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On the classification of torus manifolds with and without non-abelian symmetries

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Contents

Zusammenfassung

Es sei G eine zusammenhängende kompakte nicht-abelsche Lie-Gruppe und T ein maximaler Torus von G. Eine Torusmannigfaltigkeit mit G-Aktion ist nach Definition eine glatte zusammenhängende geschlossene orientierte Mannigfaltigkeit der Dimension $2 \dim T$, auf der G fast effektiv operiert, so dass $M^T \neq \emptyset$. In dieser Dissertation klassifizieren wir einfach zusammenhängende Torusmannigfaltigkeiten mit G-Aktion bis auf G'-äquivariante Diffeomorphie, wobei G' eine Faktorgruppe einer endlichen Überlagerungsgruppe von G ist.

Ausserdem geben wir vier neue hinreichende Bedingungen dafür an, dass zwei quasitorische Mannigfaltigkeiten schwach äquivariant homöomorph sind.

Am Ende untersuchen wir quasitorische Mannigfaltigkeiten mit verschwindender erster Pontrjagin-Klasse, die eine Operation einer zusammenhängenden kompakten nicht-abelschen Lie-Gruppe zulassen, die die Torusoperation nicht fortsetzt.

Abstract

Let G be a connected compact non-abelian Lie-group and T a maximal torus of G. A torus manifold with G-action is defined to be a smooth connected closed oriented manifold of dimension $2\dim T$ with an almost effective action of G such that $M^T \neq \emptyset$. In this thesis we classify simply connected torus manifolds with G-action up to G'-equivariant diffeomorphism, where G' is a factor group of a finite covering group of G.

Furthermore we give four new sufficient conditions for two quasitoric manifolds to be weakly equivariantly homeomorphic.

At the end we study quasitoric manifolds with vanishing first Pontrjagin-class admitting an action of a connected compact non-abelian Lie-group which does not extend the torus action.

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CHAPTER 1

Introduction

Quasitoric manifolds were introduced by Davis and Januszkiewicz [**13**] in 1991 as "topological approximations" to algebraic non-singular projective toric varieties. They are defined as follows.

Let M be a smooth closed connected orientable $2n$-dimensional manifold on which the n-dimensional torus T acts. We say that the T-action on M is *locally standard* if it is locally isomorphic to the standard action on $\mathbb{C}^n$ up to an automorphism of T. If the T-action on M is locally standard then the orbit space M/T is locally homeomorphic to the cone

$$\mathbb{R}^n_{\geq 0} = \{(x_1, \dots, x_n) \in \mathbb{R}^n;\, x_i \geq 0\}\,.$$

Therefore it is a manifold with corners [**14**, p. 303-304]. M is called *quasitoric* if the T-action on M is locally standard and the orbit space M/T is face preserving homeomorphic to a simple n-dimensional polytope.

The notion of a torus manifold is a generalisation of a quasitoric manifold. It was introduced by Masuda [**41**] and Hattori and Masuda [**26**]. A *torus manifold* is a $2n$-dimensional smooth closed connected orientable manifold on which a n-dimensional torus acts effectively such that the fixed point set M^T is non-empty.

A closed, connected submanifold M_i of codimension two of a torus manifold M which is pointwise fixed by a one dimensional subtorus $\lambda(M_i)$ of T and which contains a T-fixed point is called *characteristic submanifold* of M.

All characteristic submanifolds M_i are orientable and an orientation of M_i determines a complex structure on the normal bundle $N(M_i, M)$ of M_i.

We denote the set of unoriented characteristic submanifolds of M by $\mathfrak{F}$. If M is quasitoric the characteristic submanifolds of M are given by the preimages of the facets of $P = M/T$. In this case we identify $\mathfrak{F}$ with the set of facets of P. We call the map

$$\lambda : \mathfrak{F} \to \{\text{one-dimensional subtori of } T\}$$

the *characteristic map* of M.

Now assume that M is quasitoric over the polytope P and $\mathfrak{F} = \{F_1, \dots, F_m\}$. Let N be the integer lattice of one-parameter circle subgroups in T, so we have $N \cong \mathbb{Z}^n$. Given a facet F_i of P we denote by $\bar{\lambda}(F_i) \in N$ the primitive vector that spans $\lambda(F_i)$. Then $\bar{\lambda}(F_i)$ is determined up to sign. The map

$$\bar{\lambda} : \mathfrak{F} \to N$$

is called the *characteristic function* of M. The identification of T with the standard torus $\mathbb{R}^n/\mathbb{Z}^n$ induces an identification of N with $\mathbb{Z}^n$. This allows us to write $\bar{\lambda}$ as an integer matrix,

$$\Lambda = \begin{pmatrix} \lambda_{1,1} & \dots & \lambda_{1,m} \\ \vdots & & \vdots \\ \lambda_{n,1} & \dots & \lambda_{n,m} \end{pmatrix}.$$

Here we have

$$\lambda(F_i) = \left\{ t \begin{pmatrix} \lambda_{1,i} \\ \vdots \\ \lambda_{n,i} \end{pmatrix} \in \mathbb{R}^n/\mathbb{Z}^n; t \in \mathbb{R} \right\}.$$

We have the following strong relations between the topology of M and the combinatorics of P. At first the odd Betti-numbers of M vanish and the even Betti-numbers are given by the components of the h-vector of P (see section 2.3).

Second let $u_i \in H^2(M)$ be the Poincaré-dual of the characteristic manifold M_i. Then the cohomology ring $H^*(M)$ is generated by $u_1, \dots, u_m$. The u_i are subject to the following relations:

(1) $\forall I \subset \{1, \dots, m\}$ $\prod_{i\in I} u_i = 0 \Leftrightarrow \bigcap_{i\in I} F_i = \emptyset$
(2) For $i = 1, \dots, n$ $\sum_{j=1}^{m} \lambda_{i,j} u_j = 0$.

Furthermore it was proved by Davis and Januszkiewicz [**13**] that the T-equivariant homeomorphism type of M is determined by P and λ.

So at first glance it seems that a quasitoric manifold is a very special object. But it was shown by Buchstaber and Ray [**11**] and Buchstaber, Panov and Ray [**10**] that in dimension greater than two every complex cobordism class contains a quasitoric manifold.

There are two main classification problems for quasitoric manifolds:

- the equivariant (i.e. up to (weakly) equivariant homeomorphism / diffeomorphism)
- the topological (i.e. up to homeomorphism / diffeomorphism).

Here two torus manifolds M, M' are called *weakly equivariantly homeomorphic* if there are an automorphism $\theta : T \to T$ and a homeomorphism $f : M \to M'$ such that for all $x \in M$ and $t \in T$ we have

$$f(tx) = \theta(t)f(x).$$

If two quasitoric manifolds are weakly equivariantly homeomorphic then obviously their orbit polytopes are face-preserving homeomorphic. This implies that they are combinatorially equivalent. Therefore by the result of Davis and Januszkiewicz cited above the equivariant classification problem reduces to the classification of all characteristic maps over a given polytope.

The topological classification is often more complicated. But there is the following result. By a result of Panov and Ray [**51**] all quasitoric manifolds are formal. Because the proof of Theorem 12.5 of [**56**] only uses that compact Kähler-manifolds are formal it holds more generally for all formal manifolds. Therefore we have

THEOREM 1.1. *The diffeomorphism type of a quasitoric manifold of dimension greater than four is determined up to a finite number of possibilities by*

- *the integral cohomology ring,*
- *the (rational) Pontrjagin-classes.*

For more results on these classification problems see [**12**, p. 82-83].

A variant of the equivariant classification is the following: Assume that the T-action on the quasitoric manifold (or torus manifold) M extends to an action of the connected compact non-abelian Lie-group G. We call such M quasitoric manifolds (or torus manifolds) with G-action. Now the problem is to classify all quasitoric manifolds with G-action up to G-equivariant diffeomorphism.

This was done by Kuroki [**36, 39, 37, 38**] in the case where $\dim M/G \leq 1$. To be precise Kuroki proved the following.

THEOREM 1.2 ([**39**]). *Let M be a torus manifold with G-action such that G acts transitively on M. Then (M,G) is essentially isomorphic to*

$$\left(\prod_{i=1}^{a}\mathbb{C}P^{l_i}\times\frac{\prod_{j=1}^{b}S^{2m_j}}{\mathcal{A}},\prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)\right),$$

where $\mathcal{A}$ is a subgroup of the intersection of

$$\prod_{j=1}^{b}\{I_{2m_j+1},-I_{2m_j+1}\}\subset\prod_{j=1}^{b}O(2m_j+1)$$

and $SO(2m_1+\cdots+2m_b+b)$. Here $\prod_{i=1}^{a}PU(l_i+1)\times\prod_{j=1}^{b}SO(2m_j+1)$ acts in the natural way.

Here two transformation groups (M,G) and (M',G') are called *essentially isomorphic* if there is an isomorphism $\phi : G/N \to G'/N'$ and a diffeomorphism $f : M \to M'$ such that for all $g \in G/N$ and $x \in M$ we have

$$f(gx)=\phi(g)f(x),$$

where $N=\bigcap_{x\in M}G_x$ and $N'=\bigcap_{x\in M'}G'_x$ denote the kernels of the actions of G, G' on M, M', respectively.

THEOREM 1.3 ([**37**]). *Let M be a quasitoric manifold with G-action such that $\dim M/G = 1$. Then (M,G) is essentially isomorphic to*

$$\left(\prod_{i=1}^{a-1}S^{2l_i+1}\times_{T^{a-1}}P(\mathbb{C}_\alpha^{k_1}\oplus\mathbb{C}^{k_2}),\prod_{i=1}^{a-1}SU(l_i+1)\times S(U(k_1)\times U(k_2))\right).$$

Here $SU(l_i+1)$ acts on S^{2l_i+1} in the usual way and $S(U(k_1)\times U(k_2))$ acts in the usual way on $P(\mathbb{C}_\alpha^{k_1}\oplus\mathbb{C}^{k_2})$. The action of the torus $T^{a-1}=(S^1)^{a-1}$ on $\prod_{i=1}^{a-1}S^{2l_i+1}$ is given by the diagonal action. The action of T^{a-1} on $\mathbb{C}^{k_2}$ is trivial and its action on $\mathbb{C}_\alpha^{k_1}$ factors through $\chi^\alpha : T^{a-1}\to S^1$.

In [**38**] Kuroki gives a list with seven types of torus manifolds with G-action such that every torus manifold with G-action with $\dim M/G = 1$ is essentially isomorphic to one of the given types.

The general case of the classification problem for torus manifolds with G-action is studied in chapter 4 of this thesis. We have the following results.

Let M be a torus manifold with G-action. Then the G-action on M induces an action of the Weyl-group $W(G)$ of G on $\mathfrak{F}$ and the T-equivariant cohomology of M. Results of Masuda [**41**] and Davis and Januszkiewicz [**13**] make a comparison of these actions possible. From this comparison we get a description of the action on $\mathfrak{F}$ and the isomorphism type of $W(G)$. Namely there is a partition of $\mathfrak{F}=\mathfrak{F}_0\amalg\cdots\amalg\mathfrak{F}_k$ and a finite covering group $\tilde{G}=\prod_{i=1}^{k}G_i\times T^{l_0}$ of G such that each G_{i_0} is non-abelian and $W(G_{i_0})$ acts transitively on $\mathfrak{F}_{i_0}$ and trivially on $\mathfrak{F}_i$, $i\neq i_0$, and the orientation of each $M_j\in\mathfrak{F}_i$, $i\neq i_0$, is preserved by $W(G_{i_0})$ (see section 4.1).

We call such G_i the *elementary factors* of $\tilde{G}$.

By looking at the orbits of the T-fixed points we find that all elementary factors are isomorphic to $SU(l_i+1)$, $\mathrm{Spin}(2l_i)$ or $\mathrm{Spin}(2l_i+1)$. Furthermore the action of an elementary factor of $\tilde{G}$ which is isomorphic to $\mathrm{Spin}(l)$ factors through $SO(l)$.

Therefore we may replace $\tilde{G}$ by one of its factor groups $\tilde{G}'$ of the form

$$\tilde{G}'=\prod_{i=1}^{k}G'_i\times T^{l_0}$$

such that all G_i' are elementary and are isomorphic to $SU(l_i + 1)$, $SO(2l_i)$ or $SO(2l_i + 1)$ (see section 4.2). If M is quasitoric then all elementary factors are isomorphic to $SU(l_i + 1)$. If the G-action on M is effective, then $\tilde{G}'$ is a finite covering group of G. Therfore in the following we do not distinguish between $\tilde{G}$ and $\tilde{G}'$ and denote $\tilde{G}'$ also by $\tilde{G}$.

Now assume $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1)$ elementary. Then the restriction of the action of G_1 to $U(l_1)$ has the same orbits as the G_1-action (see section 4.5). The following theorem shows that the classification of simply connected torus manifolds with $\tilde{G}$-action reduces to the classification of torus manifolds with $U(l_1) \times G_2$-action.

THEOREM 1.4 (Lemma 4.44). *Let M, M' be two simply connected torus manifolds with $\tilde{G}$-action, $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1)$ elementary. Then M and M' are $\tilde{G}$-equivariantly diffeomorphic if and only if they are $U(l_1) \times G_2$-equivariantly diffeomorphic.*

By applying a blow up construction along the fixed points of an elementary factor of $\tilde{G}$ isomorphic to $SU(l_i + 1)$ or $SO(2l_i + 1)$ we get a fiber bundle over a complex or real projective space with some torus manifold as fiber.

This construction may be reversed and we call the inverse construction a blow down. With this notation we get:

THEOREM 1.5 (Corollaries 4.30, 4.38, 4.47, Theorem 4.53). *Let $\tilde{G} = G_1 \times G_2$, M a torus manifold with G-action such that G_1 is elementary and $l_2 = \operatorname{rank} G_2$.*

- *If $G_1 = SU(l_1 + 1)$ and $\#\mathfrak{F}_1 = 2$ in the case $l_1 = 1$ then M is the blow down of a fiber bundle $\tilde{M}$ over $\mathbb{C}P^{l_1}$ with fiber some $2l_2$-dimensional torus manifold with G_2-action along an invariant submanifold of codimension two. Here the G_1-action on $\tilde{M}$ covers the standard action of $SU(l_i + 1)$ on $\mathbb{C}P^{l_1}$.*
- *If $G_1 = SO(2l_1+1)$ and $\#\mathfrak{F}_1 = 1$ in the case $l_1 = 1$ then M is a blow down of a fiber bundle $\tilde{M}$ over $\mathbb{R}P^{2l_1}$ with fiber some $2l_2$-dimensional torus manifold with G_2-action along an invariant submanifold of codimension one or a Cartesian product of a $2l_1$-dimensional sphere and a $2l_2$-dimensional torus manifold with G_2-action. In the first case the G_1-action on $\tilde{M}$ covers the standard action of $SO(2l_1 + 1)$ on $\mathbb{R}P^{2l_1}$. In the second case G_1 acts in the usual way on S^{2l_1}.*

If all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l_i + 1)$ or $SU(l_i + 1)$ then we may iterate this construction. By this iteration we get a complete classification of torus manifolds with $\tilde{G}$-action up to $\tilde{G}$-equivariant diffeomorphism in terms of admissible 5-tuples (Theorem 4.58). For general G we have $\tilde{G} = \prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod SO(2l_i) \times T^{l_0}$. We may restrict the action of $\tilde{G}$ to $\prod SU(l_i + 1) \times \prod SO(2l_i + 1) \times \prod U(l_i) \times T^{l_0}$. Therefore we get invariants for torus manifolds with G-action from the above classification. With Theorem 1.4 we see that these invariants determine the $\tilde{G}$-equivariant diffeomorphism type of simply connected torus manifolds with $\tilde{G}$-action.

We apply our classification to get more explicit results in special cases. These are:

For the special case $G_2 = \{1\}$ we get:

COROLLARY 1.6 (Corollary 4.15). *Assume that G is elementary and M a torus manifold with G-action. Then M is equivariantly diffeomorphic to S^{2l} or $\mathbb{C}P^l$ if $G = SO(2l + 1), SO(2l)$ or $G = SU(l + 1)$, respectively.*

We recover Kuroki's results on the classification of torus manifolds with G-action and $\dim M/G \leq 1$ (see Corollaries 4.63 and 4.64).

For quasitoric manifolds we have the following result.

THEOREM 1.7 (Corollary 4.62). *If G is semi-simple and M a quasitoric manifold with G-action then*

$$\tilde{G} = \prod_{i=1}^{k} SU(l_i + 1)$$

and M is equivariantly diffeomorphic to a product of complex projective spaces.

Furthermore we give an explicit classification of simply connected torus manifolds with G-action such that $\tilde{G}$ is semi-simple and has two simple factors.

THEOREM 1.8 (Corollaries 4.15, 4.65, 4.67). *Let $\tilde{G} = G_1 \times G_2$ with G_i simple and M a simply connected torus manifold with G-action. Then M is one of the following:*

$$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}, \;\; \mathbb{C}P^{l_1} \times S^{2l_2}, \;\; \#_i(S^{2l_1} \times S^{2l_2})_i, \;\; S^{2l_1+2l_2}$$

If $G_1, G_2 \neq Spin(8)$, then the $\tilde{G}$-actions on these spaces is unique up to equivariant diffeomorphism. Otherwise the $\tilde{G}$-actions on these spaces is unique up to weakly equivariant diffeomorphism.

Furthermore we give in chapter 3 three criteria for quasitoric manifolds to be weakly T-equivariantly homeomorphic.

The first criterion gives a condition on the cohomology of M and M':

THEOREM 1.9 (Theorem 3.2). *Let M, M' be quasitoric manifolds of dimension n. Let $u_1, \dots, u_m \in H^2(M)$ the Poincaré-duals of the characteristic submanifolds of M and $u'_1, \dots, u'_{m'} \in H^2(M')$ the Poincaré-duals of the characteristic submanifolds of M'. If there is a ring isomorphism $f : H^*(M) \to H^*(M')$ with $f(u_i) = u'_i$, $i = 1, \dots, m$, then M and M' are weakly T-equivariantly homeomorphic.*

The stable tangent bundle of a quasitoric manifold M splits as a sum of complex line bundles. This induces a BT^m-structure on the stable tangent bundle of M. We show that two BT^m-bordant quasitoric manifolds are weakly equivariantly homeomorphic.

Furthermore we show that two quasitoric manifolds having the same GKM-graphs are equivariantly homeomorphic.

In chapters 5 and 6 we study torus manifolds M which admit actions of connected compact non-abelian Lie-groups G which do not necessarily extend the action of the torus T on M. To be more precise in chapter 5 we assume that both G and T preserve a given stable almost complex structure on M. We show that there is a compact connected Lie-group G' and an embedding of G in G' as a subgroup such that M is a torus manifold with G'-action.

In chapter 6 we assume that the first Pontrjagin-class of a quasitoric manifold vanishes. Under this condition we give the diffeomorphism type of all quasitoric manifolds admitting an action of a connected compact non-abelian Lie-group such that $\dim M/G \leq 1$. We have the following results:

THEOREM 1.10 (Theorem 6.1). *Let M be a quasitoric manifold which is homeomorphic (or diffeomorphic) to a homogeneous space G/H with G a compact connected Lie-group and has vanishing first Pontrjagin-class. Then M is homeomorphic (diffeomorphic) to $\prod S^2$.*

THEOREM 1.11 (Theorem 6.2). *Let M be a quasitoric manifold with $p_1(M) = 0$. Assume that the compact connected Lie-group G acts smoothly and almost effectively on M such that* $\dim M/G = 1$. *Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is a S^2-bundle over a product of two-spheres.*

In chapter 2 we give an overview about the theory of toric varieties and their generalisations.

CHAPTER 2

From toric varieties to torus manifolds

In this chapter we give an overview about the theory of toric varieties and their generalisations.

2.1. Toric varieties and fans

Toric varieties were introduced in the beginning of the 1970's independently by Demazure [**16**], Miyake and Oda [**48**], Mumford et al. [**34**] and Satake [**54**]. Here we describe their construction from combinatorial objects called fans as complex analytical spaces following Oda [**47**]. For background information on complex analytical spaces see [**25**, Chapter V].

We begin with the definition of a fan. Let N be a free $\mathbb{Z}$-module of rank r and M its dual module. Then there is a canonical $\mathbb{Z}$-bilinear form

$$\langle\,\cdot\,,\,\cdot\,\rangle : M \times N \to \mathbb{Z}.$$

We have the r-dimensional vector spaces $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ with a canonical $\mathbb{R}$-bilinear form

$$\langle\,\cdot\,,\,\cdot\,\rangle : M_\mathbb{R} \times N_\mathbb{R} \to \mathbb{R}.$$

Definition 2.1. A subset σ of $N_\mathbb{R}$ is called a *strongly convex rational polyhedral cone* if there exists a finite number of elements $n_1, \dots, n_s \in N$ such that

$$\sigma = \mathbb{R}_{\geq 0} n_1 + \dots + \mathbb{R}_{\geq 0} n_s$$

and $\sigma \cap (-\sigma) = \{0\}$.

The dimension $\dim \sigma$ of a strongly convex rational polyhedral cone σ is by definition the dimension of the smallest $\mathbb{R}$-subspace of $N_\mathbb{R}$ containing σ. The *dual cone* of σ in $M_\mathbb{R}$ is defined to be

$$\sigma^\vee = \{x \in M_\mathbb{R};\ \langle x, y\rangle \geq 0 \text{ for all } y \in \sigma\}.$$

Because $\sigma \cap (-\sigma) = \{0\}$ we have $\sigma^\vee + (-\sigma^\vee) = M_\mathbb{R}$ hence $\dim \sigma^\vee = r$. A subset τ of σ is called a *face* of σ if there is a $m_0 \in \sigma^\vee$ such that

$$\tau = \sigma \cap \{m_0\}^\perp = \{y \in \sigma;\ \langle m_0, y\rangle = 0\}.$$

A face τ of σ is also a strongly convex rational polyhedral cone and there is a $m_0' \in M \cap \sigma^\vee$ such that

$$\tau = \sigma \cap \{m_0'\}^\perp.$$

Now we define a fan.

Definition 2.2. A *fan* in N is a non-empty collection Δ of strongly convex rational polyhedral cones in $N_\mathbb{R}$ satisfying the following conditions:

- Every face of any $\sigma \in \Delta$ is again in Δ.
- For $\sigma, \sigma' \in \Delta$ the intersection $\sigma \cap \sigma'$ is a face of both σ and σ'.

The union $|\Delta| = \bigcup_{\sigma\in\Delta} \sigma$ is called the support of Δ.

Now let σ be a strongly convex rational polyhedral cone in $N_{\mathbb{R}}$. Then we define the subsemigroup $\mathcal{S}_\sigma$ of M associated to σ as follows:

$$\mathcal{S}_\sigma = M \cap \sigma^\vee = \{m \in M;\ \langle m, y\rangle \geq 0 \text{ for all } y \in \sigma\}$$

It has the following properties:

- $\mathcal{S}_\sigma$ is a finitely generated additive subsemigroup of M containing 0.
- $\mathcal{S}_\sigma$ generates M as a group.
- $\mathcal{S}_\sigma$ is saturated, i.e. $cm \in \mathcal{S}_\sigma$ for $m \in M$ and a positive integer c implies $m \in \mathcal{S}_\sigma$.
- If τ is a face of σ defined by $\tau = \sigma \cap \{m_0\}^\perp$ for $m_0 \in M \cap \sigma^\vee$ then we have
$$\mathcal{S}_\tau = \mathcal{S}_\sigma + \mathbb{Z}_{\geq 0}(-m_0).$$

Now we a ready to define our local models for toric varieties. Let $\mathcal{S}_\sigma = M \cap \sigma^\vee = \mathbb{Z}_{\geq 0}m_1 + \cdots + \mathbb{Z}_{\geq 0}m_p$ be the finitely generated subsemigroup of M associated to the strongly convex rational polyhedral cone σ in $N_{\mathbb{R}}$. We define

$$U_\sigma = \{u : \mathcal{S}_\sigma \to \mathbb{C};\ u(0) = 1, u(m + m') = u(m)u(m') \text{ for all } m, m' \in \mathcal{S}_\sigma\}.$$

We have the following lemma

Lemma 2.3 ([**47**, p. 4-5]). *Let $\mathfrak{e}(m)(u) = u(m)$ for $m \in \mathcal{S}_\sigma$ and $u \in U_\sigma$. Then the map*

$$(\mathfrak{e}(m_1), \dots, \mathfrak{e}(m_p)) : U_\sigma \to \mathbb{C}^p$$

is injective. Identified with its image under this map U_σ is an algebraic subset of $\mathbb{C}^p$ defined as the set of solutions of a system of equations of the form (monomial) = (monomial). The structure of an r-dimensional irreducible normal complex analytical space on U_σ induced by the usual complex analytical structure on $\mathbb{C}^p$ is independent of the system $\{m_1, \dots, m_p\}$ of semigroup generators chosen. Each $m \in \mathcal{S}_\sigma$ gives rise to a polynomial function $\mathfrak{e}(m)$ on U_σ which is a holomorphic function with respect to the above structure.

If τ is a face of σ such that $\tau = \sigma \cap \{m_0\}^\perp$ with $m_0 \in \mathcal{S}_\sigma$ then U_τ may be identified with the open subset $\{u \in U_\sigma;\ u(m_0) \neq 0\}$ of U_σ. This enables us to construct toric varieties.

Theorem 2.4 ([**47**, p. 7]). *For a fan Δ in N, we can naturally glue $\{U_\sigma;\ \sigma \in \Delta\}$ together to obtain a Hausdorff complex analytical space*

$$T_N^{\mathbb{C}} emb(\Delta) = \bigcup_{\sigma \in \Delta} U_\sigma,$$

which is irreducible and normal with dimension equal to $r = \operatorname{rank} N$. We call it the toric variety associated to the fan Δ.

$T_N^{\mathbb{C}}\mathrm{emb}(\Delta)$ is called toric variety for the following reasons: The fan Δ always contains $\{0\}$. Furthermore we have $\mathcal{S}_{\{0\}} = M$ and $U_{\{0\}} = T_N^{\mathbb{C}} = \hom(M, \mathbb{C}^*)$. Here $T_N^{\mathbb{C}} = \hom(M, \mathbb{C}^*)$ is the r-dimensional algebraic torus $(\mathbb{C}^*)^r$. Because $\{0\}$ is a face of each cone in Δ, $T_N^{\mathbb{C}}$ is an open subset of each U_σ. Therefore $T_N^{\mathbb{C}}\mathrm{emb}(\Delta)$ contains $T_N^{\mathbb{C}}$ as an open subset.

$T_N^{\mathbb{C}}$ acts on $T_N^{\mathbb{C}}\mathrm{emb}(\Delta)$ as follows. Let $t \in T_N^{\mathbb{C}}$ and $u \in U_\sigma$. Then we define $tu : \mathcal{S}_\sigma \to \mathbb{C}$ by

$$(tu)(m) = t(m)u(m) \text{ for } m \in \mathcal{S}_\sigma.$$

Obviously tu is an element of U_σ. So we obtain an action of $T_N^{\mathbb{C}}$ on U_σ and by the natural gluing on $T_N^{\mathbb{C}}\mathrm{emb}(\Delta)$.

We have the following converse of this construction.

THEOREM 2.5 ([**47**, p. 10]). *Suppose the algebraic torus $T_N^{\mathbb{C}}$ acts algebraically on an irreducible normal algebraic variety X locally of finite type over $\mathbb{C}$. If X contains an open orbit isomorphic to $T_N^{\mathbb{C}}$ then there exists a unique fan Δ in N such that X is equivariantly isomorphic to $T_N^{\mathbb{C}}emb(\Delta)$.*

Now let $T_N^{\mathbb{C}}\mathrm{emb}(\Delta)$ be the toric variety associated to the fan Δ in N. Then $T_N^{\mathbb{C}}\mathrm{emb}(\Delta)$ is non-singular if and only if each $\sigma \in \Delta$ is non-singular in the following sense: There exists a $\mathbb{Z}$-basis of N, $n_1, \ldots, n_r$ and $s \leq r$ such that $\sigma = \mathbb{R}_{\geq 0}n_1 + \cdots + \mathbb{R}_{\geq 0}n_s$. We call Δ non-singular in this case.

Furthermore $T_N^{\mathbb{C}}\mathrm{emb}(\Delta)$ is compact if and only if Δ is a finite and complete fan, i.e. Δ is a finite set and $|\Delta| = N_{\mathbb{R}}$.

Our next goal is to describe those fans which correspond to compact projective toric varieties, i.e. to those varieties which can be embedded holomorphically into a complex projective space as a closed subvariety. To do so we need the following definition.

DEFINITION 2.6. Let Δ be a finite complete fan. A real valued function $h : N_{\mathbb{R}} \to \mathbb{R}$ is said to be a Δ-*linear support function* if it is $\mathbb{Z}$-valued on N and is linear on each $\sigma \in \Delta$.

This means that there is a $l_\sigma \in M$ for each $\sigma \in \Delta$ such that $h(y) = \langle l_\sigma, y\rangle$ for $y \in \sigma$.

If σ has dimension r then l_σ is uniquely determined by this construction. Namely let $l_\sigma, l'_\sigma \in M$ such that

$$h(y) = \langle l_\sigma, y\rangle = \langle l'_\sigma, y\rangle \text{ for all } y \in \sigma.$$

Then we have $l_\sigma - l'_\sigma \in M \cap \sigma^\perp = \{0\}$.

We call h *strictly upper convex* with respect to Δ if for any r-dimensional $\sigma \in \Delta$ and all $y \in N_{\mathbb{R}}$ we have

$$\langle l_\sigma, y\rangle \geq h(y)$$

with equality holding if and only if $y \in \sigma$.

Now we have the following theorem.

THEOREM 2.7 ([**47**, p. 84]). *A compact toric variety $X = T_N^{\mathbb{C}}emb(\Delta)$ is a projective variety if and only if there is a Δ-linear support function h which is strictly upper convex with respect to Δ.*

Next we want to explain how an absolutely simple integral polytope in $M_{\mathbb{R}}$ gives rise to a non-singular compact projective toric variety. Here a polytope P is called *absolutely simple integral* if all vertices of P belong to M and at each vertex v meet exactly r edges such that $\{m^{(1)} - v, \ldots, m^{(r)} - v\}$ is a $\mathbb{Z}$-basis of M, where $m^{(1)}, \ldots, m^{(r)} \in M$ on the r edges are right next to the vertex v. We should mention that all non-singular compact projective toric varieties arise in this form.

THEOREM 2.8 ([**47**, p. 93-94]). *Let P be a r-dimensional absolutely simple integral convex polytope in $M_{\mathbb{R}}$. Then: There exists a unique finite complete fan Δ in N such that the support function $h : N_{\mathbb{R}} \to \mathbb{R}$ for P defined by*

$$h(y) = \inf\{\langle x, y\rangle;\, x \in P\} \text{ for } y \in N_{\mathbb{R}}$$

is a Δ-linear support function strictly upper convex with respect to Δ. We denote the corresponding r-dimensional toric projective variety by

$$X_P = T_N^{\mathbb{C}}emb(\Delta).$$

By parallel translation with respect to $m \in M$ we have $X_{m+P} = X_P$. Furthermore X_P is non-singular and X_P/T is homeomorphic to P, where T is the maximal compact subtorus of the algebraic torus $T_N^{\mathbb{C}}$.

Now we describe the cohomology of a non-singular compact toric variety in terms of its fan. Let Δ be a non-singular finite complete fan. For each one-dimensional cone $\rho \in \Delta$ introduce a variable u_ρ and consider the polynomial ring

$$R = \mathbb{Z}[u_\rho;\ \rho \in \Delta \text{ one-dimensional}] \qquad\qquad \deg u_\rho = 2.$$

Let I be the ideal in R generated by the set

$$\{u_{\rho_1} u_{\rho_2} \dots u_{\rho_s};\ \text{distinct } \rho_1, \dots, \rho_s \in \Delta \text{ one-dimensional with } \rho_1 + \dots + \rho_s \notin \Delta\}$$

Furthermore define J to be the ideal of R generated by

$$\left\{ \sum_{\rho \in \Delta \text{ one-dimensional}} \langle m, n_\rho \rangle u_\rho;\ m \in M \right\}.$$

Here for a one-dimensional cone ρ n_ρ is the unique primitive element of $N \cap \rho$ such that $\rho = \mathbb{R}_{\geq 0} n_\rho$. We have the following theorem.

Theorem 2.9 ([**47**, p. 134]). *Let $X = T_N^{\mathbb{C}} emb(\Delta)$ be a non-singular compact toric variety. Then there is an isomorphism of rings:*

$$H^*(X; \mathbb{Z}) \cong R/(I + J).$$

2.2. Hamiltonian group actions on symplectic manifolds

In this section we state the basic properties of hamiltonian group actions on symplectic manifolds. The results of this section are taken from [**4**] and [**15**]. Let us start with a definition.

Definition 2.10. A *symplectic manifold* is a pair (M, ω) where M is an even-dimensional smooth manifold without boundary and ω is a closed non-degenerated two-form on M.

Let (M, ω) be a symplectic manifold of dimension $2n$, then the n-fold product $\omega \wedge \dots \wedge \omega$ never vanishes because ω is non-degenerated. Therefore M is orientable.

Because ω is closed it represents a cohomology class $a = [\omega] \in H^2(M, \mathbb{R})$. If M is closed then the cohomology class a^n is represented by ω^n and the integral of this form over M does not vanish. Therefore we have that ω is not exact and $a^n \neq 0$.

On every symplectic manifold there are so called calibrated almost complex structures J. Here *calibrated* means that

$$\omega(J\,\cdot\,, J\,\cdot\,) = \omega(\,\cdot\,, \,\cdot\,)$$

and that the symmetric bilinear form $\omega(J\,\cdot\,, \,\cdot\,)$ is positive definite at each point.

Because ω is non-degenerated it induces a pairing between the tangent and cotangent spaces of M. Therefore we may define the *symplectic gradient* X_H of a function $H : M \to \mathbb{R}$ by

$$\iota(X_H)\omega = dH.$$

X_H is also called the *hamiltonian vector field* associated with H. H is called a *hamiltonian* for X_H.

Definition 2.11. A vector field X on M is *hamiltonian* if $\iota(X)\omega$ is an exact form, *locally hamiltonian* if it is closed. One writes $\mathcal{H}(M)$ and $\mathcal{H}_{\text{loc}}(M)$, respectively, for the space of hamiltonian and locally hamiltonian vector fields.

There is an exact sequence

$$0 \longrightarrow \mathcal{H}(M) \longrightarrow \mathcal{H}_{\text{loc}}(M) \longrightarrow H^1(M; \mathbb{R}) \longrightarrow 0$$

In particular on a simply connected symplectic manifold all locally hamiltonian vector fields are hamiltonian.

As the next step towards the introduction of hamiltonian group actions we introduce the Poisson bracket on a symplectic manifold. It defines a Lie-algebra structure on $C^\infty(M)$ and is defined as follows.

DEFINITION 2.12. Let (M, ω) be a symplectic manifold. Then the *Poisson bracket* of two functions $F, H : M \to \mathbb{R}$ is defined by

$$\{F, H\} = \omega(X_F, X_H).$$

Let G be a Lie-group and LG its Lie-algebra. If G acts smoothly on the manifold M then we may associate to an $X \in LG$ a *fundamental vector field* $\underline{X}$. It is the vector field on M with the flow

$$g_t(x) = \exp(tX)x \qquad\qquad (x \in M).$$

A G-action on a symplectic manifold M is called *symplectic* if all $g \in G$ preserve the symplectic form ω, that means $g^*\omega = \omega$. By considering the Lie-derivative of ω associated to a fundamental vector field of the G-action one finds the following:

LEMMA 2.13. *If the G-action on M preserves the symplectic form ω then all fundamental vector fields of the action are locally hamiltonian.*

From the lemma we get the following diagram

$$\begin{array}{ccccccccc}
 & & C^\infty(M) & & LG & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{H}(M) & \longrightarrow & \mathcal{H}_{\mathrm{loc}}(M) & \longrightarrow & H^1(M;\mathbb{R}) & \longrightarrow & 0
\end{array}$$

DEFINITION 2.14. The symplectic G-action on M is called *hamiltonian* if there is a Lie-algebra morphism $\tilde{\mu} : LG \to C^\infty(M)$ making the diagram commute.

