Non-negatively curved torus manifolds

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Outline

1. Non-negative curvature and torus manifolds
   - Definitions
   - Previous Work
2. Main results
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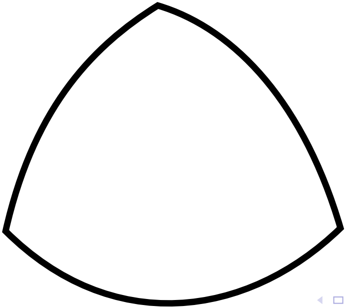
   - Structure Results for torus manifolds
   - Proof of the main result
3. Applications

Torus manifolds and non-negative curvature

- A torus manifold is a $2n$-dimensional orientable connected manifold M together with a action of an n-dimensional torus such that $M^T \neq \emptyset$.
- A Riemannian manifold M is non-negatively curved if all triangles in M are not "thinner" than a triangle in the Euclidean plane



Goal

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Classify torus manifolds which admit an invariant metric of non-negative curvature.

Previous Results

Theorem (Grove and Searle (1994))

A simply connected torus manifold with an invariant metric of positive sectional curvature is diffeomorphic to S^{2n} or $\mathbb{C}P^n$.

Theorem (Hsiang and Kleiner (1989))

A 4-dimensional simply connected Riemannian manifold with positive sectional curvature and an isometric S^1-action is homeomorphic to S^4 or $\mathbb{C}P^2$.

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Theorem (Kleiner (1990) and Searle and Yang (1994))

A 4-dimensional simply connected Riemannian manifold with non-negative sectional curvature and an isometric S^1-action is homeomorphic to S^4, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$ or $S^2 \times S^2$.

- Grove and Wilking (2013) classified 4-dimensional simply connected Riemannian manifolds with non-negative curvature and isometric S^1-action up to equivariant diffeomorphism.
- In particular, a 4-dimensional simply connected non-negatively curved torus manifold has at most four fixed points.

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Main Theorem

Theorem (W.)

Let M be a simply connected torus manifold with $H^{odd}(M;\mathbb{Q}) = 0$ such that one of the following two conditions holds:

- *M admits an invariant metric of non-negative sectional curvature.*
- *M is rationally elliptic.*

Then M has the same rational cohomology as a quotient of a free linear torus action on a product of spheres.If, moreover, $H^*(M;\mathbb{Z})$ is torsion-free or $H^{odd}(M;\mathbb{Z}) = 0$, then M is homeomorphic to such a quotient.

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Discusssion of assumptions

Definition

A simply connected topological space X is called rationally elliptic, if

$$\sum_{i=0}^{\infty} \dim H^i(X;\mathbb{Q}) < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \dim \pi_i(X) \otimes \mathbb{Q} < \infty.$$

Conjecture (Bott)

A non-negatively curved manifold is rationally elliptic.

Theorem (Spindeler (2013))

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Discussion of assumptions

- A rationally elliptic torus manifold M has $\chi(M) = \chi(M^T) > 0$ and therefore $H^{\text{odd}}(M;\mathbb{Q}) = 0$.
- Hence, the assumption on the cohomology is not necessary in the main theorem.

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Towards a proof of the conjecture

Theorem

The conjecture holds for locally standard torus manifolds M which satisfy

- *The intersection of any collection of facets of M/T is connected or empty, or*
- $\dim M = 6$.

Proof.

- We first use the geometry of M/T to show that all faces are contractible.
- Results of Masuda and Panov imply that $H^{\text{odd}}(M;\mathbb{Z}) = 0$.
- Hence, the statement follows from the main theorem.

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Strucure results for torus manifolds

Masuda and Panov (2006) proved the following structure results for torus manifolds M with $H^{\mathrm{odd}}(M;\mathbb{Z}) = 0$:

- The torus action is locally standard, i.e. each $p \in M$ has an invariant neighborhood which is equivariantly diffeomorphic to an open subset of $\mathbb{C}^n$.
- M/T is a manifold with corners.
- All faces F of M/T are acyclic, i.e. $\tilde{H}^*(F) = 0$. Therefore all F are homology discs.

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Canonical models

- Denote by $\lambda(F)$ the isotropy group of a generic orbit in F.
- There is an equivariant homeomorphism

$$(M/T \times T)/\sim \rightarrow M,$$

where $(x_1, t_1) \sim (x_2, t_2) \Leftrightarrow x_1 = x_2 \wedge t_1^{-1}t_2 \in \lambda(F(x_1))$

- Therefore there is a principal torus bundle $Z_{M/T} \rightarrow M$, where $Z_{M/T}$ is the moment angle complex associated to M/T:

$$Z_{M/T} = (M/T \times T^{\mathfrak{F}})/\sim,$$

where $(x_1, t_1) \sim (x_2, t_2) \Leftrightarrow x_1 = x_2 \wedge t_1^{-1}t_2 \in T^{\mathfrak{F}(F(x_1))}$ with $\mathfrak{F}(F) =$ set of facets containing F.

