 0.79\,w\,\sqrt{m} \tag{27}$$

The latter expression is valid for $m >> 1$. Since the transverse extent of a Gaussian mode of order m scales as $\sqrt{m}w$ [2], this indicates that the Gaussian mode behaves as if propagating in a waveguide defined by its own envelope. Reducing either the beam radius or increasing the mode order results in a finer transverse scale and a larger phase lag per axial distance. Note that the tilt, and hence the presence of a nonzero transverse wavevector, implies a reduction of the longitudinal wavevector and, as such, an increase in the longitudinal phase scale in real space (cf. Fig. 1). The latter is responsible for the phase lag.

Hariharana and Robinson [10] arrive at a similar conclusion (restricting the discussion to the fundamental mode). They focus on the average longitudinal momentum of a photon in a Gaussian beam and calculate this by differentiating the complex field with respect to the propagation direction. They find a reduction of the longitudinal momentum in the vicinity of the beam waist which they relate to the Gouy phase shift. They also (as others did already before, e.g. [2]) recognize the similarity between this problem and the situation in a waveguide. The main difference with our treatment is that we perform all calculations in Fourier space since we are interested in discussing how the

Gouy phase arises through the superposition of the tilted waves which compose the beam.

3.5 The Gouy phase shift outside the beam waist plane

In section 3.2 it was shown that the phase lag per unit length of a tilted wave, representing the average wavevector of a Gaussian beam, coincides with the Gouy shift at the beam waist. The total phase lag from minus to plus infinity,

$$< \Delta\phi_0(-\infty \rightarrow +\infty) >= \int_{-\infty}^{+\infty} dz \, \frac{< \delta\phi_0 >}{\delta z}, \tag{28}$$

however, obviously diverges, whereas it is finite for a Gaussian mode. On the other hand, within the collimation range we obtain $< \Delta\phi_0(-z_r \rightarrow z_r) >= 1$, which has still the correct order of magnitude compared to the true value for a Gaussian beam (i.e., $\pi/4 \approx 0.79$). The exact agreement close to the beam waist, together with the approximate validity within the collimation range and the divergence at infinity, indicates that the diffractive spreading of the envelope is at the heart of the discrepancy. Indeed, the phase and amplitude structure of a single tilted wave or a pair of complex conjugates is self-similar (modulo the effective wavelength in longitudinal direction) whereas the propagation of a Gaussian mode is only self-similar up to a scaling factor of $\sqrt{1+(z/z_r)^2}$. In the "waveguide" interpretation this corresponds to an increase of the diameter of the waveguide which will obviously reduce the phase lag per unit length and can render the total phase shift finite.

The diffractive spreading in real space is caused by the dephasing between tilted waves with different q due to the q-dependent phase factor in the paraxial wave equation. Therefore not all contributions add up constructively and the effective averaged $< q_{eff}^2(z) >$ is reduced. Thus, we consider

$$< q_{eff}^2(z) >= Re\left\{ \frac{1}{N} \left(\int_0^{\infty} dq \, q^2 \cdot e^{-(\frac{wq}{2})^2} \, e^{\frac{i}{2k_0} q^2 z} \right) \right\}, \tag{29}$$

since the real part of q^2 describes the phase behaviour. Rewriting the argument of the exponential one obtains $-(wq/2)^2(1-i(z/z_r))$. Since substitution yields the correction factor $Re\{1-i(z/z_r)\} = 1/(1+(z/z_r)^2)$ (as expected from the diffractive spreading in real space), one finds

$$< q_{eff}^2(z) >= \frac{2}{w^2\,(1+(z/z_r)^2)}. \tag{30}$$

Taking into account this scaling factor the phase shift between two arbitrary planes is then given by

$$< \Delta\phi(z_1 \to z_2) > = \int_{z_1}^{z_2} dz \, \frac{1}{2z_r} \frac{1}{1+(z/z_r)^2}, \tag{31}$$

$$= \frac{1}{2}\left(\arctan\left(\frac{z_2}{z_r}\right) - \arctan\left(\frac{z_1}{z_r}\right)\right). \tag{32}$$

This is exactly the Gouy phase shift for the fundamental mode. For high order modes these values have to be multiplied by $(2m+1)$.