Associated to $\tilde{\mu}$ is its moment map

$$\begin{aligned} \mu : M &\to LG^* \\ x &\mapsto (X \mapsto \tilde{\mu}(X)(x)). \end{aligned}$$

For the fundamental vector field $\underline{X}$ associated to $X \in LG$ we have

$$\iota(\underline{X})\omega = d\langle \mu(\,\cdot\,), X\rangle.$$

That means that $x \mapsto \langle \mu(x), X\rangle$ is a hamiltonian for $\underline{X}$.

Now we restrict ourselves to the case where M is compact and connected and $G = T$ is a compact torus. Then we have $\dim T \leq \frac{1}{2}\dim M$. Furthermore we have the following famous theorem of Atiyah [**3**] and Guillemin and Sternberg [**24**].

THEOREM 2.15. *Let M be a compact connected symplectic manifold with a hamiltonian action of the torus T. Then $\mu(M)$ is a convex polytope.*

We are interested in the special case $\dim T = \frac{1}{2}\dim M$. For this case we have the following results of Delzant.

THEOREM 2.16. *Let M_1, M_2 be two closed connected symplectic manifolds of dimension $2n$ and T a torus of dimension n such that T acts effectively and hamiltonian on M_1 and M_2. Furthermore let μ_i be the corresponding moment maps. If $\mu_1(M_1) = \mu_2(M_2)$ then there is a symplectic T-equivariant diffeomorphism $\phi : M_1 \to M_2$.*

Furthermore there is the following lemma:

LEMMA 2.17. *Let (M, ω) be a closed connected manifold of dimension $2n$ with a hamiltonian effective action of the n-dimensional torus T. Furthermore let $\mu : M \to LT^*$ be the corresponding moment map. Then we have:*

(1) *μ induces a homeomorphism $\bar{\mu} : M/T \to \mu(M)$.*
(2) *For $\bar{x} \in \mu(M)$ let F be the face of $\mu(M)$ which contains $\bar{x}$ in its relative interior then $\mu^{-1}(\bar{x})$ is a torus of dimension $\dim F$.*
(3) *Let $x \in \mu^{-1}(\bar{x})$. Then the isotropy group of x is the connected subgroup of T whose Lie-algebra is the annihilator of $F - \bar{x}$.*

The following theorem due to Delzant describes those polytopes in LT^* which arise as images of moment maps for hamiltonian T-actions on a closed connected symplectic manifold.

THEOREM 2.18. *A convex polytope P in LT^* is the image of the moment map for some symplectic manifold (M, ω) with hamiltonian T-action if and only if for each vertex $v \in P$, there are n points q_i lying on the rays obtained by extending the edges emanating from v, so that n vectors $\{q_i - p\}$ constitute a basis of $(\mathbb{Z}^n)^* \subset LT^*$.*

The following fact is a byproduct of Delzant's work: Every symplectic $2n$-manifold with a hamiltonian T-action is equivariantly diffeomorphic to a toric variety.

2.3. Quasitoric manifolds

Quasitoric manifolds were introduced by Davis and Januszkiewicz [**13**]. It can be shown that non-singular projective toric varieties with the natural action of the compact torus and symplectic manifolds with a hamiltonian action of a half-dimensional torus are examples for quasitoric manifolds.

Now let us recall the definition of a quasitoric manifold. Let M be a smooth closed connected orientable $2n$-dimensional manifold on which the n-dimensional torus T acts. We say that the T-action on M is *locally standard* if it is locally isomorphic to the standard action on $\mathbb{C}^n$ up to an automorphism of T. If the T-action on M is locally standard then the orbit space M/T is locally homeomorphic to the cone

$$\mathbb{R}^n_{\geq 0} = \{(x_1, \dots, x_n) \in \mathbb{R}^n; \, x_i \geq 0\}.$$

Therefore it is a manifold with corners [**14**, p. 303-304].

Simple polytopes are examples of manifolds with corners. A n-dimensional convex polytope is called *simple* if it has exactly n facets incident with each of its vertices.

DEFINITION 2.19. In the above situation M is called *quasitoric* if the T-action on M is locally standard and the orbit space M/T is face preserving homeomorphic to a simple n-dimensional convex polytope P.

Now let M be a quasitoric manifold and $\pi : M \to P$ the orbit map. Then P is determined by M up to face preserving homeomorphism or equivalently up to combinatorial equivalence. By definition two polytopes are called combinatorially equivalent if and only if their face posets are isomorphic.

Denote by $\mathfrak{F}$ the set of facets of P. Then for $F_i \in \mathfrak{F}$, $M_i = \pi^{-1}(F_i)$ is a closed connected submanifold of codimension two in M which is fixed pointwise by a one-dimensional subtorus $\lambda(F_i) = \lambda(M_i)$ of T. We call these M_i the *characteristic submanifolds* of M. The map

$$\lambda : \mathfrak{F} \to \{\text{one-dimensional subtori of } T\}$$

is called the *characteristic map* for M. This map extends uniquely to a map from the face poset of P to the poset of subtori of T. We denote this extension also by λ. For $p \in P$ denote by $F(p)$ the face of P which contains p in its relative interior. Davis and Januszkiewicz proved the following result.

LEMMA 2.20 ([**13**, p. 424]). *Let M be a quasitoric manifold over the simple convex polytope P and let λ be its characteristic map. Then M is T-equivariantly homeomorphic to*

$$M(P,\lambda) = P \times T/\sim,$$

where for $(p_1,t_1),(p_2,t_2) \in P \times T$ we have

$$(p_1,t_1) \sim (p_2,t_2)$$

if and only if $p_1 = p_2$ and $t_1 t_2^{-1} \in \lambda(F(p_1))$. In particular M is determined up to T-equivariant homeomorphism by the combinatorial type of P and λ.

Let N be the lattice of one-parameter circle subgroups of T. Then for a facet F_i of P $\lambda(F_i)$ is determined by a primitive vector $\bar{\lambda}(F_i) \in N$. This $\bar{\lambda}(F_i)$ is determined up to sign by F_i. We call

$$\bar{\lambda} : \mathfrak{F} \to N$$

the *characteristic function* for M. An identification of T with the standard n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ induces an identification of N with $\mathbb{Z}^n$. With this identification understood we may write $\bar{\lambda}$ in matrix form

$$\Lambda = \begin{pmatrix} \lambda_{1,1} & \dots & \lambda_{1,m} \\ \vdots & & \vdots \\ \lambda_{n,1} & \dots & \lambda_{n,m} \end{pmatrix},$$

where m is the number of facets of P. We call Λ the *characteristic matrix* for M.

Next we want to describe the Betti-numbers of a quasitoric manifold. To do so we first introduce the h-vector of a simple polytope. Let P be a simple n-dimensional polytope. For $0 \leq i \leq n-1$ denote by f_i the number of codimension $i+1$ faces of P.

Define the polynomial $\Psi_P(t)$ as follows

$$\Psi_P(t) = (t-1)^n + \sum_{i=0}^{n-1} f_i (t-1)^{n-1-i}$$

Then the h-vector $(h_0, \dots, h_n)$ of P is defined by the following equation

$$\Psi_P(t) = \sum_{i=0}^{n} h_i t^{n-i}$$

Obviously we have $h_0 = 1$, $h_1 = f_0 - n$, $\sum_{i=0}^{n} h_i = f_{n-1}$.

THEOREM 2.21 ([**13**, p. 430,432]). *Let M be a quasitoric manifold over the simple convex polytope P. Then the homology of M vanishes in odd degrees and is free abelian in even degrees. Let $b_{2i}(M)$ denote the rank of $H_{2i}(M;\mathbb{Z})$. Then*

$$b_{2i}(M) = h_i.$$

Furthermore M is simply connected.

Our next goal is the description of the T-equivariant cohomology of a quasitoric manifold M. To do so we first introduce the Borel-construction. Let $ET \to BT$ be a universal principal T-bundle. Then ET is a contractible free right T-space. The *Borel-construction* M_T of M is defined as

$$M_T = ET \times_T M.$$

The *T-equivariant cohomology* of M is defined to be the cohomology of M_T:

$$H_T^*(M) = H^*(M_T)$$

It turns out that for quasitoric manifolds over the simple convex polytope P the homotopy-type of M_T depends only on the combinatorial type of P [**13**, p. 434]. For a simple convex polytope P with facets $F_1, \dots, F_m$ let

$$R = \mathbb{Z}[u_1, \dots, u_m] \qquad \deg u_i = 2.$$

Furthermore let I be the ideal in R generated by

$$\{u_{i_1} \dots u_{i_k};\ F_{i_1}, \dots, F_{i_k} \text{ distinct with } \bigcap_{j=1}^{k} F_{i_j} = \emptyset\}.$$

Then the *face ring* or *Stanley-Reisner-ring* of P is defined as

$$R(P) = R/I.$$

Theorem 2.22 ([**13**, p. 436]). *Let M be a quasitoric manifold over the simple convex polytope P. Then the T-equivariant cohomology $H_T^*(M;\mathbb{Z}) = H^*(M_T;\mathbb{Z})$ is isomorphic as a ring to the face ring $R(P)$ of P.*

Now let

$$\Lambda = \begin{pmatrix} \lambda_{1,1} & \dots & \lambda_{1,m} \\ \vdots & & \vdots \\ \lambda_{n,1} & \dots & \lambda_{n,m} \end{pmatrix},$$

be the characteristic matrix for M and denote by J the ideal of R generated by

$$\{\lambda_{1,1}u_1 + \dots + \lambda_{1,m}u_m, \dots, \lambda_{n,1}u_1 + \dots + \lambda_{n,m}u_m\}.$$

Then we have the following:

Theorem 2.23 ([**13**, p. 439]). *Let M be a quasitoric manifold over P. Then $H^*(M;\mathbb{Z})$ is isomorphic to $R/(I+J)$.*

This theorem shows that the cohomology of a quasitoric manifold has a similar structure as the cohomology of a non-singular toric variety.

It can be shown that under the isomorphism given in the theorem u_i is the Poincaré-dual of the characteristic submanifold $M_i = \pi^{-1}(F_i)$. Now denote by L_i, $1 \leq i \leq m$, the complex line bundle over M with first Chern-class equal to u_i in $H^*(M;\mathbb{Z}) \cong R/(I+J)$. Then the restriction of L_i to M_i equals the oriented normal bundle of M_i in M for an appropriately chosen orientation of M_i. Furthermore the stable tangent bundle of M is isomorphic to

$$L_1 \oplus \dots \oplus L_m$$

This induces a stable almost complex structure on M. These stable almost complex structures are very rich. For example it was shown by Buchstaber and Ray [**11**] and Buchstaber, Panov and Ray [**10**] that every complex cobordism class in dimension greater than two contains a quasitoric manifold.

2.4. Torus manifolds

Another generalisation of a non-singular toric variety is a torus manifold introduced by Masuda [**41**] and Hattori and Masuda [**26**]. It is defined as follows.

Definition 2.24. A *torus manifold* is a $2n$-dimensional closed connected orientable smooth manifold M with an effective smooth action of a n-dimensional torus T such that $M^T \neq \emptyset$.

The fixed point set of a torus manifold necessarily consists out of a finite number of isolated points. A closed connected submanifold M_i of codimension two of a torus manifold M which is fixed pointwise by a circle subgroup $\lambda(M_i)$ of T and contains a T-fixed point is called a *characteristic submanifold* of M. We denote the set of characteristic submanifolds of a torus manifold by $\mathfrak{F}$. Since M is compact $\mathfrak{F}$ is a finite set and we denote its elements by M_i, $i = 1, \dots, m$. Because M is orientable each M_i is also orientable. We call a choice of orientations for each M_i together with an orientation of M a *omniorientation* for M.

In contrast to quasitoric manifolds the intersection of characteristic submanifolds of a torus manifold is not necessary connected. But there is the following lemma.

LEMMA 2.25 ([**42**, p. 719]). *Suppose that $H^*(M;\mathbb{Z})$ is generated in degree two. Then all non-empty multiple intersections of characteristic submanifolds are connected and have cohomology generated in degree two.*

In the following we discuss some properties of torus manifolds with vanishing odd degree cohomology. There is the following theorem.

THEOREM 2.26 ([**42**, p. 720]). *A torus manifold M with $H^{odd}(M;\mathbb{Z}) = 0$ is locally standard.*

The theorem implies that the orbit space of a torus manifold with vanishing odd degree cohomology is a nice manifold with corners. Here a manifold with corners is called *nice* if every codimension-k face is contained in exactly k facets. For each pair of faces G, H of a nice manifold with corners with non-empty intersection there is a unique minimal face $G \vee H$ containing both G and H.

We next generalise the notion of a face ring of a simple polytope to nice manifolds with corners.

Let Q be a nice manifold with corners. Denote by R the ring

$$R = \mathbb{Z}[u_F;\ F \text{ face of } Q] \qquad\qquad \deg u_F = 2\operatorname{codim} F.$$

Let I be the ideal of R generated by

$$\left\{ u_Q - 1, u_\emptyset, u_G u_H - u_{G\vee H} \sum_{E \text{ component of } G\cap H} u_E \right\}$$

Then the face ring of Q is defined as $R(Q) = R/I$.

THEOREM 2.27 ([**42**, p. 735]). *For a torus manifold M with vanishing odd degree cohomology there is an isomorphism of rings $R(M/T) \cong H^*_T(M;\mathbb{Z})$.*

If M is a torus manifold with vanishing odd degree cohomology and Q its orbit space then each face of Q contains a vertex. Therefore we may identify the set of facets of Q with $\mathfrak{F}$. As for quasitoric manifolds we have a characteristic map

$$\lambda : \mathfrak{F} \to \{\text{one-dimensional subtori of } T\}$$

and a characteristic matrix

$$\Lambda = \begin{pmatrix} \lambda_{1,1} & \dots & \lambda_{1,m} \\ \vdots & & \vdots \\ \lambda_{n,1} & \dots & \lambda_{n,m} \end{pmatrix}.$$

Let J be the ideal of R generated by

$$\{\lambda_{1,1}u_{Q_1} + \dots + \lambda_{1,m}u_{Q_m}, \dots, \lambda_{n,1}u_{Q_1} + \dots + \lambda_{n,m}u_{Q_m}\}$$

where Q_i are the facets of Q.

THEOREM 2.28 ([**42**, p. 736]). *For a torus manifold with vanishing odd degree cohomology we have an isomorphism of rings $H^*(M;\mathbb{Z}) \cong R/(I + J)$.*

Now we want to give a characterisation of torus manifolds with vanishing odd degree cohomology. To do so we first introduce some notation.

A space X is called *acyclic* if $\tilde{H}_i(X;\mathbb{Z}) = 0$ for all i. We say that a manifold with corners Q is *face acyclic* if all faces of Q are acyclic. We call Q a *homology polytope* if all faces of Q are acyclic and all intersections of faces of Q are connected. With this notation we have the following theorem.

THEOREM 2.29 ([**42**, p. 738,742]). *Let M be a torus manifold. Then:*

- *$H^*(M;\mathbb{Z})$ is generated by its degree two part if and only if the torus action on M is locally standard and M/T is a homology polytope.*
- *$H^{odd}(M,\mathbb{Z}) = 0$ if and only if the torus action on M is locally standard and M/T is face acyclic.*

Associated to a torus manifold is a multi-fan, which is a generalisation of a fan (for the precise definition see [**26**, p. 7-8,41]). Important topological invariants of a torus manifold are determined by its multi-fan. For example if M possesses a T-invariant stable almost complex structure then the T_y-genus of M is determined by its multi-fan. But in contrast to fans and toric varieties it may happen that different torus manifolds have the same multi-fan associated to them.

CHAPTER 3

Classification of quasitoric manifolds up to equivariant homeomorphism

In this chapter we give three sufficient criteria for two quasitoric manifolds M, M' to be (weakly) equivariantly homeomorphic. The first criterion gives a condition on the cohomology of M and M' (see section 3.1).

The stable tangent bundle of a quasitoric manifold M splits as a sum of complex line bundles. This induces a BT^m-structure on the stable tangent bundle of M. We show in section 3.2 that two BT^m-bordant quasitoric manifolds are weakly equivariantly homeomorphic.

In section 3.3 we show that two quasitoric manifolds having the same GKM-graphs are equivariantly homeomorphic.

In this chapter we take all cohomology groups with coefficients in $\mathbb{Z}$.

3.1. Isomorphisms of cohomology rings

At first we introduce some notations concerning quasitoric manifolds and their characteristic functions. We follow [**43**] for this description. Let M be a quasitoric manifold over the simple polytope P. We denote the orbit map by $\pi : M \to P$. Furthermore we denote the set of facets of P by $\mathfrak{F} = \{F_1, \dots, F_m\}$. The characteristic submanifolds $M_i = \pi^{-1}(F_i)$, $i = 1, \dots, m$, of M are the preimages of the facets of P. Each M_i is fixed pointwise by a one-dimensional subtorus $\lambda(F_i) = \lambda(M_i)$ of T.

The following lemma was proved by Davis and Januszkiewicz [**13**, p. 424]:

Lemma 3.1. *A quasitoric manifold M with $P = M/T$ is determined up to equivariant homeomorphism by the combinatorial type of P and the function λ.*

Let N be the integer lattice of one-parameter circle subgroups in T, so we have $N \cong \mathbb{Z}^n$. We denote by $\bar{\lambda} : \mathfrak{F} \to N$ the characteristic function of M. Then for a given facet F_i of P $\bar{\lambda}(F_i)$ is a primitive vector that spans $\lambda(F_i)$. $\bar{\lambda}(F_i)$ is determined up to sign by this condition.

An omniorientation of M helps to eliminate the indeterminateness in the definition of a characteristic function. This is done as follows: An omniorientation of M determines orientations for all normal bundles of the characteristic submanifolds of M. The action of a one-parameter circle subgroup of T also determines orientations for these bundles. We choose the primitive vectors $\bar{\lambda}(F_i)$ in such a way that the two orientations on $N(M_i, M)$ coincide.

A characteristic function satisfies the following non-singularity condition. For pairwise distinct facets $F_{j_1}, \dots, F_{j_n}$ of P,

$$\bar{\lambda}(F_{j_1}), \dots, \bar{\lambda}(F_{j_n})$$

forms a basis of N whenever the intersection

$$F_{j_1} \cap \dots \cap F_{j_n}$$

is non-empty. After reordering the facets we may assume that

$$F_1 \cap \dots \cap F_n \neq \emptyset.$$

Therefore $\bar{\lambda}(F_1), \dots, \bar{\lambda}(F_n)$ is a basis of N. This allows us to identify N with $\mathbb{Z}^n$ and the torus T with the standard n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$.

With this identifications understood we may write $\bar{\lambda}$ as an integer matrix of the form

$$\Lambda = \begin{pmatrix} 1 & & & \lambda_{1,n+1} & \dots & \lambda_{1,m} \\ & \ddots & & \vdots & & \vdots \\ & & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix} \tag{3.1}$$

With this notation $\lambda(F_i)$, $i = 1, \dots, m$ is given by

$$\left\{ t \begin{pmatrix} \lambda_{1,i} \\ \vdots \\ \lambda_{n,i} \end{pmatrix} \in \mathbb{R}^n/\mathbb{Z}^n; t \in \mathbb{R} \right\}.$$

Let $u_i \in H^2(M)$ be the Poincaré-dual of the characteristic submanifold M_i. Then the cohomology ring $H^*(M)$ is generated by $u_1, \dots, u_m$. The u_i are subject to the following relations [**13**, p. 439]:

(1) $\forall I \subset \{1, \dots, m\}$ $\prod_{i \in I} u_i = 0 \Leftrightarrow \bigcap_{i \in I} F_i = \emptyset$
(2) For $i = 1, \dots, n$ $-u_i = \sum_{j=n+1}^m \lambda_{i,j} u_j$.

Two quasitoric manifolds M, M' are weakly T-equivariantly homeomorphic if there is an automorphism $\theta : T \to T$ and a homeomorphism $f : M \to M'$ such that for all $x \in M$ and $t \in T$:

$$f(tx) = \theta(t) f(x).$$

Because the identification of T with $\mathbb{R}^n/\mathbb{Z}^n$ depends on a choice of a basis in N a quasitoric manifold M is determined by the combinatorial type of P and the characteristic matrix Λ only up to weakly equivariant homeomorphism.

Now we are in the position to prove our first theorem.

THEOREM 3.2. *Let M, M' be quasitoric manifolds of dimension n. Furthermore let $u_1, \dots, u_m \in H^2(M)$ be the Poincaré-duals of the characteristic submanifolds of M and $u_1', \dots, u_{m'}' \in H^2(M')$ the Poincaré-duals of the characteristic submanifolds of M'. If there is a ring isomorphism $f : H^*(M) \to H^*(M')$ with $f(u_i) = u_i'$, $i = 1, \dots, m$, then M and M' are weakly T-equivariantly homeomorphic.*

PROOF. At first notice that f preserves the grading of $H^*(M)$ and

$$m = b_2(M) + n = b_2(M') + n = m'.$$

For $I \subset \{1, \dots, m\}$ we have

$$\begin{aligned} & \bigcap_{i \in I} F_i = \emptyset \\ \Leftrightarrow & \prod_{i \in I} u_i = 0 \\ \Leftrightarrow & \prod_{i \in I} u_i' = \prod_{i \in I} f(u_i) = 0 \\ \Leftrightarrow & \bigcap_{i \in I} F_i' = \emptyset \end{aligned}$$

Here F_i, F_i' denote the facets of M/T and M'/T, respectively. Therefore M/T and M'/T are combinatorially equivalent.

Now we show that the characteristic matrices of M and M' are equal. We may assume that $F_1 \cap \dots \cap F_n \neq \emptyset \neq F_1' \cap \dots \cap F_n'$. Then $u_{n+1}, \dots, u_m$ forms a basis of $H^2(M)$ and $u_{n+1}', \dots, u_m'$ a basis of $H^2(M')$.

If we write the characteristic matrices Λ, Λ' for M, M' in the form (3.1) then we have

$$-u_i = \sum_{j=n+1}^{m} \lambda_{i,j} u_j$$
$$-u'_i = \sum_{j=n+1}^{m} \lambda'_{i,j} u'_j$$

for $i = 1, \dots, n$. Therefore we have

$$\sum_{j=n+1}^{m} \lambda'_{i,j} u'_j = -u'_i = f(-u_i) = \sum_{j=n+1}^{m} \lambda_{i,j} f(u_j) = \sum_{j=n+1}^{m} \lambda_{i,j} u'_j.$$

It follows that $\lambda'_{i,j} = \lambda_{i,j}$, $i = 1, \dots, n$, $j = n+1, \dots, m$. Therefore the characteristic matrices are the same. □

3.2. Bordism

To state our second theorem we first fix some notation. Let M be a omnioriented quasitoric manifold. By [**13**, p. 446] and [**12**, p. 71] there is an isomorphism of real vector bundles

$$TM \oplus \mathbb{R}^{2(m-n)} \cong L_1 \oplus \dots \oplus L_m$$

where the L_i are complex line bundles with

$$c_1(L_i) = u_i.$$

This isomorphism corresponds to a reduction of structure group in the stable tangent bundle of M from $O(2m)$ to T^m.

Let $g : M \to BO(2m)$ be a classifying map for the stable tangent bundle of M. Furthermore let $f_i : M \to BT^1$ be the classifying map of the line bundle L_i. Then the following diagram commutes up to homotopy:

$$\begin{array}{ccc} & & BT^m = BT^1 \times \dots \times BT^1 \\ & \overset{\prod f_i}{\nearrow} & \downarrow p_m \\ M & \longrightarrow & BO(2m) \end{array}$$

where p_m is the natural fibration [**30**, p. 77]. We may replace $\prod f_i$ by a homotopic map f which makes the above diagram commutative. By f there is given a (BT^m, p_m)-structure on the stable tangent bundle of M [**55**, p. 14]. We denote by $\Omega_n(BT^\infty, p)$ the bordism groups of the sequence

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & BT^m & & \longrightarrow & & BT^{m+1} & \longrightarrow & \cdots \\ & & \downarrow p_m & & & & \downarrow p_{m+1} & & \\ \cdots & \longrightarrow & BO(2m) & \longrightarrow & BO(2m+1) & \longrightarrow & BO(2m+2) & \longrightarrow & \cdots \end{array}$$

THEOREM 3.3. *Let M, M' be omnioriented quasitoric manifolds with $[M] = [M'] \in \Omega_n(BT^\infty, p)$. Then M and M' are weakly T-equivariantly homeomorphic.*

PROOF. We use the following notation. Let $f : M \to BT^\infty, L_1, \dots, L_m$ as above and $f' : M' \to BT^\infty, L'_1, \dots, L'_{m'}$ analogous. Let $\{F_1, \dots, F_m\}$ and $\{F'_1, \dots, F'_{m'}\}$ be the set of facets of $M/T = P$ and $M'/T = P'$, respectively.

Furthermore let

$$H^*(BT^\infty) = \mathbb{Z}[x_1, x_2, x_3, \dots].$$

Then be have

$$f^*(x_i) = \begin{cases} c_1(L_i) & \text{if } i = 1, \dots, m \\ 0 & \text{else.} \end{cases} \tag{3.2}$$

Without loss of generality we may assume that $m' \geq m$. Because bordant manifolds have the same characteristic numbers, for all $i_1, \dots, i_n \in \{1, \dots, m'\}$ we get

$$f^*(x_{i_1} \dots x_{i_n})[M] = f'^*(x_{i_1} \dots x_{i_n})[M'].$$

If the i_j are pairwise distinct then we have by (3.2)

$$f^*(x_{i_1} \dots x_{i_n})[M] = \begin{cases} \pm 1 & \text{if } i_j \leq m \text{ and } F_{i_1} \cap \dots \cap F_{i_n} \neq \emptyset \\ 0 & \text{else.} \end{cases}$$

Since this holds analogously for M' we get

$$m = m', \tag{3.3}$$

$$F_{i_1} \cap \dots \cap F_{i_n} = \emptyset \Leftrightarrow F'_{i_1} \cap \dots \cap F'_{i_n} = \emptyset. \tag{3.4}$$

By (3.4) P and P' are combinatorially equivalent. An equivalence is given by

$$\bigcap_{i \in I} F_i \mapsto \bigcap_{i \in I} F'_i, \quad (I \subset \{1, \dots, m\}).$$

Without loss of generality we may assume that $F_1 \cap \dots \cap F_n$ is non-empty. Then $f^*(x_{n+1}), \dots, f^*(x_m)$ form a basis of $H^2(M)$. Similarly $f'^*(x_{n+1}), \dots, f'^*(x_m)$ form a basis of $H^2(M')$. Therefore there is an isomorphism

$$\begin{aligned} \psi : H^2(M) &\to H^2(M') \\ f^*(x_i) &\mapsto f'^*(x_i), \quad i > n \end{aligned}$$

We claim that the following diagram commutes.

$$\begin{array}{ccc} H^{2n-2}(M) & \xrightarrow{\cong} & \hom(H^2(M), \mathbb{Z}) \\ \uparrow{\scriptstyle f^*} & & \\ H^{2n-2}(BT^\infty) & & \cong \uparrow {\scriptstyle \psi^*} \\ \downarrow{\scriptstyle f'^*} & & \\ H^{2n-2}(M') & \xrightarrow[x \mapsto \langle \cdot \cup x, [M'] \rangle]{\cong} & \hom(H^2(M'), \mathbb{Z}) \end{array} \tag{3.5}$$

Let $x \in H^{2n-2}(BT^\infty)$. Then for $i > n$ we have

$$\begin{aligned} \psi^*(\langle \cdot \cup f'^*(x), [M'] \rangle)(f^*(x_i)) &= \langle f'^*(x_i) \cup f'^*(x), [M'] \rangle \\ &= \langle f^*(x_i) \cup f^*(x), [M] \rangle && \text{by bordism} \\ &= (\langle \cdot \cup f^*(x), [M] \rangle)(f^*(x_i)). \end{aligned}$$

Therefore the diagram commutes. Now we have for $i = 1, \dots, n$ and $x \in H^{2n-2}(BT^\infty)$:

$$\begin{aligned} \langle \psi(f^*(x_i)) \cup f'^*(x), [M'] \rangle &= \psi^*(\langle \cdot \cup f'^*(x), [M'] \rangle)(f^*(x_i)) \\ &= (\langle \cdot \cup f^*(x), [M] \rangle)(f^*(x_i)) && \text{by (3.5)} \\ &= \langle f^*(x_i) \cup f^*(x), [M] \rangle \\ &= \langle f'^*(x_i) \cup f'^*(x), [M'] \rangle && \text{by bordism} \end{aligned}$$

Because $f'^* : H^{2n-2}(BT^\infty) \to H^{2n-2}(M')$ is surjective, it follows that

$$f'^*(x_i) = \psi(f^*(x_i)), \quad \text{for } i = 1, \dots, n.$$

As in the proof of Theorem 3.2 one sees that the characteristic matrices for M and M' are equal. Therefore M and M' are weakly equivariantly homeomorphic. □

3.3. GKM-Graphs

Now we introduce the notion of a *GKM-graph* of a torus manifold following [**23**].

Let M^{2n} be a torus manifold and $M^1 = \{x \in M; \dim Tx = 1\}$. Then M^T consists of isolated points and M^1 has dimension two.

Let also

$$V = \{p_1, \dots, p_e\} = M^T$$
$$E = \{e_1, \dots, e_N\} = \{\text{components of } M^1\}$$

and for $i = 1, \dots, N$ let $\bar{e}_i$ be the closure of e_i in M. Then we have:

(1) $\bar{e}_i$ is an equivariantly embedded copy of $\mathbb{C}P^1$.
(2) $\bar{e}_i - e_i$ consists of two points out of V.
(3) for $p \in V$ we have $\#\{e_i; p \in \bar{e}_i\} = n$.

Therefore V and E are the vertices and edges of a graph Γ_M.

We get a labeling of the edges of Γ_M by elements of the weight lattice of T as follows: Let $p, q \in V \cap \bar{e}_i$ then the weights α_p, α_q of $T_p\bar{e}_i, T_q\bar{e}_i$ coincide up to sign and we define

$$\alpha : e_i \mapsto \alpha_p.$$

Then α is determined up to sign and is called the *axial function* on Γ_M.

We call Γ_M together with the axial function α the GKM-graph of M.

Now let M be a quasitoric manifold over the polytope P. Let Γ_P be the graph which consists of the edges and vertices of P. Then we have

$$\Gamma_M = \Gamma_P.$$

THEOREM 3.4. *Let M be a quasitoric manifold. Then M is determined up to equivariant homeomorphism by (Γ_M, α).*

PROOF. At first we introduce some notation. For a Lie-group G we denote its identity component of by G^0.

By [**5**, p. 287,296] the combinatorial type of P is uniquely determined by Γ_M. So we have to show that the function λ is determined by α.

Let F be a facet of P then we define

$$\lambda'(F) = \left(\bigcap_{e \subset F; e \text{ edge of } P} \ker \chi^{\alpha(e)} \right)^0,$$

where $\chi^{\alpha(e)}$ denotes the one-dimensional T-representation with weight $\alpha(e)$. We claim that $\lambda'(F) = \lambda(F)$. It follows immediately from the definition of λ that $\lambda(F) \subset \lambda'(F)$. Therefore we have to show that $\lambda'(F)$ is at most one-dimensional.

Let $x \in \pi^{-1}(F)^T$. Then we have

$$T_x\pi^{-1}(F) = \bigoplus_{\pi(x) \in e; e \subset F} \chi^{\alpha(e)}$$
$$N_x(\pi^{-1}(F), M) = \bigoplus_{\pi(x) \in e; e \not\subset F} \chi^{\alpha(e)}$$

Therefore we have

$$\ker T_x\pi^{-1}(F) = \bigcap_{\pi(x) \in e; e \subset F} \ker \chi^{\alpha(e)}$$

But if

$$\dim \ker T_x \pi^{-1}(F) \geq 2$$

then the intersection

$$\ker T_x \pi^{-1}(F) \cap \ker N_x(\pi^{-1}(F), M)$$

is at least one-dimensional. This contradicts with the effectiveness of the torus-action on M. □

CHAPTER 4

Torus manifolds with non-abelian symmetries

In this chapter we study torus manifolds for which the T-action may be extended by an action of a connected compact non-abelian Lie-group G.

Let G be a connected compact non-abelian Lie-group. We call a smooth connected closed oriented G-manifold M a *torus manifold with G-action* if G acts almost effectively on M, $\dim M = 2 \operatorname{rank} G$ and $M^T \neq \emptyset$ for a maximal torus T of G. That means that M with the action of T is a torus manifold. For technical reasons we assume in this chapter that the torus action on a torus manifold is almost effective instead of assuming that the torus action is effective.

The chapter is organised as follows. In section 4.1 we investigate the action of the Weyl-group of G on $\mathfrak{F}$ and $H_T^*(M)$. In section 4.2 we determine the orbit-types of the T-fixed points in M and the isomorphism types of the elementary factors of G. In section 4.3 the basic properties of the blow up construction are established. In section 4.4 actions with elementary factor $G_1 = SU(l_1+1)$ are studied. In section 4.5 we give an argument which reduces the classification problem for actions with an elementary factor $G_1 = SO(2l_1)$ to that with an elementary factor $SU(l_1)$. In section 4.6 we classify torus manifolds with G-action with elementary factor $G_1 = SO(2l_1+1)$. In section 4.7 we iterate the classification results of the previous sections and illustrate them with some applications.

4.1. The action of the Weyl-group on $\mathfrak{F}$

Let M be a torus manifold with G-action. That means G is a compact connected Lie-group of rank n which acts almost effectively on the $2n$-dimensional smooth closed connected oriented manifold M such that $M^T \neq \emptyset$ for a maximal torus T of G. If g is an element of the normaliser N_GT of T in G, then, for every characteristic submanifold M_i, gM_i is also a characteristic submanifold. Therefore there is an action of N_GT and the Weyl-group of G on the set $\mathfrak{F}$ of characteristic submanifolds of M.

In this section we describe this action of the Weyl-group of G on $\mathfrak{F}$. At first we recall the definition of the equivariant cohomology of a G-space X. Let $EG \to BG$ be a universal principal G-bundle. Then EG is a contractible free right G-space. If T is a maximal torus of G then we may identify $ET = EG$ and $BT = EG/T$. The Borel-construction X_G of X is the orbit space of the right action $((e,x),g) \mapsto (eg, g^{-1}x)$ on $EG \times X$. The equivariant cohomology $H_G^*(X)$ of X is defined as the cohomology of X_G.

In this section we take all cohomology groups with coefficients in $\mathbb{Q}$.

The G-action on $EG \times X$ induces a right action of the normaliser of T on X_T and therefore a left action of the Weyl-group on the T-equivariant cohomology of X.

Now let $X = M$ be a torus manifold with G-action. Denote the characteristic submanifolds of M by M_i, $i = 1, \dots, m$. Then for any $g \in N_GT$ $M_{g(i)} = gM_i$ is also a characteristic submanifold which depends only on the class $w = [g] \in W(G) = N_GT/T$. Therefore we get an action of the Weyl-group of G on $\mathfrak{F}$.

If we fix an omniorientation for M then the T-equivariant Poincaré-dual τ_i of M_i is well defined.

It is the image of the Thom-class of $N(M_i, M)_T$ under the natural map

$$\psi : H^2(N(M_i, M)_T, N(M_i, M)_T - (M_i)_T) \to H^2(M_T, M_T - (M_i)_T) \to H^2_T(M).$$

Because of the uniqueness of the Thom-class [**45**, p.110] and because ψ commutes with the action of $W(G)$, we have

$$\tau_{g(i)} = \pm g^* \tau_i. \tag{4.1}$$

Here the minus-sign occurs if and only if $g|_{M_i} : M_i \to M_{g(i)}$ is orientation reversing. We say that the class $[g] \in W(G)$ acts orientation preserving at M_i if this map is orientation preserving. If $[g]$ acts orientation preserving at all characteristic submanifolds then we say that $[g]$ preserves the omniorientation of M.