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Structure results for torus manifolds

- The face poset $\mathcal{P}(M/T)$ is defined to be the set of all faces of M/T together with the ordering given by inclusion.

Theorem (W.)

Let M_1 and M_2 be two simply connected torus manifolds with $H^{odd}(M_i, \mathbb{Z}) = 0$. Then M_1 and M_2 are homeomorphic if $(\mathcal{P}(M_1/T), \lambda_1)$ and $(\mathcal{P}(M_2/T), \lambda_2)$ are isomorphic.

Structure results for torus manifolds

Proof.

- If all faces of M_i/T, $i = 1, 2$ are contractible, then the statement follows, because every homeomophism of the boundary of a contractible manifold extends to a homeomorphism of the contractible manifold.
- If not all faces are contractible, then one can change the torus action on M_i in such a way that all faces become contractible without effecting $(\mathcal{P}(M_i/T), \lambda_i)$.

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Corollary

Let M be a torus manifold homotopy equivalent to $\mathbb{C}P^n$. Then M is homeomorphic to $\mathbb{C}P^n$.

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Strategy

By the structure results for torus manifolds, for the proof of the main theorem it is sufficient to determine the combinatorial type of M/T and then to realize these combinatorial types by a simply connected torus manifold.

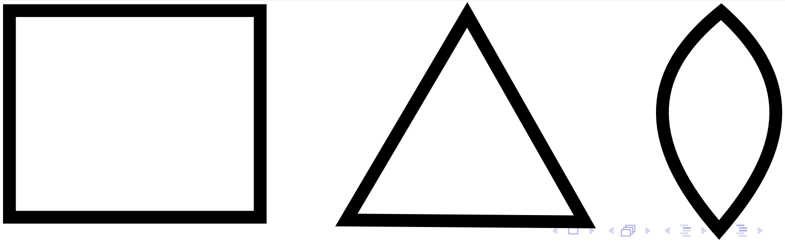
Lemmas needed for the proof of the main result.

Lemma

Let M be a torus manifold with $H^{odd}(M;\mathbb{Q}) = 0$ such that

- *M admits an invariant metric of non-negative sectional curvature, or*
- *M is rationally elliptic.*

Then all two-dimensional faces of M/T have at most four vertices.



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Let M be a torus manifold with $H^{odd}(M;\mathbb{Q}) = 0$ such that all two-dimensional faces of M/T have at most four vertices. Then M/T is combinatorially equivalent to a product $\prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i}$, where

- *Δ^{n_i} is an n_i-dimensional simplex and Σ^{n_i} is S^{2n_i}/T.*

- Note that $Z_{\Sigma^n} = S^{2n}$ and $Z_{\Delta^n} = S^{2n+1}$ and $Z_{Q_1 \times Q_2} = Z_{Q_1} \times Z_{Q_2}$.
- Therefore the theorem follows.

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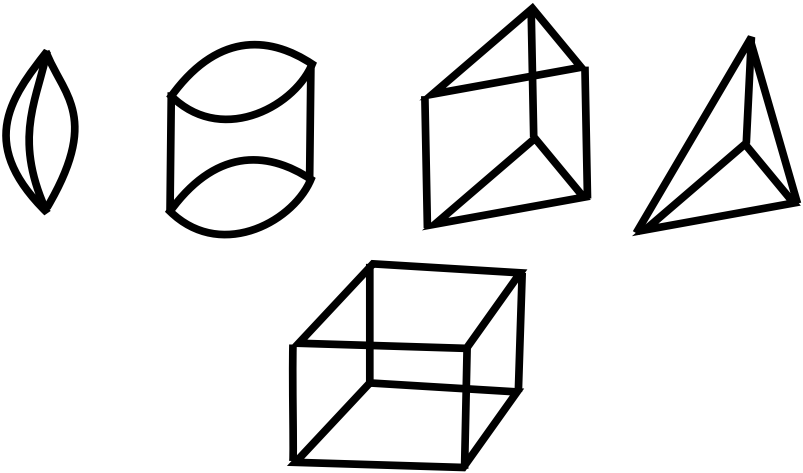
Orbit spaces in dimension 6.



Rigidity problem

Definition

A polytope P is called rigid if the following holds:

- There is a quasitoric manifold M_1 with $M_1/T = P$.
- If M_2 is another quasitoric manifold with $H^*(M_2) \cong H^*(M_1)$ and $M_2/T = Q$, then P and Q are combinatorially equivalent.

Theorem (Choi, Panov and Suh (2010))

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Applications

$\prod_i \Sigma^{n_i} \times \prod_i \Delta^{n_i}$ is rigid in the following sense:

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Corollary

Let M be a torus manifold homotopy equivalent to $\prod_i \mathbb{C}P^{n_i}$, $n_i > 1$. Then M is homeomorphic to $\prod_i \mathbb{C}P^{n_i}$.

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