4 Connection to pattern formation in optical cavities

The spontaneous appearance of spatial structures due to self-organization phenomena in nonlinear resonators, i.e. resonators which contain an optical nonlinear material, is a well established fact. The basic form of the patterns close to threshold is mainly determined by the symmetries imposed by the boundary conditions and does not depend on the details of the nonlinearity. For example, plano-planar cavities with an infinite extent in the transverse plane possess Euclidian, in particular translational, symmetry and the appropriate base functions are Fourier modes. The archetypical order parameter equation for the emergence of periodic patterns with nonzero transverse wavevector is the Swift-Hohenberg equation and it was indeed shown that close to threshold uniform field models for plano-planar resonators can be reduced to such a form for lasers [13], dispersive passive cavities [24], absorptive passive cavities [25] and photorefractive oscillators [26]. In the optical context, the emergent Fourier modes have a natural interpretation as *tilted waves*, whose properties have been discussed above. For plano-planar cavities there exists also analytical proofs that single tilted waves are also the exact weakly nonlinear solution close to threshold in a laser [27]. In a passive cavity there will be some amount of harmonics, but the nonlinear solution will still be strongly dominated by a superposition of the fundamental tilted waves [28,25]. Furthermore, for a purely absorptive nonlinearity the resonance is exactly given by the empty cavity selection criterion [25]. For a dispersive medium there are additional contributions due to the refractive index of the medium [28,29].

In experiments on optical pattern formation in resonators with curved mirrors, the translational symmetry is broken and Laguerre- or Hermite-Gaussian modes are the appropriate base functions. As a consequence the observed patterns are typically Hermite- or Laguerre-Gaussian modes or a superposition of a few of them [14,15,30,31,16] and they appear for a cavity tuning at which they are resonant with the linear resonator or by an effective detuning given by the nonlinear phase shift (homogeneous or lens-like, e.g. [30,32]). Even many dynamically aspects can be understood in first approximation by the beating between modes of different (linear) frequency [31,16].

Obviously, this possibility of describing patterns in terms of the eigenmodes of the empty cavity (a plano-planar or curved-mirror one) breaks down at some point, if the nonlinear effects are too strong and/or the nonlinear refractive index change is too inhomogeneous (and cannot be approximated by a parabolic – lens-like – profile). A prominent example might be localized (soliton-like) structures [33–35]. Another effect is cooperative frequency locking in active systems [36,16].

Fourier-like modes [37–41] as well as Gaussian modes [14,15,30,31,16] were demonstrated in a variety of nonlinear cavities. The question of the selection between Fourier modes and Gaussian modes has recently gained importance for the spatial mode structure of modern vertical cavity surface emitting semiconductor lasers. These devices have nominally plano-planar resonators which, in reality, acquire a small curvature due to self-lensing or thermal lensing. Thus, depending on the details of the structure and on the operating conditions either tilted waves or Gaussian modes can be observed (e.g. [40–43]), although a full clarification on the necessary and sufficient condition for one of the kind of modes has not been established, yet [41].

We think that our approach shows that the selection mechanism in both types of cavities is essentially the same since optical patterns are strongly determined by the resonance condition, i.e. the longitudinal boundary conditions. The transverse boundary conditions (plane or curved mirrors) determine mainly whether the transverse Fourier spectrum is sharp (Fourier modes) or delocalized (Gaussian modes).

5 Conclusion

In this paper we have proposed a new interpretation of the phase anomaly of Gaussian modes, called Gouy phase shift, by showing that it can be obtained by summing up the weighted contributions of the different tilted plane wave components present in the mode. This result reveals that the resonance condition in curved mirror resonators can be qualitatively and even in some sense quantitatively understood by the same tilted wave mechanism which was demonstrated to act in plano-planar cavities. Thus, it represents a further step in a unified treatment of optical resonators independent of the details of the boundary conditions. We believe that it also helps to clarify the connection between spontaneously emerging patterns in plano-planar and curved-mirror nonlinear resonators.

Acknowledgment

T.A. gratefully acknowledges the support by a Feodor-Lynen Fellowship of the Alexander-von-Humboldt Foundation. We thank W. Lange for stimulating and fruitful discussions carried out over several years. G.L.L. is grateful to J.-L. Meunier for discussions on some mathematical points.

References

[1] H. Kogelnik and T. Li, Appl.Opt. **5**, 1550 (1966).

[2] A. E. Siegman, *Lasers* (University Science Books, Mill Valley, California, 1986).

[3] L. G. Gouy, C. R. Acad. Sci. Paris **110**, 1251 (1890).

[4] M. Born and E. Wolf, *Principles of Optics*, 6^{th} ed. (Pergamon, Oxford, 1980).

[5] A. B. Ruffin *et al.*, in *Quantum Electronics and Laser Science Conference 1999, OSA Technical Digest Series* (Optical Society of America, Washington, DC, Baltimore, 1999), pp. 396–397, paper JThB3.