Let $S = H^{>0}(BT)$ and $\hat{H}^*_T(M) = H^*_T(M)/S$-torsion. Because $M^T \neq \emptyset$ there is an injection $H^2(BT) \hookrightarrow H^2_T(M)$ and

$$H^2(BT) \cap S\text{-torsion} = \{0\}. \tag{4.2}$$

By [**41**, p. 240-241] the τ_i are linearly independent in $\hat{H}^*_T(M)$. By Lemma 3.2 of [**41**, p. 246] they form a basis of $\hat{H}^2_T(M)$.

The Lie-algebra LG of G may be endowed with an Euclidean inner product which is invariant for the adjoint representation. This allows us to identify the Weyl-group $W(G)$ of G with a group of orthogonal transformations on the Lie-algebra LT of T. It is generated by reflections in the walls of the Weyl-chambers of G [**9**, p. 192-193]. An element $w \in W(G)$ is such a reflection if and only if it acts as a reflection on $H^2(BT)$.

LEMMA 4.1. *Let $w \in W(G)$ be a reflection. Then there are the following possibilities for the action of w on $\mathfrak{F}$:*

(1) *w acts orientation preserving at all characteristic submanifolds and fixes all except exactly two of them.*
(2) *w fixes all except exactly two characteristic submanifolds and acts orientation preserving at them. The action of w at the two other submanifolds is orientation reversing.*
(3) *w fixes all characteristic submanifolds and acts orientation reversing at exactly one.*

PROOF. We have the following commutative diagram of $W(G)$-representations with exact rows and columns

$$\begin{array}{ccccccc}
 & & & & S\text{-torsion in } H^2_T(M) & & \\
 & & & & \downarrow & & \\
0 & \longrightarrow & H^2(BT) & \longrightarrow & H^2_T(M) & \xrightarrow{\phi} & H^2(M) \\
 & & & & \downarrow & & \\
 & & & & \hat{H}^2_T(M) & & \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

Because G is connected the $W(G)$-action on $H^2(M)$ is trivial. By (4.2) the S-torsion in $H^2_T(M)$ injects into $H^2(M)$. Therefore $W(G)$ acts trivially on the S-torsion in $H^2_T(M)$.

Now we have

$$\begin{aligned}\operatorname{trace}(w, H^2_T(M)) &= \operatorname{trace}(w, H^2(BT)) + \operatorname{trace}(w, \operatorname{im}\phi)\\ &= \dim_{\mathbb{Q}} H^2(BT) - 2 + \dim_{\mathbb{Q}} \operatorname{im}\phi\\ &= \dim_{\mathbb{Q}} H^2_T(M) - 2.\end{aligned}$$

Similarly we get

$$\begin{aligned}\operatorname{trace}(w, \hat{H}^2_T(M)) &= \operatorname{trace}(w, H^2_T(M)) - \operatorname{trace}(w, S\text{-torsion in } H^2_T(M))\\ &= \dim_{\mathbb{Q}} \hat{H}^2_T(M) - 2.\end{aligned}$$

Now the statement follows from (4.1) because the τ_i form a basis of $\hat{H}^2_T(M)$. □

LEMMA 4.2. *$w \in W(G)$ acts as a reflection on $\hat{H}^2_T(M)$ if and only if it is a reflection.*

PROOF. Let L be a $\mathbb{Q}$-vector space and $W \subset \mathrm{Gl}(L)$ a finite group. Then there is a scalar product on L such that W acts on L by orthogonal transformations. Let $A \in W$. Then A is a reflection if and only if $\operatorname{ord} A = 2$ and $\operatorname{trace} A = \dim_{\mathbb{Q}} L - 2$. To see that notice that for $A \in W$ with $\operatorname{ord} A = 2$ there is a decomposition $L = L_+ \oplus L_-$ such that $A|_{L_\pm} = \pm \mathrm{Id}$. Then we have

$$\operatorname{trace} A = \dim_{\mathbb{Q}} L_+ - \dim_{\mathbb{Q}} L_- = \dim_{\mathbb{Q}} L - 2\dim_{\mathbb{Q}} L_-$$

and A is a reflection if and only if $\dim_{\mathbb{Q}} L_- = 1$.

If $w \in W(G)$ with $\operatorname{ord} w = 2$ then as in the proof of Lemma 4.1 we see that

$$\dim_{\mathbb{Q}} H^2(BT) - \operatorname{trace}(w, H^2(BT)) = \dim_{\mathbb{Q}} \hat{H}^2_T(M) - \operatorname{trace}(w, \hat{H}^2_T(M)).$$

Therefore w is a reflection if and only if it acts as a reflection on $\hat{H}^2_T(M)$. □

Let $\mathfrak{F}_0$ be the set of characteristic submanifolds which are fixed by the $W(G)$-action and at which $W(G)$ acts orientation preserving. Furthermore let $\mathfrak{F}_i$, $i = 1, \dots, k$, be the other orbits of the $W(G)$-action on $\mathfrak{F}$ and V_i the subspace of $\hat{H}^2_T(M)$ spanned by the τ_j with $M_j \in \mathfrak{F}_i$. Then $W(G)$ acts trivially on V_0. For $i > 0$ let W_i be the subgroup of $W(G)$ which is generated by the reflections which act non-trivially on V_i. Then by Lemma 4.1 W_i acts trivially on V_j, $j \neq i$. Furthermore we have $W(G) = \prod_{i=1}^k W_i$ because the action of $W(G)$ on $\hat{H}^2_T(M)$ is effective. This follows because by (4.2) $H^2(BT)$ injects into $\hat{H}^2_T(M)$.

LEMMA 4.3. *For each pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ there is a reflection $w \in W_i$ with $w(M_{j_1}) = M_{j_2}$.*

PROOF. Because $\mathfrak{F}_i$ is an orbit of the $W(G)$-action on $\mathfrak{F}$ there is a $M'_{j_1} \in \mathfrak{F}_i$ with $M'_{j_1} \neq M_{j_2}$ and a reflection $w \in W_i$ with $w(M'_{j_1}) = M_{j_2}$.

Because W_i is generated by reflections and acts transitively on $\mathfrak{F}_i$ the natural map $W_i \to S(\mathfrak{F}_i)$ to the permutation group of $\mathfrak{F}_i$ is a surjection by Lemma 4.1 and Lemma 3.10 of [**2**, p. 51]. Therefore there is a $w' \in W_i$ with

$$w'(M_{j_1}) = M'_{j_1}, \qquad w'(M'_{j_1}) = M_{j_1}, \qquad w'(M_{j_2}) = M_{j_2}.$$

Now $w'^{-1}ww' \in W_i$ is a reflection with the required properties. □

It follows from Lemma 4.1 that for each pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ there are at most two reflections which map M_{j_1} to M_{j_2}. As in the proof of Lemma 4.3 one sees that the number of these reflections does not depend on the choice of the pair M_{j_1}, M_{j_2} in $\mathfrak{F}_i$.

LEMMA 4.4. *Assume $\#\mathfrak{F}_i > 1$. If for each pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ there is exactly one reflection in W_i which maps M_{j_1} to M_{j_2}, then W_i is isomorphic to $S(\mathfrak{F}_i) \cong W(SU(l_i+1))$ with $l_i + 1 = \#\mathfrak{F}_i$.*

PROOF. First note that there is no reflection of the third type as described in Lemma 4.1 in W_i.

We have to show that the kernel of the natural map $W_i \to S(\mathfrak{F}_i)$ is trivial. Let w be an element of this kernel. Then for each $\tau_j \in V_i$ we have

$$w\tau_j = \pm\tau_j.$$

If we have $w\tau_j = \tau_j$ for all $\tau_j \in V_i$, then $w = \mathrm{Id}$.

Now assume that $w\tau_{j_0} = -\tau_{j_0}$ for a $\tau_{j_0} \in V_i$. Then there are reflections $w_1, \dots, w_n \in W_i$, $n \geq 2$, with $-\tau_{j_0} = w\tau_{j_0} = w_1 \dots w_n\tau_{j_0}$. After removing some of the w_i we may assume that

$$\begin{aligned} w_i \dots w_n \tau_{j_0} &\neq \pm\tau_{j_0} && \text{for } 2 \leq i \leq n \\ w_{i+1} \dots w_n \tau_{j_0} &\neq \pm w_i \dots w_n \tau_{j_0} && \text{for } 2 \leq i \leq n \end{aligned}$$

Therefore we have $w_i\tau_{j_0} = \tau_{j_0}$ for $2 \leq i < n$. This implies:

$$w_n \dots w_2 w_1 w_2 \dots w_n \tau_{j_0} = -w_n\tau_{j_0}$$

and therefore $w_n \dots w_2 w_1 w_2 \dots w_n M_{j_0} = w_n M_{j_0}$.

But $w_n \dots w_2 w_1 w_2 \dots w_n$ is a reflection. Therefore we have

$$w_n \dots w_2 w_1 w_2 \dots w_n = w_n$$

and

$$w_n\tau_{j_0} = w_n w_{n-1} \dots w_2 w_1 w_2 \dots w_n \tau_{j_0} = -w_n\tau_{j_0}.$$

Because $w_n\tau_{j_0} \neq 0$ this is impossible and, hence, contradicting our assumption.

Therefore the kernel is trivial. □

To get the isomorphism type of W_i in the case, where there is a pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ and exactly two reflections in W_i which map M_{j_1} to M_{j_2}, we first give a description of the Weyl-groups of some Lie-groups.

Let L be an l-dimensional $\mathbb{Q}$-vector space with basis $e_1, \dots, e_l$. For $1 \leq i < j \leq l$ let $f_{ij\pm}, g_i \in \mathrm{Gl}(L)$ such that

$$f_{ij\pm}e_k = \begin{cases} \pm e_i & \text{if } k = j \\ \pm e_j & \text{if } k = i \\ e_k & \text{else,} \end{cases}$$

$$g_i e_k = \begin{cases} -e_i & \text{if } k = i \\ e_k & \text{else.} \end{cases}$$

Then we have the following isomorphisms of groups [**9**, p. 171-172]:

$$\begin{aligned} W(SU(l-1)) \cong S(l) &\cong \langle f_{ij+}; 1 \leq i < j \leq l \rangle, \\ W(SO(2l)) &\cong \langle f_{ij\pm}; 1 \leq i < j \leq l \rangle, \\ W(SO(2l+1)) \cong W(Sp(l)) &\cong \langle f_{ij\pm}, g_1; 1 \leq i < j \leq l \rangle. \end{aligned}$$

From this description and Lemma 4.1 we get:

LEMMA 4.5. *If for each pair $M_{j_1}, M_{j_2} \in \mathfrak{F}_i$, $i > 0$, with $M_{j_1} \neq M_{j_2}$ there are exactly two reflections in W_i which map M_{j_1} to M_{j_2} then with $l_i = \#\mathfrak{F}_i$ we have*

(1) *$W_i \cong W(SO(2l_i))$ if there is no reflection of the third type as described in Lemma 4.1 in W_i.*

(2) $W_i \cong W(SO(2l_i+1)) \cong W(Sp(l_i))$ *if there is a reflection of the third type in* W_i.

By [**9**, p. 233] G has a finite covering group $\tilde{G}$ such that $\tilde{G} = \prod_i G_i \times T^{l_0}$ where the G_i are simple simply connected compact Lie-groups. The Weyl-group of G is given by $W(G) = \prod_i W(G_i)$. Because the Dynkin-diagram of a simple Lie-group is connected, each $W(G_i)$ is generated by reflections in such a way that for each pair of reflections w_1, w_2 in the generating set there is a sequence of reflections connecting w_1, w_2 such that subsequent reflections do not commute. Therefore each $W(G_i)$ is contained in a W_j. Therefore we get $W_i = \prod_{j\in J_i} W(G_j)$. Using Lemmas 4.4 and 4.5 we deduce:

$$W_i = \begin{cases} W(G_j) & \text{for some } j \text{ if } W_i \not\cong W(SO(4)) \\ W(G_{j_1}) \times W(G_{j_2}) & \text{with } G_{j_1} \cong G_{j_2} \cong SU(2) \text{ if } W_i \cong W(SO(4)) \end{cases}$$

Therefore we may write $\tilde{G} = \prod_i G_i \times T^{l_0}$ with $W_i = W(G_i)$ and G_i simple and simply connected or $G_i = \mathrm{Spin}(4)$. In the following we will call these G_i the *elementary factors* of $\tilde{G}$.

We summarise the above discussion in the following lemma.

LEMMA 4.6. *Let M be a torus manifold with G-action and $\tilde{G}$ as above. Then all G_i are non-exceptional, i.e. $G_i = SU(l_i+1), Spin(2l_i), Spin(2l_i+1), Sp(l_i)$.*

The Weyl-group of an elementary factor G_i of $\tilde{G}$ acts transitively on $\mathfrak{F}_i$ and trivially on $\mathfrak{F}_j$, $j \neq i$, and there are the following relations between the G_i and $\#\mathfrak{F}_i$:

G_i	$\#\mathfrak{F}_i$
$SU(2) = Spin(3) = Sp(1)$	$1, 2$
$Spin(4)$	2
$Spin(5) = Sp(2)$	2
$SU(4) = Spin(6)$	$3, 4$
$SU(l_i+1)$, $l_i \neq 1, 3$	$l_i + 1$
$Spin(2l_i+1)$, $l_i > 2$	l_i
$Spin(2l_i)$, $l_i > 3$	l_i
$Sp(l_i)$, $l_i > 2$	l_i

LEMMA 4.7. *Let M be a quasitoric manifold with G-action. Then there is a covering group $\tilde{G}$ of G with $\tilde{G} = \prod_{i=1}^{k_1} SU(l_i+1) \times T^{l_0}$.*

PROOF. First we show for $i > 0$:

$$W_i \cong S(\mathfrak{F}_i) \tag{4.3}$$

To do so it is sufficient to prove that there is an omniorientation on M which is preserved by the action of $W(G)$. This is true if for every characteristic submanifold M_i and $g \in N_G T$ such that $gM_i = M_i$, g preserves the orientation of M_i. Since G is connected, g preserves the orientation of M and acts trivially on $H^2(M)$. Because every fixed point of the T-action is the transverse intersection of n characteristic submanifolds and $M_i \cap M^T \neq \emptyset$, the Poincaré-dual of M_i is non-zero. Therefore g preserves the orientation of M_i.

This establishes (4.3). Recall that all simple compact simply connected Lie-groups having a Weyl-group isomorphic to some symmetric group are isomorphic to some $SU(l+1)$. Therefore all elementary factors of $\tilde{G}$ are isomorphic to $SU(l_i+1)$. From this the statement follows. □

REMARK 4.8. In [**42**] Masuda and Panov show that the cohomology with coefficients in $\mathbb{Z}$ of a torus manifold M is generated by its degree-two part if and only if the torus action on M is locally standard and the orbit space M/T is a homology polytope. That means that all faces of M/T are acyclic and all intersections of facets of M/T are connected. In particular each T-fixed point is the transverse intersection of n characteristic submanifolds. Therefore the above lemma also holds in this case.

For a characteristic submanifold M_i of M let $\lambda(M_i)$ denote the one-dimensional subtorus of T which fixes M_i pointwise. The normaliser N_GT of T in G acts by conjugation on the set of one-dimensional subtori of T. The following lemma shows that

$$\lambda : \mathfrak{F} \to \{\text{one-dimensional subtori of } T\}$$

is N_GT-equivariant.

LEMMA 4.9. *Let M be a torus manifold with G-action, $g \in N_GT$ and $M_i \subset M$ be a characteristic submanifold. Then we have:*

(1) $\lambda(gM_i) = g\lambda(M_i)g^{-1}$
(2) *If $gM_i = M_i$ then g acts orientation preserving on M_i if and only if*

$$\lambda(M_i) \to \lambda(M_i) \quad t \mapsto gtg^{-1}$$

is orientation preserving.

PROOF. (1) Let $x \in M_i$ be a generic point. Then the identity component T_x^0 of the stabiliser of x in T is given by $T_x^0 = \lambda(M_i)$. Therefore we have

$$\lambda(gM_i) = T_{gx}^0 = gT_x^0g^{-1} = g\lambda(M_i)g^{-1}.$$

(2) An orientation of M_i induces a complex structure on $N(M_i, M)$. We fix an isomorphism $\rho : \lambda(M_i) \to S^1$ such that the action of $t \in \lambda(M_i)$ on $N(M_i, M)$ is given by multiplication with $\rho(t)^m$, $m > 0$. The differential $Dg : N(M_i, M) \to N(M_i, M)$ is orientation preserving if and only if it is complex linear. Otherwise it is complex anti-linear. Therefore for $v \in N(M_i, M)$ we have

$$\begin{aligned}\rho(gtg^{-1})^m v &= (Dg)(Dt)(Dg)^{-1}v = (Dg)\rho(t)^m(Dg)^{-1}v \\ &= \rho(t)^{\pm m}(Dg)(Dg)^{-1}v = \rho(t^{\pm 1})^m v.\end{aligned}$$

From this $gtg^{-1} = t^{\pm 1}$ follows, where the plus-sign arises if and only if g acts orientation preserving on M_i.

□

4.2. G-action on M

In this section we consider torus manifolds with G-action, such that $\tilde{G}$ has only one elementary factor G_1. The action of an arbitrary G induces such an action by restricting the $\tilde{G}$-action to $G_1 \times T^{l_0'}$, where $T^{l_0'}$ is a maximal torus of $\prod_{i>1} G_i \times T^{l_0}$. There are two cases

(1) There is a T-fixed point which is not fixed by G_1.
(2) There is a G-fixed point.

LEMMA 4.10. *Let $\tilde{G} = G_1 \times T^{l_0}$ with G_1 elementary,* $\operatorname{rank} G_1 = l_1$ *and M a torus manifold with G-action of dimension $2n = 2(l_0 + l_1)$. If there is an $x \in M^T$ which is not fixed by the action of G_1, then*

(1) *$G_1 = SU(l_1 + 1)$ or $G_1 = Spin(2l_1 + 1)$ and the stabiliser of x in G_1 is conjugated to $S(U(l_1) \times U(1))$ or $Spin(2l_1)$, respectively.*
(2) *The G_1-orbit of x equals the component of $M^{T^{l_0}}$ which contains x.*

Moreover if $G_1 = SU(4)$ one has $\#\mathfrak{F}_1 = 4$.

PROOF. The G_1-orbit of x is contained in the component N of $M^{T^{l_0}}$ containing x. Therefore we have

$$\operatorname{codim} G_{1x} = \dim G_1/G_{1x} = \dim G_1x \leq \dim N \leq 2l_1.$$

Furthermore the stabiliser G_{1x} of x has maximal rank l_1.

At first we consider the case $G_1 \neq \mathrm{Spin}(4)$. From the classification of closed connected maximal rank subgroups of a compact Lie-group given in [**7**, p. 219] we get the following connected maximal rank subgroups H of maximal dimension:

G_1	H	$\operatorname{codim} H$
$SU(2) = \mathrm{Spin}(3) = Sp(1)$	$S(U(1) \times U(1))$	2
$\mathrm{Spin}(5) = Sp(2)$	$\mathrm{Spin}(4)$	4
$SU(4) = \mathrm{Spin}(6)$	$S(U(3) \times U(1))$	6
$SU(l_1 + 1)$, $l_1 \neq 1, 3$	$S(U(l_1) \times U(1))$	$2l_1$
$\mathrm{Spin}(2l_1 + 1)$, $l_1 > 2$	$\mathrm{Spin}(2l_1)$	$2l_1$
$\mathrm{Spin}(2l_1)$, $l_1 > 3$	$\mathrm{Spin}(2l_1 - 2) \times \mathrm{Spin}(2)$	$4l_1 - 4$
$Sp(l_1)$, $l_1 > 2$	$Sp(l_1 - 1) \times Sp(1)$	$4l_1 - 4$

Because H is unique up to conjugation, $G_1 = SU(l_1 + 1)$ or $G_1 = \mathrm{Spin}(2l_1 + 1)$ and G_{1x} is conjugated to a subgroup of G_1 which contains $S(U(l_1) \times U(1))$ or $\mathrm{Spin}(2l_1)$, respectively. Because $S(U(l_1) \times U(1))$ is a maximal subgroup of $SU(l_1+1)$ if $l_1 > 1$ by Lemma A.1, G_{1x} is conjugated to $S(U(l_1) \times U(1))$ if $G_1 = SU(l_1 + 1)$, $l_1 > 1$. Because $\operatorname{codim} S(U(l_1) \times U(1)) = 2l_1 \geq \dim N$ we have $G_1x = N$ in this case.

If $G_1 = \mathrm{Spin}(2l_1 + 1)$, $l_1 \geq 1$, then by Lemma A.4 there are two proper subgroups of G_1 which contain $\mathrm{Spin}(2l_1)$, $\mathrm{Spin}(2l_1)$ and its normaliser H_0. Because of dimension reasons we have $N = G_1x$. Because $\mathrm{Spin}(2l_1 + 1)/H_0$ is not orientable and $M^{T^{l_0}}$ is orientable, $G_{1x} = \mathrm{Spin}(2l_1)$ follows.

If $G_1 = SU(4)$, then G_1x is G_1-equivariantly diffeomorphic to $\mathbb{C}P^3$. Because $\mathbb{C}P^3$ has four characteristic submanifolds with pairwise non-trivial intersections, by Lemmas A.7 and A.8 there are four characteristic submanifolds $M_1, \dots, M_4$ which intersect transversally with G_1x. Because G_1x is a component of $M^{T^{l_0}}$, we have by Lemma A.6 that $\lambda(M_i) \not\subset T^{l_0}$. Therefore $\lambda(M_i)$ is not fixed pointwise by the action of $W(G_1)$. Now it follows with Lemma 4.9 that $M_1, \dots, M_4$ belong to $\mathfrak{F}_1$.

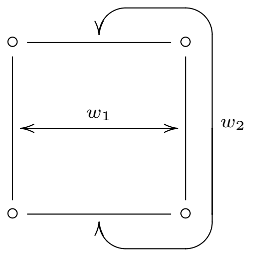
Now we turn to the case $G_1 = \mathrm{Spin}(4)$.

Then there are the following proper closed connected maximal rank subgroups H of G_1 of codimension at most 4:

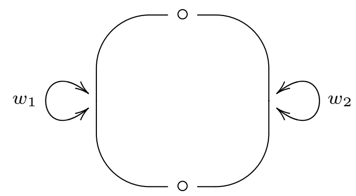
$$SU(2) \times S(U(1) \times U(1)), \quad S(U(1) \times U(1)) \times S(U(1) \times U(1)).$$

At first assume that G_1x has dimension four. Then G_1x is G_1-equivariantly diffeomorphic to $S^2 \times S^2$, $S^2 \times_{\mathbb{Z}_2} S^2$, $\mathbb{R}P^2 \times S^2$ or $\mathbb{R}P^2 \times \mathbb{R}P^2$. Since $G_1x = M^{T^{l_0}}$ is orientable, the latter two do not occur.

For $N = G_1x = S^2 \times S^2, S^2 \times_{\mathbb{Z}_2} S^2$ let $N^{(1)}$ be the union of the T-orbits in N of dimension less than or equal to one. Then $W(G_1) = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on the orbit space $N^{(1)}/T$. This space is given by one of the following graphs:



$(S^2 \times S^2)^{(1)}/T$



$(S^2 \times_{\mathbb{Z}_2} S^2)^{(1)}/T$

Where the edges correspond to orbits of dimension one and the vertices to the fixed points. The arrows indicate the action of the generators $w_1, w_2 \in W(G_1)$ on this space. Let M_1, M_2 be the two characteristic submanifolds of M which intersect transversely with N in x. Because N is a component of $M^{T^{l_0}}$, $\lambda(M_i)$ is not a subgroup of T^{l_0} for $i = 1, 2$ by Lemma A.6. Therefore $\lambda(M_i)$ is not fixed pointwise by $W(G_1)$. By Lemma 4.9 this implies $M_1, M_2 \in \mathfrak{F}_1$. Therefore there is a $w \in W(G_1)$ with $w(M_1) = M_2$. But from the pictures above we see that M_1 and M_2 are not in the same $W(G_1)$-orbits.

Now assume that G_1x has dimension two. Then we may assume without loss of generality that G_1x is a component of $M^{S(U(1)\times U(1))\times 1\times T^{l_0}}$. Therefore by Lemmas A.6 and A.8 there are characteristic submanifolds $M_2, \dots, M_{l_0+2}$ of M such that G_1x is a component of $\bigcap_{i=2}^{l_0+2} M_i$. Furthermore we may assume that $\lambda(M_2) \not\subset T^{l_0}$. Therefore by Lemma 4.9 we have $M_2 \in \mathfrak{F}_1$.

But there is also a characteristic submanifold M_1 of M which intersects G_1x transversely in x. With the Lemmas A.6 and 4.9 we see $M_1 \in \mathfrak{F}_1$.

Therefore there is a $w \in W(G_1)$ with $w(M_2) = M_1$. But this is impossible because $M_2 \supset G_1x \not\subset M_1$.

Therefore $G_1 \neq \mathrm{Spin}(4)$ and the lemma is proved. □

REMARK 4.11. If $T \cap G_1$ is the standard maximal torus of G_1 then it follows by Proposition 2 of [**28**, p. 325] that G_{1x} is conjugated to the given groups by an element of the normaliser of the maximal torus.

LEMMA 4.12. *In the situation of the previous lemma x is contained in the intersection of exactly l_1 characteristic submanifolds belonging to $\mathfrak{F}_1$.*

PROOF. Because $N = G_1x$ has dimension $2l_1$ x is contained in exactly l_1 characteristic submanifolds of N. By Lemma A.7 we know that they are components of intersections of characteristic submanifolds $M_1, \dots, M_{l_1}$ of M with N.

Because G_1x is a component of $M^{T^{l_0}}$, $\lambda(M_i)$ is not a subgroup of T^{l_0} for $i = 1, \dots, l_1$ by Lemmas A.6 and A.8. Therefore $\lambda(M_i)$ is not fixed pointwise by $W(G_1)$. By Lemma 4.9 this implies that M_i belongs to $\mathfrak{F}_1$.

By Lemmas A.8 and A.6 G_1x is the intersection of l_0 characteristic submanifolds $M_{l_1+1}, \dots, M_n$ of M. We show that these manifolds do not belong to $\mathfrak{F}_1$. Assume that there is an $i \geq l_1 + 1$ such that M_i belongs to $\mathfrak{F}_1$. Because $W(G_1)$ acts transitively on $\mathfrak{F}_1$, there is a $w \in W(G_1)$ with $w(M_i) = M_j$, $j \leq l_1$. But this is impossible because $M_i \supset G_1x \not\subset M_j$. □

LEMMA 4.13. *Let $\tilde{G} = G_1 \times T^{l_0}$ with G_1 elementary,* $\operatorname{rank} G_1 = l_1$ *and M a torus manifold with G-action of dimension $2n = 2(l_0 + l_1)$. If there is a T-fixed point $x \in M^T$ which is fixed by G_1, then $G_1 = SU(l_1 + 1)$ or $G_1 = Spin(2l_1)$.*

Moreover if $G_1 \neq Spin(8)$ one has

$$T_xM = V_1 \oplus V_2 \otimes_{\mathbb{C}} W_1 \text{ if } G_1 = SU(l_1+1) \text{ and } \#\mathfrak{F}_1 = 4 \text{ in the case } l_1 = 3, \tag{4.4}$$

$$T_xM = V_3 \oplus W_2 \text{ if } G_1 = Spin(2l_1) \text{ and } \#\mathfrak{F}_1 = 3 \text{ in the case } l_1 = 3, \tag{4.5}$$

where W_1 is the standard complex representation of $SU(l_1+1)$ or its dual, W_2 is the standard real representation of $SO(2l_1)$ and the V_i are complex T^{l_0}-representations.

In the case $G_1 = Spin(8)$ one may change the action of G_1 on M by an automorphism of G_1 which is independent of x to reach the situation described above.

Furthermore we have $x \in \bigcap_{M_i \in \mathfrak{F}_1} M_i$. If $l_1 = 1$ we have $\#\mathfrak{F}_1 = 2$.

Proof. Let $M_1, \dots, M_n$ be the characteristic submanifolds of M which intersect in x. Then the weight spaces of the $\tilde{G}$-representation T_xM are given by

$$N_x(M_1, M), \dots, N_x(M_n, M).$$

For $g \in N_GT$ we have $M_i = gM_j$ if and only if $N_x(M_i, M) = gN_x(M_j, M)$. Because G_1 acts non-trivially on T_xM there is at least one M_i, $i \in \{1, \dots, n\}$, such that $M_i \in \mathfrak{F}_1$. Because $W(G_1)$ acts transitively on $\mathfrak{F}_1$ and x is a G-fixed point, we have

$$\frac{1}{2}\#\{\text{oriented weight spaces of } T_xM \text{ which are not fixed by } W(G_1)\} = \#\mathfrak{F}_1 \tag{4.6}$$

and $x \in \bigcap_{M_i \in \mathfrak{F}_1} M_i$.

For the $\tilde{G}$-representation T_xM we have

$$T_xM = N_x(M^{T^{l_0}}, M) \oplus T_xM^{T^{l_0}}. \tag{4.7}$$

If $l_0 = 0$ then we have $N_x(M^{T^{l_0}}, M) = \{0\}$. Otherwise the action of T^{l_0} induces a complex structure on $N_x(M^{T^{l_0}}, M)$ and we have

$$N_x(M^{T^{l_0}}, M) = \bigoplus_i V_i \otimes_{\mathbb{C}} W_i. \tag{4.8}$$

Here the V_i are one-dimensional complex T^{l_0}-representations and the W_i are irreducible complex G_1-representations. Since T^{l_0} acts almost effectively on M, there are at least $n - l_1$ summands in this decomposition. Therefore we get

$$\dim_{\mathbb{C}} W_i = \dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) - \sum_{j \neq i} \dim_{\mathbb{C}} V_j \otimes_{\mathbb{C}} W_j \leq n - (n - l_1 - 1) = l_1 + 1.$$

Furthermore $\dim_{\mathbb{R}} T_xM^{T^{l_0}} \leq 2(n - l_0) = 2l_1$.

If there is a W_{i_0} with $\dim_{\mathbb{C}} W_{i_0} = l_1 + 1$, then from equation (4.8) we get for all other W_i

$$\dim_{\mathbb{C}} W_i = \dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) - \dim_{\mathbb{C}} V_{i_0} \otimes_{\mathbb{C}} W_{i_0} - \sum_{j \neq i, i_0} \dim_{\mathbb{C}} V_j \otimes_{\mathbb{C}} W_j \leq 1.$$

So they are one-dimensional and therefore trivial. Furthermore we have

$$\dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) = \sum_i \dim_{\mathbb{C}} V_i \otimes_{\mathbb{C}} W_i \geq n.$$

Therefore $T_xM^{T^{l_0}}$ is zero-dimensional in this case.

If $\dim_{\mathbb{R}} T_xM^{T^{l_0}} = 2l_1$, then we have

$$\dim_{\mathbb{C}} W_i = \dim_{\mathbb{C}} N_x(M^{T^{l_0}}, M) - \sum_{j \neq i} \dim_{\mathbb{C}} V_j \otimes_{\mathbb{C}} W_j \leq 1.$$

Therefore all W_i are one dimensional and therefore trivial in this case.

There are the following lower bounds $d_{\mathbb{R}}, d_{\mathbb{C}}$ for the dimension of real and complex non-trivial irreducible representations of G_1 [**49**, p. 53-54]:

G_1	$d_{\mathbb{R}}$	$d_{\mathbb{C}}$
$SU(2) = \text{Spin}(3) = Sp(1)$	3	2
$\text{Spin}(4)$	3	2
$\text{Spin}(5) = Sp(2)$	5	4
$SU(4) = \text{Spin}(6)$	6	4
$SU(l_1+1)$, $l_1 \neq 1, 3$	$2l_1 + 2$	$l_1 + 1$
$\text{Spin}(2l_1+1)$, $l_1 > 2$	$2l_1 + 1$	$2l_1 + 1$
$\text{Spin}(2l_1)$, $l_1 > 3$	$2l_1$	$2l_1$
$Sp(l_1)$, $l_1 > 2$	$2l_1 + 1$	$2l_1$

Because G_1 acts non-trivially on T_xM, we have $d_{\mathbb{R}} \leq 2l_1$ or $d_{\mathbb{C}} \leq l_1 + 1$. Therefore $G_1 \neq Sp(l_1)$, $l_1 > 1$ and $G_1 \neq \text{Spin}(2l_1+1)$, $l_1 > 1$.

If $G_1 = \text{Spin}(2l_1), l_1 > 3$, then all W_i are trivial and $T_xM^{T^{l_0}}$ has dimension $2l_1$. Therefore it is the standard real $SO(2l_1)$-representation if $l_1 > 4$. If $l_1 = 4$ then there are three eight-dimensional real representations of Spin(8), the standard real $SO(8)$-representation and the two half spinor representations. They have three different kernels. Notice that the kernel of the G_1-representation $T_xM^{T^{l_0}}$ is equal to the kernel of the G_1-action on M. Therefore, if one of them is isomorphic to $T_xM^{T^{l_0}}$, then it is isomorphic to $T_yM^{T^{l_0}}$ for all $y \in M^T$. So we may – after changing the action of Spin(8) on M by an automorphism – assume that $T_xM^{T^{l_0}}$ is the standard real $SO(8)$-representation.

If $G_1 = SU(l_1+1), l_1 \neq 1, 3$, then only one W_i is non-trivial and $T_xM^{T^{l_0}}$ has dimension zero. The non-trivial W_i is the standard representation of $SU(l_1+1)$ or its dual depending on the complex structure of $N_x(M^{T^{l_0}}, M)$.

If $G_1 = SU(4)$ then there are one real representation of dimension 6 and two complex representations of dimension 4. If the first representation occurs in the decomposition of T_xM, then by (4.6) we have $\#\mathfrak{F}_1 = 3$. If one of the others occurs, then $\#\mathfrak{F}_1 = 4$.

If $G_1 = SU(2)$, then there is one non-trivial W_i of dimension 2. Therefore one has $\#\mathfrak{F}_1 = 2$.

If $G_1 = \text{Spin}(4)$, then T_xM is an almost faithful representation. Because all almost faithful complex representations of Spin(4) have at least dimension four, there is no W_i of dimension three.

If there is one W_{i_0} of dimension two, then all other W_i and $T_xM^{T^{l_0}}$ have dimension less than or equal to two. Because there is no two-dimensional real Spin(4)-representation, there is another W_i of dimension two. But this contradicts (4.6) because $\#\mathfrak{F}_1 = 2$.

Therefore all W_i are one-dimensional and therefore trivial. $T_xM^{T^{l_0}}$ has to be the standard four-dimensional real representation of Spin(4). □

With the Lemmas 4.10 and 4.13 we see that there is no elementary factor of $\tilde{G}$ which is isomorphic to $Sp(l_1)$ for $l_1 > 2$.

Now let $G_1 = \text{Spin}(2l)$ and $\#\mathfrak{F}_1 = 3$ in the case $l = 3$. Then by looking at the G_1-representation T_xM one sees with Lemma 4.13 that the G_1-action factors through $SO(2l)$.

Now let $G_1 = \text{Spin}(2l+1)$, $l > 1$. Then by Lemma 4.10 we have $G_{1x} = \text{Spin}(2l)$. Because the G_{1x}-action on $N_x(G_1x, M)$ is trivial by Lemma 4.13 the G_1-action factors through $SO(2l+1)$.

In the case $G_1 = \text{Spin}(3)$ and $\#\mathfrak{F}_1 = 1$ we have $G_1x = S^2$. The characteristic submanifold $M_1 \in \mathfrak{F}_1$ intersects G_1x transversely in x. Because $\#\mathfrak{F}_1 = 1$, $\lambda(M_1)$ is invariant under the action of $W(G_1)$ on the maximal torus of G. Because by Lemma 4.9 the non-trivial element of $W(G_1)$ reverses the orientation of $\lambda(M_1)$, it is a maximal torus of G_1. Therefore the center of G_1 acts trivially on M and the G_1-action on M factors through $SO(3)$.

If in the case $G_1 = \mathrm{Spin}(3)$ and $\#\mathfrak{F}_1 = 2$ the principal orbit type of the G_1-action is given by $\mathrm{Spin}(3)/\mathrm{Spin}(2)$, then the G_1-action factors through $SO(3)$.

Therefore in the following we may replace an elementary factor G_i of $\tilde{G}$ isomorphic to $\mathrm{Spin}(l)$ which satisfies the above conditions by $SO(l)$.

Convention 4.14. If we say that an elementary factor G_i is isomorphic to $SU(2)$ or $SU(4)$, then we mean that $\#\mathfrak{F}_i = 2, 4$ respectively. Conversely, if we say that G_i is isomorphic to $SO(3)$, we mean that $\#\mathfrak{F}_i = 1$ or $\#\mathfrak{F}_1 = 2$ and the $SO(3)$-action has principal orbit type $SO(3)/SO(2)$. If we say $G_i = SO(6)$, then we mean $\#\mathfrak{F}_i = 3$.

Corollary 4.15. *Assume that G is elementary. Then M is equivariantly diffeomorphic to $\mathbb{C}P^{l_1}$ or $M = S^{2l_1}$, if $\tilde{G} = SU(l_1+1)$ or $\tilde{G} = SO(2l_1+1), SO(2l_1)$, respectively.*

Proof. If G is elementary we may assume that $G = \tilde{G} = SO(2l_1), SO(2l_1 + 1), SU(l_1 + 1)$.

If $G = SO(2l_1)$, then by Lemmas 4.10 and 4.13 the principal orbit type of the $SO(2l_1)$-action is given by $SO(2l_1)/SO(2l_1 - 1)$ which has codimension one in M.

$S(O(2l_1 - 1) \times O(1))$ is the only proper subgroup of $SO(2l_1)$ which contains $SO(2l_1 - 1)$ properly. Because $SO(2l_1)/S(O(2l_1 - 1) \times O(1)) = \mathbb{R}P^{2l_1-1}$ is orientable, all orbits of the $SO(2l_1)$-action are of types $SO(2l_1)/SO(2l_1-1)$ or $SO(2l_1)/SO(2l_1)$ by [**8**, p. 185].

By [**8**, p. 206-207] we have

$$M = D_1^{2l_1} \cup_\phi D_2^{2l_1},$$

where $SO(2l_1)$ acts on the disks $D_i^{2l_1}$ in the usual way and

$$\phi : S^{2l_1-1} = SO(2l_1)/SO(2l_1 - 1) \to S^{2l_1-1} = SO(2l_1)/SO(2l_1 - 1)$$

is given by $gSO(2l_1 - 1) \mapsto gnSO(2l_1 - 1)$ where $n \in N_{SO(2l_1)}SO(2l_1 - 1) = S(O(2l_1 - 1) \times O(1))$.

Therefore $\phi = \pm \mathrm{Id}_{S^{2l_1-1}}$ and $M = S^{2l_1}$.

If $G = SO(2l_1 + 1)$, then

$$M = SO(2l_1 + 1)/SO(2l_1) = S^{2l_1}$$

follows directly from Lemmas 4.10 and 4.13.

If $G = SU(l_1 + 1)$, then $\dim M = 2l_1$. Therefore the intersection of $l_1 + 1$ pairwise distinct characteristic submanifolds of M is empty. By Lemma 4.13 no T-fixed point is fixed by G. Therefore from Lemma 4.10 we get

$$M = SU(l_1 + 1)/S(U(l_1) \times U(1)) = \mathbb{C}P^{l_1}.$$

□

Remark 4.16. Another proof of this statement follows from the classification given in section 4.7.

4.3. Blowing up

In this section we describe blow ups of torus manifolds with G-action. They are used in the following sections to construct from a torus manifold M with G-action another torus manifold $\tilde{M}$ with G-action, such that an elementary factor of the covering group $\tilde{G}$ of G has no fixed point in $\tilde{M}$.

References for this construction are [**19**, p. 602-611] and [**44**, p. 269-270].

As before we write $\tilde{G} = \prod_{i=1}^k G_i \times T^{l_0}$ with G_i elementary and T^{l_0} a torus.

We will see in sections 4.4 and 4.6 that there are the following two cases:

(1) a component N of M^{G_1} has odd codimension in M.
(2) a component N of M^{G_1} has even codimension in M and there is a $x \in Z(\tilde{G})$ such that x acts trivially on N and x^2 acts as $-\operatorname{Id}$ on $N(N, M)$.

In the second case the action of x on $N(N,M)$ induces a G-invariant complex structure and we equip $N(N,M)$ with this structure. Let $E = N(N,M) \oplus \mathbb{K}$ where $\mathbb{K} = \mathbb{R}$ in the first case and $\mathbb{K} = \mathbb{C}$ in the second case.

Lemma 4.17. *The projectivication $P_{\mathbb{K}}(E)$ is orientable.*

Proof. Because M is orientable the total space of the normal bundle of N in M is orientable. Therefore

$$E = N(N,M) \oplus \mathbb{K} = N(N,M) \times \mathbb{K}$$

and the associated sphere bundle $S(E)$ are orientable.

Let $Z_{\mathbb{K}} = \mathbb{Z}/2\mathbb{Z}$ if $\mathbb{K} = \mathbb{R}$ and $Z_{\mathbb{K}} = S^1$ if $\mathbb{K} = \mathbb{C}$. Then $Z_{\mathbb{K}}$ acts on E and $S(E)$ by multiplication on the fibers. Now $P_{\mathbb{K}}(E)$ is given by $S(E)/Z_{\mathbb{K}}$. If $\mathbb{K} = \mathbb{C}$ then $Z_{\mathbb{K}}$ acts orientation preserving on $S(E)$.

If $\mathbb{K} = \mathbb{R}$ then $\dim E$ is even. Therefore the restriction of the $Z_{\mathbb{K}}$-action to a fiber of E is orientation preserving and, hence, it preserves the orientation of $S(E)$.

Because the action of $Z_{\mathbb{K}}$ is orientation preserving on $S(E)$ $P_{\mathbb{K}}(E)$ is orientable. □

Choose a G-invariant Riemannian metric on $N(N,M)$ and a G-equivariant closed tubular neighbourhood B around N. Then one may identify

$$B = \{z_0 \in N(N,M); |z_0| \leq 1\} = \{(z_0 : 1) \in P_{\mathbb{K}}(E); |z_0| \leq 1\}.$$

We orient $P_{\mathbb{K}}(E)$ in such way that this identification is orientation preserving.

By gluing the complements of the interior of B in M and $P_{\mathbb{K}}(E)$ along the boundary of B we get a new torus manifold with G-action $\tilde{M}$, the *blow up* of M along N. It is easy to see using isotopies of tubular neighbourhoods that the G-equivariant diffeomorphism-type of $\tilde{M}$ does not depend on the choices of the Riemannian metric and the tubular neighbourhood.

$\tilde{M}$ is oriented in such a way that the induced orientation on $M - \mathring{B}$ coincides with the orientation induced from M. This forces the inclusion of $P_{\mathbb{K}}(E) - \mathring{B}$ to be orientation reversing. Because G_1 is elementary there is no one-dimensional G_1-invariant subbundle of $N(N,M)$. Therefore we have $\#\pi_0(\tilde{M}^{G_1}) = \#\pi_0(M^{G_1}) - 1$.

So by iterating this process over all components of M^{G_1} one ends up at a torus manifold $\tilde{M}'$ with G-action without G_1-fixed points. In the following we will call $\tilde{M}'$ the blow up of M along M^{G_1}.

Lemma 4.18. *There is a G-equivariant map $F : \tilde{M} \to M$ which maps the exceptional submanifold $M_0 = P_{\mathbb{K}}(N(N,M) \oplus \{0\})$ to N and is the identity on $M - B$. Moreover F restricts to a diffeomorphism $\tilde{M} - M_0 \to M - N$ and is the bundle projection on M_0.*

Proof. The G-equivariant map

$$f : P_{\mathbb{K}}(E) - \mathring{B} \to B \quad (z_0 : z_1) \mapsto (z_0 \bar{z}_1 : |z_0|^2) \quad (z_0 \in N(N,M), z_1 \in \mathbb{K})$$

is the identity on ∂B. Therefore it may be extended to a continuous map $h : \tilde{M} \to M$ which is the identity outside of $P_{\mathbb{K}}(E) - \mathring{B}$.

Because $f|_{P_{\mathbb{K}}(E) - \mathring{B} - M_0} : P_{\mathbb{K}}(E) - \mathring{B} - M_0 \to B - N$ is a diffeomorphism there is a G-equivariant diffeomorphism $F' : \tilde{M} - M_0 \to M - N$ which is the identity outside $P_{\mathbb{K}}(E) - \mathring{B} - M_0$ and coincides with f near M_0 by [**33**, p. 24-25]. Therefore F' extends to a differentiable map $F : \tilde{M} \to M$. □

LEMMA 4.19. *Let H be a closed subgroup of G. Then there is a bijection*

$$\{\text{components of } M^H \not\subset N\} \to \{\text{components of } \tilde{M}^H \not\subset M_0\}$$

$$N' \mapsto \tilde{N}' = \left(P_{\mathbb{K}}(N(N \cap N', N') \oplus \mathbb{K}) - \mathring{B}\right) \cup_{\partial B \cap N'} \left(N' - \mathring{B}\right)$$

$$F(N'') \leftarrow N''.$$

For a component N' of M^H we call $\tilde{N}'$ the proper transform of N'.

PROOF. At first we calculate the fixed point set of the H-action on $\tilde{M}$.

$$\begin{aligned}\tilde{M}^H &= \left(\left(P_{\mathbb{K}}(E) - \mathring{B}\right) \cup_{\partial B} \left(M - \mathring{B}\right)\right)^H \\ &= \left(P_{\mathbb{K}}(E) - \mathring{B}\right)^H \cup_{\partial B^H} \left(M - \mathring{B}\right)^H\end{aligned}$$

There are pairwise distinct i-dimensional non-trivial irreducible H-representations V_{ij} and H-vector bundles E_{ij} over N^H such that

$$N(N,M)|_{N^H} = N(N,M)|_{N^H}^H \oplus \bigoplus_i \bigoplus_j E_{ij},$$

and the H-representation on each fiber of E_{ij} is isomorphic to $\mathbb{K}^{d_{ij}} \otimes_{\mathbb{K}} V_{ij}$ where $\mathbb{K}^{d_{ij}}$ denotes the trivial H-representation.

Now the H-fixed points in $P_{\mathbb{K}}(E)$ are given by

$$\begin{aligned}P_{\mathbb{K}}(E)^H &= P_{\mathbb{K}}(N(N,M) \oplus \mathbb{K})|_{N^H}^H \\ &= P_{\mathbb{K}}(N(N,M)|_{N^H}^H \oplus \mathbb{K}) \amalg \coprod_j P_{\mathbb{K}}(E_{1j} \oplus \{0\}),\end{aligned}$$

Because $N(N,M)|_{N^H}^H = N(N^H, M^H)$ we get

$$\begin{aligned}\tilde{M}^H &= \left(\left(P_{\mathbb{K}}(N(N^H,M^H) \oplus \mathbb{K}) - \mathring{B}^H\right) \cup_{\partial B^H} \left(M - \mathring{B}\right)^H\right) \\ &\qquad \amalg \coprod_j P_{\mathbb{K}}(E_{1j} \oplus \{0\}) \\ &= \coprod_{N' \subset M^H} \tilde{N}' \amalg \coprod_j P_{\mathbb{K}}(E_{1j} \oplus \{0\}),\end{aligned}$$

where N' runs through the connected components of M^H which are not contained in N. From this the statement follows. □

By replacing H by an one-dimensional subtorus of T we get:

COROLLARY 4.20. *There is a bijection between the characteristic submanifolds of M and the characteristic submanifolds of $\tilde{M}$ which are not contained in M_0.*

PROOF. The only thing what is to prove here is, that for a characteristic submanifold M_i of M $\tilde{M}_i^T$ is non-empty. If $(M_i - N)^T \neq \emptyset$ then this is clear.

If $p \in (M_i \cap N)^T$ then $P_{\mathbb{K}}(N(M_i \cap N, M_i) \oplus \{0\})|_p$ is a T-invariant submanifold of $\tilde{M}_i$ which is diffeomorphic to $\mathbb{C}P^k$ or $\mathbb{R}P^{2k}$. Therefore it contains a T-fixed point. □

This bijection is compatible with the action of the Weyl-group of G on the sets of characteristic submanifolds of $\tilde{M}$ and M.

In the real case the exceptional submanifold M_0 has codimension one in $\tilde{M}$ and is G-invariant. Because there is no S^1-representation of real dimension one, M_0 does not contain a characteristic submanifold of $\tilde{M}$ in this case.

In the complex case M_0 is G-invariant and may be a characteristic submanifold of $\tilde{M}$.

Therefore there is a bijection between the non-trivial orbits of the $W(G)$-actions on the sets of characteristic submanifolds of M and $\tilde{M}$. Therefore we get the same elementary factors for the actions on $\tilde{M}$ and M.

COROLLARY 4.21. *Let H be a closed subgroup of G and N' a component of M^H such that $N \cap N'$ has codimension one –in the real case– or two –in the complex case– in N'. Then F induces a $(N_GH)^0$-equivariant diffeomorphism of $\tilde{N}'$ and N'.*

PROOF. Because of the dimension assumption the map

$$f|_{P_{\mathbb{K}}(N(N\cap N',N')\oplus\mathbb{K})-\mathring{B}\cap N'} : P_{\mathbb{K}}(N(N \cap N', N') \oplus \mathbb{K}) - \mathring{B} \cap N' \to B \cap N'$$

from the proof of Lemma 4.18 is a diffeomorphism. Because the restriction of F to $\tilde{M} - M_0$ is an equivariant diffeomorphism the restriction $F|_{\tilde{N}'-M_0} : \tilde{N}' - M_0 \to N' - N$ is a diffeomorphism. Therefore $F|_{\tilde{N}'} : \tilde{N}' \to N'$ is a diffeomorphism. □

LEMMA 4.22. *In the complex case let $\bar{E} = N(N, M)^* \oplus \mathbb{C}$ where $N(N, M)^*$ is the normal bundle of N in M equipped with the dual complex structure. Then there is a G-equivariant diffeomorphism*

$$\tilde{M} \to P_{\mathbb{C}}(\bar{E}) - \mathring{B} \cup_{\partial B} M - \mathring{B}.$$

That means that the diffeomorphism type of $\tilde{M}$ does not change if we replace the complex structure on $N(N, M)$ by its dual.

PROOF. We have $P_{\mathbb{C}}(E) = E/\sim$ and $P_{\mathbb{C}}(\bar{E}) = E/\sim'$ where

$$(z_0, z_1) \sim (z_0', z_1') \Leftrightarrow \exists t \in \mathbb{C}^* \quad (tz_0, tz_1) = (z_0', z_1'),$$
$$(z_0, z_1) \sim' (z_0', z_1') \Leftrightarrow \exists t \in \mathbb{C}^* \quad (tz_0, \bar{t}z_1) = (z_0', z_1').$$

Therefore

$$E \to E \qquad\qquad (z_0, z_1) \mapsto (z_0, \bar{z}_1)$$

induces a G-equivariant diffeomorphism $P_{\mathbb{C}}(E) - \mathring{B} \to P_{\mathbb{C}}(\bar{E}) - \mathring{B}$ which is the identity on ∂B. By [**33**, p. 24-25] the result follows. □

LEMMA 4.23. *If in the complex case $G_1 = SU(l_1 + 1)$ and $\operatorname{codim} N = 2l_1 + 2$ or in the real case $G_1 = SO(2l_1 + 1)$ and $\operatorname{codim} N = 2l_1 + 1$ then $F : \tilde{M} \to M$ induces a homeomorphism $\bar{F} : \tilde{M}/G_1 \to M/G_1$.*

PROOF. Because $F|_{\tilde{M}-M_0} : \tilde{M} - M_0 \to M - N$ is a equivariant diffeomorphism and $\tilde{M}/G_1$,M/G_1 are compact Hausdorff-spaces the only thing that has to be checked is that

$$F|_{P_{\mathbb{K}}(N(N,M))} : P_{\mathbb{K}}(N(N, M)) \to N$$

induces a homeomorphism of the orbit spaces. But this map is just the bundle map $P_{\mathbb{K}}(N(N, M)) \to N$. Because of dimension reasons the G_1-action on the fibers of this bundle is transitive [**49**, p. 53-54]. Therefore the statement follows. □

REMARK 4.24. All statements proved above also hold for non-connected groups of the form $G \times K$ where K is a finite group and G is connected if we replace N by a K-invariant union of components of M^{G_1}.

Now we want to reverse the construction of a blow up. Let A be a closed G-manifold and $E \to A$ be a G-vector bundle such that G_1 acts trivially on A. If E is even dimensional we assume that there is a $x \in Z(G)$ such that x acts trivially on A and x^2 acts on E as $-\operatorname{Id}$. In this case we equip E with the complex structure induced by the action of x.

Assume that $\tilde{M}$ is a G-manifold and there is a G-equivariant embedding of $P_{\mathbb{K}}(E) \hookrightarrow \tilde{M}$ such that the normal bundle of $P_{\mathbb{K}}(E)$ is isomorphic to the tautological bundle over $P_{\mathbb{K}}(E)$.

Then one may identify a closed G-equivariant tubular neighbourhood B^c of $P_{\mathbb{K}}(E)$ in $\tilde{M}$ with

$$B^c = \{(z_0 : 1) \in P_{\mathbb{K}}(E \oplus \mathbb{K}); |z_0| \geq 1\} \cup \{(z_0 : 0) \in P_{\mathbb{K}}(E \oplus \mathbb{K})\}.$$

By gluing the complements of the interior of B^c in $\tilde{M}$ and $P_{\mathbb{K}}(E \oplus \mathbb{K})$ we get a G-manifold M such that A is G-equivariantly diffeomorphic to a union of components of M^{G_1}.

We call M the *blow down* of $\tilde{M}$ along $P_{\mathbb{K}}(E)$.

It is easy to see, that the G-equivariant diffeomorphism type of M does not depend on the choices of a metric on E and the tubular neighbourhood of $P_{\mathbb{K}}(E)$ in $\tilde{M}$, if G_1 acts transitively on the fibers of $P_{\mathbb{K}}(E) \to A$.

It is also easy to see that the blow up and blow down constructions are inverse to each other.

4.4. The case $G_1 = SU(l_1 + 1)$

In this section we discuss actions of groups which have a covering group of the form $G_1 \times G_2$ where $G_1 = SU(l_1+1)$ is elementary and G_2 acts effectively on M. It turns out that the blow up of M along M^{G_1} is a fiber bundle over $\mathbb{C}P^{l_1}$. This fact leads to our first classification result.

The assumption on G_2 is no restriction on G, because one may replace any covering group $\tilde{G}$ by the quotient $\tilde{G}/H$ where H is a finite subgroup of G_2 acting trivially on M. Following Convention 4.14 we also assume $\#\mathfrak{F}_1 = 2, 4$ in the cases $G_1 = SU(2)$ or $G_1 = SU(4)$, respectively. Furthermore we assume after conjugating T with some element of G_1 that $T_1 = T \cap G_1$ is the standard maximal torus of G_1.

We have the following lemma:

Lemma 4.25. *Let M be a torus manifold with G-action. Suppose $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$ elementary. Then the $W(S(U(l_1) \times U(1)))$-action on $\mathfrak{F}_1$ has an orbit $\mathfrak{F}_1'$ with l_1 elements and there is a component N_1 of $\bigcap_{M_i \in \mathfrak{F}_1'} M_i$ which contains a T-fixed point.*

Proof. We know that $W(SU(l_1+1)) = S_{l_1+1} = S(\mathfrak{F}_1)$ and $W(S(U(l_1) \times U(1))) = S_{l_1} \subset S_{l_1+1}$. Therefore the first statement follows. Let $x \in M^T$. Then by Lemmas 4.12 and 4.13 x is contained in the intersection of l_1 characteristic submanifolds of M belonging to $\mathfrak{F}_1$. Because $W(G_1) = S(\mathfrak{F}_1)$ there is a $g \in N_{G_1}T_1$ such that $gx \in \bigcap_{M_i \in \mathfrak{F}_1'} M_i$. □

Remark 4.26. We will see in Lemma 4.34 that $\bigcap_{M_i \in \mathfrak{F}_1'} M_i$ is connected.

Lemma 4.27. *Let M be a torus manifold with G-action. Suppose $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$ elementary. Furthermore let N_1 as in Lemma 4.25. Then there is a group homomorphism $\psi_1 : S(U(l_1) \times U(1)) \to Z(G_2)$ such that, with*

$$H_0 = SU(l_1+1) \times \operatorname{im}\psi_1,$$
$$H_1 = S(U(l_1) \times U(1)) \times \operatorname{im}\psi_1,$$
$$H_2 = \{(g, \psi_1(g)) \in H_1; g \in S(U(l_1) \times U(1))\},$$

(1) *$\operatorname{im}\psi_1$ is the projection of $\lambda(M_i)$, for all $M_i \in \mathfrak{F}_1$, to G_2,*
(2) *N_1 is a component of M^{H_2},*
(3) *N_1 is invariant under the action of G_2,*
(4) *$M = G_1 N_1 = H_0 N_1$.*

PROOF. Denote by T_2 the maximal torus $T \cap G_2$ of G_2. Let $x \in N_1^T$. If $x \in M^{SU(l_1+1)}$ we have by Lemma 4.13 the $SU(l_1+1) \times T_2$-representation

$$T_xM = W \otimes_{\mathbb{C}} V_1 \oplus \bigoplus_{i=2}^{n-l_1} V_i,$$

where W is the standard complex representation of $SU(l_1+1)$ or its dual and the V_i are one-dimensional complex representations of T_2 whose weights form a basis of the integral lattice in LT_2^*. From the description of the weight spaces of T_xM given in the proof of Lemma 4.13 we get that T_xN_1 is $S(U(l_1) \times U(1))$-invariant and that there is a one-dimensional complex representation W_1 of $S(U(l_1) \times U(1))$ such that

$$T_xN_1 = W_1 \otimes_{\mathbb{C}} V_1 \oplus \bigoplus_{i=2}^{n-l_1} V_i.$$

Now assume that x is not fixed by $SU(l_1+1)$. Because by Lemma 4.10 $G_1x \subset M^{T_2}$ is G_1-equivariantly diffeomorphic to $\mathbb{C}P^{l_1}$ we see by the definition of N_1 that $G_{1x} = S(U(l_1) \times U(1))$.

At the point x we get a representation of $S(U(l_1) \times U(1)) \times T_2$ of the form

$$T_xM = T_xN_1 \oplus T_xG_1x.$$

Since T_2 acts effectively on M and trivially on G_1x, there is a decomposition

$$T_xN_1 = \bigoplus_{i=1}^{n-l_1} V_i \otimes_{\mathbb{C}} W_i,$$

where the W_i are one-dimensional complex $S(U(l_1)\times U(1))$-representations and the V_i are one-dimensional complex T_2-representations whose weights form a basis of the integral lattice in LT_2^*.

Therefore in both cases there is a homomorphism $\psi_1 : S(U(l_1) \times U(1)) \to S^1 \to T_2$ such that, for all $g \in S(U(l_1) \times U(1))$, $(g, \psi_1(g))$ acts trivially on $T_xN_1 = \bigoplus_{i=1}^{n-l_1} V_i \otimes_{\mathbb{C}} W_i$.

The component of the identity of the isotropy subgroup of the torus T for generic points in N_1 is given by

$$H_3 = \{(t, \psi_1(t)) \in T_1 \times T_2\} = \langle \lambda(M_i); M_i \in \mathfrak{F}_1, M_i \supset N_1 \rangle. \tag{4.9}$$

Because the Weyl-group of G_2 acts trivially and orientation preserving on $\mathfrak{F}_1$, H_3 is pointwise fixed by the action of $W(G_2)$ by Lemma 4.9. Therefore the image of ψ_1 is contained in the center of G_2. Furthermore $\operatorname{im}\psi_1$ is the projection of $\lambda(M_i)$, $M_i \in \mathfrak{F}_1$, to T_2.

Because H_3 commutes with G_2 it follows that N_1 is G_2-invariant. So we have proved the first and the third statement.

Now we turn to the second and fourth part.

Because $T_xN_1 = (T_xM)^{H_3} = (T_xM)^{H_2}$, N_1 is a component of M^{H_2}. Because by Lemma A.2 H_1 is the only proper closed connected subgroup of H_0 which contains H_2 properly, for $y \in N_1$, there are the following possibilities:

- $H_{0y}^0 = H_0$,
- $H_{0y}^0 = H_1$ and $\dim H_0y = 2l_1$,
- $H_{0y}^0 = H_2$ and $\dim H_0y = 2l_1 + 1$,

where H_{0y}^0 is the identity component of the stabiliser of y in H_0. If $g \in H_0$ such that $gy \in N_1$ then we have $H_{0gy}^0 = gH_{0y}^0 g^{-1} \in \{H_0, H_1, H_2\}$. Therefore

$$g \in N_{H_0} H_{0y}^0 = \begin{cases} H_0 & \text{if } y \in M^{H_0}, \\ H_1 & \text{if } y \notin M^{H_0} \text{ and } l_1 > 1, \\ N_{G_1}T_1 \times \operatorname{im}\psi_1 & \text{if } H_{0y}^0 = H_1 \text{ and } l_1 = 1, \\ T_1 \times \operatorname{im}\psi_1 & \text{if } H_{0y}^0 = H_2, l_1 = 1 \text{ and } \operatorname{im}\psi_1 \neq \{1\}. \end{cases}$$

Now for $y \in N_1$ which is not fixed by the action of H_0 we have

$$\begin{aligned} \dim T_y N_1 \cap T_y H_0 y &\leq \dim N_1 \cap H_0 y \leq \dim H_1 y \\ &= \dim H_1/H_{0y}^0 = \begin{cases} 0 & \text{if } H_{0y}^0 = H_1, \\ 1 & \text{if } H_{0y}^0 = H_2 \text{ and } \operatorname{im}\psi_1 \neq \{1\}. \end{cases} \end{aligned}$$

Therefore N_1 intersects $H_0 y$ transversely in y and $GN_1 - N_1^{H_0} = H_0 N_1 - N_1^{H_0}$ is an open subset of M by Lemma A.5.

Because M is connected and $\operatorname{codim} M^{H_0} \geq 4$, $M - M^{H_0}$ is connected. Since $(M - M^{H_0}) \cap H_0 N_1 = H_0 N_1 - N_1^{H_0}$ is closed in $M - M^{H_0}$, we have $M - M^{H_0} = H_0 N_1 - N_1^{H_0}$. This implies

$$\begin{aligned} M &= \left(M - M^{H_0}\right) \amalg M^{H_0} = \left(H_0 N_1 - N_1^{H_0}\right) \amalg M^{H_0} \\ &= \left(H_0 N_1 - N_1^{H_0}\right) \amalg \left(M^{H_0} \cap N_1\right) \amalg \left(M^{H_0} - N_1^{H_0}\right) \\ &= H_0 N_1 \amalg \left(M^{H_0} - N_1^{H_0}\right). \end{aligned}$$

Because N_1 is a component of M^{H_2}, $N_1^{H_0}$ is a union of components of M^{H_0}. Therefore $M^{H_0} - N_1^{H_0}$ is closed in M. Because $H_0 N_1$ is closed in M, it follows that $M = GN_1 = H_0 N_1 = G_1 N_1$. □

The following lemma guarantees together with Lemma A.3 that, if $l_1 > 1$, the homomorphism ψ_1 is independent of all choices made in its construction, namely the choice of N_1 and of $x \in N_1^T$.

LEMMA 4.28. *In the situation of Lemma 4.27 let $T' = T_2$ or $T' = \operatorname{im}\psi_1$. Then the principal orbit type of the $G_1 \times T'$-action on M is given by $(G_1 \times T')/H_2$.*

PROOF. Let $H \subset G_1 \times T'$ be a principal isotropy subgroup. Then by Lemma 4.27 we may assume $H \supset H_2$. Consider the projection

$$\pi_1 : G_1 \times T' \to G_1$$

on the first factor.

At first we show that the restriction of π_1 to H is injective. Because $(G_1 \times T')_x \cap T' = T'_x$ for all $x \in M$ and the T'-action on M is effective there is an $x \in M$ such that

$$(G_1 \times T')_x \cap T' = \{1\}.$$

Furthermore there is an $g \in G_1 \times T'$ such that $(G_1 \times T')_x \supset gHg^{-1}$.

Because T' is contained in the center of $G_1 \times T'$ we get

$$\begin{aligned} gHg^{-1} \cap T' &= \{1\}, \\ H \cap g^{-1}T'g &= \{1\}, \\ H \cap T' &= \{1\}. \end{aligned}$$

Therefore the restriction of π_1 to H is injective.

Furthermore $\pi_1(H) \supset \pi_1(H_2) = S(U(l_1) \times U(1))$. Therefore we have

$$\pi_1(H) = \begin{cases} SU(l_1+1), S(U(l_1) \times U(1)) & \text{if } l_1 > 1, \\ SU(l_1+1), S(U(l_1) \times U(1)), N_{G_1}T_1 & \text{if } l_1 = 1. \end{cases}$$

There is a section $\phi : \pi_1(H) \to H \hookrightarrow G_1 \times T'$. Because T' is abelian and the center of $S(U(l_1) \times U(1))$ is one-dimensional we get

$$H = \phi(\pi_1(H)) = \begin{cases} G_1 & \text{if } \pi_1(H) = SU(l_1+1), \\ N_{G_1}T_1 & \text{if } \pi_1(H) = N_{G_1}T_1, \\ H_2 & \text{if } \pi_1(H) = S(U(l_1) \times U(1)). \end{cases}$$

The first case does not occur because G_1 acts non-trivially on M. If $l_1 = 1$ we see with Lemmas 4.10 and 4.13 that there are G_1-orbits of type $SU(2)/S(U(1) \times U(1))$ or of type $SU(2)/\{1\}$. Therefore $SU(2)/N_{G_1}T_1$ is not a principal orbit type of the $SU(2)$-action. Therefore $(SU(2) \times T')/N_{G_1}T_1$ is not a principal orbit type of the $SU(2) \times T'$-action. This proves the statement. □

If $l_1 = 1$, we have $\#\mathfrak{F}_1 = 2$ and $W(S(U(l_1) \times U(1))) = \{1\}$. Therefore there are two choices for N_1. Denote them by M_1 and M_2.

LEMMA 4.29. *In the situation described above let ψ_i be the homomorphism constructed for M_i, $i = 1, 2$. Then we have $\psi_1 = \psi_2^{-1}$.*

PROOF. By (4.9) we have

$$\lambda(M_i) = \{(t, \psi_i(t)) \in H_1;\ t \in S(U(1) \times U(1))\}.$$

Now with Lemma 4.9 we see

$$\begin{aligned} \lambda(M_1) = g\lambda(M_2)g^{-1} &= \{(t^{-1}, \psi_2(t)) \in H_1;\ t \in S(U(1) \times U(1))\} \\ &= \{(t, \psi_2(t)^{-1}) \in H_1;\ t \in S(U(1) \times U(1))\} \end{aligned}$$

where $g \in N_{G_1}T_1 - T_1$. Therefore the result follows. □

COROLLARY 4.30. *If in the situation of Lemma 4.27 the G_1-action on M has no fixed point, then M is the total space of a G-equivariant fiber bundle over $\mathbb{C}P^{l_1}$ with fiber some torus manifold; more precisely $M = H_0 \times_{H_1} N_1$.*

PROOF. $H_0 \times_{H_1} N_1$ is defined to be the space $H_0 \times N_1 / \sim_1$ where

$$\begin{aligned} & (g_1, y_1) \sim_1 (g_2, y_2) \\ \Leftrightarrow \quad & \exists h \in H_1 \quad g_1 h^{-1} = g_2 \text{ and } h y_1 = y_2. \end{aligned}$$

By Lemma 4.27 we have that $M = H_0 N_1 = (H_0 \times N_1)/\sim_2$ where

$$\begin{aligned} & (g_1, y_1) \sim_2 (g_2, y_2) \\ \Leftrightarrow \quad & g_1 y_1 = g_2 y_2. \end{aligned}$$

We show that the two equivalence relations $\sim_1, \sim_2$ are equal.

For $(g_1, y_1), (g_2, y_2) \in H_0 \times N_1$ we have

$$\begin{aligned} & g_1 y_1 = g_2 y_2 \\ \Leftrightarrow \quad & \exists h \in N_{H_0} H^0_{0y_1} \quad g_1 h^{-1} = g_2 \text{ and } h y_1 = y_2 \\ \Leftrightarrow \quad & \exists h \in H_1 \quad g_1 h^{-1} = g_2 \text{ and } h y_1 = y_2. \end{aligned}$$

For the last equivalence we have to show the implication from the second to the third line. If $l_1 > 1$, $N_{H_0} H^0_{0y_1}$ is equal to H_1 because y_1 is not a H_0-fixed point. So we have $h \in H_1$.

If $l_1 = 1$, then N_1 is a characteristic submanifold of M belonging to $\mathfrak{F}_1$. If $H^0_{0y_1} = H_2$, we are done because $N_{H_0} H^0_{0y_1} = H_1$.

Now assume that $H^0_{0y_1} = H_1$ and there is an $h \in N_{G_1}T_1 \times \operatorname{im}\psi_1 - T_1 \times \operatorname{im}\psi_1$ such that $y_2 = hy_1 \in N_1$. Then $y_2 \in N_1 \cap N_2 \subset M^{T_1 \times \operatorname{im}\psi_1}$ where N_2 is the other characteristic submanifold of M belonging to $\mathfrak{F}_1$.

As shown in the proof of Lemma 4.27 N_1 intersects H_0y_2 transversely in y_2. Therefore one has

$$T_{y_2}N_1 \oplus T_{y_2}H_0y_2 = T_{y_2}M = T_{y_2}N_2 \oplus T_{y_2}H_0y_2$$

as $T_1 \times \operatorname{im}\psi_1$-representations. This implies

$$T_{y_2}N_1 = T_{y_2}N_2$$

as $T_1 \times \operatorname{im}\psi_1$-representations. Therefore $T_1 \times \operatorname{im}\psi_1$ acts trivially on both N_1 and N_2. Therefore we have $\operatorname{im}\psi_1 = \{1\}$ and $\lambda(N_1) = \lambda(N_2) = T_1$. This gives a contradiction because the intersection of N_1 and N_2 is non-empty. □

COROLLARY 4.31. *In the situation of Lemma 4.27 we have* $M^{G_1} = M^{H_0} = \bigcap_{M_i \in \mathfrak{F}_1} M_i$.

PROOF. At first let $l_1 > 1$. By Lemma 4.27 we know $M^{H_0} \subset M^{G_1} \subset N_1$. Therefore $M^{G_1} \subset \bigcap_{g \in N_{G_1}T_1} gN_1 = \bigcap_{M_i \in \mathfrak{F}_1} M_i$. There is a $g \in N_{G_1}T_1 - T_1$ with $\dim\langle H_2, gH_2g^{-1}\rangle > \dim H_2$ and $gH_2g^{-1} \not\subset H_1$. Therefore $\langle H_2, gH_2g^{-1}\rangle = H_0$ follows. Because H_2 acts trivially on N_1 this implies

$$M^{H_0} \supset \bigcap_{g \in N_{G_1}T_1} gN_1 = \bigcap_{M_i \in \mathfrak{F}_1} M_i.$$

Now let $l_1 = 1$. Then $\mathfrak{F}_1$ contains two characteristic submanifolds M_1 and M_2. As in the first case one can show that $M^{H_0} \subset M^{G_1} \subset M_1 \cap M_2$.

So $M^{H_0} \supset M_1 \cap M_2$ remains to be shown. The assumption that there is an $y \in M_1 \cap M_2 - M^{H_0} \subset M^{H_1}$ leads to a contradiction as in the proof of Corollary 4.30. □

COROLLARY 4.32. *If in the situation of Lemma 4.27 ψ_1 is trivial then M^{G_1} is empty. Otherwise the normal bundle of $M^{G_1} = M^{H_0} = \bigcap_{M_i \in \mathfrak{F}_1} M_i$ possesses a G-invariant complex structure. It is induced by the action of some element $g \in \operatorname{im}\psi_1$. Furthermore it is unique up to conjugation.*

PROOF. If ψ_1 is trivial, $\langle \lambda(M_i); M_i \in \mathfrak{F}_1\rangle$ is contained in the l_1-dimensional maximal torus of G_1 by Lemma 4.27. By Corollary 4.31 and Lemma A.6 it follows that M^{H_0} is empty.