[6] W. Demtröder, *Laser Spectroscopy — Basic Concepts and Instrumentation*, 3^{rd} ed. (Springer Verlag, Berlin, Heidelberg, New York, 1989).

[7] R. W. Boyd, J. Opt. Soc. Am. **70**, 877 (1980).

[8] R. Simon and N. Mukunda, Phys. Rev. Lett. **70**, 880 (1993).

[9] D. Subbarao, Opt. Lett. **20**, 2162 (1995).

[10] P. Hariharan and P. A. Robinson, J. Mod. Opt. **43**, 219 (1996).

[11] L. A. Lugiato, M. Brambilla, and A. Gatti, Adv. Atom. Mol. Opt. Phys. **40**, 229 (1999).

[12] F. T. Arecchi, S. Boccaletti, and P. L. Ramazza, Phys. Rep. **318**, 1 (1999).

[13] J. Lega, J. Moloney, and A. Newell, Phys. Rev. Lett. **73**, 2978 (1994).

[14] M. Brambilla *et al.*, Phys. Rev. A **43**, 5090 (1991).

[15] G. L. Lippi *et al.*, Phys. Rev. A **48**, R4043 (1993).

[16] E. Louvergneaux, G. Slekys, D. Dangoisse, and P. Glorieux, Phys. Rev. A **57**, 4899 (1998).

[17] G. J. de Valcárcel, Phys. Rev. A **56**, 1542 (1997).

[18] G. K. Harkness, J. C. Lega, and G.-L. Oppo, in *European Quantum Electronics Conference - EQEC 1996, Technical Digest* (Optical Society of America, Washington, DC, Hamburg, 1996), paper QTuB5.

[19] P. Lorrain, D. R. Corson, and F. Lorrain, *Electromagnetic Fields and Waves*, 3rd ed. (W. H. Friedman and Company, New York, 1988).

[20] H. Weber, Opt. Quantum Electron. **24**, S1027 (1992).

[21] I. S. Gradshteyn and I. M. Ryzhik, *Table of integrals, series, and products* (Academic Press, San Diego, 1980).

[22] D. Z. Anderson, Appl.Opt. **23**, 2944 (1984).

[23] J. D. Jackson, *Classical electrodynamics*, 2nd ed. (Wiley, New York, 1975).

[24] M. Tlidi and P. Mandel, Chaos, Solitons & Fractals **4**, 1475 (1994).

[25] W. J. Firth and A. J. Scroggie, Europhys. Lett. **26**, 521 (1994).

[26] K. Staliunas, M. F. H. Tarroja, G. Slekys, and C. O. Weiss, Phys. Rev. A **51**, 4140 (1995).

[27] P. Jakobsen, J. Moloney, A. Newell, and R. Indik, Phys. Rev. A **45**, 8129 (1992).

[28] M. Tlidi, M. Georgiou, and P. Mandel, Phys. Rev. A **48**, 4605 (1993).

[29] L. Spinelli *et al.*, Phys. Rev. A **58**, 2542 (1998).

[30] G. L. Lippi, T. Ackemann, L. M. Hoffer, and W. Lange, Chaos, Solitons & Fractals **4**, 1409 (1994).

[31] A. B. Coates *et al.*, Phys. Rev. A **49**, 1452 (1994).

[32] G. L. Lippi, T. Ackemann, L. M. Hoffer, and W. Lange, Chaos, Solitons & Fractals **4**, 1433 (1994).

[33] M. Brambilla, L. A. Lugiato, and M. Stefani, Europhys. Lett. **34**, 109 (1996).

[34] W. J. Firth and A. J. Scroggie, Phys. Rev. Lett. **76**, 1623 (1996).

[35] M. Kreuzer, B. Thüring, and T. Tschudi, Asian J. Phys. **7**, 678 (1998).

[36] L. A. Lugiato *et al.*, J. Opt. Soc. Am. B **5**, 879 (1988).

[37] M. Kreuzer, H. Gottschilk, T. Tschudi, and R. Neubecker, Mol. Cryst. Liquid Cryst. **207**, 219 (1991).

[38] A. V. Mamaev and M. Saffman, Opt.Commun. **128**, 281 (1996).

[39] K. Staliunas, G. Slekys, and C. O. Weiss, Phys. Rev. Lett. **79**, 2658 (1997).

[40] S. Hegarty, G. Huyet, J. G. McInerney, and K. D. Choquette, Phys. Rev. Lett. **82**, 1434 (1999).

[41] T. Ackemann *et al.*, J. Opt. B: Quantum Semiclass. Opt. **2**, 406 (2000).