If ψ_1 is non-trivial then for $y \in M^{H_0}$ we have

$$N_y(M^{H_0}, M) = V_{\mathbb{C}} \oplus V_{\mathbb{R}},$$

where $\operatorname{im}\psi_1$ acts non-trivially on $V_{\mathbb{C}}$ and trivially on $V_{\mathbb{R}}$. Clearly $V_{\mathbb{C}}$ has at least real dimension two and the action of $\operatorname{im}\psi_1$ induces a H_0-invariant complex structure on $V_{\mathbb{C}}$. Because M^{H_0} has codimension $2l_1 + 2$ by Lemmas 4.31 and A.6 the dimension of $V_{\mathbb{R}}$ is at most $2l_1$. So it follows from [**49**, p. 53-54] that $V_{\mathbb{R}}$ is trivial, if $l_1 \neq 3$.

If $l_1 = 3$, we have $SU(4) = \mathrm{Spin}(6)$ and there are two possibilities:

(1) $V_{\mathbb{R}}$ is trivial.
(2) $V_{\mathbb{R}}$ is the standard representation of $SO(6)$ and $V_{\mathbb{C}}$ a one-dimensional complex representation of $\operatorname{im}\psi_1$.

In the second case the principal orbit type of the H_0 action is given by $\mathrm{Spin}(6) \times S^1/\mathrm{Spin}(5) \times \{1\}$. Therefore we see with Lemma 4.28 that the second case does not occur.

Because of dimension reasons we get

$$N_y(M^{H_0}, M) = V_{\mathbb{C}} = W \otimes_{\mathbb{C}} V,$$

where W is the standard complex representation of $SU(l_1+1)$ or its dual and V is a complex one-dimensional $\operatorname{im}\psi_1$ representation. Because $\operatorname{im}\psi_1 \subset Z(G)$ we see that $N(M^{H_0}, M)$ has a G-invariant complex structure which is induced by the action of some $g \in \operatorname{im}\psi_1$.

Next we prove the uniqueness of this complex structure. Assume that there is another $g' \in Z(G) \cap G_y$ whose action induces a complex structure on $N_y(M^{H_0}, M)$. Then g' induces a – with respect to the complex structure induced by g – complex linear H_0-equivariant map

$$J : N_y(M^{H_0}, M) \to N_y(M^{H_0}, M)$$

with $J^2 + \mathrm{Id} = 0$. Because $N_y(M^{H_0}, M)$ is an irreducible H_0-representation it follows by Schur's Lemma that J is multiplication with $\pm i$. Therefore g' induces up to conjugation the same complex structure as g. □

COROLLARY 4.33. *If in the situation of Lemma 4.27* $M^{G_1} = M^{H_0} \neq \emptyset$ *then* $\ker\psi_1 = SU(l_1)$.

PROOF. Let $y \in M^{H_0}$. Then by the proof of Corollary 4.32 we have

$$N_y(M^{H_0}, M) = W \otimes_{\mathbb{C}} V,$$

where W is the standard complex $SU(l_1+1)$-representation or its dual and V is a one-dimensional complex $\operatorname{im}\psi_1$-representation. Furthermore $\operatorname{im}\psi_1$ acts effectively on M.

Therefore a principal isotropy subgroup of the H_0-action is given by

$$H = \left\{ (g, g_{l+1}^{\pm 1}) \in H_1; g = \begin{pmatrix} A & 0 \\ 0 & g_{l_1+1} \end{pmatrix} \in S(U(l_1) \times U(1)) \text{ with } A \in U(l_1) \right\}.$$

Now the statement follows by the uniqueness of the principal orbit type and Lemmas 4.28 and A.3. □

LEMMA 4.34. *In the situation of Lemma 4.25* $\bigcap_{M_i \in \mathfrak{F}_1'} M_i = N_1$ *is connected.*

PROOF. Let $\tilde{M}$ be the blow up of M along M^{G_1} and $\tilde{N}_1$ the proper transform of N_1 in $\tilde{M}$. By Corollary 4.30 we have $\tilde{M} = H_0 \times_{H_1} \tilde{N}_1$ which is a fiber bundle over $\mathbb{C}P^{l_1}$. The characteristic submanifolds of $\tilde{M}$ which are permuted by $W(G_1)$ are given by the preimages of the characteristic submanifolds of $\mathbb{C}P^{l_1}$ under the bundle map. Because l_1 characteristic submanifolds of $\mathbb{C}P^{l_1}$ intersect in a single point we see $\bigcap_{M_i \in \mathfrak{F}_1'} \tilde{M}_i = \tilde{N}_1$. Therefore this intersection is connected. Because $\bigcap_{M_i \in \mathfrak{F}_1'} \tilde{M}_i$ is mapped by F to $\bigcap_{M_i \in \mathfrak{F}_1'} M_i$, we see that $\bigcap_{M_i \in \mathfrak{F}_1'} M_i = N_1$ is connected. □

By blowing up a torus manifold M with G-action along M^{G_1} one gets a torus manifold $\tilde{M}$ without G_1-fixed points.

Denote by $\tilde{N}_1$ the proper transform of N_1 as defined in Lemma 4.25. Then by Corollary 4.21 there is a $\langle H_1, G_2 \rangle$-equivariant diffeomorphism $F : \tilde{N}_1 \to N_1$.

Because $M_0 \cap \tilde{N}_1$ is mapped by this diffeomorphism to $M^{G_1} = M^{H_0} = N_1^{H_0}$, H_1 acts trivially on $M_0 \cap \tilde{N}_1$. By Corollary 4.30 we know that $\tilde{M}$ is diffeomorphic to $H_0 \times_{H_1} \tilde{N}_1 = H_0 \times_{H_1} N_1$.

A natural question arising here is: When is a torus manifold of this form a blow up of another torus manifold with G-action?

We claim that this is the case if and only if N_1 has a codimension two submanifold which is fixed by the H_1-action and $\ker\psi_1 = SU(l_1)$.

LEMMA 4.35. *Let N_1 be a torus manifold with G_2-action, A a closed codimension two submanifold of N_1,* $\psi_1 \in \operatorname{Hom}(S(U(l_1) \times U(1)), Z(G_2))$ *such that* $\operatorname{im}\psi_1$

acts trivially on A and $\ker\psi_1 = SU(l_1)$. *Let also*

$$\begin{aligned} H_0 &= SU(l_1+1)\times \operatorname{im}\psi_1,\\ H_1 &= S(U(l_1)\times U(1))\times \operatorname{im}\psi_1,\\ H_2 &= \{(g,\psi_1(g)); g\in S(U(l_1)\times U(1))\}.\end{aligned}$$

(1) *Then H_1 acts on N_1 by $(g,t)x = \psi_1(g)^{-1}tx$, where $x\in N_1$ and $(g,t)\in H_1$.*
(2) *Assume that $Z(G_2)$ acts effectively on N_1 and let $y\in A$ and V the one-dimensional complex H_1-representation $N_y(A,N_1)$. Then V extends to an l_1+1-dimensional complex representation of H_0. Therefore there is an l_1+1-dimensional complex G-vector bundle E' over A which contains $N(A,N_1)$ as a subbundle.*
(3) *Then the normal bundle of $H_0/H_1\times A$ in $H_0\times_{H_1} N_1$ is isomorphic to the dual of the normal bundle of $P_{\mathbb{C}}(E'\oplus\{0\})$ in $P_{\mathbb{C}}(E'\oplus\mathbb{C})$.*

The lemma guarantees that one can remove $H_0/H_1\times A$ from $H_0\times_{H_1} N_1$ and replace it by A to get a torus manifold with G-action M, such that $M^{H_0} = A$. The blow up of M along A is $H_0\times_{H_1} N_1$.

Proof. (1) is trivial.
(2) For $i = 1,\dots,l_1+1$ let

$$\lambda_i : T_1 \to S^1 \quad \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_{l_1+1}\end{pmatrix} \mapsto g_i$$

and $\mu : \operatorname{im}\psi_1 \to S^1$ the character of the $\operatorname{im}\psi_1$ representation $N_y(A,N_1)$. Then μ is an isomorphism.

And by [**9**, p. 176] the character ring of the maximal torus $T_1\times\operatorname{im}\psi_1$ of $H_1 = S(U(l_1)\times U(1))\times\operatorname{im}\psi_1$ is given by

$$R(T_1\times\operatorname{im}\psi_1) = \mathbb{Z}[\lambda_1,\dots,\lambda_{l_1+1},\mu,\mu^{-1}]/(\lambda_1\cdots\lambda_{l_1+1}-1)$$

With this notation the character of V is given by $\mu\lambda_{l_1+1}^{\pm1}$. Therefore the H_0-representation W with the character $\mu\sum_{i=1}^{l_1+1}\lambda_i^{\pm1}$ is l_1+1-dimensional and $V\subset W$.

Let $G_2 = G_2'\times\operatorname{im}\psi_1$ and $E'' = N(A,N_1)$ equipped with the action of G_2', but without the action of H_1. Then $E' = E''\otimes_{\mathbb{C}} W$ is a G-vector bundle with the required features.

Now we turn to (3). The normal bundle of $H_0/H_1\times A$ in $H_0\times_{H_1} N_1$ is given by $H_0\times_{H_1} N(A,N_1)$. The normal bundle of $P_{\mathbb{C}}(E'\oplus\{0\})$ in $P_{\mathbb{C}}(E'\oplus\mathbb{C})$ is the dual of the tautological bundle. Let $B = \{(z_0:1)\in P_{\mathbb{C}}(E'\oplus\mathbb{C}); |z_0|\leq 1\}$. Because the inclusion of $P_{\mathbb{C}}(E) - B$ in $\tilde{M}$ in the construction of the blow up was orientation reversing we have to find an isomorphism of $H_0\times_{H_1} N(A,N_1)$ and the tautological bundle over $P_{\mathbb{C}}(E'\oplus\{0\})$.

Consider the following commutative diagram

$$\begin{array}{ccc} H_0\times_{H_1} N(A,N_1) & \xrightarrow{f} & P_{\mathbb{C}}(E'\oplus\{0\})\times E' \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ H_0/H_1\times A & \xrightarrow{g} & P_{\mathbb{C}}(E'\oplus\{0\}) \end{array}$$

where the vertical maps are the natural projections and f,g are given by

$$f([(h_1,h_2):m]) = ([m\otimes h_2h_1e_1], m\otimes h_2h_1e_1)$$

and

$$g([h_1,h_2],q) = [m_q\otimes h_2h_1e_1],$$

where $e_1 \in W - \{0\}$ is fixed such that for all $g \in S(U(l_1) \times U(1))$ $\psi_1(g)ge_1 = e_1$ and $m_q \neq 0$ some element of the fiber of $N(A, N_1)$ over $q \in A$.

f induces the sought-after isomorphism. □

Now we are in the position to state our first classification theorem. To do so we need the following definition.

DEFINITION 4.36. Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. Then a triple (ψ, N, A) with

- $\psi \in \operatorname{Hom}(S(U(l_1) \times U(1)), Z(G_2))$,
- N a torus manifold with G_2-action,
- A the empty set or a closed codimension two submanifold of N, such that $\operatorname{im}\psi$ acts trivially on A and $\ker\psi = SU(l_1)$ if $A \neq \emptyset$,

is called *admissible for* $(\tilde{G}, G_1)$. We say that two admissible triples (ψ, N, A), (ψ', N', A') for $(\tilde{G}, G_1)$ are equivalent if there is a G_2-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A) = A'$ and

$$\psi = \begin{cases} \psi' & \text{if } l_1 > 1 \\ \psi'^{\pm 1} & \text{if } l_1 = 1. \end{cases}$$

THEOREM 4.37. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. There is a one-to-one-correspondence between the $\tilde{G}$-equivariant diffeomorphism classes of torus manifolds with $\tilde{G}$-action such that G_1 is elementary and the equivalence classes of admissible triples for $(\tilde{G}, G_1)$.*

PROOF. Let M be a torus manifold with $\tilde{G}$-action such that G_1 is elementary. Then by Corollaries 4.31 and 4.33 (ψ_1, N_1, M^{H_0}) is an admissible triple, where ψ_1 is defined as in Lemma 4.27 and N_1 is defined as in Lemma 4.25.

Let (ψ, N, A) be an admissible triple for $(\tilde{G}, G_1)$. If $A \neq \emptyset$ by Lemma 4.35 the blow down of $H_0 \times_{H_1} N$ along $H_0/H_1 \times A$ is a torus manifold with $\tilde{G}$-action. If $A = \emptyset$ then we have the torus manifold $H_0 \times_{H_1} N$.

We show that these two operations are inverse to each other. Let M be a torus manifold with $\tilde{G}$-action. If $M^{H_0} = \emptyset$ then by Corollary 4.30 we have $M = H_0 \times_{H_1} N_1$. If $M^{H_0} \neq \emptyset$ then by the discussion before Lemma 4.35 M is the blow down of $H_0 \times_{H_1} N_1$ along $H_0/H_1 \times M^{H_0}$.

Now assume $l_1 > 1$ and let (ψ, N, A) be an admissible triple with $A \neq \emptyset$ and M the blow down of $H_0 \times_{H_1} N$ along $H_0/H_1 \times A$. Then by the remark after Lemma 4.35 we have $A = M^{H_0}$. By Lemma 4.34 and Corollary 4.21 we have $N = N_1$. With Lemmas 4.28 and A.3 one sees that $\psi = \psi_1$, where ψ_1 is the homomorphism defined in Lemma 4.27 for M.

Now let $(\psi, N, \emptyset)$ be an admissible triple and $M = H_0 \times_{H_1} N$. Then we have $M^{H_0} = \emptyset$. By Lemma 4.34 we have $N = N_1$. As in the first case one sees $\psi = \psi_1$.

Now assume $l_1 = 1$ and let (ψ, N, A) be an admissible triple with $A \neq \emptyset$ and M the blow down of $H_0 \times_{H_1} N$ along $H_0/H_1 \times A$. Then by the remark after Lemma 4.35 $A = M^{H_0}$. By Lemma 4.29 we have two choices for N_1 and $\psi = \psi_1^{\pm 1}$. Because the two choices for N_1 lead to equivalent admissible triples we recover the equivalence class of (ψ, N, A). In the case $A = \emptyset$ a similar argument completes the proof of the theorem. □

COROLLARY 4.38. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$. Then the torus manifolds with $\tilde{G}$-action such that G_1 is elementary and $M^{G_1} \neq \emptyset$ are given by blow downs of fiber bundles over $\mathbb{C}P^{l_1}$ with fiber some torus manifold with G_2-action along a submanifold of codimension two.*

Now we specialise our classification result to special classes of torus manifolds.

THEOREM 4.39. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$, M a torus manifold with $\tilde{G}$-action and (ψ, N, A) the admissible triple for $(\tilde{G}, G_1)$ corresponding to M. Then $H^*(M;\mathbb{Z})$ is generated by its degree two part if and only if $H^*(N;\mathbb{Z})$ is generated by its degree two part and A is connected.*

PROOF. To make the notation simpler we omit the coefficients of the cohomology in the proof. If $H^*(M)$ is generated by its degree two part then $H^*(N)$ is generated by its degree two part by [**42**, p. 716] and A is connected by [**42**, p. 738] and Corollary 4.31.

Now assume that $H^*(N)$ is generated by its degree two part and $A = \emptyset$. Then by Poincaré duality $H_{\text{odd}}(N) = 0$. Therefore by an universal coefficient theorem $H^*(N) = \text{Hom}(H_*(N), \mathbb{Z})$ is torsion free. By Corollary 4.30 M is a fiber bundle over $\mathbb{C}P^{l_1}$ with fiber N. Because the Serre-spectral sequence of this fibration degenerates we have

$$H^*(M) \cong H^*(\mathbb{C}P^{l_1}) \otimes H^*(N)$$

as a $H^*(\mathbb{C}P^{l_1})$-modul. Because $H^*(N)$ is generated by its degree two part it follows that the cohomology of M is generated by its degree two part.

Now we turn to the general case $A \neq \emptyset$. Then by [**42**, p. 716] $H^*(A)$ is generated by its degree two part and $H^*(N) \to H^*(A)$ surjective. Let $\tilde{M}$ be the blow up of M along A and $F : \tilde{M} \to M$ the map defined in section 4.3.

Because by Lemma 4.18 F is the identity outside some open tubular neighbourhood of $A \times \mathbb{C}P^{l_1}$, $F^* : H^*(M, A) \to H^*(\tilde{M}, A \times \mathbb{C}P^{l_1})$ is an isomorphism by excision. Furthermore the push forward $F_! : H^*(\tilde{M}) \to H^*(M)$ is a section to $F^* : H^*(M) \to H^*(\tilde{M})$. Therefore $F^* : H^*(M) \to H^*(\tilde{M})$ is injective and $H^{\text{odd}}(M)$ vanishes.

Because A is connected we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & & H^2(\tilde{M}, A \times \mathbb{C}P^{l_1}) & \longrightarrow & H^2(N, A) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(\mathbb{C}P^{l_1}) & \longrightarrow & H^2(\tilde{M}) & \longrightarrow & H^2(N) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^2(\mathbb{C}P^{l_1}) & \longrightarrow & H^2(A \times \mathbb{C}P^{l_1}) & \longrightarrow & H^2(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & H^3(\tilde{M}, A \times \mathbb{C}P^{l_1}) & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Now from the snake lemma it follows that

$$H^2(M, A) \cong_{F^*} H^2(\tilde{M}, A \times \mathbb{C}P^{l_1}) \cong H^2(N, A)$$

and

$$H^3(M, A) \cong_{F^*} H^3(\tilde{M}, A \times \mathbb{C}P^{l_1}) \cong 0.$$

Because $\iota_{NM} = F \circ \iota_{N\tilde{M}}$, where $\iota_{NM}, \iota_{N\tilde{M}}$ are the inclusions of N in M and $\tilde{M}$, the left arrow in the following diagram is an isomorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^2(M,A) & \longrightarrow & H^2(M) & \longrightarrow & H^2(A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota_{NM}^*} & & \downarrow{\scriptstyle \iota_{NM}^*} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & H^2(N,A) & \longrightarrow & H^2(N) & \longrightarrow & H^2(A) & \longrightarrow & 0 \end{array}$$

Therefore it follows from the five lemma that

$$H^2(M) \cong H^2(N)$$

and

$$H^2(\tilde{M}) \cong H^2(\mathbb{C}P^{l_1}) \oplus H^2(N) \cong H^2(\mathbb{C}P^{l_1}) \oplus H^2(M).$$

Let $t \in H^2(\mathbb{C}P^{l_1})$ be a generator of $H^*(\mathbb{C}P^{l_1})$ and $x \in H^*(M)$. Then because $H^*(\tilde{M})$ is generated by its degree two part there are sums of products $x_i \in H^*(M)$ of elements of $H^2(M)$ such that

$$x = F_!F^*(x) = F_! \left(\sum F^*(x_i)t^i \right) = \sum x_i F_!(t^i).$$

Therefore it remains to show that $F_!(t^i)$ is a product of elements of $H^2(M)$.

The l_1+1 characteristic submanifolds $\tilde{M}_1, \dots, \tilde{M}_{l_1+1}$ of $\tilde{M}$ which are permuted by $W(G_1)$ are the preimages of the characteristic submanifolds of $\mathbb{C}P^{l_1}$ under the projection $\tilde{M} \to \mathbb{C}P^{l_1}$. Therefore they can be oriented in such a way that t is the Poincaré-dual of each of them.

Because F restricts to a diffeomorphism $\tilde{M} - A \times \mathbb{C}P^{l_1} \to M - A$ and $F(\tilde{M}_i) = M_i$, $F_!(t^i)$, $i \leq l_1$, is the Poincaré-dual of the intersection $\bigcap_{1\leq k\leq i} M_k$ of characteristic submanifolds of M which belong to $\mathfrak{F}_1$. Therefore for $i \leq l_1$ we have

$$F_!(t)^i = PD \left(\bigcap_{1\leq k\leq i} M_k \right) = F_!(t^i).$$

Because $t^i = 0$ for $i > l_1$ the statement follows. □

THEOREM 4.40. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1+1)$, M a torus manifold with $\tilde{G}$-action and (ψ, N, A) the admissible triple for $(\tilde{G}, G_1)$ corresponding to M. Then M is quasitoric if and only if N is quasitoric and A is connected.*

PROOF. At first assume that M is quasitoric. Then N is quasitoric and A connected because all intersections of characteristic submanifolds of M are quasitoric and connected.

Now assume that N is quasitoric and $A \subset N$ connected. Then by Theorem 4.39 and [**42**, p. 738] the T-action on M is locally standard and M/T is a homology polytope. We have to show that M/T is face preserving homeomorphic to a simple polytope.

The orbit space N/T^{l_0} is face preserving homeomorphic to a simple polytope P. Because A is connected A/T^{l_0} is a facet F_1 of P.

With the notation from Lemma 4.35 let

$$B = \{(z_0 : 1) \in P_{\mathbb{C}}(E' \oplus \mathbb{C}); z_0 \in E', |z_0| \leq 1\}.$$

Then the orbit space of the T-action on B is given by $F_1 \times \Delta^{l_1+1}$.

Let B' be a closed $\tilde{G}$-invariant tubular neighbourhood of $H_0/H_1 \times A$ in $H_0 \times_{H_1} N$. Then the bundle projection $\partial B' \to H_0/H_1 \times A$ extends to an equivariant map

$$H_0 \times_{H_1} N - \mathring{B}' \to H_0 \times_{H_1} N$$

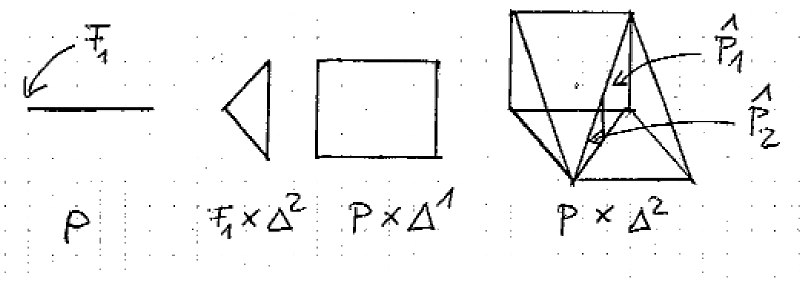


FIGURE 1. The orbit space of a blow down

which induces a face preserving homeomorphism

$$\left(H_0 \times_{H_1} N - \mathring{B}'\right)/T \cong P \times \Delta^{l_1}$$

Now M is given by gluing B and $H_0 \times_{H_1} N - \mathring{B}'$ along the boundaries $\partial B, \partial B'$. The corresponding gluing of the orbit spaces is illustrated in Figure 1 for the case $\dim N = 2$ and $l_1 = 1$. Because the gluing map $f : \partial B \to \partial B'$ is $\tilde{G}$-equivariant and G_1 acts transitive on the fibers of $\partial B \to A$ and $\partial B' \to A$ it induces a map

$$\hat{f} : F_1 \times \Delta^{l_1} = \partial B/T \to \partial B'/T = F_1 \times \Delta^{l_1} \quad , \quad (x,y) \mapsto (\hat{f}_1(x), \hat{f}_2(x,y)),$$

where $\hat{f}_1 : F_1 \to F_1$ is a face preserving homeomorphism and $\hat{f}_2 : F_1 \times \Delta^{l_1} \to \Delta^{l_1}$ such that for all $x \in F_1$ $\hat{f}_2(x, \cdot)$ is a face preserving homeomorphism of Δ^{l_1}.

Now fix embeddings

$$\Delta^{l_1+1} \hookrightarrow \mathbb{R}^{l_1+1} \text{ and } P \hookrightarrow \mathbb{R}^{n-l_1-1} \times [0,1[$$

such that $\Delta^{l_1} \subset \mathbb{R}^{l_1} \times \{1\}$ and $\Delta^{l_1+1} = \operatorname{conv}(0, \Delta^{l_1})$ and $P \cap \mathbb{R}^{n-l_1-1} \times \{0\} = F_1$.

Denote by $p_1 : \mathbb{R}^{l_1+1} \to \mathbb{R}$ and $p_2 : \mathbb{R}^{n-l_1} \to \mathbb{R}$ the projections on the last coordinate. For $\epsilon > 0$ small enough P and $P \cap \{p_2 \geq \epsilon\}$ are combinatorially equivalent. Therefore there is a face preserving homeomorphism

$$g_1 : P \to P \cap \{p_2 \geq \epsilon\}$$

such that $g_1(F_1) = P \cap \{p_2 = \epsilon\}$ and $g_1(F_i) = F_i \cap \{p_2 \geq \epsilon\}$ for the other facets of P.

$$\begin{aligned} g_2 : F_1 \times [0,1] &\to P \cap \{p_2 \leq \epsilon\} \\ (x,y) &\mapsto x(1-y) + y g_1(x) \end{aligned}$$

is a face preserving homeomorphism with $p_2 \circ g_2(x,y) = \epsilon y$ for all $(x,y) \in F_1 \times [0,1]$. Now let

$$\begin{aligned} \hat{P} &= P \times \Delta^{l_1+1} \cap \{p_1 = p_2\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}, \\ \hat{P}_1 &= P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \geq \epsilon\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}, \\ \hat{P}_2 &= P \times \Delta^{l_1+1} \cap \{p_1 = p_2 \leq \epsilon\} \subset \mathbb{R}^{n-l_1} \times \mathbb{R}^{l_1+1}. \end{aligned}$$

Then there are face preserving homeomorphisms

$$h_1 : P \times \Delta^{l_1} \to \hat{P}_1 \qquad (x,y) \mapsto (g_1(x), p_2(g_1(x))y)$$

and

$$h_2 : F_1 \times \Delta^{l_1+1} \to \hat{P}_2 \qquad (x,y) \mapsto (g_2(x, p_1(y)), \epsilon y).$$

We claim that $\hat{P}$ and M/T are face preserving homeomorphic. This is the case if

$$\hat{f}^{-1} \circ h_1^{-1} \circ h_2 : F_1 \times \Delta^{l_1} \to F_1 \times \Delta^{l_1}$$

extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$. Now for $(x, y) \in F_1 \times \Delta^{l_1}$ we have

$$\begin{aligned}\hat{f}^{-1} \circ h_1^{-1} \circ h_2(x, y) &= \hat{f}^{-1} \circ h_1^{-1}(g_2(x, p_1(y)), \epsilon y)\\ &= \hat{f}^{-1} \circ h_1^{-1}(g_2(x, 1), \epsilon y)\\ &= \hat{f}^{-1}(g_1^{-1} \circ g_2(x, 1), y)\\ &= (\hat{f}_1^{-1}(x), (\hat{f}_2(x, \cdot))^{-1}(y)).\end{aligned}$$

Because Δ^{l_1+1} is the cone over Δ^{l_1} this map extends to a face preserving homeomorphism of $F_1 \times \Delta^{l_1+1}$. □

LEMMA 4.41. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SU(l_1 + 1)$, M a torus manifold with $\tilde{G}$-action and (ψ, N, A) the admissible triple for $(\tilde{G}, G_1)$ corresponding to M. Then there is an isomorphism $\pi_1(N) \to \pi_1(M)$.*

PROOF. Let $\tilde{M}$ be the blow up of M along A. Then by [**44**, p. 270] there is an isomorphism $\pi_1(\tilde{M}) \to \pi_1(M)$

Now by Corollary 4.30 $\tilde{M}$ is the total space of a fiber bundle over $\mathbb{C}P^{l_1}$ with fiber N. Therefore there is an exact sequence

$$\pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1}) \to \pi_1(N) \to \pi_1(\tilde{M}) \to 0.$$

Because the torus action on N has fixed points there is a section in this bundle and $\pi_2(\tilde{M}) \to \pi_2(\mathbb{C}P^{l_1})$ is surjective. □

4.5. The case $G_1 = SO(2l_1)$

In this section we discuss torus manifolds with G-action where $\tilde{G} = G_1 \times G_2$ and $G_1 = SO(2l_1)$ is elementary. It turns out that the restriction of the action of G_1 to $U(l_1)$ on such a manifold has the same orbits as the action of $SO(2l_1)$. Therefore the results of the previous section may be applied to construct invariants for such manifolds. For simply connected torus manifolds with G-action these invariants determine their $\tilde{G}$-equivariant diffeomorphism type.

Let $\tilde{G} = G_1 \times G_2$ where $G_1 = SO(2l_1)$ is elementary and M a torus manifold with G-action. Then as in the proof of Corollary 4.15 one sees that the G_1-action has only two orbit types $SO(2l_1)/SO(2l_1-1)$ and $SO(2l_1)/SO(2l_1)$. The induced action of $U(l_1)$ has the same orbits which are of type $U(l_1)/U(l_1 - 1)$ and $U(l_1)/U(l_1)$.

REMARK 4.42. By Corollary 4.31 M is a special $SO(2l_1)$-, $U(l_1)$-manifold in the sense of Jänich [**32**].

Let $S = S^1$. Then there is a finite covering

$$SU(l_1) \times S \to U(l_1) \qquad\qquad (A, s) \mapsto sA.$$

So we may replace the factor G_1 of $\tilde{G}$ by $SU(l_1)$ and G_2 by $S \times G_2$ to reach the situation of the previous section.

Let $x \in M^T$ and $T_2 = T \cap G_2$. Then we may assume by Lemma 4.13 that the $G_1 \times T_2$-representation T_xM is given by

$$T_xM = V \oplus W,$$

where V is a complex representation of T_2 and W is the standard real representation of G_1. Therefore

$$T_xM = V \oplus V_0 \otimes_{\mathbb{C}} W_0$$

as a $SU(l_1)\times S\times T_2$-representation, where V_0 is the standard complex one-dimensional representation of S and W_0 is the standard complex representation of $SU(l_1)$.

Therefore the group homomorphism ψ_1 and the groups H_0, H_1, H_2 introduced in Lemma 4.27 have the following form:

$$\operatorname{im}\psi_1 = S,$$

and

$$H_0 = SU(l_1) \times S,$$
$$H_1 = S(U(l_1-1) \times U(1)) \times S,$$
$$H_2 = \left\{ (g, g_{l_1+1}^{-1}) \in H_1; g = \begin{pmatrix} A & 0 \\ 0 & g_{l_1+1} \end{pmatrix} \text{ with } A \in U(l_1-1) \right\}.$$

Let N_1 be the intersection of $l_1 - 1$ characteristic submanifolds of M belonging to $\mathfrak{F}_1$ as defined in Lemmas 4.25 and 4.34. Then by Lemma 4.27 we know that N_1 is a component of M^{H_2} and $M = H_0N_1$. Therefore we have $N_1 = M^{H_2}$ if for all H_0-orbits O O^{H_2} is connected. Because all orbits are of type H_0/H_0 or H_0/H_2 and

$$(H_0/H_2)^{H_2} = N_{H_0}H_2/H_2 = H_1/H_2,$$

it follows that $N_1 = M^{H_2}$.

The projection $H_1 \to H_1/H_2$ induces an isomorphism $S \to H_1/H_2$. Therefore S acts freely on $(H_0/H_2)^{H_2}$ and trivially on H_0/H_0. This implies that S acts semi-freely on N_1.

By Corollary 4.31 $N_1^S = M^{H_0}$ has codimension two in N_1.

Now we turn to the question under which conditions the action of $U(l_1) \times G_2$ on a torus manifold with $U(l_1) \times G_2$-action satisfying the above conditions on the $U(l_1)$-orbits extends to an action of $SO(2l_1) \times G_2$.

Let X be the orbit space of the $U(l_1)$-action on M. Then by [**32**, p. 303] X is a manifold with boundary such that the interior $\mathring{X}$ of X corresponds to orbits of type $U(l_1)/U(l_1-1)$ and the boundary ∂X to the fixed points. The action of G_2 on M induces a natural action of G_2 on X.

Following Jänich [**32**] we may construct from M a manifold $M \odot M^{U(l_1)}$ with boundary on which $U(l_1) \times G_2$ acts such that all orbits of the $U(l_1)$-action on $M \odot M^{U(l_1)}$ are of type $U(l_1)/U(l_1-1)$ and $\left(M \odot M^{U(l_1)}\right)/U(l_1) = X$. Denote by P_M the G_2-equivariant principal S^1-bundle

$$\left(M \odot M^{U(l_1)}\right)^{U(l_1-1)} \to X.$$

Lemma 4.43. *Let M be a torus manifold with $U(l_1) \times G_2$-action such that all $U(l_1)$-orbits are of type $U(l_1)/U(l_1-1)$ or $U(l_1)/U(l_1)$. Then the action of $U(l_1) \times G_2$ on M extends to an action of $SO(2l_1) \times G_2$ if and only if there is a G_2-equivariant $\mathbb{Z}_2$-principal bundle P'_M such that*

$$P_M = S^1 \times_{\mathbb{Z}_2} P'_M,$$

where the action of G_2 on S^1 is trivial.

Proof. If the action extends to a $SO(2l_1) \times G_2$-action, then $SO(2l_1) \times G_2$ acts on $M \odot M^{U(l_1)}$. Therefore $P'_M = \left(M \odot M^{U(l_1)}\right)^{SO(2l_1-1)} \to X$ is such a G_2-equivariant $\mathbb{Z}_2$-principal bundle.

If there is such a G_2-equivariant $\mathbb{Z}_2$-bundle P'_M then by a G_2-equivariant version of Jänich's Klassifikationssatz [**32**] there is a torus manifold M' with $SO(2l_1) \times G_2$-action with $M'/U(l_1) = X$ and $P_M = S^1 \times_{\mathbb{Z}_2} P'_M = P_{M'}$. Therefore M' and M are $U(l_1) \times G_2$-equivariantly diffeomorphic. □

Lemma 4.44. *Let M, M' be simply connected torus manifolds with $SO(2l_1) \times G_2$-action. Then M and M' are $SO(2l_1) \times G_2$-equivariantly diffeomorphic if and only if they are $U(l_1) \times G_2$-equivariantly diffeomorphic.*

PROOF. If M and M' are $U(l_1) \times G_2$-equivariantly diffeomorphic, then $X = M/SO(2l_1)$ and $M'/SO(2l_2)$ are G_2-equivariantly diffeomorphic. By [**8**, p. 91] X is simply connected.

Therefore the only $\mathbb{Z}_2$-bundle over X is the trivial one. The G_2-action on X lifts uniquely into it. Therefore by Jänich's Klassifikationssatz M and M' are $SO(2l_1) \times G_2$-equivariantly diffeomorphic. □

Let M be a simply connected torus manifold with $SO(2l_1) \times G_2$-action. By Theorem 4.37 there is a admissible triple (ψ, N, A) corresponding to M equipped with the action of $SU(l_1) \times S \times G_2$ as above. (ψ, N, A) determines the $SU(l_1) \times S \times G_2$-equivariant diffeomorphism type of M. With Lemma 4.44 we see that the $SO(2l_1) \times G_2$-equivariant diffeomorphism type of M is determined by (ψ, N, A).

LEMMA 4.45. *Let M be a torus manifold with $G_1 \times G_2$ action where $G_1 = SO(2l_1)$ is elementary and G_2 is a not necessary connected Lie-group. If $M^{SO(2l_1)}$ is connected then G_2 acts orientation preserving on $N(M^{SO(2l_1)}, M)$. Therefore G_2 acts orientation preserving on M if and only if it acts orientation preserving on $M^{SO(2l_1)}$.*

PROOF. Let $g \in G_2$ and $x \in M^{SO(2l_1)}$ and $y = gx \in M^{SO(2l_1)}$. Because $M^{SO(2l_1)}$ is connected there is a orientation preserving $SO(2l_1)$-invariant isomorphism

$$N_x(M^{SO(2l_1)}, M) \cong N_y(M^{SO(2l_1)}, M)$$

Therefore $g : N_x(M^{SO(2l_1)}, M) \to N_y(M^{SO(2l_1)}, M)$ induces an automorphism ϕ of the $SO(2l_1)$-representation $N_x(M^{SO(2l_1)}, M)$ which is orientation preserving if and only if g is orientation preserving.

Because by Lemma 4.13 $N_x(M^{SO(2l_1)}, M)$ is just the standard real representation of $SO(2l_1)$ $N_x(M^{SO(2l_1)}, M) \otimes_{\mathbb{R}} \mathbb{C}$ is an irreducible complex representation. Therefore by Schur's Lemma there is a $\lambda \in \mathbb{C} - \{0\}$ such that for all $a \in N_x(M^{SO(2l_1)}, M)$

$$\phi(a) \otimes 1 = \phi_{\mathbb{C}}(a \otimes 1) = a \otimes \lambda.$$

This equation implies that $\lambda \in \mathbb{R} - \{0\}$ and $\phi(a) = \lambda a$. Therefore ϕ is orientation preserving. □

4.6. The case $G_1 = SO(2l_1 + 1)$

In this section we discuss actions of groups which have a covering group whose action on M factors through $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$ elementary. In the case $G_1 = SO(3)$ we also assume $\#\mathfrak{F}_1 = 1$ or that the principal orbit type of the $SO(3)$-action on M is given by $SO(3)/SO(2)$.

It is shown that a torus manifold M with $\tilde{G}$-action is a product of a sphere and a torus manifold with G_2-action or the blow up of M along the fixed points of G_1 is a fiber bundle over a real projective space.

We assume that $T_1 = T \cap G_1$ is the standard maximal torus of G_1.

LEMMA 4.46. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$, M a torus manifold with G-action such that G_1 is elementary. If $l_1 > 1$ there is by Lemma 4.12 a component N_1 of $\bigcap_{M_i \in \mathfrak{F}_1} M_i$ with $N_1^T \neq \emptyset$. If $l_1 = 1$ let N_1 be a characteristic submanifold belonging to $\mathfrak{F}_1$. Then:*

1. *N_1 is a component of $M^{SO(2l_1)}$.*
2. *$M = G_1 N_1$.*

PROOF. Let $x \in N_1^T$. Then by Lemmas 4.10, 4.13 and Remark 4.11 $G_{1x} = SO(2l_1)$. Let T_2 be the maximal torus $T \cap G_2$ of G_2. On the tangent space of M in x we have the $SO(2l_1) \times T_2$-representation

$$T_xM = N_x(G_1x, M) \oplus T_xG_1x.$$

By Lemma 4.10 T_2 acts trivially on G_1x and $N_x(G_1x, M)$ splits as a sum of complex one dimensional representations. If $l_1 > 1$ $SO(2l_1)$ has no non-trivial one-dimensional complex representation. Therefore we have

$$T_xM = \bigoplus_i V_i \oplus W, \tag{4.10}$$

where the V_i are one-dimensional complex representations of T_2 and W is the standard real representation of $SO(2l_1)$.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 2$, then $SO(2l_1)$ acts trivially on $N_x(G_1x, M)$, because $SO(3)/SO(2)$ is the principal orbit type of the $SO(3)$-action on M [**8**, p. 181].

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$, then by the discussion leading to Convention 4.14 $SO(2)$ acts trivially on $N_x(G_1x, M)$. Therefore in these cases T_xM splits as in (4.10).

Because $N_x(G_1x, M)$ is the tangent space of N_1 in x the maximal torus T_1 of G_1 acts trivially on N_1. Therefore N_1 is the component of M^{T_1} which contains x. Because $T_xN_1 = (T_xM)^{T_1} = (T_xM)^{SO(2l_1)}$, N_1 is a component of $M^{SO(2l_1)}$.

Now we prove (2). Let $y \in N_1$. Then there are the following possibilities:

- $G_{1y} = G_1$
- $G_{1y} = S(O(2l_1) \times O(1))$ and $\dim G_1y = 2l_1$
- $G_{1y} = SO(2l_1)$ and $\dim G_1y = 2l_1$

If $g \in G_1$ such that $gy \in N_1$ then

$$gG_{1y}g^{-1} = G_{1gy} \in \{S(O(2l_1) \times O(1)), SO(2l_1), G_1\}$$

and

$$g \in N_{G_1}G_{1y} = \begin{cases} G_1 & \text{if } y \in M^{G_1}, \\ S(O(2l_1) \times O(1)) & \text{if } y \notin M^{G_1}. \end{cases}$$

Therefore $G_1y \cap N_1 \subset S(O(2l_1) \times O(1))y$ contains at most two elements. If y is not fixed by G_1, then G_1y and N_1 intersect transversely in y.

Therefore $G_1(N_1 - N_1^{G_1})$ is open in $M - M^{G_1}$ by Lemma A.5. Because M^{G_1} has codimension at least three, $M - M^{G_1}$ is connected. But

$$G_1\left(N_1 - N_1^{G_1}\right) = G_1N_1 \cap \left(M - M^{G_1}\right)$$

is also closed in $M - M^{G_1}$. This implies

$$M - M^{G_1} = G_1\left(N_1 - N_1^{G_1}\right) = G_1N_1 - N_1^{G_1}.$$

Therefore

$$M = G_1N_1 \amalg \left(M^{G_1} - N_1^{G_1}\right).$$

Because G_1N_1 and $M^{G_1} - N_1^{G_1}$ are closed in M the statement follows. □

COROLLARY 4.47. *If in the situation of Lemma 4.46 the G_1-action on M has no fixed point in M, then $M = SO(2l_1 + 1)/SO(2l_1) \times N_1$ or $M = SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$, where $\mathbb{Z}_2 = S(O(2l_1) \times O(1))/SO(2l_1)$.*

In the second case the $\mathbb{Z}_2$-action on N_1 is orientation reversing.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$ then we have $M = SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$. If $l_1 = 1$ and $\#\mathfrak{F}_1 = 2$ then we have $M = SO(2l_1 + 1)/SO(2l_1) \times N_1$.

PROOF. Let $g \in S(O(2l_1) \times O(1)) = N_{G_1}SO(2l_1)$. Then gN_1 is a component of $M^{SO(2l_1)}$. Because $N_1 \subset M^{SO(2l_1)}$ gN_1 only depends on the class

$$gSO(2l_1) \in S(O(2l_1) \times O(1))/SO(2l_1) = \mathbb{Z}_2.$$

Therefore there are two cases

(1) There is a $g \in S(O(2l_1) \times O(1))$ such that $gN_1 \neq N_1$.
(2) For all $g \in S(O(2l_1) \times O(1))$ $gN_1 = N_1$.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$, then N_1 is the only characteristic submanifold of M belonging to $\mathfrak{F}_1$. Therefore only the second case occurs in this case.

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 2$, then there is a $g_1 \in N_{G_1}T_1$ such that $N_1 \neq g_1N_1$. Therefore we are in the first case.

Furthermore we have $M = G_1 \times N_1/\sim$ with

$$\begin{aligned} & (g_1, y_1) \sim (g_2, y_2) \\ \Leftrightarrow \quad & g_1y_1 = g_2y_2 \\ \Leftrightarrow \quad & g_2^{-1}g_1y_1 = y_2 \\ \Leftrightarrow \quad & g_2^{-1}g_1 \in S(O(2l_1) \times O(1)) \text{ and } g_2^{-1}g_1y_1 = y_2. \end{aligned}$$

In the first case the last statement is equivalent to

$$g_2^{-1}g_1 \in SO(2l_1) \text{ and } g_2^{-1}g_1y_1 = y_2.$$

Therefore we get $M = SO(2l_1+1)/SO(2l_1) \times N_1$.

In the second case we have as in the proof of Corollary 4.30

$$M = SO(2l_1+1) \times_{S(O(2l_1)\times O(1))} N_1 = SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1.$$

That means that M is the orbit space of a diagonal $\mathbb{Z}_2$-action on $SO(2l_1+1)/SO(2l_1)\times N_1$. Because M is orientable this action has to be orientation preserving. But the $\mathbb{Z}_2$-action on $SO(2l_1+1)/SO(2l_1)$ is orientation reversing. Therefore the $\mathbb{Z}_2$-action on N_1 is also orientation reversing. □

COROLLARY 4.48. *In the situation of Lemma 4.46 $M^{G_1} \subset N_1$ is empty or has codimension one in N_1.*

PROOF. By Lemma 4.46 it is clear that $M^{G_1} \subset N_1$. For $y \in M^{G_1}$ consider the G_1 representation T_yM. Its restriction to $SO(2l_1)$ equals the $SO(2l_1)$-representation T_xM where $x \in N_1^T$.

Because this is a direct sum of a trivial representation and the standard real representation of $SO(2l_1)$ and $T_1 \subset SO(2l_1)$, T_yM is a sum of a trivial and the standard real representation of $SO(2l_1+1)$. Therefore $M^{G_1} \subset N_1$ has codimension one. □

As in section 4.4 we discuss the question when a manifold of the form given in Corollary 4.47 is a blow up.

If $\tilde{M}$ is the blow up of M along M^{G_1} then there is an equivariant embedding of $P_{\mathbb{R}}(N(M^{G_1}, M))$ into $\tilde{M}$. Therefore the G_1-action on $\tilde{M}$ has an orbit of type $SO(2l_1+1)/S(O(2l_1) \times O(1))$. This shows that $\tilde{M}$ is of the form $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} \tilde{N}_1$ where $\tilde{N}_1$ is the proper transform of N_1. By Lemma 4.21 $\tilde{N}_1$ and N_1 are diffeomorphic. Because M^{G_1} has codimension one in N_1, the $\mathbb{Z}_2$-action on N_1 has a fixed point component of codimension one.

The following lemma shows that these two conditions are sufficient.

LEMMA 4.49. *Let N_1 be a torus manifold with G_2-action. Assume that there are a non-trivial orientation reversing action of $\mathbb{Z}_2 = S((O(2l_1) \times O(1))/SO(2l_1))$ on N_1 which commutes with the action of G_2 and a closed codimension one submanifold A of N_1 on which $\mathbb{Z}_2$ acts trivially.*

Let $E' = N(A, N_1)$ equipped with the action of G_2 induced from the action on N_1 and the trivial action of $\mathbb{Z}_2$. Denote by W the standard real representation of $SO(2l_1+1)$. Then:

(1) *$SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ is orientable.*
(2) *The normal bundle of $SO(2l_1+1)/S(O(2l_1) \times O(1)) \times A$ in $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ is isomorphic to the normal bundle of $P_{\mathbb{R}}(E' \otimes W \oplus \{0\})$ in $P_{\mathbb{R}}(E' \otimes W \oplus \mathbb{R})$.*

The lemma guarantees that one may remove $SO(2l_1+1)/S(O(2l_1) \times O(1)) \times A$ from $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ and replace it by A to get a torus manifold with G-action M such that $M^{SO(2l_1+1)} = A$. The blow up of M along A is $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$.

Proof. The diagonal $\mathbb{Z}_2$-action on $SO(2l_1+1)/SO(2l_1) \times N_1$ is orientation preserving. Therefore $SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1$ is orientable.

The normal bundle of $SO(2l_1+1)/S(O(2l_1)\times O(1))\times A$ in $SO(2l_1+1)/SO(2l_1)\times_{\mathbb{Z}_2} N$ is given by $SO(2l_1+1)/SO(2l_1)\times_{\mathbb{Z}_2} N(A,N)$. The normal bundle of $P_{\mathbb{R}}(N(A,N)\otimes W \oplus \{0\})$ in $P_{\mathbb{R}}(N(A,N) \otimes W \oplus \mathbb{R})$ is the tautological bundle.

Consider the following commutative diagram

$$\begin{array}{ccc} SO(2l_1+1)/SO(2l_1) \times_{\mathbb{Z}_2} N(A,N) & \xrightarrow{f} & P_{\mathbb{R}}(E' \otimes W) \times E' \otimes W \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ SO(2l_1+1)/S(O(2l_1) \times O(1)) \times A & \xrightarrow{g} & P_{\mathbb{R}}(E' \otimes W) \end{array}$$

where the vertical maps are the natural projections and f, g are given by

$$f([hSO(2l_1) : m]) = ([m \otimes he_1], m \otimes he_1)$$

and

$$g(hS(O(2l_1) \times O(1)), q) = [m_q \otimes he_1],$$

where $e_1 \in W - \{0\}$ is fixed such that for all $g \in SO(2l_1)$, $ge_1 = e_1$ and $m_q \neq 0$ some element of the fiber of E' over q.

f induces the sought-after isomorphism. □

Lemma 4.50. *If $l_1 > 1$ in the situation of Lemma 4.46, then $\bigcap_{M_i \in \mathfrak{F}_1} M_i = M^{SO(2l_1)}$ has at most two components. It has two components if and only if $M = S^{2l_1} \times N_1$.*

Proof. If $M = S^{2l_1} \times N_1$ then $\bigcap_{M_i \in \mathfrak{F}_1} M_i = \{N, S\} \times N_1$ where N, S are the north and the south pole of the sphere respectively. Otherwise the blow up of M along $M^{SO(2l_1+1)}$ is given by $S^{2l_1} \times_{\mathbb{Z}_2} N_1$ which is a fiber bundle over $\mathbb{R}P^{2l_1}$. The characteristic submanifolds of $S^{2l_1} \times_{\mathbb{Z}_2} N_1$ which are permuted by $W(G_1)$ are given by the preimages of the following submanifolds of $\mathbb{R}P^{2l_1}$:

$$\mathbb{R}P_i^{2l_1-2} = \{(x_1 : x_2 : \cdots : x_{2i-2} : 0 : 0 : x_{2i+1} : \cdots : x_{2l_1+1}) \in \mathbb{R}P^{2l_1}\} \quad i = 1, \ldots, l_1$$

Because

$$\bigcap_{i=1}^{l_1} \mathbb{R}P_i^{2l_1-2} = \{(0 : 0 : \cdots : 0 : 1)\},$$

it follows that

$$\bigcap_{M_i \in \mathfrak{F}_1} \tilde{M}_i = N_1 = \tilde{M}^{SO(2l_1)}.$$

Therefore with Lemma 4.19 and Corollary 4.48

$$\bigcap_{M_i \in \mathfrak{F}_1} M_i = N_1 = M^{SO(2l_1)}$$

follows. In particular $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ is connected. □

LEMMA 4.51. *If $l_1 = 1$ in the situation of Lemma 4.46, then the following statements are equivalent:*

- *$M^{SO(2)}$ has two components*
- *$\#\mathfrak{F}_1 = 2$*
- *$M = S^2 \times N_1$.*

If $l_1 = 1$ and $\#\mathfrak{F}_1 = 1$ then $M^{SO(2)}$ is connected.

PROOF. At first we prove that all components of $M^{SO(2)}$ are characteristic submanifolds of M belonging to $\mathfrak{F}_1$. By Lemma 4.46 N_1 is a characteristic submanifold of M and a component of $M^{SO(2)}$ such that $G_1N_1 = M$. Therefore if $x \in M^{SO(2)}$ then there is a $g \in N_{G_1}SO(2)$ such that $g^{-1}x \in N_1$. This implies $x \in gN_1$. Because gN_1 is a characteristic submanifold belonging to $\mathfrak{F}_1$ and a component of $M^{SO(2)}$ it follows that $M^{SO(2)}$ is a union of characteristic submanifolds of M belonging to $\mathfrak{F}_1$.

Now assume that $\#\mathfrak{F}_1 = 1$. Then we have $M^{SO(2)} = N_1$. Therefore $M^{SO(2)}$ is connected.

Now assume that $M = SO(3)/SO(2) \times N_1$. Then it is clear that $M^{SO(2)}$ has two components.

Now assume that $M^{SO(2)}$ has two components. Because these components are characteristic submanifolds belonging to $\mathfrak{F}_1$ it follows that $\#\mathfrak{F}_1 = 2$.

Now assume that $\#\mathfrak{F}_1 = 2$. If there is no G_1-fixed point then it follows from Corollary 4.47 that $M = SO(3)/SO(2)\times N_1$. Assume that there is a G_1-fixed point in M. Then the blow up of M along M^{G_1} contains an orbit of type $SO(3)/S(O(2)\times O(1))$. Now Corollary 4.47 implies $\#\mathfrak{F}_1 = 1$. Therefore there is no G_1-fixed point if $\#\mathfrak{F}_1 = 2$. □

We are now in the position to state another classification theorem. For this we use the following definition.

DEFINITION 4.52. Let $\tilde{G} = G_1\times G_2$ with $G_1 = SO(2l_1+1)$. Then a pair (N, A) with

- N a torus manifold with $G_2 \times \mathbb{Z}_2$-action, such that the $\mathbb{Z}_2$-action is orientation-reversing or trivial,
- $A \subset N$ the empty set or a closed $G_2 \times \mathbb{Z}_2$-invariant submanifold of codimension one, such that $\mathbb{Z}_2$ acts trivially on A, such that if $A \neq \emptyset$ then $\mathbb{Z}_2$ acts non-trivially on N,

is called admissible for $(\tilde{G}, G_1)$.

We say that two admissible pairs (N, A), (N', A') are equivalent if there is a $G_2 \times \mathbb{Z}_2$-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A) = A'$.

THEOREM 4.53. *Let $\tilde{G} = G_1 \times G_2$ with $G_1 = SO(2l_1 + 1)$. There is a one-to-one correspondence between the $\tilde{G}$-equivariant diffeomorphism classes of torus manifolds with $\tilde{G}$-actions such that G_1 is elementary and the equivalence classes of admissible pairs for $(\tilde{G}, G_1)$.*

PROOF. Let M be a torus manifold with $\tilde{G}$-action. If $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ has two components and $l_1 > 1$ or $\#\mathfrak{F}_1 = 2$ and $l_1 = 1$, then we assign to M the admissible pair $\Phi(M) = (N_1, \emptyset)$, where N_1 is a component of $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ or a characteristic submanifold belonging to $\mathfrak{F}_1$ in the case $l_1 = 1$. The action of $\mathbb{Z}_2$ is trivial in this case.

If $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ is connected and $l_1 > 1$ or $\#\mathfrak{F}_1 = 1$ and $l_1 = 1$, then we assign to M the pair

$$\Phi(M) = \left(\bigcap_{M_i\in\mathfrak{F}_1} M_i, M^{SO(2l_1+1)}\right).$$

Because $\bigcap_{M_i\in\mathfrak{F}_1} M_i = M^{SO(2l_1)}$, there is a non-trivial action of

$$\mathbb{Z}_2 = S(O(2l_1)\times O(1))/SO(2l_1)$$

on $\bigcap_{M_i\in\mathfrak{F}_1} M_i$.

Now let (N, A) be a admissible pair for $(\tilde{G}, G_1)$. If the $\mathbb{Z}_2$-action on N is trivial we have $A = \emptyset$ and we assign to $(N, \emptyset)$ the torus manifold with $\tilde{G}$-action $\Psi((N, \emptyset)) = S^{2l_1} \times N$.

If the $\mathbb{Z}_2$-action on N is non-trivial we assign to (N, A) the blow down $\Psi((N, A))$ of $SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N$ along $SO(2l_1 + 1)/S(O(2l_1) \times O(1)) \times A$.

By Lemma 4.50 it is clear that this gives a one-to-one correspondence between torus manifolds with $\tilde{G}$-action such that $\bigcap_{M_i\in\mathfrak{F}_1} M_i$ has two components and $l_1 > 1$ and admissible pairs with trivial $\mathbb{Z}_2$-action. With Lemma 4.51 we see that an analogous statement holds for $l_1 = 1$ and $\#\mathfrak{F}_1 = 2$.

Now let (N, A) be an admissible pair such that $\mathbb{Z}_2$ acts non-trivially on N_1. Then the discussion after Lemma 4.49 shows that $\Phi(\Psi((N, A)))$ is equivalent to (N, A).

If M is a torus manifold with $G_1 \times G_2$-action such that G_1 is elementary and $N_1 = \bigcap_{M_i\in\mathfrak{F}_1} M_i$ is connected the blow up of M along $M^{SO(2l_1+1)}$ is given by

$$SO(2l_1 + 1)/SO(2l_1) \times_{\mathbb{Z}_2} N_1.$$

Therefore we find that $\Psi(\Phi(M))$ is equivariantly diffeomorphic to M. □

4.7. Classification

Here we use the results of the previous sections to state a classification of torus manifolds with G-action. We do not consider actions of groups which have $SO(2l_1)$ as an elementary factor because as explained in section 4.5 this factors may be replaced by $SU(l_1) \times S^1$. We get the classification by iterating the constructions given in Theorem 4.37 and Theorem 4.53.

We illustrate this iteration in the case that all elementary factors of G are isomorphic to $SU(l_i + 1)$. Let $\tilde{G} = \prod_{i=1}^k G_i \times T^{l_0}$ and M a torus manifold with $\tilde{G}$-action such that all G_i are elementary and isomorphic to $SU(l_i + 1)$.

In Theorem 4.37 we constructed a triple (ψ_1, N_1, A_1) which determines the $\tilde{G}$-equivariant diffeomorphism type of M. Here N_1 is a torus manifold with $\prod_{i=2}^k G_i \times T^{l_0}$-action. Therefore there is a triple (ψ_2, N_2, A_2) which determines the $\prod_{i=2}^k G_i \times T^{l_0}$-equivariant diffeomorphism type of N_1. Because $N_2 \subset N_1$ such that $G_2N_2 = N_1$ and A_1 is G_2-invariant we have $G_2(A_1 \cap N_2) = A_1$. Therefore the G-equivariant diffeomorphism type of M is determined by

$$(\psi_1 \times \psi_2, N_2, A_1 \cap N_2, A_2).$$

Continuing in this manner leads to a triple

$$(\psi, N, (A_1, \dots, A_k)),$$

where $\psi \in \operatorname{Hom}\left(\prod_{i=1}^k S(U(l_i) \times U(1), T^{l_0}\right)$, N is a $2l_0$-dimensional torus manifold and the A_i are codimension two submanifolds of N or empty.

We use the following definition to make this construction more formal.

DEFINITION 4.54. Let $G = \prod_{i=1}^{k} G_i \times G'$ with

$$G_i = \begin{cases} SU(l_i+1) & \text{if } i \leq k_0 \\ SO(2l_i+1) & \text{if } i > k_0 \end{cases}$$

and $k_0 \in \{0, \dots, k\}$. Then a 5-tuple

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k}, (a_{ij})_{k_0+1\leq i<j\leq k})$$

with

(1) $\psi \in \operatorname{Hom}(\prod_{i=1}^{k_0} S(U(l_i)\times U(1)), Z(G'))$ and $\psi_i = \psi|_{S(U(l_i)\times U(1))}$,
(2) N a torus manifold with $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-action,
(3) $A_i \subset N$ the empty set or a closed $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-invariant submanifold of codimension two on which $\operatorname{im}\psi_i$ acts trivially such that if $A_i \neq \emptyset$ then $\ker\psi_i = SU(l_i)$,
(4) $B_i \subset N$ the empty set or a closed $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-invariant submanifold of codimension one on which $(\mathbb{Z}_2)_i$ acts trivially such that if $B_i \neq \emptyset$ then the action of $(\mathbb{Z}_2)_i$ on N is non-trivial,
(5) $a_{ij} \in \{0,1\}$ such that
   (a) if $a_{ij} = 1$ then
      (i) the action of $(\mathbb{Z}_2)_j$ on N is trivial,
      (ii) $a_{jk} = 0$ for $k > j$,
      (iii) $B_i = \emptyset$,
   (b) if the action of $(\mathbb{Z}_2)_i$ on N is non-trivial then it is orientation preserving if and only if $\sum_{j>i} a_{ij}$ is odd,
   (c) if the action of $(\mathbb{Z}_2)_i$ on N is trivial then $\sum_{j>i} a_{ij}$ is odd or zero,

is called admissible for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ if the A_i and B_i intersect pairwise transversely.

If G' is a torus we also say that a 5-tuple is admissible for $\tilde{G}$ instead of $(\tilde{G}, \prod_{i=1}^{k} G_i)$.

We say that two admissible 5-tuples

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k}, (a_{ij})_{k_0+1\leq i<j\leq k})$$

and

$$(\psi', N', (A'_i)_{i=1,\dots,k_0}, (B'_i)_{i=k_0+1,\dots,k}, (a'_{ij})_{k_0+1\leq i<j\leq k})$$

are equivalent if

- $\psi_i = \psi'_i$ if $l_i > 1$ and $\psi_i = \psi'^{\pm 1}_i$ if $l_i = 1$,
- $a_{ij} = a'_{ij}$,
- there is a $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$-equivariant diffeomorphism $\phi : N \to N'$ such that $\phi(A_i) = A'_i$ and $\phi(B_i) = B'_i$.

REMARK 4.55. By Lemma A.6 two submanifolds A_1, A_2 of N satisfying the condition (3) intersect transversely if and only if no component of A_1 is a component of A_2.

By Lemma A.9 two submanifolds A_1, B_1 of N satisfying the conditions (3) and (4), respectively, intersect always transversely.

By Lemma A.10 two submanifolds B_1, B_2 of N satisfying the condition (4) intersect transversely if and only if no component of B_1 is a component of B_2.

LEMMA 4.56. *Let $\tilde{G}$ as above then there is a one-to-one correspondence between the equivalence classes of admissible 5-tuples for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ and the equivalence classes of admissible 5-tuples*

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k-1}, (a_{ij})_{k_0+1\leq i<j\leq k-1})$$

for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ *such that* G_k *is elementary for the* $G_k \times G'$*-action on* N.

PROOF. At first assume that $G_k = SU(l_k+1)$. Let $(\psi, N, (A_i)_{i=1,\dots,k-1}, \emptyset, \emptyset)$ be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that G_k is elementary for the $G_k \times G'$-action on N.

Let (ψ_k, N_k, A_k) be the admissible triple for $(G_k \times G', G_k)$ which corresponds to N under the correspondence given in Theorem 4.37. Then N_k is a submanifold of N. By Lemma A.6 A_i, $i = 1, \dots, k-1$, intersects N_k transversely. Therefore $N_k \cap A_i$ has codimension 2 in N_k. Because $A_i = G_k(N_k \cap A_i)$ $N_k \cap A_i$ has no component which is contained in A_k or $N_k \cap A_j$, $j \neq i$. Therefore by

$$(\psi \times \psi_k, N_k, (A_1 \cap N_k, \dots, A_{k-1} \cap N_k, A_k), \emptyset, \emptyset)$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ is given.

Now let

$$(\psi \times \psi_k, N_k, (A_1, \dots, A_k), \emptyset, \emptyset)$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$. Let $H_0 = G_k \times \operatorname{im}\psi_k$ and $H_1 = S(U(l_k) \times U(1)) \times \operatorname{im}\psi_k$. Then by Lemma 4.35 the blow down N of $\tilde{N} = H_0 \times_{H_1} N_k$ along $H_0/H_1 \times A_k$ is a torus manifold with $G_k \times G'$-action. By Lemma 4.19 $F(H_0 \times_{H_1} A_i) = G_k F(A_i)$, $i < k$, are submanifolds of N satisfying the condition (3) of Definition 4.54. Because $F(A_i)$ and $F(A_j)$, $i < j < k$, have no components in common, $G_k F(A_i)$ and $G_k F(A_j)$ intersect transversely. Therefore by

$$(\psi, N, (G_k F(A_1), \dots, G_k F(A_{k-1})), \emptyset, \emptyset)$$

an admissible triple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ is given.

As in the proof of Theorem 4.37 one sees that this construction leads to a one-to-one-correspondence.

Now assume that $G_k = SO(2l_k + 1)$. Let

$$(\psi, N, (A_i)_{i=1,\dots,k_0}, (B_i)_{i=k_0+1,\dots,k-1}, (a_{ij})_{k_0+1 \leq i < j \leq k-1}) \tag{4.11}$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that G_k is elementary for the $G_k \times G'$-action on N.

At first assume that, for the G_k-action on N, $N^{SO(2l_k)}$ is connected. Let (N_k, B_k) be the admissible pair for $(G_k \times G', G_k)$ which corresponds to N under the correspondence given in Theorem 4.53. Then N_k is a submanifold of N which is invariant under the action of $G' \times \prod_{i=k_0+1}^{k} (\mathbb{Z}_2)_i$, where $(\mathbb{Z}_2)_k = S(O(2l_k) \times O(1))/SO(2l_k)$. For $i < k$ let $a_{ik} = 0$.

We claim that by

$$(\psi, N_k, (A_1 \cap N_k, \dots, A_{k_0} \cap N_k), (B_{k_0+1} \cap N_k, \dots, B_{k-1} \cap N_k, B_k), (a_{ij})) \tag{4.12}$$

an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ is given.

At first note that for $i = 1, \dots, k-1$ the A_i and B_i intersect N_k transversely by Lemmas A.6 and A.9. Therefore $A_i \cap N_k$ and $B_i \cap N_k$ has codimension two or one, respectively, in N_k.

One sees as in the case $G_k = SU(l_k+1)$ that the $N_k \cap A_i$ and $N_k \cap B_i$ intersect pairwise transversely.

Now we verify the condition (5) of Definition 4.54 for the 5-tuple (4.12). By Lemma 4.45 $(\mathbb{Z}_2)_i$, $i < k$, acts orientation preserving on N if and only if it acts orientation preserving on N_k. This proves (5b).

Because by Lemma 4.46 $G_k N_k = N$ $(\mathbb{Z}_2)_i$, $i < k$, acts trivially on N_k if and only if it acts trivially on N. This proofs (5c) and (5(a)i).

Because $a_{ik} = 0$ (5(a)ii) and (5(a)iii) are clear.

Now assume that $N^{SO(2l_k)}$ is non-connected. Then by Lemma 4.50 and 4.51 we have

$$N = SO(2l_k+1)/SO(2l_k) \times N_k.$$

In this case the $(\mathbb{Z}_2)_i$-action, $i<k$, on N splits in a product of an action on $SO(2l_k+1)/SO(2l_k)$ and an action on N_k. We put $a_{ik}=1$ if the $(\mathbb{Z}_2)_i$-action on $SO(2l_k+1)/SO(2l_k)$ is non-trivial and $a_{ik}=0$ otherwise. Because there is only one non-trivial action of $\mathbb{Z}_2$ on $SO(2l_k+1)/SO(2l_k)$ which commutes with the action of $SO(2l_k+1)$ we may recover the action of $(\mathbb{Z}_2)_i$ on N from the action on N_k and a_{ik}.

We identify $SO(2l_k)/SO(2l_k)\times N_k$ with N_k and equip it with the trivial action of $(\mathbb{Z}_2)_k = S(O(2l_k)\times O(1))/SO(2l_k)$. We claim that by

$$(\psi, N_k, (A_1\cap N_k,\dots,A_{k_0}\cap N_k),(B_{k_0+1}\cap N_k,\dots,B_{k-1}\cap N_k,\emptyset),(a_{ij})) \tag{4.13}$$

an admissible 5-tuple for $(\tilde{G},\prod_{i=1}^k G_i)$ is given.

The conditions (3) and (4) of Definition 4.54 and the transversality condition are verified as in the previous cases.

Therefore we only have to verify condition (5). Because the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k+1)/SO(2l_k)$ is orientation reversing the $(\mathbb{Z}_2)_i$-action on N_k has the same orientation behaviour as the action on N if and only if the $(\mathbb{Z}_2)_i$-action on $SO(2l_k+1)/SO(2l_k)$ is trivial. This proofs (5b).

If the $(\mathbb{Z}_2)_i$-action on N_k is trivial and non-trivial on $SO(2l_k+1)/SO(2l_k)$ then the $(\mathbb{Z}_2)_i$-action on N is orientation reversing. Therefore $\sum_{j>i}a_{ij}$ is odd.

The $(\mathbb{Z}_2)_i$-actions on N_k and $SO(2l_k+1)/SO(2l_k)$ are trivial if and only if the $(\mathbb{Z}_2)_i$-action on N is trivial. Therefore $\sum_{j>i}a_{ij}$ is odd or trivial. This verifies (5c).

If there is a $j<i$ such that $a_{ji}=1$ then $(\mathbb{Z}_2)_i$ acts trivially on N and therefore $a_{ik}=0$. This proves (5(a)ii).

If the $(\mathbb{Z}_2)_i$-action on $SO(2l_k+1)/SO(2l_k)$ is non-trivial the action on N has no fixed points. Therefore $B_i=\emptyset$. This proves (5(a)iii). (5(a)i) is clear.

Now let

$$(\psi, N_k, (A_1,\dots,A_{k_0}),(B_{k_0+1},\dots,B_k),(a_{ij}))$$

be an admissible 5-tuple for $(\tilde{G},\prod_{i=1}^k G_i)$. At first assume that $(\mathbb{Z}_2)_k$ acts non-trivially on N_k. Then the blow down N of $\tilde{N} = SO(2l_k+1)/SO(2l_k)\times_{\mathbb{Z}_2} N_k$ along $SO(2l_k+1)/SO(2l_k)\times_{\mathbb{Z}_2} B_k$ is a torus manifold with $G_k\times G'\times\prod_{i=k_0+1}^{k-1}(\mathbb{Z}_2)_i$-action. As in the case $G_k=SU(l_k+1)$ one sees that

$$(\psi, N, (G_kF(A_1),\dots,G_kF(A_{k_0})),(G_kF(B_{k_0+1}),\dots,G_{k-1}F(B_{k-1})),(a_{ij}))$$

is an admissible 5-tuple for $(\tilde{G},\prod_{i=1}^{k-1}G_i)$.

If $(\mathbb{Z}_2)_k$ acts trivially on N_k then put

$$N = SO(2l_k+1)/SO(2l_k)\times N_k.$$

Here $(\mathbb{Z}_2)_i$, $i<k$, acts by the product action of the non-trivial $\mathbb{Z}_2$-action on $SO(2l_k+1)/SO(2l_k)$ and the action on N_k if $a_{ik}=1$. Otherwise $(\mathbb{Z}_2)_i$ acts by the product action of the trivial action on $SO(2l_k+1)/SO(2l_k)$ and the action on N_k. Now by

$$\begin{aligned}(\psi, N, (&SO(2l_k+1)/SO(2l_k)\times A_1,\dots,SO(2l_k+1)/SO(2l_k)\times A_{k_0}),\\ &(SO(2l_k+1)/SO(2l_k)\times B_{k_0+1},\dots,SO(2l_k+1)/SO(2l_k)\times B_k),(a_{ij}))\end{aligned}$$

an admissible 5-tuple for $(\tilde{G},\prod_{i=1}^{k-1}G_i)$ is given.

As in the proof of Theorem 4.53 one sees that this construction leads to a one-to-one-correspondence. ☐

Let $\tilde{G} = \prod_i G_i \times T^{l_0}$ and

$$(\psi, M, (A_i), (B_i), (a_{ij}))$$

be an admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k-1} G_i)$ such that G_k is an elementary factor of $\prod_{i\geq k} G_i \times T^{l_0}$ for the action on N. Furthermore let

$$(\psi', N, (A'_i), (B'_i), (a'_{ij}))$$

be the admissible 5-tuple for $(\tilde{G}, \prod_{i=1}^{k} G_i)$ corresponding to $(\psi, M, (A_i), (B_i), (a_{ij}))$. Then the following lemma shows that G_i, $i > k$, is an elementary factor of $\prod_{i\geq k} G_i \times T^{l_0}$ for the action on M if and only if it is an elementary factor of $\prod_{i\geq k+1} G_i \times T^{l_0}$ for the action on N.

LEMMA 4.57. *Let $\tilde{G} = G_1 \times G' \times G''$, M a torus manifold with $\tilde{G}$-action and N a component of an intersection of characteristic submanifolds of M which is $G_1 \times G'$ invariant and contains a T-fixed point x such that G_1 acts non-trivially on N. Furthermore assume that G'' is a product of elementary factors for the action on M.*

Then N is a torus manifold with $G_1 \times G' \times T^{l_0}$-action for some $l_0 \geq 0$ and G_1 is an elementary factor of $\tilde{G}$, with respect to the action on M, if and only if it is an elementary factor of $G_1 \times G' \times T^{l_0}$ with respect to the action on N.

PROOF. Assume that G_1 is an elementary factor for one of the two actions. Then G_1 is isomorphic to a simple group or Spin(4). If G_1 is simple and not isomorphic to $SU(2)$ then the statement is clear.

Therefore there are two cases $G_1 = SU(2), \mathrm{Spin}(4)$.