[42] H. Li, T. L. Lucas, J. G. McInerney, and R. A. Morgan, Chaos, Solitons & Fractals **4**, 1619 (1994).

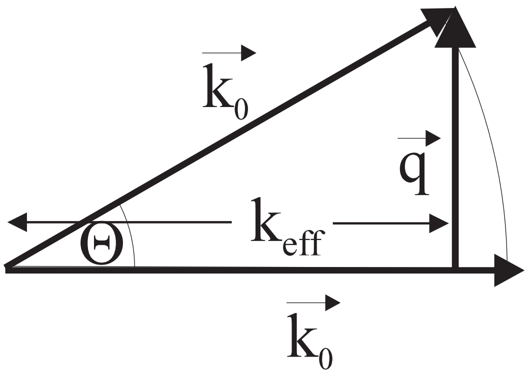
[43] C. Degen, I. Fischer, and W. Elsäßer, Opt.Express **5**, 38 (1999).

Fig. 1. Resonance condition in a plano-planar cavity and tilted waves: a) Fourier space, b) real space.

Fig. 2. Modal spectrum of a plano-planar resonator (top) and of a close-to-planar resonator (bottom).

Fig. 3. Ratio of the numerical evaluated second moment, Eq. (23)), for the lowest order Hermite-Gaussian modes to the asymptotic value, Eq. ((22)). Beam radius $w = 1$. Squares denote even modes, triangles odd ones.

a) **b)**



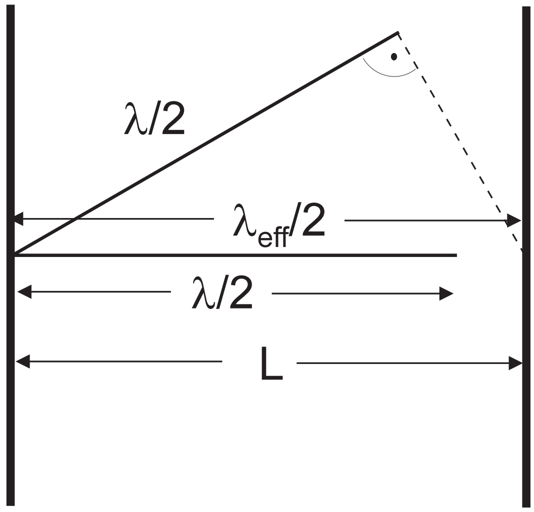


Fig. 1 of Ackemann et al.: Tilted waves, the Gouy phase shift ...

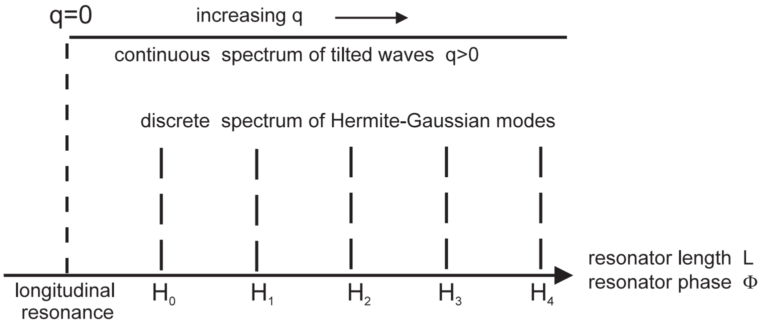


Fig. 2 of Ackemann et al.: Tilted waves, the Gouy phase shift ...

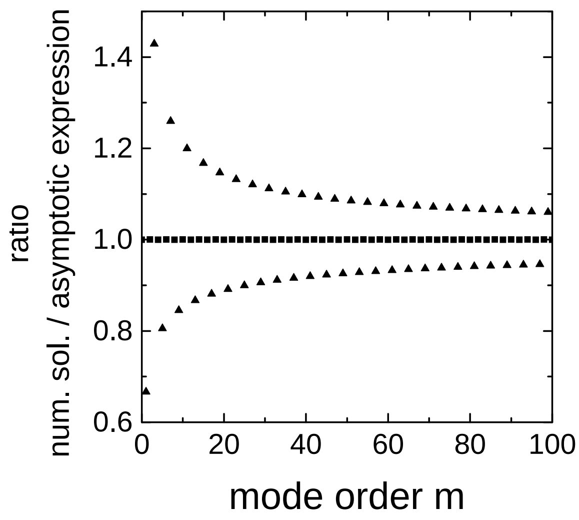


Fig. 3 of Ackemann et al.: Tilted waves, the Gouy phase shift ...