If x is not fixed by G_1 then $G_1 = SU(2)$ is elementary for both actions on N and M by Lemma 4.10. Therefore we may assume that $x \in N^{G_1} \subset M^{G_1}$. Then there is a bijection

$$\mathfrak{F}_{xM} \to \mathfrak{F}_{xN} \amalg \mathfrak{F}_N^{\perp}$$

where

$$\mathfrak{F}_{xM} = \{\text{characteristic submanifolds of } M \text{ containing } x\},$$
$$\mathfrak{F}_{xN} = \{\text{characteristic submanifolds of } N \text{ containing } x\},$$
$$\mathfrak{F}_N^{\perp} = \{\text{characteristic submanifolds of } M \text{ containing } N\}.$$

This bijection is compatible with the action of the Weyl-group of G_x.

At first assume that $G_1 = SU(2)$ is elementary for the action on M but not for the action on N. Then there is another simple factor $G_2 = SU(2)$ of $G_1 \times G' \times T^{l_0}$ such that $G_1 \times G_2$ is elementary for the action on N. At first assume that G_2 is elementary for the action on M.

Let $w_i \in W(G_i)$, $i = 1, 2$, be generators. Then there are two non-trivial $W(G_1 \times G_2)$-orbits $\mathfrak{F}_1, \mathfrak{F}_2$ in $\mathfrak{F}_{xM}$. We have:

- $\#\mathfrak{F}_i = 2$, $i = 1, 2$,
- w_i, $i = 1, 2$, acts non-trivially on $\mathfrak{F}_i$ and trivially on the other orbit.

But because $G_1 \times G_2$ is elementary for the action on N there is exactly one non-trivial $W(G_1 \times G_2)$-orbit $\mathfrak{F}'_1$ in $\mathfrak{F}_{xN}$. We have:

- $\#\mathfrak{F}'_1 = 2$,
- w_i, $i = 1, 2$, acts non-trivially on $\mathfrak{F}'_1$.

This is a contradiction.

If G_2 is not elementary then G_2 is a simple factor of an elementary factor. In this case the action of $W(G_1 \times G_2)$ on $\mathfrak{F}_{xM}$ behaves as in the first case. Therefore we get a contradiction in this case, too.

Under the assumption that $G_1 = \mathrm{Spin}(4)$ is elementary for the action on M a similar argument shows that G_1 is elementary for the action on N.

Therefore G_1 is elementary for the action on N if it is elementary for the action on M.

If G_1 is elementary for the action on N but not elementary for the action on M then it is a simple factor of an elementary factor $G_1' \neq G_1$ of $\tilde{G}$ or a product of elementary factors $G_2' \times G_3'$ of $\tilde{G}$. But because G'' is a product of elementary factors G_1', G_2' and G_3' are subgroups of $G_1 \times G'$. Therefore G_1' or $G_2' \times G_3'$ are elementary for the action on N. This is a contradiction to the assumption that G_1 is elementary for the action on N. □

Recall from section 4.2 that if M is a torus manifold with G-action then we may assume that all elementary factors of G are isomorphic to $SU(l_i+1)$, $SO(2l_i+1)$ or $SO(2l_i)$. That means $\tilde{G} = \prod SU(l_i+1) \times \prod SO(2l_i+1) \times \prod SO(2l_i) \times T^{l_0}$. Because as described in section 4.5 we may replace elementary factors isomorphic to $SO(2l_i)$ by $SU(l_i) \times S^1$ the following theorem may be used to construct invariants of torus manifolds with $\tilde{G}$-action. By Lemma 4.44 these invariants determine the $\tilde{G}$-equivariant diffeomorphism type of simply connected torus manifolds with $\tilde{G}$-action.

Theorem 4.58. *Let* $\tilde{G} = \prod_{i=1}^{k} G_i \times T^{l_0}$ *with*

$$G_i = \begin{cases} SU(l_i+1) & \text{if } i \leq k_0 \\ SO(2l_i+1) & \text{if } i > k_0 \end{cases}$$

and $k_0 \in \{0, \dots, k\}$. *Then there is a one-to-one correspondence between the equivalence classes of admissible* 5*-tuples for* $\tilde{G}$ *and the* $\tilde{G}$*-equivariant diffeomorphism classes of torus manifolds with* $\tilde{G}$*-action such that all* G_i *are elementary.*

Proof. This follows from Lemma 4.56 and Lemma 4.57 by induction. □

Using Lemma 4.7 and Theorem 4.40 we get the following result for quasitoric manifolds.

Theorem 4.59. *Let* $\tilde{G} = \prod_{i=1}^{k} G_i \times T^{l_0}$ *with* $G_i = SU(l_i+1)$ *Then there is a one-to-one correspondence between the equivalence classes of admissible* 5*-tuples for* $\tilde{G}$ *of the form*

$$(\psi, N, (A_i)_{1 \leq i \leq k}, \emptyset, \emptyset)$$

with N *quasitoric and* A_i, $1 \leq i \leq k$, *connected and the* $\tilde{G}$*-equivariant diffeomorphism classes of quasitoric manifolds with* $\tilde{G}$*-action.*

Remark 4.60. Remark 4.8 and Theorem 4.39 lead to a similar result for torus manifolds with G-actions whose cohomologies are generated by their degree two parts.

Corollary 4.61. *Let* $\tilde{G} = \prod_{i=1}^{k_1} G_i \times T^{l_0}$ *with* G_i *elementary and* M *a torus manifold with* G*-action. Then* M/G *has dimension* $l_0 + \#\{G_i;\ G_i = SO(2l_i)\}$.

Proof. Because – as discussed in section 4.5 – the orbits of the G action does not change if we replace all elementary factors isomorphic to $SO(2l)$ by $SU(l) \times S^1$ we may assume that all elementary factors of $\tilde{G}$ are isomorphic to $SO(2l+1)$ or $SU(l+1)$. By Lemma 4.23 replacing M by the blow up $\tilde{M}$ of M along the fixed points of G_1 does not change the orbit space. Therefore by Corollaries 4.30 and

4.47 we have up to finite coverings

$$M/G = (M/G_1)/(\prod_{i\geq 2} G_i \times T^{l_0}) = (\tilde{M}/G_1)/(\prod_{i\geq 2} G_i \times T^{l_0})$$
$$= ((H_0 \times_{H_1} N_1)/G_1)/(\prod_{i\geq 2} G_i \times T^{l_0}) = N_1/(\prod_{i\geq 2} G_i \times T^{l_0})$$

where N_1 is the $\prod_{i\geq 2} G_i \times T^{l_0}$-manifold from the admissible 5-tuple for $(\tilde{G}, G_1)$ corresponding to M. Here H_0, H_1 are defined as in Lemma 4.27 if $G_1 = SU(l_1+1)$. If $G_1 = SO(2l_1+1)$ we have $H_0 = SO(2l_1+1)$ and $H_1 = S(O(2l_1) \times O(1))$.

By iterating this argument we find that $M/G = N/T^{l_0}$ up to finite coverings where N is the T^{l_0}-manifold from the admissible 5-tuple for $\tilde{G}$ corresponding to M. □

Corollary 4.62. *If G is semi-simple and M is a torus manifold with G-action such that $H^*(M;\mathbb{Z})$ is generated by its degree two part, then*

$$\tilde{G} = \prod_{i=1}^{k} SU(l_i+1)$$

and

$$M = \prod_{i=1}^{k} \mathbb{C}P^{l_i},$$

where each $SU(l_i+1)$ acts in the usual way on $\mathbb{C}P^{l_i}$ and trivially on $\mathbb{C}P^{l_j}$, $j \neq i$.

Proof. By Lemma 4.7 and Remark 4.8 all elementary factors of $\tilde{G}$ are isomorphic to $SU(l_i+1)$. Because G is semi-simple, there is only one admissible 5-tuple for $\tilde{G}$, namely $(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, \emptyset)$. It corresponds to a product of complex projective spaces. □

With Theorem 4.58 we recover the following two results of S. Kuroki [**36**]:

Corollary 4.63. *Let M be a simply connected torus manifold with G-action such that M is a homogeneous G-manifold. Then M is a product of even-dimensional spheres and complex projective spaces.*

Proof. By Corollary 4.61 we have that the center of G is zero-dimensional and all elementary factors of G are isomorphic to $SU(l_i+1)$ or $SO(2l_i+1)$. Therefore the admissible 5-tuple corresponding to M is given by

$$(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, (a_{ij}))$$

In particular no elementary factor of G has a fixed point in M. Therefore by Corollaries 4.30 and 4.47 M splits into a direct product of complex projective spaces and even dimensional spheres. □

Corollary 4.64. *If the G-action on the simply connected torus manifold M has an orbit of codimension one then M is the projectivication of a complex vector bundle or a sphere bundle over a product of complex projective spaces and even-dimensional spheres.*

Proof. By Corollary 4.61 we may assume that there is a covering group $\tilde{G} = S^1 \times \prod_i G_i$ of G with G_i elementary and $G_i = SU(l_i+1)$ or $G_i = SO(2l_i+1)$. By Corollaries 4.30 and 4.47 we may assume that all elementary factors of G have fixed points in M. Let $(\psi, N', (A_i), (B_i), (a_{ij}))$ be the admissible 5-tuple for G of M. Then we have

$$N' = S^2 \qquad A_i \neq \emptyset \qquad B_i \neq \emptyset$$

Because the S^1-action on S^2 has only two fixed points, N and S, there are at most two elementary factors isomorphic to $SU(l_i+1)$. The orientation reversing involutions of S^2 which commute with the S^1-action and have fixed points are given by "reflections" at S^1-orbits. Therefore there is at most one elementary factor isomorphic to $SO(2l_i+1)$. If there is such a factor then there is at most one G_i isomorphic to $SU(l_i+1)$ because N is mapped to S by such a reflection. Let

$$\phi_i : S(U(l_i)\times U(1)) \to U(1) \qquad \begin{pmatrix} A & 0 \\ 0 & g \end{pmatrix} \mapsto g \qquad (A \in U(l_i), g \in U(1)).$$

Then we have the following admissible 5-tuples:

$\tilde{G}$	5-tuple	M
S^1	$(\emptyset, S^2, \emptyset, \emptyset, \emptyset)$	S^2
$S^1 \times SU(l_1+1)$	$(\phi_1^{\pm 1}, S^2, \{N\}, \emptyset, \emptyset)$	$\mathbb{C}P^{l_1+1}$
	$(\phi_1^{\pm 1}, S^2, \{N, S\}, \emptyset, \emptyset)$	S^{2l_1+2}
$S^1 \times SO(2l_1+1)$	$(\emptyset, S^2, \emptyset, S^1, \emptyset)$	S^{2l_1+2}
$S^1 \times SU(l_1+1) \times SU(l_2+1)$	$(\phi_1^{\pm 1}\phi_2^{\pm 1}, S^2, (\{N\}, \{S\}), \emptyset, \emptyset)$	$\mathbb{C}P^{l_1+l_2+1}$
$S^1 \times SU(l_1+1) \times SO(2l_2+1)$	$(\phi_1^{\pm 1}, S^2, \{N, S\}, S^1, \emptyset)$	$S^{2l_1+2l_2+2}$

Therefore the statement follows. □

COROLLARY 4.65. *Let $\tilde{G} = G_1 \times G_2 \neq SO(2l_1) \times SO(2l_2)$ with G_1 and G_2 elementary of rank l_1, l_2, respectively, and M a torus manifold with G-action then M is one of the following:*

$$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}, \mathbb{C}P^{l_1} \times S^{2l_2}, S^{2l_1} \times S^{2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_1^{2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_2^{2l_2}, S^{2l_1+2l_2}.$$

Here S_1^l denotes the l-sphere together with the $\mathbb{Z}_2$-action generated by the antipodal map and S_2^l the l-sphere together with the $\mathbb{Z}_2$-action generated by a reflection at a hyperplane.

Furthermore the $\tilde{G}$-actions on these spaces is unique up to equivariant diffeomorphism.

PROOF. First assume that $G_1, G_2 \neq SO(2l)$. Then we have the following possibilities for the admissible 5-tuple of M:

G_1	G_2	5-tuple	M
$SU(l_1+1)$	$SU(l_2+1)$	$(\text{const}, \text{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^{l_1} \times \mathbb{C}P^{l_2}$
$SU(l_1+1)$	$SO(2l_2+1)$	$(\text{const}, \text{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^{l_1} \times S^{2l_2}$
$SO(2l_1+1)$	$SO(2l_2+1)$	$(\emptyset, \text{pt}, \emptyset, \emptyset, a_{12}=0)$	$S^{2l_1} \times S^{2l_2}$
		$(\emptyset, \text{pt}, \emptyset, \emptyset, a_{12}=1)$	$S_1^{2l_1} \times_{\mathbb{Z}_2} S_1^{2l_2}$

If $G_1 = SU(l_1+1)$ and $G_2 = SO(2l_2)$ then by Corollary 4.15 there is one admissible triple for (G, G_1) namely $(\text{const}, S^{2l_2}, \emptyset)$ which corresponds to $\mathbb{C}P^{l_1} \times S^{2l_2}$.

Now assume that $G_1 = SO(2l_1+1)$ and $G_2 = SO(2l_2)$ and let (N, B) be the admissible pair for (G, G_1) corresponding to M. Then by Corollary 4.15 we have $N = S^{2l_2}$. Up to equivariant diffeomorphism there are two orientation reversing involutions on S^{2l_2} which commute with the action of G_2, the anti-podal map and a reflection at an hyperplane in $\mathbb{R}^{2l_2+1}$. Therefore we have four possibilities for M:

$$S^{2l_1} \times S^{2l_2}, S^{2l_1+2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_1^{2l_2}, S_1^{2l_1} \times_{\mathbb{Z}_2} S_2^{2l_2}$$

□

LEMMA 4.66. *Let $\tilde{G} = SO(2l_1) \times S^1$ and M a simply connected torus manifold with G-action such that $SO(2l_1)$ is an elementary factor of $\tilde{G}$ and S^1 acts semi-freely on M and M^{S^1} has codimension two in M.*

Then M is equivariantly diffeomorphic to $\#_i(S^2 \times S^{2l_1})_i$ or S^{2l_1+2}.

Here the action on S^{2l} is given by the restriction of the usual $SO(2l+1)$-action to S^1, $SO(2l_1)$ or $SO(2l_1) \times S^1$, respectively.

PROOF. As described in section 4.5 we may replace $\tilde{G}$ by $SU(l_1) \times S \times S^1$. Let (ψ, N, A) be the admissible triple corresponding to M. Then ψ is completely determined by the discussion in section 4.5 and $A = N^S = M^{SU(l_1)}$. Furthermore S and S^1 act semi-freely on N and all components of N^S and N^{S^1} have codimension two in N.

By Lemma 4.41 N is simply connected.

Denote by $\tilde{M}$ the blow up of M along A. Because all T-fixed points of M are contained in A we have $l_1\#M^T = \#\tilde{M}^T$. On the other hand $\tilde{M}$ is a fiber bundle with fiber N over $\mathbb{C}P^{l_1-1}$. Therefore we have $l_1\#N^{S\times S^1} = \#\tilde{M}^T$.

From this $\#M^T = \#N^{S\times S^1}$ follows.

Because S and S^1 act both semi-freely on N such that their fixed point sets have codimension two it follows from the classification of simply connected four-dimensional T^2-manifolds given in [**50**, p. 547,549] that the T-equivariant diffeomorphism type of N is determined by $\#M^T$ and that $\#M^T$ is even.

Therefore the $S\times S^1\times SU(l_1)$-equivariant diffeomorphism type of M is uniquely determined by $\#M^T = \chi(M)$. It follows from Lemma 4.44 that the $SO(2l_1) \times S^1$-equivariant diffeomorphism type of M is uniquely determined by $\chi(M)$. Because

$$M_k = \begin{cases} \#_{i=1}^k(S^2 \times S^{2l_1})_i & \text{if } k \geq 1 \\ S^{2l_1+2} & \text{if } k = 0 \end{cases}$$

possesses an action of $\tilde{G}$ and $\chi(M_k) = 2k$ the statement follows. □

COROLLARY 4.67. *Let $\tilde{G} = SO(2l_1)\times SO(2l_2)$ and M a simply connected torus manifold with G-action such that $SO(2l_1)$, $SO(2l_2)$ are elementary factors of $\tilde{G}$.*

Then M is equivariantly diffeomorphic to $\#_i(S^{2l_1} \times S^{2l_2})_i$ or $M = S^{2l_1+2l_2}$.

Here the action on S^{2l} is given by the restriction of the usual $SO(2l+1)$-action to $SO(2l)$.

PROOF. As described in section 4.5 we may replace $\tilde{G}$ by $SU(l_1)\times S\times SO(2l_2)$. Let (ψ, N, A) be the admissible triple corresponding to M. Then ψ is completely determined by the discussion in section 4.5 and $A = N^S$. Furthermore S acts semi-freely on N such that N^S has codimension two.

By Lemma 4.41 N is simply connected. Therefore by Lemma 4.66 the equivariant diffeomorphism-type of M is uniquely determined by $\chi(M) = \chi(N) \in 2\mathbb{Z}$.

As in the proof of Lemma 4.66 the statement follows. □

COROLLARY 4.68. *Let M be a four dimensional torus manifold with G-action, G a non-abelian Lie-group of rank two. Then M is one of the following*

$$\mathbb{C}P^2,\ \mathbb{C}P^1 \times \mathbb{C}P^1,\ S^4,\ S_1^2 \times_{\mathbb{Z}_2} S_1^2,\ S_1^2 \times_{\mathbb{Z}_2} S_2^2$$

or a S^2-bundle over $\mathbb{C}P^1$. Here S_1^2 denotes the two-sphere together with the $\mathbb{Z}_2$-action generated by the antipodal map and S_2^2 the two-sphere together with the $\mathbb{Z}_2$-action generated by a reflection at a hyperplane.

PROOF. Let $\tilde{G}$ be a covering group of G. Then there are the following possibilities using Convention 4.14:

$$\tilde{G} = SU(3),\ SU(2) \times SU(2),\ SU(2) \times S^1,$$
$$SU(2) \times SO(3),\ SO(3) \times SO(3),\ SO(3) \times S^1,\ \mathrm{Spin}(4)$$

If $\tilde{G} = \mathrm{Spin}(4)$ we replace it by $SU(2) \times S^1$ as before.

Then we have the following admissible 5-tuples:

$\tilde{G}$	5-tuple	M
$SU(3)$	$(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^2$
$SU(2) \times SU(2)$	$(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^1 \times \mathbb{C}P^1$
$SU(2) \times S^1$	$(\psi, S^2, \emptyset, \emptyset, \emptyset)$	S^2-bundle over $\mathbb{C}P^1$
	$(\psi, S^2, N, \emptyset, \emptyset)$	$\mathbb{C}P^2$
	$(\psi, S^2, \{N, S\}, \emptyset, \emptyset)$	S^4
$SU(2) \times SO(3)$	$(\mathrm{const}, \mathrm{pt}, \emptyset, \emptyset, \emptyset)$	$\mathbb{C}P^1 \times S^2$
$SO(3) \times SO(3)$	$(\emptyset, \mathrm{pt}, \emptyset, \emptyset, a_{12} = 1)$	$S^2_1 \times_{\mathbb{Z}_2} S^2_1$
	$(\emptyset, \mathrm{pt}, \emptyset, \emptyset, a_{12} = 0)$	$S^2 \times S^2$
$SO(3) \times S^1$	$(\emptyset, S^2, \emptyset, \emptyset, \emptyset)$	$S^2 \times S^2$
	$(\emptyset, S^2_1, \emptyset, \emptyset, \emptyset)$	$S^2_1 \times_{\mathbb{Z}_2} S^2_1$
	$(\emptyset, S^2_2, \emptyset, \emptyset, \emptyset)$	$S^2_1 \times_{\mathbb{Z}_2} S^2_2$
	$(\emptyset, S^2_2, \emptyset, S^1, \emptyset)$	S^4

Here ψ is a group homomorphism $S(U(1) \times U(1)) \to S^1$. □

CHAPTER 5

Torus manifolds with stable almost complex structures

In this chapter we discuss torus manifolds which possess T-equivariant stable almost complex structures. We show in section 5.2 that a diffeomorphism between such manifolds which preserves these structures may be replaced by a weakly equivariant diffeomorphism. In section 5.3 we show that if a compact connected non-abelian Lie-group G acts on a torus manifold M preserving the stable almost complex structure then there is a compact connected Lie-group G' and a homomorphism $G \to G'$ such that M is a torus manifold with G'-action. In section 5.4 we give an example of a torus manifold which does not admit an equivariant stable almost complex structure. We begin with a discussion of the automorphism group of a stable almost complex structure in section 5.1.

5.1. The automorphism group of a stable almost complex structure

In this section we introduce stable almost complex structures on smooth manifolds. We prove that the automorphism group of a stable almost complex structure on a compact manifold is a finite dimensional Lie-group. At first we introduce some notations.

Definition 5.1. Let M^{2n} be a manifold. A *stable $Gl(\mathbb{C}, m)$-structure* for M is a $Gl(\mathbb{C}, m)$-structure on the stable tangent bundle of M which is a reduction of structure group from $Gl(\mathbb{R}, 2m)$ to $Gl(\mathbb{C}, m)$, that means it is a $Gl(\mathbb{C}, m)$-principle bundle $P_{Gl(\mathbb{C},m)} \to M$ with

$$\begin{array}{ccc} P_{Gl(\mathbb{C},m)} & \hookrightarrow & P_{TM\oplus\mathbb{R}^{2m-2n}} \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

where $P_{TM\oplus\mathbb{R}^{2m-2n}}$ denotes the frame bundle of $TM \oplus \mathbb{R}^{2m-2n}$.

Definition 5.2. Let M^{2n}, M'^{2n} be manifolds with stable $Gl(\mathbb{C}, m)$-structures $P_{Gl(\mathbb{C},m)}$, $P'_{Gl(\mathbb{C},m)}$. We say that a diffeomorphism $f : M \to M'$ preserves the stable almost complex structures if

$$\begin{array}{ccc} P_{Gl(\mathbb{C},m)} & \hookrightarrow & P_{TM\oplus\mathbb{R}^{2m-2n}} \\ {\scriptstyle Df\oplus\mathrm{Id}|_{P_{Gl(\mathbb{C},m)}}}\downarrow & & \downarrow{\scriptstyle Df\oplus\mathrm{Id}} \\ P'_{Gl(\mathbb{C},m)} & \hookrightarrow & P'_{TM\oplus\mathbb{R}^{2m-2n}} \end{array}$$

commutes.

Definition 5.3. Let M^{2n} be a manifold with a stable $Gl(\mathbb{C}, m)$-structure $P_{Gl(\mathbb{C},m)}$. We denote by $\mathrm{Aut}(M, P_{Gl(\mathbb{C},m)})$ the group of all diffeomorphisms of M which preserve the given stable almost complex structure.

The following lemma was first proven by Kosniowski and Ray in the 1980's [**53**].

LEMMA 5.4. *Let M^{2n} be a manifold with stable $Gl(\mathbb{C}, m)$-structure P. If M is compact then* $\operatorname{Aut}(M, P)$ *is a finite dimensional Lie-group.*

PROOF. A complex structure P for $TM \oplus \mathbb{R}^{2m-2n}$ induces an almost complex structure P' for $M \times T^{2m-2n}$ and

$$\begin{aligned}\operatorname{Aut}(M,P) = \{&g \in \operatorname{Aut}(M \times T^{2m-2n}, P'); \forall x \in M \times T^{2m-2n} \Rightarrow p_2(g(x)) = p_2(x);\\ &\forall x, y \in M \times T^{2m-2n}, p_1(x) = p_1(y) \Rightarrow p_1(g(x)) = p_1(g(y))\}\\ = &\bigcap_{x \in M \times T^{2m-2n}} \{g \in \operatorname{Aut}(M \times T^{2m-2n}, P'); p_2(g(x)) = p_2(x)\}\\ \cap &\bigcap_{x,y \in M \times T^{2m-2}, p_1(x) = p_1(y)} \{g \in \operatorname{Aut}(M \times T^{2m-2n}, P'); p_1(g(x)) = p_1(g(y))\}\end{aligned}$$

where $p_1 : M \times T^{2m-2n} \to M$ and $p_2 : M \times T^{2m-2n} \to T^{2m-2n}$ denote the projections on the first and second factor.

Since the map

$$\operatorname{Aut}(M \times T^{2m-2n}, P') \to M \times T^{2m-2n} \qquad g \mapsto g(x)$$

is continuous for all $x \in M \times T^{2m-2n}$ it follows that $\operatorname{Aut}(M, P)$ is a closed subgroup of $\operatorname{Aut}(M \times T^{2m-2n}, P')$. But by [**35**, p. 19] $\operatorname{Aut}(M \times T^{2m-2n}, P')$ is a finite dimensional Lie-group. Therefore $\operatorname{Aut}(M, P)$ is a finite dimensional Lie-group. □

5.2. Stable almost complex structures and weakly equivariant diffeomorphism

The purpose of this section is to prove the following theorem.

THEOREM 5.5. *Let M, M' be torus manifolds endowed with T-equivariant stable almost complex structures P, P'. If there is a diffeomorphism $\phi : M \to M'$ preserving these structures then there is a diffeomorphism $\psi : M \to M'$ preserving the stable almost complex structures and an automorphism $\gamma : T \to T$ with*

$$\psi(tx) = \gamma(t)\psi(x)$$

for all $t \in T$ and $x \in M$.

PROOF. The T-actions on M and M' induce group homomorphisms $\gamma_1 : T \to \operatorname{Aut}(M, P)$ and $\gamma_2 : T \to \operatorname{Aut}(M', P')$. We denote the images of this homomorphisms by T_M and $T_{M'}$. Then T_M and $T_{M'}$ are maximal tori in a maximal compact subgroup of $\operatorname{Aut}^0(M, P)$ and $\operatorname{Aut}^0(M', P')$, respectively. Here $\operatorname{Aut}^0(*, *)$ denotes the identity component of $\operatorname{Aut}(*, *)$.

Furthermore there is an isomorphism

$$\phi_* : \operatorname{Aut}^0(M, P) \to \operatorname{Aut}^0(M', P') \qquad f \mapsto \phi \circ f \circ \phi^{-1}$$

By [**31**, p. 530] and [**9**, p. 159] $T_{M'}$ and $\phi_* T_M$ are conjugated in $\operatorname{Aut}^0(M', P')$, that means there is a $\psi' \in \operatorname{Aut}^0(M', P')$ with $T_{M'} = \psi' \phi T_M \phi^{-1} \psi'^{-1}$.

Let

$$\gamma(t) = \gamma_2^{-1}(\psi' \phi \gamma_1(t) \phi^{-1} \psi'^{-1})$$

for all $t \in T$ and $\psi = \psi' \circ \phi$. Then we have

$$\gamma(t)\psi(x) = \psi(\gamma_1(t)(x)) = \psi(tx).$$

□

REMARK 5.6. There are torus manifolds which do not possess a T-equivariant stable almost complex structure (see section 5.4). But on quasitoric manifolds such structures always exist [**13**, p. 446].

5.3. Stable almost complex structures and non-abelian Lie-groups

In this section we discuss some properties of Lie-group actions on torus manifolds which preserve a given stable almost complex structure.

LEMMA 5.7. *Let M be a torus manifold endowed with a T-equivariant stable almost complex structure P. If the compact connected non-abelian Lie-group G acts effectively on M preserving P then there is a compact connected non-abelian Lie-group G', such that M is a torus-manifold with G'-action.*

Furthermore there is an homomorphism $\iota : G \to G'$ and a diffeomorphism ϕ of M such that for $x \in M$ and $g \in G$

$$\iota(g)x = \phi(g\phi^{-1}(x)).$$

PROOF. The actions of G and T on M induce homomorphisms $\iota' : G \to \operatorname{Aut}^0(M, P)$ and $T \to \operatorname{Aut}^0(M, P)$. Denote the images of these homomorphisms by G_M and T_M.

Let $G_M \subset G'' \subset \operatorname{Aut}^0(M, P)$ and $T_M \subset G' \subset \operatorname{Aut}^0(M, P)$ be maximal compact Lie-groups in $\operatorname{Aut}^0(M, P)$. By [**31**, p. 530] G'' and G' are connected and conjugated in $\operatorname{Aut}^0(M, P)$. In particular there is a $\phi \in \operatorname{Aut}^0(M, P)$ such that $\phi G_M \phi^{-1} \subset G'$. With

$$\iota : G \xrightarrow{\iota'} G_M \xrightarrow{\phi * \phi^{-1}} G'$$

the claim follows. □

LEMMA 5.8. *Let G be a compact connected Lie-group and M a torus manifold with G-action. Furthermore assume that M possesses a stable almost complex structure which is preserved by the G-action. Then all elementary factors of G are isomorphic to $SU(l_i + 1)$.*

PROOF. At first assume that there is an elementary factor G_1 of G which is isomorphic to $SO(2l_1 + 1)$, $l_1 \geq 2$, or $SO(2l_1)$, $l_1 \geq 2$. Let $x \in M^T$. Then we have $G_{1x} \cong SO(2l_1)$ and by Lemmas 4.10 and 4.13

$$T_xM = W \oplus V,$$

where W is the standard real representation of G_{1x} and V a trivial G_{1x}-representation of even dimension.

By assumption $T_xM \oplus V'$, where V' is a trivial even-dimensional G_{1x}-representation, is a complex G_{1x}-representation. Therefore $W \oplus V \oplus V'$ is contained in the image of the natural homomorphism

$$\Phi : R(G_{1x}, \mathbb{C}) \to R(G_{1x}, \mathbb{R}).$$

This contradicts the fact that W is not contained in the image of Φ and $V \oplus V'$ is contained in the image of Φ.

By Convention 4.14 we have to exclude also the case $G_1 = SO(3)$ and $\#\mathfrak{F}_1 = 1$. Assume that this case occurs. Let $N \in \mathfrak{F}_1$ and $x \in M^T$. Then we have by the discussion before Convention 4.14:

$$x \in N \subset M^{G_{1x}} \qquad\qquad G_{1x} \cong SO(2)$$

and

$$T_xM = N_x(N, M) \oplus T_xN = N_x(N, M) \oplus V$$

as G_{1x}-representations. Here V is again a trivial even-dimensional G_{1x}-representation. Let $g \in N_{G_1}G_{1x} - G_{1x}$. Then $gx \in N$.

Because $N(N,M)$ is orientable we have $N_x(N,M) = N_{gx}(N,M)$ as complex G_{1x}-representations. Because the G_1-action on M preserves the stable almost complex structure

$$g^{-1} : T_{gx}M \oplus V' \to T_xM \oplus V'$$

is complex linear and for $h \in G_{1x}$, $y \in T_{gx}M \oplus V'$ we have

$$hg^{-1}y = g^{-1}(ghg^{-1})y.$$

Therefore we have

$$g^*(T_{gx}M \oplus V') = T_xM \oplus V'$$

as complex G_{1x}-representations. Here, for a G_{1x}-representation W, g^*W denotes the representation of G_{1x} corresponding to the group homomorphism

$$G_{1x} \xrightarrow{g*g^{-1}} G_{1x} \xrightarrow{\phi} \mathrm{Aut}(W)$$

where ϕ is the homomorphism corresponding to the representation W. It follows that

$$\begin{aligned} g^*(N_x(N,M) \oplus V \oplus V') &= g^*N_x(N,M) \oplus V \oplus V' \\ &= g^*N_{gx}(N,M) \oplus V \oplus V' \\ &= N_x(N,M) \oplus V \oplus V' \end{aligned}$$

This contradicts the fact that $N_x(N,M)$ is a non-trivial complex one-dimensional representation of G_{1x}. Now the statement follows from Lemmas 4.10 and 4.13. □

5.4. A torus manifold which does not admit an equivariant stable almost complex structure

In this section we prove the following theorem.

THEOREM 5.9. *There is a 12-dimensional torus manifold M which does not admit an equivariant stable almost complex structure.*

PROOF. Let M_0 be the homogeneous space $G_2/SO(4)$. Then there is an effective action of a two-dimensional torus on M_0. Furthermore M_0 does not admit any stable almost complex structure [**57**].

Let $M_1 = M_0 \times T^4$. Then M_1 is a 12-dimensional manifold with an effective action of a six-dimensional torus. This action does not have a fixed point. Therefore M_1 is not a torus manifold. Let $\iota : M_0 \to M_1$, $\iota(x) = (x, 1)$ be an inclusion. Then ι^*TM_1 is stably isomorphic to TM_0. Therefore M_1 does not admit a stable almost complex structure.

Let M_2 be a 12-dimensional torus manifold, for example $M_2 = \mathbb{C}P^6$. Let $(\mathring{D}^6 \times T^6)_i$ be equivariant open tubular neighbourhoods of principal orbits in M_i, $i = 1, 2$. We glue the complements of $(\mathring{D}^6 \times T^6)_i$ in M_i to get a torus manifold

$$M = \left(M_1 - (\mathring{D}^6 \times T^6)_1\right) \cup_{S^5 \times T^6} \left(M_2 - (\mathring{D}^6 \times T^6)_2\right).$$

Assume that M admits an equivariant stable almost complex structure. Then $M_1 - (\mathring{D}^6 \times T^6)_1$ admits an equivariant stable almost complex structure.

The restriction of the tangent bundle of M_1 to $S^5 \times T^6$ may be trivialised in such a way that for all $t \in T^6$ and $(x, y, z) \in S^5 \times T^6 \times \mathbb{R}^{12} = TM_1|_{S^5 \times T^6}$ we have

$$t(x, y, z) = (x, ty, z).$$

Let P be the frame bundle of the stable tangent bundle of M_1. Then we have $P|_{S^5 \times T^6} = S^5 \times T^6 \times Gl(\mathbb{R}, 2m)$ with m large. T^6 acts in the following way on $P/Gl(\mathbb{C}, m)|_{S^5 \times T^6} = S^5 \times T^6 \times Gl(\mathbb{R}, 2m)/Gl(\mathbb{C}, m)$:

$$t(x, y, z) = (x, ty, z) \quad ((x, y, z) \in P/Gl(\mathbb{C}, m)|_{S^5 \times T^6}, t \in T^6). \tag{5.1}$$

Because there is an equivariant stable almost complex structure on $M_1 - (\mathring{D}^6 \times T^6)_1$ there is an equivariant section

$$\sigma : M_1 - (\mathring{D}^6 \times T^6)_1 \to P/Gl(\mathbb{C}, m)|_{M_1 - (\mathring{D}^6 \times T^6)_1}.$$

Because $P/Gl(\mathbb{C}, m)|_{S^5 \times T^6}$ is a trivial $Gl(\mathbb{R}, 2m)/Gl(\mathbb{C}, m)$-bundle over $S^5 \times T^6$ such that the T-action is given by (5.1) there is a map $g : S^5 \to Gl(\mathbb{R}, 2m)/Gl(\mathbb{C}, m)$ such that

$$\sigma(x, y) = (x, y, g(x)) \ \ ((x, y) \in S^5 \times T^6).$$

Because $\pi_5(Gl(\mathbb{R}, 2m)/Gl(\mathbb{C}, m)) = \pi_5(O/U) = 0$ this map g may be extended to a map $\tilde{g} : D^6 \to Gl(\mathbb{R}, 2m)/Gl(\mathbb{C}, m)$. Because the tangent bundle of $D^6 \times T^6$ is trivial $\tilde{g}$ may be used to extend σ to a section of $P/Gl(\mathbb{C}, m)$. This contradicts the fact that there is no stable almost complex structure on M_1.

Therefore there is no equivariant stable almost complex structure on M. □

COROLLARY 5.10. *Let $n \in \mathbb{N}$, $n \geq 6$. Then there is a torus manifold of dimension $2n$ which does not admit an equivariant stable almost complex structure.*

PROOF. Let M as in Theorem 5.9 and M' a torus manifold of dimension $2n-12$. Let $x \in M'^T$. Consider the inclusion $\iota : M \to M \times M'$, $\iota(y) = (y, x)$. Then TM and $\iota^* T(M \times M')$ are stably equivariantly isomorphic. Therefore an equivariant stable almost complex structure on $M \times M'$ induces an equivariant stable almost complex structure on M. Because M does not admit such a structure there is no equivariant stable almost complex structure on $M \times M'$. □

CHAPTER 6

Quasitoric manifolds homeomorphic to homogeneous spaces

In [**36**] Kuroki studied quasitoric manifolds M which admit an extension of the torus action to an action of some compact connected Lie-group G such that $\dim M/G \leq 1$. Here we drop the condition that the G-action extends the torus action in the case where the first Pontrjagin-class of M vanishes. We have the following two results.

Theorem 6.1. *Let M be a quasitoric manifold which is homeomorphic (or diffeomorphic) to a homogeneous space G/H with G a compact connected Lie-group and has vanishing first Pontrjagin-class. Then M is homeomorphic (diffeomorphic) to $\prod S^2$.*

Theorem 6.2. *Let M be a quasitoric manifold with $p_1(M) = 0$. Assume that the compact connected Lie-group G acts smoothly and almost effectively on M such that $\dim M/G = 1$. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2-bundle over a product of two-spheres.*

In this chapter all cohomology groups are taken with coefficients in $\mathbb{Q}$. The proofs of these theorems are based on Hauschild's study [**27**] of spaces of q-type. A space of *q*-type is defined to be a topological space X satisfying the following cohomological properties:

- The cohomology ring $H^*(X)$ is generated as a $\mathbb{Q}$-algebra by elements of degree two, i.e. $H^*(X) = \mathbb{Q}[x_1, \dots, x_n]/I_0$ and $\deg x_i = 2$.
- The defining ideal I_0 contains a definite quadratic form Q.

The chapter is organised as follows. In section 6.1 we establish some properties of rationally elliptic spaces. In section 6.2 we show that a quasitoric manifold with vanishing first Pontrjagin-class is of *q*-type. In section 6.3 we prove Theorem 6.1. In section 6.4 we recall some properties of cohomogeneity one manifolds. In section 6.5 we prove Theorem 6.2.

6.1. Rationally elliptic quasitoric manifolds

A simply connected space X is called rationally elliptic if it satisfies

$$\dim_{\mathbb{Q}} H_*(X) < \infty \qquad\qquad \dim_{\mathbb{Q}} \pi_*(X) \otimes \mathbb{Q} < \infty.$$

Examples of rationally elliptic spaces are simply connected homogeneous spaces and simply connected closed manifolds admitting a smooth action by a compact Lie-group with a codimension one orbit [**20**]. For more examples of rationally elliptic spaces see [**17**]. In this section we discuss some properties of rationally elliptic quasitoric manifolds.

Lemma 6.3. *Let M_1, M_2 be quasitoric manifolds over the same polytope P. Then M_1 is rationally elliptic if and only if M_2 is rationally elliptic.*

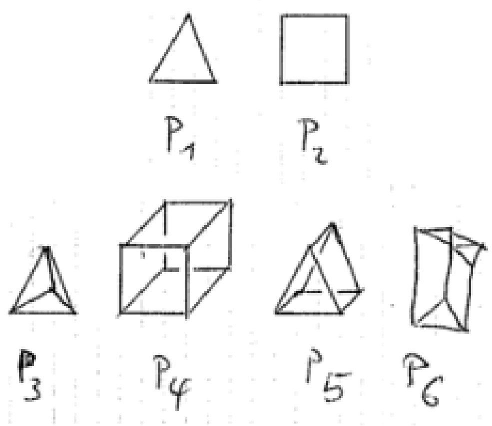


FIGURE 1. Some polytopes

PROOF. Because $\dim_{\mathbb{Q}} \pi_*(BT) \otimes \mathbb{Q} < \infty$ it follows from the exact homotopy sequence for the fibration $M_i \to M_{iT} \to BT$ that M_i is rationally elliptic if and only if $\dim_{\mathbb{Q}} \pi_*(M_{iT}) \otimes \mathbb{Q} < \infty$. But by [**13**, p. 434] the homotopy type of M_{iT} depends only on the combinatorial type of P. Therefore M_1 is rationally elliptic if and only if M_2 is rationally elliptic. □

COROLLARY 6.4. *Let M be a quasitoric manifold over a product P of simplices. Then M is rationally elliptic.*

PROOF. There is a product M' of complex projective spaces such that M' has P as its orbit polytope. Because a complex projective space is rationally elliptic M' is rationally elliptic. Now the statement follows from Lemma 6.3 □

LEMMA 6.5. *Let M be a rationally elliptic quasitoric manifold over the n-dimensional polytope P. Then the number m of facets of P is smaller or equal to $2n$.*

PROOF. Because $\chi(M) > 0$ we have

$$\dim_{\mathbb{Q}} \pi_{\text{odd}}(M) \otimes \mathbb{Q} = \dim_{\mathbb{Q}} \pi_{\text{even}}(M) \otimes \mathbb{Q}$$

by [**18**, p. 447-448]. Furthermore we have $\dim_{\mathbb{Q}} \pi_*(M) \otimes \mathbb{Q} \leq 2n$. With the Hurewicz-isomorphism and Theorem 3.1 of [**13**, p. 430] it follows that

$$\begin{aligned} 2(m-n) &= 2\dim_{\mathbb{Q}} \pi_2(M) \otimes \mathbb{Q} \leq 2\dim_{\mathbb{Q}} \pi_{\text{even}}(M) \otimes \mathbb{Q} \\ &= \dim_{\mathbb{Q}} \pi_*(M) \otimes \mathbb{Q} \leq 2n. \end{aligned}$$

This implies $2m \leq 4n$. □

REMARK 6.6. The bound given in Lemma 6.5 is sharp because a product of n copies of S^2 is a rationally elliptic quasitoric manifold over I^n which has $2n$ facets.

COROLLARY 6.7. *Let M be a quasitoric manifold over the n-dimensional polytope P. If $n \leq 3$ then M is rationally elliptic if and only if P is a product of simplices.*

PROOF. At first assume that M is rational elliptic. If $n = 2$ then by Lemma 6.5 and [**22**, p. 98] P is P_1 or P_2, as drawn in Figure 1, which are both products of simplices.

If $n = 3$ then by Lemma 6.5 and [**22**, p. 113] P is P_3, P_4, P_5 or P_6, as drawn in Figure 1. The first three polytopes are products of simplices. $M' = (\mathbb{C}P^2 \times \mathbb{C}P^1)\#\mathbb{C}P^3$ is a quasitoric manifold over P_6. By [**40**, p. 206] and [**58**, p. 416] M' is not rationally elliptic. With Lemma 6.3 and Corollary 6.4 the statement follows. □

LEMMA 6.8. *Let M be a quasitoric manifold and N a characteristic submanifold of M. If M is rationally elliptic then N is rationally elliptic.*

PROOF. This follows from the fact that the components of the fixed point set of a smooth torus action on a rationally elliptic manifold are rationally elliptic [**1**, p. 155]. □

COROLLARY 6.9. *Let M be a rationally elliptic quasitoric manifold over the polytope P. Then all two and three dimensional faces of P are products of simplices.*

6.2. Quasitoric manifolds with vanishing first Pontrjagin-class

In this section we study quasitoric manifolds with vanishing first Pontrjagin-class. To do so we first introduce some notations from [**27**] and [**29**, Chapter VII]. For a topological space X we define the topological degree of symmetry of X as

$$N_t(X) = \max\{\dim G; G \text{ compact Lie-group, } G \text{ acts effectively on } X\}$$

Similarly one defines the semi-simple degree of symmetry of X as

$$N_t^{ss}(X) = \max\{\dim G; G \text{ compact semi-simple Lie-group, } G \text{ acts effectively on } X\}$$

and the torus-degree of symmetry as

$$T_t(X) = \max\{\dim T; T \text{ torus, } T \text{ acts effectively on } X\}.$$

In the above definitions we assume that all groups act continuously.

Another imported invariant of a topological space X used in [**27**] is the so called embedding dimension of its rational cohomology ring. For a local $\mathbb{Q}$-algebra A we denote by $\operatorname{edim} A$ the embedding dimension of A. By definition we have $\operatorname{edim} A = \dim_{\mathbb{Q}} \mathfrak{m}_A/\mathfrak{m}_A^2$ where $\mathfrak{m}_A$ is the maximal ideal of A. In case that $A = \bigoplus_{i\geq 0} A^i$ is a positively graded local $\mathbb{Q}$-algebra $\mathfrak{m}_A$ is the augmentation ideal $A_+ = \bigoplus_{i>0} A^i$. If furthermore A is generated by its degree two part then $\mathfrak{m}_A^2 = \bigoplus_{i>2} A^i$. Therefore for a quasitoric manifold M over the polytope P we have $\operatorname{edim} H^*(M) = \dim_{\mathbb{Q}} H^2(M) = m - n$ where m is the number of facets of P and n is its dimension.

LEMMA 6.10. *Let M be a quasitoric manifold with $p_1(M) = 0$. Then M is a manifold of q-type.*

PROOF. The discussion at the beginning of section 3.1 together with Corollary 6.8 of [**13**, p. 448] shows that there are a basis $u_{n+1}, \dots, u_m$ of $H^2(M)$ and $\lambda_{i,j} \in \mathbb{Z}$ such that

$$0 = p_1(M) = \sum_{i=n+1}^{m} u_i^2 + \sum_{j=1}^{n} \left(\sum_{i=n+1}^{m} \lambda_{i,j} u_i \right)^2 .$$

Because

$$\sum_{i=n+1}^{m} X_i^2 + \sum_{j=1}^{n} \left(\sum_{i=n+1}^{m} \lambda_{i,j} X_i \right)^2$$

is a positive definite bilinear form the statement follows. □

REMARK 6.11. The above lemma also holds if we assume that $p_1(M)$ does not vanish but is equal to $-\sum_i a_i^2$ for some $a_i \in H^2(M)$.

Corollary 6.12. *Let M be a quasitoric manifold of q-type over the n-dimensional polytope P. Then we have for the number m of facets of P:*

$$m \geq 2n$$

Proof. By Theorem 3.2 of [**27**, p. 563] we have

$$n \leq T_t(M) \leq \operatorname{edim} H^*(M) = m - n.$$

Therefore we have $2n \leq m$. □

Remark 6.13. The inequality in the above lemma is sharp, because for $M = S^2 \times \cdots \times S^2$ we have $m = 2n$ and $p_1(M) = 0$.

The following corollary follows with Theorem 5.13 of [**27**, p. 573].

Corollary 6.14. *Let M as in Corollary 6.12. Then we have*

$$N_t^{ss}(M) \leq 2n + m - n = n + m.$$

Remark 6.15. The inequality in the above corollary is sharp because for $M = S^2 \times \cdots \times S^2$ we have $m = 2n$ and $SU(2) \times \cdots \times SU(2)$ acts on M and has dimension $3n$.

6.3. Quasitoric manifolds which are also homogeneous spaces

In this section we prove Theorem 6.1. Recall from Lemma 6.10 that a quasitoric manifold with vanishing first Pontrjagin-class is a manifold of q-type.

Let M be a quasitoric manifold over the polytope P which is also a homogeneous space and is of q-type. Then by Lemma 6.5 and Corollary 6.12 the number of facets of P is equal to $2n$ where n is the dimension of P. Therefore by Corollary 6.14 we have $N_t^{ss}(M) \leq 3n$.

Let G be a compact connected Lie-group and $H \subset G$ a closed subgroup such that M is homeomorphic or diffeomorphic to G/H. Because $\chi(M) > 0$ and M is simply connected we have $\operatorname{rank} G = \operatorname{rank} H$ and H is connected. Therefore we may assume that G is semi-simple and simply connected. This implies $\dim G \leq 3n$.

Let T be a maximal torus of G then $(G/H)^T$ is non-empty. Therefore it follows from Theorem 5.9 of [**27**, p. 572] that H is a maximal torus of G.

Now it follows from Theorem 3.3 of [**27**, p. 563] that

$$n \leq T_t(G/H) = \operatorname{rank} G$$

and therefore

$$\dim G \leq 3 \operatorname{rank} G.$$

For a simple simply connected Lie-group G' we have $\dim G' \geq 3 \operatorname{rank} G'$ and $\dim G' = 3 \operatorname{rank} G'$ if and only if $G' = SU(2)$. Therefore we have $G = \prod SU(2)$ and $M = \prod SU(2)/T^1 = \prod S^2$. This proves Theorem 6.1.

6.4. Cohomogeneity one manifolds

Here we discuss some facts about closed cohomogeneity one Riemannian G-manifolds M with orbit space a compact interval $[-1, 1]$. We follow [**21**, p. 39-44] in this discussion.

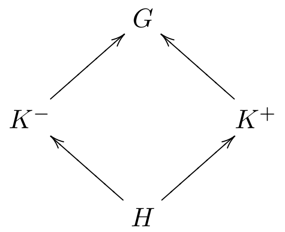
We fix a normal geodesic $c : [-1, 1] \to M$ perpendicular to all orbits. We denote by H the principal isotropy group $G_{c(0)}$, which is equal to the isotropy group $G_{c(t)}$ for $t \in]-1, 1[$, and by $K^{\pm}$ the isotropy groups of $c(\pm 1)$.

Then M is the union of tabular neighbourhoods of the non-principal orbits $Gc(\pm 1)$ glued along their boundary, i.e., by the slice theorem we have

$$M = G \times_{K^-} D_- \cup G \times_{K^+} D_+, \tag{6.1}$$

where $D_{\pm}$ are discs. Furthermore $K^{\pm}/H = \partial D_{\pm} = S_{\pm}$ are spheres.

Note that the diagram of groups



determines M. Conversely such a group diagram with $K^{\pm}/H = S_{\pm}$ spheres defines a cohomogeneity one G-manifold. We also write these diagrams as $H \subset K^-, K^+ \subset G$.

Now we give a criterion for two group diagrams yielding up to G-equivariant diffeomorphism the same manifold M.

LEMMA 6.16. *The group diagrams $H \subset K^-, K_1^+ \subset G$ and $H \subset K^-, K_2^+ \subset G$ yield the same cohomogeneity one manifold up to equivariant diffeomorphism if there is a $a \in N_G(H)^0$ with $K_1^+ = aK_2^+a^{-1}$.*

6.5. Quasitoric manifolds with cohomogeneity one actions

In this section we study quasitoric manifolds M which admit a smooth action of a compact connected Lie-group G which has an orbit of codimension one. As before we do not assume that the G-action on M extends the torus action. We have the following lemma:

LEMMA 6.17. *Let M be a quasitoric manifold of dimension $2n$ which is of q-type. Assume that the compact connected Lie-group G acts almost effectively and smoothly on M such that $\dim M/G = 1$. Then we have:*

1. *The singular orbits are given by G/T where T is a maximal torus of G.*
2. *The Euler-characteristic of M is $2\#W(G)$.*
3. *The principal orbit type is given by G/S where $S \subset T$ is a subgroup of codimension one.*
4. *The center Z of G has dimension at most one.*
5. $\dim G/T = 2n - 2$.

PROOF. At first note that M/G is an interval $[-1, 1]$ and not a circle because M is simply connected. We start with proving (1). Let T be a maximal torus of G. Without loss of generality we may assume $G = G' \times Z'$ with G' a compact connected semi-simple Lie-group and Z' a torus. Let $x \in M^T$. Then the isotropy group G_x has maximal rank in G. Therefore G_x splits as $G'_x \times Z'$.

By Theorem 5.9 of [**27**, p. 572] G'_x is a maximal torus of G'. Therefore we have $G_x = T$.

Because $\dim G - \dim T$ is even, x is contained in a singular orbit. In particular we have

$$\chi(M) = \chi(M^T) = \chi(G/K^+) + \chi(G/K^-), \tag{6.2}$$

where $G/K^{\pm}$ are the singular orbits. Furthermore we may assume that G/K^+ contains a T-fixed point. This implies

$$\chi(G/K^+) = \chi(G/T) = \#W(G) = \#W(G'). \tag{6.3}$$

Now assume that all T-fixed points are contained in the singular orbit G/K^+. Then we have $(G/K^-)^T = \emptyset$. This implies

$$\chi(M) = \chi(G/K^+) = \#W(G').$$

Now Theorem 5.11 of [**27**, p. 573] implies that M is the homogeneous space $G'/G' \cap T = G/T$. This contradicts our assumption that $\dim M/G = 1$.

Therefore both singular orbits contain T-fixed points. This implies that they are of type G/T. This proves (1). (2) follows from (6.2) and (6.3).

Now we prove (3) and (5). Let $S \subset T$ be a minimal isotropy group. Then T/S is a sphere of dimension $\operatorname{codim}(G/T, M) - 1$. Therefore S is a subgroup of codimension one in T and $\operatorname{codim}(G/T, M) = 2$.

If the center of G has dimension greater than one then $\dim Z' \cap S \geq 1$. That means that the action is not almost effective. Therefore (4) holds. □

By Lemma 6.17 we have with the notation of the previous section that $K^{\pm}$ are maximal tori of G containing $H = S$. In the following we will write $G = G' \times Z'$ with G' a compact connected semi-simple Lie-group and Z' a torus.

LEMMA 6.18. *Let M and G as in the previous lemma. Then we have*

$$T_t(M) \leq \operatorname{rank} G' + 1.$$

PROOF. At first we recall the rational cohomology of G/T. By [**6**, p. 67] we have

$$H^*(G/T) \cong H^*(BT)/I$$

where I is the ideal generated by the elements of positive degree which are invariant under the action of the Weyl-group of G. Therefore it follows that

$$\dim_{\mathbb{Q}} H^1(G/T) = 0 \qquad \dim_{\mathbb{Q}} H^2(G/T) = \operatorname{rank} G'.$$

There is a S^1-bundle $G/S \to G/T$. We consider two cases:

- the rational Euler-class χ of this bundle vanishes
- the rational Euler-class χ of this bundle is non-zero.

At first we assume that χ vanishes. Then we have the following Gysin-sequence:

$$0 \longrightarrow H^1(G/S) \longrightarrow H^0(G/T) \xrightarrow{0} H^2(G/T) \longrightarrow H^2(G/S) \longrightarrow 0$$

Therefore we have

$$\dim_{\mathbb{Q}} H^1(G/S) = 1 \qquad \dim_{\mathbb{Q}} H^2(G/S) = \dim_{\mathbb{Q}} H^2(G/T).$$

Now we look at the Mayer-Vietoris-sequence induced by the decomposition (6.1) of M:

$$0 \longrightarrow H^1(G/S) \longrightarrow H^2(M) \longrightarrow H^2(G/T) \oplus H^2(G/T) \longrightarrow H^2(G/S) \longrightarrow 0$$

From this sequence we get

$$\begin{aligned} \dim_{\mathbb{Q}} H^2(M) - 1 &= 2 \dim_{\mathbb{Q}} H^2(G/T) - \dim_{\mathbb{Q}} H^2(G/S) \\ &= \dim_{\mathbb{Q}} H^2(G/T) = \operatorname{rank} G' \end{aligned}$$

and therefore

$$\dim_{\mathbb{Q}} H^2(M) = \operatorname{rank} G' + 1.$$

Because M is quasitoric $H^*(M)$ is generated by its degree two part. Therefore we have

$$T_t(M) \leq \operatorname{edim} H^*(M) = \dim_{\mathbb{Q}} H^2(M) = \operatorname{rank} G' + 1.$$

Now assume that χ does not vanish. Then we have the Gysin-sequence

$$0 \longrightarrow H^1(G/S) \longrightarrow H^0(G/T) \xrightarrow{\chi} H^2(G/T) \longrightarrow H^2(G/S) \longrightarrow 0$$

Here the map in the middle is injective. Therefore we get

$$\dim_{\mathbb{Q}} H^1(G/S) = 0 \qquad \dim_{\mathbb{Q}} H^2(G/S) = \dim_{\mathbb{Q}} H^2(G/T) - 1.$$

Now the Mayer-Vietoris-sequence induced by the decomposition (6.1) of M:

$$0 \longrightarrow H^2(M) \longrightarrow H^2(G/T) \oplus H^2(G/T) \longrightarrow H^2(G/S) \longrightarrow 0$$

implies

$$\begin{aligned}\dim_{\mathbb{Q}} H^2(M) &= 2\dim_{\mathbb{Q}} H^2(G/T) - \dim_{\mathbb{Q}} H^2(G/S)\\ &= \dim_{\mathbb{Q}} H^2(G/T) + 1 = \operatorname{rank} G' + 1.\end{aligned}$$

As in the first case we see $T_t(M) \leq \operatorname{rank} G' + 1$. □

THEOREM 6.19. *Let M and G as in the previous lemmas. Then G has a finite covering group of the form $\prod SU(2)$ or $\prod SU(2) \times S^1$. Furthermore M is diffeomorphic to a S^2-bundle over a product of two-spheres.*

PROOF. Because M is quasitoric we have $n \leq T_t(M)$. By Lemma 6.17 we have

$$\dim G' - \operatorname{rank} G' = \dim G/T = 2n - 2.$$

Now Lemma 6.18 implies

$$\dim G' = 2n - 2 + \operatorname{rank} G' \leq 3 \operatorname{rank} G'.$$

Therefore $\prod SU(2)$ is a finite covering group of G'. This implies the statement about the finite covering group of G.

Because $K^{\pm}$ are maximal tori of the identity component $Z_G(S)^0$ of the centraliser of S, there is some $a \in Z_G(S)^0$ such that $K^- = aK^+a^{-1}$. By Lemma 6.16 we may assume that $K^+ = K^- = T$. Now from Theorem 4.1 of [**52**, p. 198] it follows that M is a fiber bundle over G/T with fiber the cohomogeneity one manifold with group diagram $S \subset T, T \subset T$. Therefore it is a S^2-bundle over $\prod S^2$. □

Now Theorem 6.2 follows from Theorem 6.19 and Lemma 6.10.

APPENDIX A

Generalities on Lie-groups and torus manifolds

A.1. Lie-groups

LEMMA A.1. *Let $l > 1$. Then $S(U(l) \times U(1))$ is a maximal subgroup of $SU(l+1)$.*

PROOF. Let H be a subgroup of $SU(l+1)$ with $S(U(l)\times U(1)) \subset H \subsetneq SU(l+1)$.

Because $S(U(l) \times U(1))$ is a maximal connected subgroup of $SU(l + 1)$ the identity component of H has to be $S(U(l) \times U(1))$. Therefore H is contained in the normaliser of $S(U(l) \times U(1))$. Because

$$N_{SU(l+1)}S(U(l) \times U(1))/S(U(l) \times U(1)) \\ = (SU(l+1)/S(U(l) \times U(1)))^{S(U(l)\times U(1))} = \left(\mathbb{C}P^l\right)^{S(U(l)\times U(1))}$$

is just one point $H = S(U(l) \times U(1))$ follows. □

LEMMA A.2. *Let $\psi : S(U(l)\times U(1)) \to S^1$ be a non-trivial group homomorphism and*

$$H_0 = SU(l + 1) \times S^1,$$
$$H_1 = S(U(l) \times U(1)) \times S^1,$$
$$H_2 = \{(g, \psi(g)), g \in S(U(l) \times U(1))\}.$$

Then H_1 is the only connected proper closed subgroup of H_0 which contains H_2 properly.

PROOF. Let $H_2 \subset H \subset H_0$ be a closed connected subgroup. Then we have

$$\operatorname{rank} H_0 \geq \operatorname{rank} H \geq \operatorname{rank} H_2 = \operatorname{rank} H_0 - 1.$$

At first assume that $\operatorname{rank} H = \operatorname{rank} H_0$. Then we have by [**46**, p. 297]

$$H = H' \times S^1,$$

where H' is a subgroup of maximal rank of $SU(l+1)$. Let $\pi_1 : H_0 \to SU(l+1)$ be the projection on the first factor. Because $H' = \pi_1(H) \supset \pi_1(H_2) = S(U(l) \times U(1))$ we have by A.1 that $H = H_1, H_0$.

Now assume that $\operatorname{rank} H = \operatorname{rank} H_2$. Then there is a non-trivial group homomorphism $H \to S^1$. Therefore locally H is a product $H' \times S^1$ where H' is a simple group which contains $SU(l)$ as a maximal rank subgroup. By [**7**, p. 219] we have

$$H' = E_7, E_8, G_2, SU(l)$$

If $H' = SU(l)$ then we have $H = H_2$. Therefore we have to show that the other cases do not occur. These groups have the following dimensions:

l	$\dim H_0$	$\dim H' \times S^1$
8	81	$\dim E_7 \times S^1 = 134$
9	100	$\dim E_8 \times S^1 = 249$
3	16	$\dim G_2 \times S^1 = 15$

Therefore the first two cases do not occur. Because there is no G_2-representation of dimension less than seven the third case does not occur. □

LEMMA A.3. *Let T be a torus and $\psi_1, \psi_2 : S(U(l) \times U(1)) \to T$ be two group homomorphisms. Furthermore let for $i = 1, 2$*

$$H_i = \{(g, \psi_i(g)) \in SU(l+1) \times T; g \in S(U(l) \times U(1))\}$$

be the graph of ψ_i.

(1) *If $l > 1$ then H_1 and H_2 are conjugated in $SU(l+1) \times T$ if and only if $\psi_1 = \psi_2$.*
(2) *If $l = 1$ then H_1 and H_2 are conjugated in $SU(l+1) \times T$ if and only if $\psi_1 = \psi_2^{\pm 1}$.*

PROOF. At first assume that H_1 and H_2 are conjugated in $SU(l+1) \times T$. Let $g' \in SU(l+1) \times T$ such that

$$H_1 = g' H_2 g'^{-1}.$$

Because T is contained in the center of $SU(l+1) \times T$ we may assume that $g' = (g, 1) \in SU(l+1) \times \{1\}$. Let $\pi_1 : SU(l+1) \times T \to SU(l+1)$ be the projection on the first factor. Then:

$$S(U(l) \times U(1)) = \pi_1(H_1) = g\pi_1(H_2)g^{-1} = gS(U(l) \times U(1))g^{-1}$$

By Lemma A.1 it follows that

$$g \in N_{SU(l+1)}S(U(l) \times U(1)) = \begin{cases} S(U(l) \times U(1)) & \text{if } l > 1 \\ N_{SU(2)}S(U(1) \times U(1)) & \text{if } l = 1. \end{cases}$$

Now for $h \in S(U(l) \times U(1))$ we have

$$(h, \psi_1(h)) = g'(g^{-1}hg, \psi_1(h))g'^{-1}$$

Now $(g^{-1}hg, \psi_1(h))$ lies in H_2. Therefore we may write:

$$g'(g^{-1}hg, \psi_1(h))g'^{-1} = g'((g^{-1}hg, \psi_2(g^{-1}hg))g'^{-1} = (h, \psi_2(g^{-1}hg))$$

If $l > 1$ we have

$$\psi_2(g^{-1}hg) = \psi_2(g)^{-1}\psi_2(h)\psi_2(g) = \psi_2(h).$$

Otherwise we have

$$\psi_2(g^{-1}hg) = \psi_2(h^{\pm 1}) = \psi_2(h)^{\pm 1}.$$

The other implications are trivial. Therefore the statement follows. □

LEMMA A.4. *Let $l \geq 1$. $Spin(2l)$ is a maximal connected subgroup of $Spin(2l+1)$. Its normaliser consists out of two components.*

PROOF. By [**7**, p. 219] Spin$(2l)$ is a maximal connected subgroup of Spin$(2l+1)$.

$$N_{\mathrm{Spin}(2l+1)}\mathrm{Spin}(2l)/\mathrm{Spin}(2l) = (\mathrm{Spin}(2l+1)/\mathrm{Spin}(2l))^{\mathrm{Spin}(2l)} = \left(S^{2l}\right)^{\mathrm{Spin}(2l)}$$

consists out of two points. Therefore the second statement follows. □

LEMMA A.5. *Let G be a Lie-group, which acts on the manifold M. Furthermore let $N \subset M$ be a submanifold. If the intersection of Gx and N is transverse in x for all $x \in N$, then GN is open in M.*

PROOF. We will show that $f : G \times N \to M$ $(h,x) \mapsto hx$ is a submersion. Because a submersion is an open map it follows that $GN = f(G \times N)$ is open in M. For $g \in G$ let

$$\begin{aligned} l_g : G \times N &\to G \times N \\ (h,x) &\mapsto (gh,x) \end{aligned}$$

and

$$\begin{aligned} l'_g : M &\to M \\ x &\mapsto gx. \end{aligned}$$

Then we have for all $g \in G$

$$f = l'_g \circ f \circ l_{g^{-1}}.$$

Now for $(g,x) \in G \times N$ we have

$$D_{(g,x)} f = D_x l'_g \, D_{(e,x)} f \, D_{(g,x)} l_{g^{-1}}.$$

Because $D_{(e,x)}f$ is surjective by assumption and l'_g, $l_{g^{-1}}$ are diffeomorphisms, it follows that $D_{(g,x)}f$ is surjective. Therefore f is a submersion. □

A.2. Generalities on torus manifolds

LEMMA A.6. *Let M be a torus-manifold and $M_1, \dots, M_k$ pairwise distinct characteristic submanifolds of M with $N = M_1 \cap \dots \cap M_k \neq \emptyset$. Then the M_i intersect transversely. Therefore N is a submanifold of M with* $\operatorname{codim} N = 2k$ *and* $\dim\langle \lambda(M_1), \dots, \lambda(M_k)\rangle = k$. *Furthermore N is the union of components of $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$.*

PROOF. Let $x \in N$. Then we have

$$T_x M = \bigcap_{i=1}^{k} T_x M_i \oplus \bigoplus_j V_j,$$

where the V_j are one-dimensional complex $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$-representations. Since the M_i have codimension two in M each $\lambda(M_i)$ acts non-trivially on exactly one V_{j_i}.

If $\operatorname{codim} \bigcap_{i=1}^k T_x M_i < 2k$ then there are i_1 and i_2, such that $V_{j_{i_1}} = V_{j_{i_2}}$. Therefore

$$T_x M_{i_1} = T_x M_{i_2} = T_x M^{\langle \lambda(M_{i_1}), \lambda(M_{i_2})\rangle}$$

has codimension two.

Since $\langle \lambda(M_{i_1}), \lambda(M_{i_2})\rangle$ has dimension two, it does not act almost effectively on M. This is a contradiction. Therefore $\bigcap_{i=1}^k T_x M_i$ has codimension $2k$. This implies that the M_i, $i = 1, \dots, k$, intersect transversely. Therefore N is a submanifold of M of codimension $2k$.

If $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$ has dimension smaller than k then the weights of the V_j are linear dependent. Therefore there is $(a_1, \dots, a_k) \in \mathbb{Z}^k - \{0\}$, such that

$$\mathbb{C} = V_1^{a_1} \otimes \dots \otimes V_k^{a_k},$$

where $\mathbb{C}$ denotes the trivial $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$-representation. This gives a contradiction because each $\lambda(M_i)$ acts non-trivially on exactly one V_j.

Because $\langle \lambda(M_1), \dots, \lambda(M_k)\rangle$ has dimension k, $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$ has dimension at most $n-2k$. But N is contained in $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$ and has dimension $n-2k$. Therefore it is the union of components of $M^{\langle \lambda(M_1), \dots, \lambda(M_k)\rangle}$. □

Lemma A.7. *Let M be a torus manifold of dimension $2n$ and N a component of the intersection of $k(\leq n)$ characteristic submanifolds $M_1,\dots,M_k$ of M with $N^T\neq\emptyset$. Then N is a torus manifold and the characteristic submanifolds of N are given by the components of intersections of characteristic submanifolds $M_i\neq M_1,\dots,M_k$ of M with N which contain a T-fixed point.*

Proof. Let $M_i\neq M_1,\dots,M_k$ be a characteristic submanifold of M with $(M_i\cap N)^T\neq\emptyset$. Then by Lemma A.6 each component of $M_i\cap N$ which contains a T-fixed point has codimension two in N. That means that they are characteristic submanifolds of N.

Now let $N_1\subset N$ be a characteristic submanifold and $x\in N_1^T$. Then we have

$$T_xM=T_xN_1\oplus V_0\oplus N_x(N,M)$$

as T-representations with V_0 a one dimensional complex T-representation. Let M_i be the characteristic submanifold of M which corresponds to V_0. Then N_1 is the component of the intersection $M_i\cap N$ which contains x. □

Lemma A.8. *Let M be a $2n$-dimensional torus-manifold and T' a subtorus of T. If N is a component of $M^{T'}$ which contains a T-fixed point x then N is a component of the intersection of some characteristic submanifolds of M.*

Proof. By Lemma A.6 the intersection of the characteristic submanifolds $M_1,\dots M_k$ is a union of components of $M^{\langle\lambda(M_1),\dots,\lambda(M_k)\rangle}$.

Therefore we have to show that there are characteristic manifolds $M_1,\dots,M_k$ of M such that

$$T_xN=T_xM_1\cap\dots\cap M_k.$$

There are n characteristic submanifolds $M_1,\dots,M_n$ which intersect transversely in x. Therefore we have

$$T_xM=N_x(M_1,M)\oplus\dots\oplus N_x(M_n,M).$$

We may assume that there is a $1\leq k\leq n$ such that T' acts trivially on $N_x(M_i,M)$ for $i>k$ and non-trivially on $N_x(M_i,M)$ for $i\leq k$. Then we have

$$T_xN=(T_xM)^{T'}=N_x(M_{k+1},M)\oplus\dots\oplus N_x(M_n,M)=T_xM_1\cap\dots\cap M_k.$$

□

Lemma A.9. *Let M be a torus manifold with $T^n\times\mathbb{Z}_2$-action, such that $\mathbb{Z}_2$ acts non-trivially on M. Furthermore let $B\subset M$ be a submanifold of codimension one on which $\mathbb{Z}_2$ acts trivially and N the intersection of characteristic submanifolds $M_1,\dots,M_k$ of M. Then B and N intersect transversely.*

Proof. Let $x\in B\cap N$ then we have the $\langle\lambda(M_1),\dots,\lambda(M_k)\rangle\times\mathbb{Z}_2$-representation T_xM. It decomposes as the sum of the eigenspaces of the non-trivial element of $\mathbb{Z}_2$. Because B has codimension one the eigenspace to the eigenvalue -1 is one dimensional. Because the irreducible non-trivial torus representations are two-dimensional we have

$$\begin{aligned}T_xN=(T_xM)^{\langle\lambda(M_1),\dots,\lambda(M_k)\rangle}&=T_xM^{\langle\lambda(M_1),\dots,\lambda(M_k)\rangle\times\mathbb{Z}_2}\oplus N_x(B,M)^{\langle\lambda(M_1),\dots,\lambda(M_k)\rangle}\\&=T_xM^{\langle\lambda(M_1),\dots,\lambda(M_k)\rangle\times\mathbb{Z}_2}\oplus N_x(B,M).\end{aligned}$$

That means that the intersection is transverse. □

Lemma A.10. *Let M^{2n} be a $(\mathbb{Z}_2)_1\times(\mathbb{Z}_2)_2$-manifold such that $(\mathbb{Z}_2)_i$ acts non-trivially on M. Furthermore let $B_i\subset M$, $i=1,2$, connected submanifolds of codimension one such that $(\mathbb{Z}_2)_i$ acts trivially on B_i. Then the following statements are equivalent:*

(1) *B_1,B_2 intersect transversely*

(2) $B_1 \neq B_2$
(3) $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$ *acts effectively on* M *or* $B_1 \cap B_2 = \emptyset$

PROOF. Denote by V_i the non-trivial real irreducible representation of $(\mathbb{Z}_2)_i$. Let $x \in B_1 \cap B_2$. Then for the $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$-representation T_xM there are two possibilities:

$$T_xM = \begin{cases} \mathbb{R}^{2n-1} \oplus V_1 \otimes V_2 \\ \mathbb{R}^{2n-2} \oplus V_1 \oplus V_2 \end{cases}$$

In the first case B_i, $i = 1, 2$, is the component of $M^{(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2}$ containing x and $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$ acts non-effectively on M. In the second case $(\mathbb{Z}_2)_1 \times (\mathbb{Z}_2)_2$ acts effectively on M and B_1, B_2 intersect transversely in x.

All conditions given in the lemma imply that we are in the second case or $B_1 \cap B_2 = \emptyset$. Therefore they are equivalent. □

REMARK A.11. Lemmas A.6, A.9 also hold if we do not require that a characteristic manifold contains a T-fixed point.

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